Blowup and solitary wave solutions with ring profiles of two-component nonlinear Schrödinger systems

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Abstract

Blowup ring profiles have been investigated by finding non-vortex blowup solutions of nonlinear Schrödinger equations (NLSE) (cf. [5] and [6]). However, those solutions have infinite L^2 norm so one may not maintain the ring profile all the way up to the singularity. To find H^1 non-vortex blowup solutions with ring profiles, we study blowup solutions of two-component systems of NLSE with nonlinear coefficients β and ν_j, $j = 1, 2$. When $\beta < 0$ and $\nu_1 \gg \nu_2 > 0$, the two-component system can be transformed into a multi-scale system with fast and slow variables which may produce H^1 blowup solutions with non-vortex ring profiles. We use the localized energy method with symmetry reduction to construct these solutions rigorously. On the other hand, these solutions may describe steady non-vortex bright ring solitons. Various types of ring profiles including m-ring and ring-ring profiles are presented by numerical solutions.

Keywords: blowup, solitary wave, ring profile, two-component systems of NLSE

1 Introduction

Blowup solutions of nonlinear Schrödinger equations (NLSE) may describe nonlinear wave collapse which is universal to many areas of physics including nonlinear optics (cf. [12]), plasma physics (cf. [24]), and Bose-Einstein condensates (BEC) (cf. [25]). The spatial profile of a collapsing wave may evolve into a universal, self-similar, circularly symmetric shape with a single peak known as the Townes profile which has been observed experimentally by amplified laser beams (cf. [18]). Theoretically, one may find the Townes profile by investigating self-similar solutions of self-focusing cubic NLSE as follows:

$$\begin{cases} i\partial_t \Psi + \triangle \Psi + \nu |\Psi|^2 \Psi = 0\,, \\ \Psi = \Psi(x,t) \in \mathbb{C}\,, \quad x = (x_1, x_2) \in \mathbb{R}^2\,,\ t > 0\,, \\ \Psi(\cdot, t) \in H^1(\mathbb{R}^2)\,,\ t > 0\,, \end{cases} \tag{1.1}$$

where ν is a positive constant. It is well-known that the equation (1.1) has self-similar H^1 solutions with the Townes profile to express finite-time blowup behavior and have the singularity at $t = T$ (i.e. $\|\Psi(\cdot,t)\|_{L^\infty} < \infty$ for $0 < t < T$ and $\|\Psi(\cdot,t)\|_{L^\infty} \to \infty$ as $t \uparrow T < \infty$) (cf. [26]). Hence the Townes profile can be maintained all the way up to the singularity.

In high-power laser beams, different collapsing behaviors may develop blowup ring profiles which break into filaments with multi-Townes profiles under the effect of noise (cf. [10]). It would be naive to think that ring profiles can be obtained by finding blowup solutions of the equation (1.1). One may find blowup solutions of (1.1) with ring profiles in [5] and [6]. However, those solutions have infinite L^2 norm so one may not maintain the ring profile all the way up to the singularity. Recently, H^1 vortex blowup solutions with ring profiles have been found (cf. [7]). However, until now, it is still an open issue whether there exist H^1 non-vortex blowup solutions with ring profiles.

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In order to find H^1 non-vortex blowup solutions with ring profiles, we study two-component systems of self-focusing cubic NLSE given by

$$\begin{cases} i\partial_t\Phi + \triangle\Phi + \nu_1|\Phi|^2\Phi + \beta|\Psi|^2\Phi = 0\,, \\ i\partial_t\Psi + \triangle\Psi + \nu_2|\Psi|^2\Psi + \beta|\Phi|^2\Psi = 0\,, \\ \Phi = \Phi(x,t)\,, \Psi = \Psi(x,t) \in \mathbb{C}\,, \quad x = (x_1,x_2) \in \mathbb{R}^2\,,\ t > 0\,, \\ \Phi(\cdot,t)\,, \Psi(\cdot,t) \in H^1(\mathbb{R}^2)\,,\ t > 0\,, \end{cases} \tag{1.2}$$

under the condition $\nu_1 \gg \nu_2 > 0$, where $\triangle = \partial_{x_1}^2 + \partial_{x_2}^2$, ν_j's are positive constants and $\beta \neq 0$ is a coupling constant. The system (1.2) is a well-known model for photorefractive media in nonlinear optics (cf. [1]). Besides, the system (1.2) may also describe two-component BEC in the limit of strong transverse confinement (cf. [8]). Physically, the coefficients ν_j's and β satisfy $\nu_j \sim -a_{jj}$, $j = 1,2$, and $\beta \sim -a_{12}$, where a_{ij}'s are the scattering lengths. Due to Feshbach resonance, a_{ij}'s can be tuned over a very large range by adjusting the externally applied magnetic field (cf. [14]). Consequently, we may let $a_{jj} < 0$ i.e. $\nu_j > 0$, $j = 1,2$, and $a_{12} > 0$ i.e. $\beta < 0$. Recently, a small and negative scattering length has been achieved by experiments (cf. [23]) so we may assume $0 < -a_{22} \ll -a_{11}$ i.e. $0 < \nu_2 \ll \nu_1$.

To study the system (1.2) with $\nu_1 \gg \nu_2 > 0$ and $\beta < 0$, we may set $\nu_1 = h\,\mu_1$, $\nu_2 = h^{-1}\mu_2$, $\Phi(x,t) = \phi(x,t)$, and $\Psi(x,t) = \sqrt{h}\,\psi(x,t)$, where μ_j's positive constants and $h \sim \sqrt{\nu_1/\nu_2} \gg 1$ a large parameter. Then the system (1.2) can be transformed into the following system

$$\begin{cases} i\varepsilon^2\partial_t\phi + \varepsilon^2\triangle\phi + \mu_1|\phi|^2\phi + \beta|\psi|^2\phi = 0\,, \\ i\partial_t\psi + \triangle\psi + \mu_2|\psi|^2\psi + \beta|\phi|^2\psi = 0\,, \\ \phi = \phi(x,t)\,, \psi = \psi(x,t) \in \mathbb{C}\,, \quad x = (x_1,x_2) \in \mathbb{R}^2\,,\ t > 0\,, \\ \phi(\cdot,t)\,, \psi(\cdot,t) \in H^1(\mathbb{R}^2)\,,\ t > 0\,, \end{cases} \tag{1.3}$$

where $\varepsilon = h^{-1/2} > 0$ a small parameter and $\beta < 0$ a coupling constant. Similar systems of NLSE with trap potentials and different dispersion coefficients can be found in [22]. Note that due to the small parameter ε, the system (1.3) can be regarded as a multi-scale system having fast and slow variables. In this paper, we want to prove that the system (1.3) may have H^1 blowup solutions with ring profiles.

Pseudo-conformal transformations are useful to construct explicit blowup solutions of the equation (1.1) (cf. [29]). To get blowup solutions of the system (1.3), as for [15], we consider the following pseudo-conformal transformations

$$\phi(x,t) = A_1(x,t)\,e^{i\theta_1(x,t)}\,, \quad \psi(x,t) = A_2(x,t)\,e^{i\theta_2(x,t)}\,, \tag{1.4}$$

where

$$A_1(x,t) = u(\xi)\,\exp\left(-\int_0^t a(\tau)d\tau\right)\,, \quad A_2(x,t) = v(\xi)\,\exp\left(-\int_0^t a(\tau)d\tau\right)\,, \tag{1.5}$$

$$\theta_j(x,t) = a(t)\frac{|x|^2}{4} + \gamma_j(t)\,, \quad j = 1,2\,, \tag{1.6}$$

and

$$\gamma_1'(t) = \frac{\lambda_1}{\varepsilon^2}\exp\left(-2\int_0^t a(\tau)d\tau\right)\,, \quad \gamma_2'(t) = \lambda_2\exp\left(-2\int_0^t a(\tau)d\tau\right)\,. \tag{1.7}$$

Here u and v are real-valued functions, λ_j's are positive constants, $\xi = (\xi_1,\xi_2) \in \mathbb{R}^2$ is defined by

$$\xi = x\exp\left(-\int_0^t a(\tau)d\tau\right)\,, \quad x = (x_1,x_2) \in \mathbb{R}^2\,, \tag{1.8}$$

and $a(\cdot)$ is defined by an ordinary differential equation given by

$$a'(t) + a^2(t) = 0\,, \quad \forall\, t > 0\,, \tag{1.9}$$

with initial data

$$a(0) = a_0 < 0\,. \tag{1.10}$$

By (1.4)-(1.9), we may transform the system (1.3) into

$$\begin{cases} \varepsilon^2 \triangle u - \lambda_1 u + \mu_1 u^3 + \beta v^2 u = 0 & \text{in } \mathbb{R}^2 \,, \\ \triangle v - \lambda_2 v + \mu_2 v^3 + \beta u^2 v = 0 & \text{in } \mathbb{R}^2 \,, \\ u, \, v \in H^1(\mathbb{R}^2) \,, \quad u, v > 0 & \text{in } \mathbb{R}^2 \,, \end{cases} \tag{1.11}$$

where $0 < \varepsilon \ll 1$ is a small parameter, λ_j's and μ_j's are positive constants, and β is a negative constant. Here $\triangle$ is the Laplacian corresponding to ξ-coordinates denoted as $\triangle = \sum_{j=1}^{2} \partial^2_{\xi_j}$. To get non-vortex solutions, we only consider positive solutions of (1.11) i.e. $u, v > 0$ in $\mathbb{R}^2$. Moreover, (1.9) and (1.10) imply

$$a(t) = \frac{a_0}{a_0 t + 1} \to -\infty \quad \text{as } t \uparrow T = -1/a_0 \,, \tag{1.12}$$

and then both $|\phi|$ and $|\psi|$ blowup at the same time $T = -1/a_0$ by (1.4) and (1.5). One may remark that the sign of β may affect the blowup profile of $|\phi|$ and $|\psi|$. Suppose $\beta > 0$. Then the system (1.11) becomes cooperative and only provides the Towne profile for $|\phi|$ and $|\psi|$. Hence we must assume $\beta < 0$ in order to obtain blowup ring-profile from the system (1.3).

Blowup profiles for blowup solutions of (1.2) and (1.3) are governed by the system (1.11). Here we prove that as $\varepsilon > 0$ sufficiently small, there are two kinds of H^1 positive solutions $(u_\varepsilon, v_\varepsilon)$'s of (1.11) having different asymptotic behaviors. One is that u_ε concentrates at vertices of a regular k-polygon (for any $k \geq 2$) and v_ε concentrates at the origin (see Theorem 2.1 in Section 2). The other is that u_ε concentrates on a circle away from the origin and v_ε concentrates at the origin (see Theorem 2.2 in Section 2). Now we fix $\varepsilon > 0$ as a small enough constant. Then the graph of u_ε may approach to a single ring profile without any vortex. Hereafter, the single ring profile is defined as the graph of a positive function $f = f(r)$ ($r = |x|$ is the radial variable for $x \in \mathbb{R}^2$) such that $f(\infty) = 0$, and f is increasing on $(0, r_1)$ but decreasing on (r_1, ∞), where r_1 a positive constant. Hence by (1.4), (1.5) and (1.12), we may obtain H^1 non-vortex blowup solutions (ϕ, ψ)'s of (1.3) i.e. (Φ, Ψ)'s of (1.2) blowing up at $T = -1/a_0$, and the blowup profile of Φ is of ring profiles. This may provide non-vortex ring profiles which can be maintained all the way up to the singularity.

Another motivation of the system (1.11) may come from bright ring solitons which exist as stationary localized states observed in self-focusing Kerr media modelled by NLSE (cf. [27]). One may find quantized vortices corresponding to bright ring solitons by solving vortex solutions of the equation (1.1) (cf. [3]). However, until now, steady non-vortex bright ring solitary wave solutions of (1.1) have not yet been found. To learn steady non-vortex bright ring solitons, we study steady solitary wave solutions of the system (1.3) by setting $\phi(x,t) = e^{i\lambda_1 t/\varepsilon^2} u(x)$ and $\psi(x,t) = e^{i\lambda_2 t} v(x)$ for $x = (x_1, x_2) \in \mathbb{R}^2, t > 0$, where both u and v are positive functions. Then the system (1.3) can be transformed into (1.11) with $\triangle = \partial^2_{x_1} + \partial^2_{x_2}$. Hence Theorem 2.2 may also provide steady solitary wave solutions of the system (1.3) to describe non-vortex bright ring solitons.

In addition, non-vortex ring profiles can be obtained by numerical simulations on the system (1.11) with $\triangle = \partial^2_{x_1} + \partial^2_{x_2}$. Setting $u = u(r), v = v(r)$, and $r = |x|$ for $x \in \mathbb{R}^2$, we may rewrite the system (1.11) as follows:

$$\begin{cases} \varepsilon^2 \left(u'' + \frac{1}{r} u'\right) - \lambda_1 u + \mu_1 u^3 + \beta v^2 u = 0 \,, & \text{for } r > 0 \,, \\ \left(v'' + \frac{1}{r} v'\right) - \lambda_2 v + \mu_2 v^3 + \beta u^2 v = 0 \,, & \text{for } r > 0 \,, \\ u, v > 0 \,, & \text{for } r > 0 \,, \\ u'(0) = v'(0) = 0, \; u(\infty) = v(\infty) = 0 \,. \end{cases} \tag{1.13}$$

We may use a singular boundary value problem solver BVP4C in MATLAB to solve (1.13) and obtain numerical solutions with ring profiles as described in Theorem 2.2 and Remark 3 (see also Figs 1-3 in Section 5). Our numerical scheme is reliable since it produces numerical solutions of (1.13) with computational errors of order $\mathcal{O}(10^{-15})$ (see, e.g., Fig 1(c)). Besides, numerical solutions with m-ring and ring-ring profiles can be shown in Fig 4 and Fig 5, respectively. Here the m-ring profile is the graph of a positive function $g = g(r)$ ($r = |x|$ is the radial variable for $x \in \mathbb{R}^2$) with m bumps. The ring-ring profile means that both u and v have ring profiles. Until now, we have no theoretical result to support the existence of the ring-ring profile.

The rest of this paper is organized as follows: We state Theorem 2.1 and 2.2 in Section 2. In Section 3 and 4, we give rigorous arguments to prove Theorem 2.1 and 2.2 using the localized energy method with symmetry reduction. In Section 5, various numerical solutions of (1.13) are given.

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2 Main Results

Let ω_j be the unique positive solution of

$$\begin{cases} \triangle\omega_j - \lambda_j\omega_j + \mu_j\omega_j^3 = 0 \quad \text{in} \quad \mathbb{R}^2\,, \\ \omega_j = \omega_j(r) > 0 \quad \text{for } r = |x| > 0\,, \\ \lim_{r\to\infty}\omega_j(r) = 0\,, \quad j = 1, 2\,. \end{cases} \tag{2.1}$$

Note that the equation (2.1) is of semilinear elliptic equations. By the symmetry result of Gidas-Ni-Nirenberg [11], each w_j is radially symmetric and strictly decreasing. For the uniqueness theorem, one may refer to [21] for the uniqueness of semilinear elliptic equations covering the case of cubic nonlinearity but those are totally different from the equation (2.1). Hence we need the result of Kwong [13] to assure the uniqueness of ω_j's. Then it is easy to check that $\omega_j = \omega_j(r) = \sqrt{\frac{\lambda_j}{\mu_j}}\,\omega(\sqrt{\lambda_j}r)$, where $\omega = \omega(r)$ is the unique positive solution of $\triangle\omega - \omega + \omega^3 = 0$ in $\mathbb{R}^2$. Our first result is stated as follows:

Theorem 2.1. *Let $k \in \mathbb{N}$, $k \geq 2$. Then for ε sufficiently small, problem (1.11) has a solution $(u_\varepsilon, v_\varepsilon)$ with the following properties*

(a) $v_\varepsilon(x) = \omega_2(|x|)(1 + o_\varepsilon(1))$,

(b) $u_\varepsilon(x) = \sum_{l=1}^{k} \omega_1\left(\left|\frac{x - P_{\varepsilon,l}}{\varepsilon}\right|\right)(1 + o_\varepsilon(1))$,

where $P_{\varepsilon,l}$'s are spike centers of u_ε satisfying

$$|P_{\varepsilon,i} - P_{\varepsilon,j}| \sim \varepsilon \log\frac{1}{\varepsilon}\,, \quad P_{\varepsilon,i} = o_\varepsilon(1)\,, \tag{2.2}$$

for $i, j = 1, \cdots, k$ and $i \neq j$. Moreover, $P_{\varepsilon,l}$'s are located at vertices of a regular k-polygon in $\mathbb{R}^2$.

In this paper, we use the notation $\sim$ to denote $A_\varepsilon \sim B_\varepsilon$ which means $C_1\, B_\varepsilon \leq A_\varepsilon \leq C_2\, B_\varepsilon$ as $\varepsilon \to 0$, where C_j's are positive constant independent of ε. Besides, $o_\varepsilon(1)$ is a small quantity tending to zero as ε goes to zero.

Remark 1. We can also prove the existence of solutions with more complex structures: for example, u_ε may have spikes at concentric polygons. For the different case that $\varepsilon = 1$ and $-\beta$ is a positive but small parameter, one may refer to [15] to get multi-Townes profiles far away from the origin.

In Theorem 2.1, we rigorously prove that as ε sufficiently small, there exist $(u_\varepsilon, v_\varepsilon)$'s solutions of (1.11) with u_ε concentrating at $P_{\varepsilon,l}, l = 1, \cdots, k$ vertices of a regular k-polygon near the origin and v_ε concentrating at the origin. Now we fix $\varepsilon > 0$ as a sufficiently small constant. By

(1.4)-(1.8) and (1.12), the associated solution (Φ, Ψ) of (1.2) satisfies $|\Phi|(x,t) = \frac{T}{T-t}\, u_\varepsilon\left(\frac{T\,x}{T-t}\right) \sim \frac{T}{T-t} \sum_{l=1}^{k} w_1 \left(\left| \dfrac{Tx}{\varepsilon(T-t)} - \varepsilon^{-1}\, P_{\varepsilon,l} \right| \right)$ and $|\Psi|(x,t) = \varepsilon^{-1}\, \frac{T}{T-t}\, v_\varepsilon\left(\frac{T\,x}{T-t}\right) \sim \varepsilon^{-1}\, \frac{T}{T-t}\, w_2\left(\frac{T\,|x|}{T-t}\right)$. Thus $|\Phi|$ has the k-fold Townes profile with k peaks at $\frac{T-t}{T}\, P_{\varepsilon,l}$'s and $|\Psi|$ has the Townes profile with a single peak at the origin such that as $t \uparrow T$, $|\Phi|(x,t) \to \infty$ for $x = \frac{T-t}{T}\, P_{\varepsilon,l}$, $l = 1, \cdots, k$ and $|\Psi|(0,t) \to \infty$. Note that both $|\Phi|$ and $|\Psi|$ blow up at the same time T. Moreover, both u_ε and v_ε have finite H^1 norms which may imply that the the k-fold Townes profile of $|\Phi|$ and the Townes profile of $|\Psi|$ can be maintained all the way up to the singularity.

Next theorem shows that there exist solutions (u,v)'s with u concentrating on a circle away from the origin and v concentrating at the origin. To state the result, we need to introduce some functions. Let U be the unique homoclinic solution of

$$U'' - U + \mu_1\, U^3 = 0,\; U(y) = U(-y),\; U > 0,\; U \to 0 \text{ at } \infty. \tag{2.3}$$

Let

$$M(r) = r^{2/3}\, V(r) \quad \text{ and } \quad V(r) = \lambda_1 - \beta \omega_2^2(r). \tag{2.4}$$

Then $r^{-2/3} M'(r) = \frac{2}{3r} V(r) + V'(r)$. Due to $\beta < 0$, it is obvious that $V(r) > 0$ and $V'(r) < 0$ for $r > 0$. Moreover, $M'(r) > 0$ for r sufficiently close to zero or infinity. Suppose

$$\max_{r>0} \frac{|rV'(r)|}{V(r)} > \frac{2}{3}\,. \tag{2.5}$$

Then $M'(r_0) < 0$ for some $r_0 > 0$. Hence the function M may have two critical points $r_j, j = 1, 2$ such that $0 < r_1 < r_0 < r_2 < \infty$.

Remark 2. To fulfill the condition (2.5), we remark that due to $\beta < 0$, we have

$$V'(r) = -2\beta\omega_2(r)\omega_2'(r) = 2|\beta| \frac{\lambda_2^{3/2}}{\mu_2} \omega(\sqrt{\lambda_2} r)\omega'(\sqrt{\lambda_2} r)\,,$$

and hence

$$r\frac{V'(r)}{V(r)} = \frac{t\omega(t)\omega'(t)}{A + \omega^2(t)}\,, \tag{2.6}$$

where $t = \sqrt{\lambda_2} r$ and $A = \frac{\lambda_1 \mu_2}{|\beta| \lambda_2}$. Now we set $f(a) = \max_{t>0} \frac{|t\omega(t)\omega'(t)|}{a+\omega^2(t)}$ for $a > 0$. Then it is obvious that f is monotone decreasing in a, $\lim_{a\to 0} f(a) = \infty$ and $\lim_{a\to\infty} f(a) = 0$. Consequently, there exists a unique $A_0 > 0$ such that $f(A_0) = \frac{2}{3}$ i.e. $\max_{r>0} \frac{|rV'(r)|}{V(r)} = \frac{2}{3}$ if $A = A_0$. Therefore the condition (2.5) can be replaced by $A < A_0$ i.e.

$$-\beta > \frac{\lambda_1 \mu_2}{\lambda_2 A_0} > 0\,. \tag{2.7}$$

Now we state another main theorem as follows:

Theorem 2.2. *Assume (2.7) holds. Then the problem (1.11) has two solutions $(u_{\varepsilon,1}, v_{\varepsilon,1})$ and $(u_{\varepsilon,2}, v_{\varepsilon,2})$ such that*

(**a**) $u_{\varepsilon,i}(r) \sim \sqrt{V(r_{\varepsilon,i})}\, U \left(\sqrt{V(r_{\varepsilon,i})}\, \dfrac{|r - r_{\varepsilon,i}|}{\varepsilon} \right),\; i = 1, 2,$

(**b**) $v_{\varepsilon,i}(r) \sim \omega_2(r),\; i = 1, 2\,,$

where $r_{\varepsilon,i} \to r_i$ as $\varepsilon \to 0+$, and $r_1 < r_2$ are two critical points of $M(r)$.

Remark 3. Following the proof of [17], we can also show the existence of clustered ring solutions, i.e, $u_{\varepsilon,i}(r) \sim \sum_{j=1}^{K} \sqrt{V(r_{\varepsilon,i}^j)}\, U\left(\sqrt{V(r_{\varepsilon,i}^j)}\, \frac{|r - r_{\varepsilon,i}^j|}{\varepsilon}\right)$, where $r_{\epsilon,i}^j \to r_i, j = 1, ..., K$.

In Theorem 2.2, we rigorously prove that as ε sufficiently small, there exist $(u_{\varepsilon,i}, v_{\varepsilon,i})$'s solutions of (1.11) with $u_{\varepsilon,i}$ concentrating on a circle (with a center at the origin and radius r_i) away from the origin and $v_{\varepsilon,i}$ concentrating at the origin. Now we fix $\varepsilon > 0$ as a small enough constant. By (1.4)-(1.8), (1.12) and Theorem 2.2, the associated solution (Φ, ψ) satisfies $|\Phi|(x,t) = \frac{T}{T-t}\, u_{\varepsilon,i}\left(\frac{T\,x}{T-t}\right) \sim \frac{T}{T-t} \sqrt{V(r_{\varepsilon,i})}\, U\left(\sqrt{V(r_{\varepsilon,i})}\left|\frac{T\,|x|}{\varepsilon(T-t)} - \varepsilon^{-1} r_{\varepsilon,i}\right|\right)$ and $|\Psi|(x,t) = \varepsilon^{-1}\, \frac{T}{T-t}\, v_{\varepsilon,i}\left(\frac{T\,x}{T-t}\right) \sim \varepsilon^{-1}\, \frac{T}{T-t}\, w_2\left(\frac{T\,|x|}{T-t}\right)$. Thus as $t \uparrow T$, $|\Phi|(x,t) \to \infty$ for $x \in \Gamma_t^i$ and $|\Psi|(0,t) \to \infty$, where $\Gamma_t^i = \left\{x \in \mathbb{R}^2 : |x| = \frac{T-t}{T}\, r_{\varepsilon,i}\right\}$ is a circle shrinking to the origin as t goes to T. Note that both $|\Phi|$ and $|\Psi|$ blow up at the same time T. Furthermore, both u_ε and v_ε have finite H^1 norms which may imply that the ring profile of $|\Phi|$ and the Townes profile of $|\Psi|$ can be maintained all the way up to the singularity.

3 Proof of Theorem 2.1

In this section, we use the *method of localized energy with symmetry reduction* to prove Theorem 2.1. For an overview on localized energy method, we refer to Chapter 2 of [28]. Here we closely follow a combination of localized energy method and symmetry reduction which has been used in [15].

3.1 Symmetry Class

For $k \geq 2$, we define a class of functions with the symmetry property as follows:

$$\Sigma_1 = \left\{ u\left(\widetilde{r}\cos\left(\widetilde{\theta} + \frac{2\pi}{k}\right), \widetilde{r}\sin\left(\widetilde{\theta} + \frac{2\pi}{k}\right)\right) = u\left(\widetilde{r}\cos\widetilde{\theta}, \widetilde{r}\sin\widetilde{\theta}\right) \right\}. \tag{3.1}$$

Then we have

Lemma 3.1. *Let $p > 1$ be a fixed number. If $\phi \in W^{2,p}(\mathbb{R}^2) \cap \Sigma_1$ and*

$$L_2\phi := \triangle\phi - \lambda_2\phi + 3\mu_2 w_2^2 \phi = 0, \tag{3.2}$$

then $\phi \equiv 0$. As a consequence,

$$||\phi||_{W^{2,p}} \leq C_1 ||L_2\phi||_{L^p}, \tag{3.3}$$

for $\phi \in W^{2,p}(\mathbb{R}^2) \cap \Sigma_1$, where C_1 is a positive constant independent of ϕ. Moreover, the inverse map $L_2^{-1} : L^p(\mathbb{R}^2) \cap \Sigma_1 \to W^{2,p}(\mathbb{R}^2) \cap \Sigma_1$ exists and

$$||L_2^{-1} f||_{W^{2,p}} \leq C_1 ||f||_{L^p}, \tag{3.4}$$

for $f \in L^p(\mathbb{R}^2) \cap \Sigma_1$.

Proof. By the uniqueness result of [13], w_2 is unique in the radial class. By Lemma C of [20], the kernel of L_2 in $W^{2,p}(\mathbb{R}^2)$ consists of functions $\frac{\partial w_2}{\partial y_j}, j = 1, 2$. However, for any constants c_1 and c_2, $c_1 \frac{\partial w_2}{\partial y_1} + c_2 \frac{\partial w_2}{\partial y_2} \notin \Sigma_1$. Thus the first part of the Lemma is proved. The second part follows from Fredholm alternative. The proof is given in the appendix. □

As a consequence of Lemma 3.1, we have

Lemma 3.2. *There exists $\delta > 0$ such that if*

$$g \in \Sigma_1 \quad \text{and} \quad ||g||_{L^2(\mathbb{R}^2)} < \delta, \tag{3.5}$$

then the equation

$$\triangle v - \lambda_2 v + \mu_2 v^3 + gv = 0 \quad \text{in } \mathbb{R}^2 \tag{3.6}$$

has a unique solution $v \in H^2(\mathbb{R}^2) \cap \Sigma_1$ satisfying

$$||v - \omega_2||_{H^2(\mathbb{R}^2)} \le C_2 \|g\|_{L^2(\mathbb{R}^2)}\,, \tag{3.7}$$

where C_2 can be chosen as

$$C_2 = C_1 \left(1 + \|w_2\|_{L^\infty(\mathbb{R}^2)}\right)\,, \tag{3.8}$$

and C_1 is given in (3.3).

Remark 4. Note that we only require that $\|g\|_{L^2}$ is small. So $\lambda_2 - g$ might be negative somewhere in $\mathbb{R}^2$.

Proof. We use contraction mapping theorem to prove Lemma 3.2. Let $v = \omega_2 + \psi$, where $\psi \in H^2(\mathbb{R}^2) \cap \Sigma_1$. Then v satisfies (3.6) if and only if ψ satisfies

$$\triangle \psi - \lambda_2 \psi + 3\mu_2 \omega_2^2 \psi + N[\psi] + g\omega_2 + g\psi = 0,$$

which is equivalent to

$$\psi = L_2^{-1} \left[-N[\psi] - g\psi - g\omega_2 \right] \equiv \mathcal{A}[\psi]\,, \tag{3.9}$$

where $N[\psi] = \mu_2(3\omega_2 \psi^2 + \psi^3)$ and L_2^{-1} is the inverse map given by Lemma 3.2.

Now we choose a complete metric space

$$B := \{\psi \in H^2(\mathbb{R}^2) \cap \Sigma_1 \mid \|\psi\|_{H^2(\mathbb{R}^2)} \le C_2 \|g\|_{L^2(\mathbb{R}^2)}\}$$

with the metric $d(\varphi, \psi) = \|\varphi - \psi\|_{H^2(\mathbb{R}^2)}$. Here C_2 is defined by (3.8). Note that by Lemma 3.1 and the Sobolev embedding $H^2(\mathbb{R}^2)$ into $L^p(\mathbb{R}^2)\,, 1 < p \le \infty$ (cf. [4]),

$$\begin{aligned} \|L_2^{-1}[N[\psi]]\|_{H^2} &\le C_3 \|\psi\|_{H^2}^2 + C_4 \|\psi\|_{H^2}^3, \\ ||L_2^{-1}[g\omega_2]||_{H^2} &\le C_1 ||g\omega_2||_{L^2} \le C_1 \|w_2\|_{L^\infty} \|g\|_{L^2}\,, \\ \|L_2^{-1}[g\psi]\|_{H^2} &\le C_5 \|g\|_{L^2} \|\psi\|_{H^2}\,, \end{aligned}$$

where C_3, C_4 and C_5 are positive constants independent of ψ and g. Thus we have for $\psi \in B$,

$$\|\mathcal{A}[\psi]\|_{H^2} \le C_3 C_2^2 \|g\|_{L^2}^2 + C_4 C_2^3 \|g\|_{L^2}^3 + C_5 C_2 \|g\|_{L^2}^2 + C_1 \|w_2\|_{L^\infty} \|g\|_{L^2} \le C_2 \|g\|_{L^2}\,, \tag{3.10}$$

provided that

$$C_3 C_2^2 \|g\|_{L^2} + C_4 C_2^3 \|g\|_{L^2}^2 + C_5 C_2 \|g\|_{L^2} \le C_1. \tag{3.11}$$

Similarly, we also have

$$\|\mathcal{A}[\psi_1] - \mathcal{A}[\psi_2]\|_{H^2} \le C_6 \|g\|_{L^2} \|\psi_1 - \psi_2\|_{H^2} \le \frac{1}{2} \|\psi_1 - \psi_2\|_{H^2}\,, \tag{3.12}$$

if $\|g\|_{L^2}$ is small enough, where C_6 is a positive constant independent of ψ and g. Setting $\|g\|_{L^2} < \delta$ small enough, the map $\mathcal{A}$ becomes a contraction mapping from B to B. By the contraction mapping theorem, the unique solution of (3.9) in B is guaranteed. Therefore, we may complete the proof of Lemma 3.2. $\square$

Given g, let us denote the solution v in Lemma 3.2 as $T[g] := v$. Next we introduce a framework to solve the first equation (i.e. u) of (1.11). Let $P_0 = (\varepsilon\, l, 0)$, $P_i = R_i(\varepsilon\, l, 0)$, $i = 1, ..., k$, where

$$R_i = \begin{pmatrix} \cos\left(\frac{2\pi}{k}(i-1)\right) & -\sin\left(\frac{2\pi}{k}(i-1)\right) \\ \sin\left(\frac{2\pi}{k}(i-1)\right) & \cos\left(\frac{2\pi}{k}(i-1)\right) \end{pmatrix} \tag{3.13}$$

and l satisfies $\alpha \log\frac{1}{\varepsilon} \le l \le \gamma \log\frac{1}{\varepsilon}$, $\alpha > 1$ and γ will be chosen later. Let $\widetilde{\omega}$ be the unique positive solution of

$$\triangle\omega - \left(\lambda_1 - \beta\omega_2^2(P_0)\right)\omega + \mu_1\omega^3 = 0 \quad \text{in } \mathbb{R}^2, \tag{3.14}$$

with $\lim_{|x|\to\infty} \omega(|x|) = 0$. Note that $\omega_2^2(P_0)$=$\omega_2^2(P_i)$ for $i = 1, ..., k$. Let $\widetilde{\omega}_i(x) = \widetilde{\omega}(\frac{|x-P_i|}{\varepsilon})$ and $W(x) = \sum_{i=1}^k \widetilde{\omega}_i(x)$. Of course, $W \in \Sigma_1$ so we may choose W as an ansaz to approximate the u-component solution. Now we rescale the spatial variables by ε i.e. $y = x/\varepsilon$, and consider the following operator

$$S[u] = \triangle\, u - \lambda_1 u + \mu_1 u^3 + \beta(T[\beta u^2](\varepsilon\, y))^2 u \tag{3.15}$$

on $H^2(\mathbb{R}^2)$ with norms given by

$$||\cdot||_{**,1} = \left[\int_{\mathbb{R}^2} u^2(y)dy\right]^{1/2}, \quad ||\cdot||_{*,1} = ||\cdot||_{H^2(\mathbb{R}^2)}.$$

Now let us estimate the error introduced by W.

3.2 Error Estimate

Let us compute the error $E = S[W]$. From Lemma 3.2, we have

$$\begin{aligned} T[\beta\, W^2](\varepsilon\, y) =& \omega_2(\varepsilon\, y) + O(||W^2||_{L^2}) \\ =& \omega_2(\varepsilon\, y) + O(\varepsilon). \end{aligned} \tag{3.16}$$

Here we have used the fact that $\alpha \log\frac{1}{\varepsilon} \le l \le \gamma \log\frac{1}{\varepsilon}$ and $\alpha > 1$. Hence

$$\begin{aligned} E = S[W] =& \triangle\, W - \lambda_1 W + \mu_1 W^3 + \beta(T[\beta W^2](\varepsilon y))^2 W \\ =& \triangle W - \left(\lambda_1 - \beta\omega_2^2\right) W + \mu_1 W^3 + \beta\left[(T[\beta W^2](\varepsilon y))^2 - \omega_2^2(\varepsilon y)\right] W \\ =& E_1 + E_2, \end{aligned}$$

where $E_1 = \triangle W - \left(\lambda_1 - \beta\omega_2^2\right) W + \mu_1 W^3$ and $E_2 = \beta\left[(T[\beta W^2](\varepsilon y))^2 - \omega_2^2(\varepsilon y)\right] W$. It is easy to check that

$$\begin{aligned} E_1 =& \triangle W - \left(\lambda_1 - \beta\omega_2^2\right) W + \mu_1 W^3 \\ =& \left(\beta\omega_2^2(\varepsilon y) - \beta\omega_2^2(P_0)\right) W + \mu_1 \left[\left(\sum_{i=1}^k \widetilde{\omega}_i\right)^3 - \sum_{i=1}^k \widetilde{\omega}_i^3\right] \\ =& E_{11} + E_{12}, \end{aligned}$$

where $E_{11} = \left(\beta\omega_2^2(\varepsilon y) - \beta\omega_2^2(P_0)\right) W$ and $E_{12} = \mu_1 \left[\left(\sum_{i=1}^k \widetilde{\omega}_i\right)^3 - \sum_{i=1}^k \widetilde{\omega}_i^3\right]$.

For E_{12}, we have

$$\begin{aligned} \int_{\mathbb{R}^2} |E_{12}|^2 \le& C \int_{\mathbb{R}^n} \sum_{k\neq l} \widetilde{\omega}_k^4 \widetilde{\omega}_l^2 \\ \le& C \sum_{k\neq l} \widetilde{\omega}^2 \left(\frac{|P_k - P_l|}{\varepsilon}\right) \\ \le& C\widetilde{\omega}^2 \left(2l \sin\frac{\pi}{k}\right), \end{aligned} \tag{3.17}$$

where C is a positive constant independent of ε. For E_{11}, we have

$$E_{11} = \beta\left(\omega_2^2(|\varepsilon y|) - \omega_2^2(|P_0|)\right) W.$$

Let $\varepsilon y = P_0 + \varepsilon z$. Then we obtain

$$\begin{aligned} |E_{11}| &\le CW \cdot \left[|\omega_2'(|P_0|)| \times (|P_0+\varepsilon z| - |P_0|) + O((|P_0+\varepsilon z| - |P_0|)^2)\right] \\ &\le CW \cdot \left(\varepsilon |P_0||z| + \varepsilon^2 |z|^2\right), \end{aligned} \tag{3.18}$$

and hence

$$\int_{\mathbb{R}^2} |E_{11}|^2 \le C(\varepsilon^2|P_0|^2 + \varepsilon^4) \le C\varepsilon^4 \log^2 \frac{1}{\varepsilon}, \tag{3.19}$$

where C is a positive constant independent of ε. For E_2, we may use (3.16) to get

$$\int_{\mathbb{R}^2} |E_2|^2 \le C\varepsilon^2 \int_{\mathbb{R}^2} |W|^2 \le C\varepsilon^2. \tag{3.20}$$

Combining the estimates in (3.17)−(3.20), we obtain the following error estimates.

Lemma 3.3. *The error $E = S[W]$ satisfies*

$$||E||_{**,1} \le C\left(\varepsilon + \widetilde{w}\left(2l \sin\frac{\pi}{k}\right)\right), \tag{3.21}$$

where $\|\cdot\|_{**,1} = \|\cdot\|_{L^2(\mathbb{R}^2)}$.

3.3 Linear Theory

We consider the following linear problem

$$\begin{cases} \triangle\phi - \lambda_1\phi + 3\mu_1 W^2\phi + \beta\omega_2^2\phi = h + c\frac{\partial W}{\partial l}, \\ \int_{\mathbb{R}^2} \phi \frac{\partial W}{\partial l} = 0, \quad \phi \in \Sigma_1, \end{cases} \tag{3.22}$$

with the solution (ϕ, c), where $h \in L^2(\mathbb{R}^2) \cap \Sigma_1$. Then we may derive apriori estimates as follows:

Lemma 3.4. *For ε sufficiently small, given $||h||_{**} < \infty$, problem (3.22) has a unique solution (ϕ, c) such that*

$$||\phi||_{*,1} + |c| \le C||h||_{**,1}, \tag{3.23}$$

where $\|\cdot\|_{*,1} = \|\cdot\|_{H^2(\mathbb{R}^2)}$ *and* $\|\cdot\|_{**,1} = \|\cdot\|_{L^2(\mathbb{R}^2)}$.

Proof. Firstly, we prove (3.23). We note that

$$\begin{aligned} \frac{\partial W}{\partial l} &= \sum_{i=1}^k \widetilde{\omega}'\left(\frac{|\varepsilon y - P_i|}{\varepsilon}\right) \cdot \frac{1}{\varepsilon} \cdot \frac{-(\varepsilon y - P_i)\cdot \varepsilon (R_i e_1)}{|\varepsilon y - P_i|} \\ &= \sum_{i=1}^k \widetilde{\omega}'\left(\frac{|\varepsilon y - P_i|}{\varepsilon}\right) \frac{-(\varepsilon y - P_i)\cdot (R_i e_1)}{|\varepsilon y - P_i|}, \end{aligned} \tag{3.24}$$

where $e_1 = (1,0)$ and R_i's are defined in (3.13). Multiplying (3.22) by $\frac{\partial W}{\partial l}$ and integrating over $\mathbb{R}^2$, we obtain

$$\begin{aligned} |c| \le& \left|\int_{\mathbb{R}^2} \left(\triangle \frac{\partial W}{\partial l} - \widetilde{\lambda}_1 \frac{\partial W}{\partial l} + 3\mu_1 W^2 \frac{\partial W}{\partial l}\right)\phi\right| + \left|\int_{\mathbb{R}^2} \left(\beta\omega_2^2 - \beta\omega_2^2(P_0)\right)\phi \frac{\partial W}{\partial l}\right| + \|h\|_{**,1} \\ \le& o(||\phi||_{**,1}) + \|h\|_{**,1}, \end{aligned} \tag{3.25}$$

where $\widetilde{\lambda}_1 = \lambda_1 - \beta\omega_2^2(P_0)$ and $o(1)$ is a small quantity tending to zero as ε goes to zero. Here we have used the inequality (3.18) to deal with the second integral of (3.25). To get (3.23), it is enough to show that $||\phi||_{*,1} \le C||h||_{**,1}$. In fact, we can prove it by contradiction using a similar argument to Lemma 4.1 of [15]. We just sketch the idea: suppose not. We then have a sequence of (ϕ_n, c_n, h_n) satisfying (3.22) such that $\|\phi_n\|_{*,1} = 1, \|h_n\|_{**,1} = o(1)$. By (3.25), we derive that $|c_n| = o(1)$. Let $\tilde{\phi}_n = \phi_n(y + \frac{P_1}{\epsilon})$. We then obtain that $\tilde{\phi}_n \to \phi_0$ where ϕ_0 satisfies $\Delta\phi_0 - \lambda_1\phi_0 + 3\mu_1\omega_1^2\phi_0 = 0$. This and the fact that $\int_{\mathbb{R}^2} \tilde{\phi}_n \frac{\partial W}{\partial l} = 0$ force $\phi_0 \equiv 0$. Then we use regularity theorem to conclude that $\|\phi_n\|_{*,1} = o(1)$ which contradicts with our assumption $\|\phi_n\|_{*,1} = 1$. This proves the a priori estimate (3.23). Using (3.23), Lemma 8, Proposition 1 and Lemma 10 of [16], we may complete the proof of Lemma 3.4. □

3.4 Nonlinear reduction

From Lemma 3.4, we deduce the following Lemma.

Lemma 3.5. *For ε sufficiently small, there exist a unique solution (ϕ_l, c_l) such that*

$$S[W + \phi_l] = c_l \frac{\partial W}{\partial l}, \quad \int_{\mathbb{R}^2} \phi_l \frac{\partial W}{\partial l} = 0, \tag{3.26}$$

and

$$||\phi_l||_{*,1} \le C\left(\varepsilon + \widetilde{\omega}\left(2l \sin\frac{\pi}{k}\right)\right). \tag{3.27}$$

Proof. Let

$$B = \left\{\phi \in H^2 \cap \Sigma_1 : ||\phi||_{*,1} \le \rho\left(\varepsilon + \widetilde{\omega}\left(2l \sin\frac{\pi}{k}\right)\right)\right\},$$

where ρ is a suitable positive constant. Then by (3.15), we have

$$\begin{aligned} S[W + \phi] =& S[W] + \triangle\phi - \lambda_1\phi + 3\mu_1 W^2\phi + \beta\omega_2^2\phi + \beta\left(T[\beta(W+\phi)^2]^2 - \omega_2^2\right)\phi \\ &+ N[\phi] + \beta\left(T[\beta(W+\phi)^2]^2 - T[\beta W^2]^2\right)W, \end{aligned} \tag{3.28}$$

where $N[\phi] = \mu_1\left(3W\phi^2 + \phi^3\right)$. By (3.16), we may calculate

$$\begin{aligned} &\left\|\beta\left(T[\beta(W+\phi)^2]^2 - \omega_2^2\right)\phi\right\|_{**,1} \le C\varepsilon||\phi||_{*,1}, \\ &\|N[\phi]\|_{**,1} \le C||\phi||_{*,1}^2, \\ &\left\|\beta\left(T[\beta(W+\phi)^2]^2 - T[\beta W^2]^2\right) W\right\|_{**,1} \le C\varepsilon. \end{aligned}$$

The rest of the proof follows from standard contraction mapping theorem. One may refer to [15] for the details. □

3.5 Expansion of c_l

Let us now expand c_l as follows: Multiply the first equation in (3.26) by $\frac{\partial W}{\partial l}$ and integrate over $\mathbb{R}^2$. Then we may use (3.28) to get

$$\begin{aligned} c_l \int_{\mathbb{R}^2}\left(\frac{\partial W}{\partial l}\right)^2 dy =& \int_{\mathbb{R}^2} E\frac{\partial W}{\partial l}\,dy + \int_{\mathbb{R}^2}\left[\triangle\phi - \lambda_1\phi + 3\mu_1 W^2\phi + \beta\omega_2^2\phi\right]\frac{\partial W}{\partial l}\,dy \\ &+ \int_{\mathbb{R}^2}\beta\left(T[\beta(W+\phi)^2]^2 - \omega_2^2\right)\phi\frac{\partial W}{\partial l}\,dy \\ &+ \int_{\mathbb{R}^2} N[\phi]\frac{\partial W}{\partial l}\,dy + \int_{\mathbb{R}^2}\beta(T[\beta(W+\phi)^2]^2 - T[\beta W^2]^2)W\frac{\partial W}{\partial l}\,dy \\ =& I_1 + I_2 + I_3 + I_4 + I_5, \end{aligned}$$

where $x = P_0 + \varepsilon\, y$, $E = S[W]$, $\phi = \phi_l$, and

$$\begin{aligned}
I_1 &= \int_{\mathbb{R}^2} E \frac{\partial W}{\partial\, l}\, dy\,, \\
I_2 &= \int_{\mathbb{R}^2} \left[\triangle\phi - \lambda_1\phi + 3\mu_1\, W^2\phi + \beta\omega_2^2\phi\right] \frac{\partial W}{\partial\, l}\, dy\,, \\
I_3 &= \int_{\mathbb{R}^2} \beta \left(T[\beta(W+\phi)^2]^2 - \omega_2^2\right) \phi \frac{\partial W}{\partial\, l}\, dy\,, \\
I_4 &= \int_{\mathbb{R}^2} N[\phi] \frac{\partial W}{\partial\, l}\, dy\,, \\
I_5 &= \int_{\mathbb{R}^2} \beta (T[\beta(W+\phi)^2]^2 - T[\beta\, W^2]^2) W \frac{\partial W}{\partial\, l}\, dy
\end{aligned}$$

Using (3.16) and (3.27), it is obvious that

$$I_3 = \int_{\mathbb{R}^2} \beta \left(T[\beta(W+\phi)^2]^2 - \omega_2^2\right) \phi \frac{\partial W}{\partial\, l}\, dy = O\left(\varepsilon^2 + \widetilde{\omega}^2 \left(2l \sin\frac{\pi}{k}\right)\right)\,, \tag{3.29}$$

and

$$I_4 = \int_{\mathbb{R}^2} N[\phi] \frac{\partial W}{\partial\, l}\, dy = O\left(\varepsilon^2 + \widetilde{\omega}^2 \left(2l \sin\frac{\pi}{k}\right)\right)\,. \tag{3.30}$$

To estimate I_5, we set $\psi = T[\beta(W+\phi)^2] - T[\beta\, W^2]$. Then ψ satisfies

$$\triangle\psi - \lambda_2\psi + 3\mu_2\omega_2^2\psi = \tilde{E}_2 \tag{3.31}$$

where

$$\tilde{E}_2 = 3\mu_2(\omega_2^2 - (T[\beta\, W^2])^2)\psi - \beta\, W^2\psi + O(\psi^2) + O\left(|(2\phi\, W + \phi^2)\, T[\beta(W+\phi)^2]|\right) \tag{3.32}$$

Let us estimate each term in (3.32): By (3.16) and (3.27), it is easy to check that

$$||(2\phi\, W + \phi^2)\, T[\beta(W+\phi)^2]||_{L^2(\mathbb{R}^2)} \leq C\varepsilon^2\,, \tag{3.33}$$

and

$$||\psi^2||_{L^2(\mathbb{R}^2)} \leq C\varepsilon^2\,. \tag{3.34}$$

Using (3.16), we see that

$$\|((T[\beta W^2])^2 - \omega_2^2)\psi\|_{L^2(\mathbb{R}^2)} \leq C\epsilon \|\psi\|_{H^2(\mathbb{R}^2)}. \tag{3.35}$$

Recall that $W = \sum_{i=1}^k \tilde{\omega}(\frac{|x-P_i|}{\epsilon})$. Hence

$$\|\beta W^2\psi\|_{L^2(\mathbb{R}^2)} \leq C\epsilon^{1/2}\|\psi\|_{H^2(\mathbb{R}^2)}. \tag{3.36}$$

By Lemma 3.1 and (3.33)-(3.36), we have

$$\|\psi\|_{H^2(\mathbb{R}^2)} \leq C\|\tilde{E}_2\|_{L^2(\mathbb{R}^2)} \leq C\epsilon^{1/2}\|\psi\|_{H^2(\mathbb{R}^2)} + C\epsilon^2 \tag{3.37}$$

which gives

$$||\psi||_{H^2(\mathbb{R}^2)} \leq C\varepsilon^2\,, \tag{3.38}$$

and hence by Sobolev embedding,

$$||\psi||_{L^\infty(\mathbb{R}^2)} \leq C\varepsilon^2\,.$$

Consequently,

$$I_5 \leq C\varepsilon^2 \,.$$

For I_2, we may use integration by parts to get

$$\begin{aligned} I_2 =& \int_{\mathbb{R}^2} \left[\triangle \frac{\partial W}{\partial\, l} - \lambda_1 \frac{\partial W}{\partial\, l} + 3\mu_1\, W^2 \frac{\partial W}{\partial\, l} + \beta\omega_2^2 \frac{\partial W}{\partial\, l} \right] \phi\, dy \\ =& \int_{\mathbb{R}^2} \left[\triangle \frac{\partial W}{\partial\, l} - \widetilde{\lambda}_1 \frac{\partial W}{\partial\, l} + 3\mu_1\, W^2 \frac{\partial W}{\partial\, l} \right] \phi\, dy + \int_{\mathbb{R}^2} \beta(\omega_2^2 - \omega_2^2(P_0)) \frac{\partial W}{\partial\, l} \phi\, dy \,, \end{aligned}$$

where $\widetilde{\lambda}_1 = \lambda_1 - \beta\omega_2^2(P_0)$. By (3.24) and (3.27), it is obvious that

$$\begin{aligned} \left| \int_{\mathbb{R}^2} \beta(\omega_2^2 - \omega_2^2(P_0)) \frac{\partial W}{\partial\, l} \phi\, dy \right| \leq& C|\omega_2'(P_0)| \int_{\mathbb{R}^2} |\varepsilon\, y - P_0| \left| \frac{\partial W}{\partial\, l} \right| |\phi|\, dy \\ \leq& C\varepsilon^2, \end{aligned} \tag{3.39}$$

and

$$\begin{aligned} \left| \int_{\mathbb{R}^2} \left[\triangle \frac{\partial W}{\partial\, l} - \widetilde{\lambda}_1 \frac{\partial W}{\partial\, l} + 3\mu_1\, W^2 \frac{\partial W}{\partial\, l} \right] \phi\, dy \right| \leq& C \left(\varepsilon + \widetilde{\omega} \left(2l \sin \frac{\pi}{k} \right) \right) \varepsilon \\ \leq& C\varepsilon^2 + C\varepsilon\widetilde{\omega} \left(2l \sin \frac{\pi}{k} \right) . \end{aligned} \tag{3.40}$$

Here we have used the fact that

$$\triangle \frac{\partial W}{\partial\, l} - \widetilde{\lambda}_1 \frac{\partial W}{\partial\, l} + 3\mu_1 \sum_{i=1}^{k} \widetilde{\omega}_i^2 \frac{\partial \widetilde{\omega}_i}{\partial l} - \frac{\partial \widetilde{\lambda}_1}{\partial l} W = 0 \quad \text{ in } \mathbb{R}^2 \,,$$

and $\frac{\partial \widetilde{\lambda}_1}{\partial l} = O(\varepsilon)$ since $W = \sum_{i=1}^k \widetilde{\omega}_i$ and $\widetilde{\omega}_i$'s satisfy (3.14). Consequently,

$$I_2 = O\left(\varepsilon^2 + \widetilde{\omega}^2 \left(2l \sin \frac{\pi}{k} \right) \right) . \tag{3.41}$$

Now it remains to compute I_1:

$$I_1 = \int_{\mathbb{R}^2} \left[\triangle\, W - \left(\lambda_1 - \beta\omega_2^2 \right) W + \mu_1\, W^3 \right] \frac{\partial W}{\partial\, l}\, dy + \int_{\mathbb{R}^2} \beta[(T[\beta\, W^2])^2 - \omega_2^2]\, W \frac{\partial W}{\partial\, l}\, dy.$$

Note that

$$||T[\beta\, W^2] - \omega_2||_{H^2(\mathbb{R}^2)} \leq\ C\varepsilon \,, \tag{3.42}$$

and then by Sobolev embedding,

$$||T[\beta\, W^2] - \omega_2||_{C^1(\mathbb{R}^2)} \leq\ C\varepsilon \,. \tag{3.43}$$

Hence

$$\begin{aligned} & \int_{\mathbb{R}^2} \beta[(T[\beta\, W^2])^2 - \omega_2^2]\, W \frac{\partial W}{\partial\, l}\, dy \\ =& \int_{\mathbb{R}^2} \beta[(T[\beta\, W^2])^2(P_0) - \omega_2^2(P_0)]\, W \frac{\partial W}{\partial\, l}\, dy \\ &+ \int_{\mathbb{R}^2} \beta \left\{ [(T[\beta\, W^2])^2(x) - \omega_2^2(x)] - [(T[\beta\, W^2])^2(P_0) - \omega_2^2(P_0)] \right\}\, W \frac{\partial W}{\partial\, l}\, dy \\ =& O(\varepsilon) \int_{\mathbb{R}^2} |x - P_0|\, W \left| \frac{\partial W}{\partial\, l} \right|\, dy \\ =& O(\varepsilon^2), \end{aligned} \tag{3.44}$$

where we have used (3.43), $x = P_0 + \varepsilon\, y$, $P_0 = (\varepsilon\, l, 0)$ and $\int_{\mathbb{R}^2} W \frac{\partial W}{\partial l}\, dy = 0$. Finally, by (3.14), we obtain

$$
\begin{aligned}
&\int_{\mathbb{R}^2} \left[\triangle W - (\lambda_1 - \beta\omega_2^2)\, W + \mu_1\, W^3\right] \frac{\partial W}{\partial\, l}\, dy \\
=&\int_{\mathbb{R}^2} \left\{ \left[\beta\omega_2^2 - \beta\omega_2^2(P_0)\right]\, W\, dy + \mu_1 \left[\left(\sum_{i=1}^{k} \widetilde{\omega}_i \right)^3 - \sum_{i=1}^{k} \widetilde{\omega}_i^3 \right] \right\} \frac{\partial W}{\partial\, l}\, dy \\
=&I_{11} + I_{22},
\end{aligned}
$$

where

$$
\begin{aligned}
I_{11} &= \beta \int_{\mathbb{R}^2} [\omega_2^2 - \omega_2^2(P_0)]\, W \frac{\partial W}{\partial\, l}\, dy\,, \\
I_{12} &= \int_{\mathbb{R}^2} \mu_1 \left[\left(\sum_{i=1}^{k} \widetilde{\omega}_i \right)^3 - \sum_{i=1}^{k} \widetilde{\omega}_i^3 \right] \frac{\partial W}{\partial\, l}\, dy\,.
\end{aligned}
$$

By symmetry,

$$
\begin{aligned}
I_{11} =&\beta \int_{\mathbb{R}^2} [\omega_2^2(x) - \omega_2^2(P_0)]\, W \frac{\partial W}{\partial\, l}\, dy \\
=&\beta k \int_{\Gamma_0} [\omega_2^2(x) - \omega_2^2(P_0)]\, W \frac{\partial W}{\partial\, l}\, dy\,, \\
I_{12} =&\mu_1 k \int_{\Gamma_0} \left[\left(\sum_{i=1}^{k} \widetilde{\omega}_i \right)^3 - \sum_{i=1}^{k} \widetilde{\omega}_i^3 \right] \frac{\partial W}{\partial\, l}\, dy\,,
\end{aligned}
$$

where

$$
\Gamma_0 = \left\{ (r\cos\theta, r\sin\theta) : r \geq 0\,, -\frac{\pi}{k} < \theta < \frac{\pi}{k} \right\}\,.
$$

Thus

$$
\begin{aligned}
I_{11} =&\beta\, k \int_{\Gamma_0} 2\omega_2(P_0)\omega_2'(P_0) \left[\frac{P_0}{|P_0|} \cdot (x - P_0) \right] W \frac{\partial W}{\partial\, l}\, dy + O(\varepsilon^2) \\
=&\beta\, k \int_{\Gamma_0} 2\omega_2(0)\omega_2''(0)|P_0|\varepsilon\, y_1 \widetilde{\omega} \left(-\frac{\partial \widetilde{\omega}}{\partial\, y_1} \right) dy + O(\varepsilon^2) \\
=&\beta\, k \left(C_0 \int_{\mathbb{R}^2} \widetilde{\omega}^2\, dy \right) \omega_2(0)\omega_2''(0)\varepsilon^2 l + O(\varepsilon^2) := C_1 \varepsilon^2 l + O(\varepsilon^2),
\end{aligned}
\tag{3.45}
$$

where $x = P_0 + \varepsilon\, y$, $P_0 = (\varepsilon\, l, 0, 0)$, $C_1 := \beta\, k \left(c_0 \int_{\mathbb{R}^n} \widetilde{\omega}^2\right) \omega_2(0)\omega_2''(0) > 0$, and C_0 is a positive constant.

To estimate I_{12}, we observe that

$$
\begin{aligned}
I_{12} =&\mu_1 k \int_{\Gamma_0} 3\widetilde{\omega}_1^2 \sum_{i=2}^{k} \widetilde{\omega}_i \left(-\frac{\partial \widetilde{\omega}_1}{\partial\, y_1} \right) + O(\varepsilon^2) \\
=&\mu_1 k \int_{\Gamma_0} 3\widetilde{\omega}_1^2 (\widetilde{\omega}_2 + \widetilde{\omega}_k) \left(-\frac{\partial \widetilde{\omega}_1}{\partial\, y_1} \right) + O(\varepsilon^2).
\end{aligned}
$$

In particular,

$$
\int_{\Gamma_0} 3\widetilde{\omega}_1^2 \widetilde{\omega}_2 \left(-\frac{\partial \widetilde{\omega}_1}{\partial\, y_1} \right) dy = \int_{\mathbb{R}^2} \widetilde{\omega}_1^3 \frac{\partial \widetilde{\omega}_2}{\partial\, y_1}\, dy + O(\varepsilon^2).
$$

Note that

$$
\begin{aligned}
\widetilde{\omega}_1 =& \widetilde{\omega}\left(\frac{|x-P_1|}{\varepsilon}\right) = \widetilde{\omega}\left(\frac{|\varepsilon\, y + P_0 - P_1|}{\varepsilon}\right) = \widetilde{\omega}(|y|),\\
\frac{\partial \widetilde{\omega}_2}{\partial\, y_1} =& \widetilde{\omega}'\left(\left|y+\frac{P_0-P_2}{\varepsilon}\right|\right)\cdot \frac{y_1 + \frac{<P_0-P_2, e_1>}{\varepsilon}}{\left|y+\frac{P_0-P_2}{\varepsilon}\right|}\\
=& \widetilde{\omega}'\left(\frac{|P_0-P_2|}{\varepsilon}\right)\left(\frac{<P_0-P_2,e_1>}{|P_0-P_2|}\right)\cdot(1+O(\varepsilon|y|^2))e^{-\frac{<P_0-P_2,y>}{|P_0-P_2|}}.
\end{aligned}
$$

Consequently,

$$
\begin{aligned}
\int_{\mathbb{R}^2}\widetilde{\omega}_1^3\frac{\partial\widetilde{\omega}_2}{\partial\, y_1}\,dy =& \left(\int_{\mathbb{R}^2}\widetilde{\omega}^3(|y|)e^{-\frac{<P_0-P_2,y>}{|P_0-P_2|}}\,dy\right)\cdot\widetilde{\omega}'\left(\frac{|P_0-P_2|}{\varepsilon}\right)\frac{<P_0-P_2,e_1>}{|P_0-P_2|}\\
=& C_2\widetilde{\omega}'\left(2l\sin\frac{\pi}{2k}\right)+O(\varepsilon^2),
\end{aligned}
\tag{3.46}
$$

where

$$
C_2 = \left(\int_{\mathbb{R}^2}\widetilde{\omega}^3(|y|)e^{-\frac{<P_0-P_2,y>}{|P_0-P_2|}}\,dy\right)\cdot\frac{<P_0-P_2,e_1>}{|P_0-P_2|} > 0. \tag{3.47}
$$

Hence

$$
I_{12} = \hat{C}_2\widetilde{\omega}'\left(2l\sin\frac{\pi}{2k}\right)+O(\varepsilon^2)\,, \tag{3.48}
$$

where $\hat{C}_2$ is a positive constant independent of ε. Therefore by (3.45) and (3.48), we have

$$
I_{11}+I_{12} = C_1\varepsilon^2 l + \hat{C}_2\widetilde{\omega}'\left(2l\sin\frac{\pi}{2k}\right)+O(\varepsilon^2). \tag{3.49}
$$

In summary, we have

$$
c_l = \widetilde{C}_1\varepsilon^2 l + \widetilde{C}_2\widetilde{\omega}'\left(2l\sin\frac{\pi}{2k}\right)+O(\varepsilon^2), \tag{3.50}
$$

where $\widetilde{C}_1$, $\widetilde{C}_2>0$ are positive generic constants independent of ε.

3.6 Proof of Theorem 2.1

We prove Theorem 2.1 by a continuity argument. Note that

$$
\widetilde{\omega}'(2l\sin\frac{\pi}{2k}) = -A_0\left(2l\sin\frac{\pi}{2k}\right)^{-\frac{1}{2}}e^{-2l\sin\frac{\pi}{2k}}\left(1+O\left(\frac{1}{l}\right)\right), \tag{3.51}
$$

where $A_0>0$ is a constant independent of ε. Let $\alpha=(1-\eta)/\sin\frac{\pi}{2k}$, and $\gamma=(1+\eta)/\sin\frac{\pi}{2k}$, where $0<\eta\ll 1$ is a small constant independent of ε. Then by (3.50), we have

$$
\begin{aligned}
c_l =& \widetilde{C}_1\varepsilon^2 l - \widetilde{C}_2\,A_0\left[2\alpha\left(\sin\frac{\pi}{2k}\right)\cdot\log\frac{1}{\varepsilon}\right]^{-\frac{1}{2}}e^{-2\alpha\sin\frac{\pi}{2k}\cdot\log\frac{1}{\varepsilon}}\\
\le& -\varepsilon^{2-\eta}<0\,,
\end{aligned}
\tag{3.52}
$$

provided $l=\alpha\log\frac{1}{\varepsilon}$ and $\varepsilon>0$ is small enough. Here we have used the fact that $\alpha=(1-\eta)/\sin\frac{\pi}{2k}$. On the other hand, if $l=\gamma\log\frac{1}{\varepsilon}$, then by (3.51) and $\gamma=(1+\eta)/\sin\frac{\pi}{2k}$, we obtain

$$
\widetilde{\omega}'(2l\sin\frac{\pi}{2k}) = O\left(\varepsilon^{2(1+\eta)}\right).
$$

Moreover, (3.50) may give

$$
c_l \ge \frac{1}{2}\widetilde{C}_1\,\varepsilon^2\,l>0\,,
$$

as $\varepsilon>0$ is sufficiently small. Since c_l is continuous to l, there exists $l_\varepsilon\in(\alpha\log\frac{1}{\varepsilon},\gamma\log\frac{1}{\varepsilon})$ such that $c_{l_\varepsilon}=0$, which implies that $S[W+\phi_{l_\varepsilon}]=0$. Therefore by setting $u_\varepsilon=W+\phi_{l_\varepsilon}$ and $v_\varepsilon=T[\beta u_\varepsilon^2]$, we may complete the proof of Theorem 2.1.

4 Proof of Theorem 2.2

In this section, we prove Theorem 2.2. Let us explain the main ideas as follows: Suppose the solution $(u_\varepsilon, v_\varepsilon)$ of the system (1.11) formally having

$$v_\varepsilon \sim \omega_2(r)\,. \tag{4.1}$$

Then substituting (4.1) into the equation of u in (1.11), we find u_ε satisfies (formally)

$$\varepsilon^2 \triangle u - V(r)u + \mu_1 u^3 = 0 \quad \text{in } \mathbb{R}^2\,, \tag{4.2}$$

where $V(r) = \lambda_1 - \beta\omega_2^2(r)$. For the equation (4.2), Ambrosetti, Malchiodi and Ni [2] have showed that as long as $M(r) = r^{2/3}V(r)$ has a point of local strict maximum or minimum at some $\bar{r} > 0$, then there exists a positive solution concentrating on a circle. The main problem here is to control the error induced by v_ε.

4.1 Solving v_ε first

As for the proof of Theorem 2.1, we consider $\Sigma_2 = \{u = u(r)\}$ the class of all radial functions and we have

Lemma 4.1. *There exists a $\delta > 0$ such that if $g = g(r)$ satisfies*

$$||g||_{L^p(\mathbb{R}^2)} < \delta, \tag{4.3}$$

where $1 < p < 2$, then the equation

$$\triangle v - \lambda_2 v + \mu_2 v^3 + gv = 0 \quad \textit{in } \mathbb{R}^2\,, \quad \textit{and} \quad v \in \Sigma_2$$

has a unique solution $v = v(r) \equiv T_2[g] \in W^{2,p}(\mathbb{R}^2)$ satisfying

$$||v - \omega_2||_{W^{2,p}(\mathbb{R}^2)} \leq C||g||_{L^p(\mathbb{R}^2)}$$

where C is a positive constant independent of g and δ.

Proof. Replacing Σ_1 by Σ_2, one may follow similar arguments of Lemmas 3.1 and 3.2 to complete the proof of Lemma 4.1. Note that $p > 1$ may assure the Sobolev embedding $W^{2,p}(\mathbb{R}^2)$ into $L^\infty(\mathbb{R}^2)$. $\square$

4.2 Approximate Solutions

For $t > 0$, let

$$U_t(r) = \left(\sqrt{V(t)}\, U\left(\sqrt{V(t)}\,\frac{|r-t|}{\varepsilon}\right)\right)\eta(r), \quad \forall\, r > 0\,,$$

where U is defined in (2.3), $V(r) = \lambda_1 - \beta\omega_2^2(r)$ and $\eta(r) = 1$ for $r \in [\alpha, \gamma]$ and $\eta(r) = 0$ for $r \in [0, \alpha/2] \cup [2\gamma, +\infty)$. Here α and γ are positive constants such that $0 < \alpha < t < \gamma$. Note that by (2.3), U_t satisfies

$$\varepsilon^2\, U_t'' - V(t)U_t + \mu_1\, U_t^3 = 0\,, \quad \forall\, r \in [\alpha, \gamma]\,, \tag{4.4}$$

and for $r \notin [\alpha, \gamma]$, U_t decays to zero exponentially as ε goes to zero. For $t > 0$, let

$$Z_t(r) := 3U^2\, U'\left(\sqrt{V(t)}\frac{|r-t|}{\varepsilon}\right)\eta(r)\,, \quad \forall\, r > 0\,.$$

4.3 Linear and Nonlinear Reductions

Let

$$||u||_{*,2} = \left\| e^{\sigma|r-t|/\varepsilon} u \right\|_{L^\infty(\mathbb{R}^2)}, \ ||E||_{**,2} = \left\| e^{\sigma|r-t|/\varepsilon} E \right\|_{L^\infty(\mathbb{R}^2)},$$

where $0 < \sigma < 1$ is a small number independent of ε. Then we have

Lemma 4.2. *There exists a unique solution $(\phi_t(r), d_t)$ such that*

$$\begin{cases} S_2[U_t + \phi_t] = d_t Z_t, \\ \int_{\mathbb{R}^2} U_t Z_t = 0, \end{cases} \tag{4.5}$$

where

$$S_2[u] = \varepsilon^2 \left(u'' + \frac{1}{r} u' \right) - \lambda_1 u + \mu_1 u^3 + \beta \left(T_2[\beta\, u^2] \right)^2 u. \tag{4.6}$$

Furthermore,

$$||\phi_t||_{*,2} \le C\varepsilon^{1/p}. \tag{4.7}$$

Proof. Let $r = t + \varepsilon\, y \in [\alpha, \gamma]$. Then it is easy to compute that

$$S_2[U_t] = U'' V(t)^{3/2} - \lambda_1 V(t)^{1/2} U + \mu_1 V(t)^{3/2} U^3 + \frac{\varepsilon}{t + \varepsilon\, y} V(t) U' + \beta\, V(t)^{1/2} \left(T_2[\beta\, U_t^2] \right)^2 U\,.$$

Hence by (2.3), we have

$$S_2[U_t] = \frac{\varepsilon}{t + \varepsilon\, y} V(t) U' + \beta\, V(t)^{1/2} \left(T_2[\beta\, U_t^2]^2 (t + \varepsilon\, y) - \omega_2^2(t) \right) U\,. \tag{4.8}$$

By Lemma 4.1,

$$\begin{aligned} T_2[\beta\, U_t^2](t + \varepsilon\, y) =& \omega_2(t + \varepsilon\, y) + O(||U_t^2||_{L^p(\mathbb{R}^2)}) \\ =& \omega_2(t + \varepsilon\, y) + O(\varepsilon^{1/p}) \\ =& \omega_2(t) + O\left(\varepsilon^{1/p} + \frac{|\varepsilon\, y|}{1 + |\varepsilon\, y|} \right). \end{aligned} \tag{4.9}$$

Thus (4.8) and (4.9) give

$$\begin{aligned} S_2[U_t] =& \frac{\varepsilon}{t + \varepsilon\, y} V(t) U' + O(\varepsilon^{1/p} U) \\ =& O(\varepsilon\, U' + \varepsilon^{1/p} U), \end{aligned} \tag{4.10}$$

which implies that

$$||S_2[U_t]||_{**,2} \le C\varepsilon^{1/p}. \tag{4.11}$$

The rest of the proof is similar to [17] so we omit the details here. □

4.4 Expansion of d_t

Let $\tilde{U}(y) = V(t) U'(\sqrt{V(t)}\, y)$ and set $r = t + \varepsilon\, y$. Then $\frac{d}{dy} U_t(r) = \tilde{U}(y)$ for $r = t + \varepsilon\, y \in [\alpha, \gamma]$. Hence (4.4) implies

$$\tilde{U}'' - V(t)\tilde{U} + 3\mu_1 U_t^2 \tilde{U} = 0\,, \quad \forall\, r = t + \varepsilon\, y \in [\alpha, \gamma]\,. \tag{4.12}$$

We may multiply (4.5) by $\tilde{U}(y)\eta(r)$ and integrate it over $\mathbb{R}^2$ with respect to y variable. It is easy to calculate that

$$
\begin{aligned}
d_t \int_{\mathbb{R}^2} Z_t \tilde{U}\eta =& \int_{\mathbb{R}^2} S_2[U_t]\tilde{U}\eta + \int_{\mathbb{R}^2} \left(\varepsilon^2 \left(\frac{d^2}{dr^2} + \frac{1}{r}\frac{d}{dr}\right)\phi - \lambda_1\phi + 3\mu_1 U_t^2\phi + \beta(T_2[\beta\, U_t^2])^2\phi\right)\tilde{U}\eta \\
&+ \int_{\mathbb{R}^2} N[\phi]\tilde{U}\eta + \int_{\mathbb{R}^2} \beta\left(T_2[\beta(U_t+\phi)^2]^2 - T_2[\beta\, U_t^2]^2\right)(U_t+\phi)\,\tilde{U}\eta \\
:=& J_1 + J_2 + J_3 + J_4\,,
\end{aligned}
\tag{4.13}
$$

where $\phi = \phi_t(r)$ defined in Lemma 4.2, $N[\phi] = \mu_1\left(3U_t\phi^2 + \phi^3\right)$,

$$
\begin{aligned}
J_1 &= \int_{\mathbb{R}^2} S_2[U_t]\tilde{U}\eta\,, \\
J_2 &= \int_{\mathbb{R}^2} \left(\varepsilon^2 \left(\frac{d^2}{dr^2} + \frac{1}{r}\frac{d}{dr}\right)\phi - \lambda_1\phi + 3\mu_1 U_t^2\phi + \beta(T_2[\beta\, U_t^2])^2\phi\right)\tilde{U}\eta\,, \\
J_3 &= \int_{\mathbb{R}^2} N[\phi]\tilde{U}\eta\,, \\
J_4 &= \int_{\mathbb{R}^2} \beta\left(T_2[\beta(U_t+\phi)^2]^2 - T_2[\beta\, U_t^2]^2\right)(U_t+\phi)\,\tilde{U}\eta\,.
\end{aligned}
$$

By (4.7), we have

$$
J_3 = O(\varepsilon^{2/p})\,. \tag{4.14}
$$

As for the proof of (3.38), we may obtain

$$
\left\|T_2[\beta(U_t+\phi)^2] - T_2[\beta\, U_t^2]\right\|_{L^\infty([\alpha,\gamma])} \le C\varepsilon^{2/p}\,,
$$

and hence

$$
J_4 = O(\varepsilon^{2/p})\,. \tag{4.15}
$$

For J_2, we may use integration by parts to get

$$
\begin{aligned}
J_2 =& \int_{\mathbb{R}^2} \left[\varepsilon^2 \left(\frac{d^2}{dr^2} + \frac{1}{r}\frac{d}{dr}\right)(\tilde{U}\eta) - \lambda_1\tilde{U}\eta + 3\mu_1 U_t^2\tilde{U}\eta + \beta(T_2[\beta\, U_t^2])^2\tilde{U}\eta\right]\phi \\
=& \int_{\mathbb{R}^2} \left[\varepsilon^2 \left(\frac{d^2}{dr^2} + \frac{1}{r}\frac{d}{dr}\right)(\tilde{U}\eta) - \lambda_1\tilde{U}\eta + 3\mu_1 U_t^2\tilde{U}\eta + \beta\omega_2^2(t)\,\tilde{U}\eta\right]\phi + O(\varepsilon^{2/p}) \\
=& \int_{\mathbb{R}^2} \left[\tilde{U}''\eta + \frac{\varepsilon}{t+\varepsilon\, y}\tilde{U}'\eta - V(t)\tilde{U}\eta + 3\mu_1 U_t^2\tilde{U}\eta\right]\phi + O(\varepsilon^{2/p}) \\
=& O(\varepsilon^{2/p}).
\end{aligned}
\tag{4.16}
$$

Here we have used (4.9) and (4.12). Now it remains to estimate J_1. We may use (4.8) to get

$$
\begin{aligned}
J_1 =& \int_{\mathbb{R}^2} S_2[U_t]\tilde{U}\eta \\
=& \int_{\mathbb{R}^2} \left[\frac{\varepsilon}{t+\varepsilon\, y}V(t)U' + \beta\, V(t)^{1/2}\left(T_2[\beta\, U_t^2]^2(t+\varepsilon\, y) - \omega_2^2(t)\right)U\right]\tilde{U}\eta \\
=& \int_{\mathbb{R}^2} \left[\frac{\varepsilon}{t+\varepsilon\, y}V(t)U' - (V(t+\varepsilon\, y) - V(t))V^{1/2}(t)U\right]\tilde{U}\eta \\
&+ \int_{\mathbb{R}^2} \beta\, V(t)^{1/2}\left(T_2[\beta U_t^2]^2 - \omega_2^2\right)(t+\varepsilon\, y)U\tilde{U}\eta \\
:=& J_{11} + J_{12}\,,
\end{aligned}
$$

where

$$J_{11} = \int_{\mathbb{R}^2} \left[\frac{\varepsilon}{t+\varepsilon\, y} V(t) U' - (V(t+\varepsilon\, y) - V(t)) V^{1/2}(t) U \right] \tilde{U} \eta \,,$$
$$J_{12} = \int_{\mathbb{R}^2} \beta\, V(t)^{1/2} \left(T_2[\beta U_t^2]^2 - \omega_2^2 \right) (t+\varepsilon\, y) U \tilde{U} \eta \,.$$

For J_{12}, we have

$$\begin{aligned} J_{12} =& \int_{\mathbb{R}^2} \beta\, V(t)^{1/2} \left\{ (T_2[\beta\, U_t^2]^2 - \omega_2^2)(t+\varepsilon\, y) - (T_2[\beta\, U_t^2]^2 - \omega_2^2)(t) \right\} U \tilde{U} \eta + O(\varepsilon^2) \\ =& O\left(||T_2[\beta\, U_t^2] - \omega_2||_{W^{2,p}(\mathbb{R}^2)} \int_{\mathbb{R}^2} \varepsilon |y| |U \tilde{U} \eta| \right) + O(\varepsilon^2) \\ =& O(\varepsilon^{2/p}). \end{aligned} \tag{4.17}$$

For J_{11}, we have

$$\begin{aligned} J_{11} =& \int_{\mathbb{R}^2} \left[\frac{\varepsilon}{t+\varepsilon\, y} V(t) U' - (V(t+\varepsilon\, y) - V(t)) V^{1/2}(t) U \right] \tilde{U} \eta \\ =& \varepsilon\, V(t) \left[\frac{1}{t} V(t) \int_{\mathbb{R}^2} (U'(\sqrt{V(t)} y))^2 dy - V'(t) V^{1/2}(t) \int_{\mathbb{R}^2} U(\sqrt{V(t)} y) U'(\sqrt{V(t)} y) y dy \right] + O(\varepsilon^{2/p}) \\ =& \varepsilon \sqrt{V(t)} \left[\frac{1}{t} V(t) \int_0^\infty U'(z)^2 \, dz - V'(t) \int_0^\infty U U'(z) z \, dz \right] + O(\varepsilon^{2/p}) \\ =& \varepsilon \sqrt{V(t)} \left[\frac{V(t)}{t} \int_0^\infty U'(z)^2 dz + \frac{1}{2} V'(t) \int_0^\infty U^2(z) \, dz \right] + O(\varepsilon^{2/p}) \\ =& \varepsilon\, c_0 \sqrt{V(t)}\, t^{-2/3}\, M'(t) + o(\varepsilon) \,, \end{aligned} \tag{4.18}$$

where $c_0 = \frac{1}{2} \int_0^\infty U^2(z)\, dz > 0$ and $M(t) = t^{2/3}\, V(t)$. Here we have used the following identity:

$$\int_0^\infty (U'(z))^2 dz = \frac{1}{3} \int_0^\infty U^2(z) dz. \tag{4.19}$$

Combining (4.13)-(4.18), we may obtain

$$d_t = \varepsilon \widetilde{c}_0\, t^{-2/3} M'(t) + o(\varepsilon) \,, \tag{4.20}$$

where $\tilde{c}_0 \neq 0$ and $o(1)$ is a small quantity tending to zero as ε goes to zero.

4.5 Proof of Theorem 2.2

Let $H(t) = t^{-2/3} M'(t)$ for $t > 0$. Then it is obvious that

$$\lim_{t \to 0^+} H(t) > 0 \quad \text{and} \quad \lim_{t \to \infty} H(t) > 0 \,.$$

By (2.5) (see Remark 1), there exists $r_0 > 0$ such that $H(r_0) < 0$. Hence there exists $[\alpha_2, \beta_2] \subset (0, r_0)$ such that $H(\alpha_2) > 0 > H(\beta_2)$. By (4.20) and the continuity of $H(t)$, there exists $\gamma_{\varepsilon,1} \in (0, r_0)$ such that $d_{\gamma_{\varepsilon,1}} = 0$. Thus

$$S_2[U_{\gamma_{\varepsilon,1}} + \phi_{\gamma_{\varepsilon,1}}] = 0$$

and $(U_{\gamma_{\varepsilon,1}} + \phi_{\gamma_{\varepsilon,1}}, T_2[\beta(U_{\gamma_{\varepsilon,1}} + \phi_{\gamma_{\varepsilon,1}})^2])$ satisfies the properties of Theorem 2.2. Similarly, we can find $\gamma_{\varepsilon,2} \in (r_0, +\infty)$ such that $d_{\gamma_{\varepsilon,2}} = 0$ and $(U_{\gamma_{\varepsilon,2}} + \phi_{\gamma_{\varepsilon,2}}, T_2[\beta_j(U_{\gamma_{\varepsilon,2}} + \phi_{\gamma_{\varepsilon,2}})^2])$ becomes the second solution. Therefore we may complete the proof of Theorem 2.2.

5 Numerical Investigations

We use the solver BVP4C in MATLAB to find solutions of (1.13) with ring profiles including a single ring profile, a double ring profile and m-ring profiles for $m \geq 3$. A single ring profile is the graph of a positive function $f = f(r)$ ($r = |x|$ is the radial variable for $x \in \mathbb{R}^2$) with $f(\infty) = 0$ and one bump which means that f is increasing on $(0, r_1)$ but decreasing on (r_1, ∞), where r_1 is a positive constant. A double ring profile is the graph of a positive function $g = g(r)$ with $g(\infty) = 0$ and two bumps which means that g is increasing on $(0, r_2) \cup (r_3, r_4)$ but decreasing on $(r_2, r_3) \cup (r_4, \infty)$, for some positive constants $r_j, j = 2, 3, 4$ with $r_2 < r_3 < r_4$. Similarly, the m-ring profile is the graph of a positive function $h = h(r)$ with $h(\infty) = 0$ and m bumps for $m \geq 3$.

For notation convenience, we may denote the solution of (1.13) as $(u, v) = (u(r), v(r))$ for $r \in [0, \infty)$. Due to the limitation of numerical computations, we can only approximate solutions of (1.13) on a bounded interval $[0, R]$ ($R > 0$). To implement the solver, we firstly need to transform (1.13) into a first-order ODE system by setting $\zeta = u'$ and $\eta = v'$. We want to find the positive solutions (u, v)'s (i.e. $u(r), v(r) > 0$ for $r \geq 0$) with a ring profile, i.e., either u or v has a ring profile. It is necessary to have "good" initial guesses in order to obtain solutions as desired. Otherwise, the solver may generate either an unwanted solution (e.g., a solution (u^*, v^*) with $u^* \equiv 0$) or no solution if an initial guess is not "good" enough. To obtain "good" initial guesses, we firstly choose the initial guess (u_0, v_0) so that $u_0 = u_0(r)$ has a single ring profile away from the origin and $v_0 = v_0(r)$ has a single peak at the origin (see Fig. 1(b)). If (u_0, v_0) is not "good" enough, then we may replace $u_0(r)$ and $v_0(r)$ by $C_1 r^2 e^{-k_1(r - a\, r_{max})^2} u_0(r)$ and $C_2\, e^{-k_2 r^2} v_0(r)$, respectively, where a, C_i's and k_j's are positive constants, and $r_{max} = \arg\max(u_0)$ is the maximum point of u_0. With $\varepsilon^2 = 0.02, \lambda_1 = 2, \lambda_2 = 1, \mu_1 = \mu_2 = 0.5, \beta = -0.05$, we may adjust a, C_i's and k_j's to get the numerical solution (u_1, v_1) of (1.13) with a single ring profile of u_1 and the Townes profile of v_1 (see Fig. 1(a)). Similarly, we may set another "good" initial guess to find the numerical solution (u_2, v_2) with a single ring profile of u_2 and the Townes profile of v_2 (see Fig. 2(a)). Moreover, the numerical solution (u_3, v_3) with a double ring profile of u_3 and the Townes profile of v_3 (see Fig. 2(c)) can be obtained such that $u_3 \approx u_1 + u_2$ and the profiles of v_i's ($i = 1, 2, 3$) are indistinguishable. Our numerical experiments may support Theorem 2.2 and Remark 3. On the other hand, the solver BVP4C also provides the first and second derivatives of numerical solutions which can be substituted into (1.13) to check the computational errors of order $\mathcal{O}(10^{-15})$ (see Fig. 1(c)). This may assure the reliability of our numerical scheme so we may use it to produce further solutions beyond those of Theorem 2.2 and Remark 3.

Besides solutions (u_j, v_j), $j = 1, 2, 3$, we may find the solution (u_4, v_4) with a single ring profile of u_4 and the Townes profile of v_4 (see Fig. 3(a)) under the same numerical parameters as those of (u_j, v_j), $j = 1, 2, 3$. We also obtain the solution (u_5, v_5) with a double ring profile of u_5 and the Townes profile of v_5 (see Fig. 3(b)). The ring profile of u_4 may almost fit the outer ring profile of u_5, and the profiles of v_4 and v_5 are indistinguishable (see Fig. 3(c)). Hence there exist at least two solutions (u, v)'s of (1.13) with a double ring profile of u and the Townes profile of v. Such a result of nonuniqueness can not be obtained from Theorem 2.2 and Remark 3. Further numerical solutions (u, v)'s with m-ring profiles of u and Townes profiles of v are sketched in Fig 4, wherein the same numerical parameters are used as those in Figs. 1-3 except $\varepsilon^2 = 0.01$.

Finally, a new type of numerical solution (u, v) to (1.13) with ring-ring profiles (i.e. the graphs of both u and v are of ring profiles) on the interval $[0, 20]$ is shown in Fig. 5 with $\varepsilon^2 = 0.05, \lambda_1 = 2, \lambda_2 = 1, \mu_1 = \mu_2 = 0.5, \beta = -1$. The ring profile of u concentrates in a narrow region due to the small ε. However, the ring profile of v spreads on a much wider region than that of u. That would make it very difficult to find solutions of (1.13) with ring-ring profiles on the interval $[0, 8]$. On the other hand, until now, there is no theoretical argument to prove the existence of solutions with ring-ring profiles. It would be a nice problem to study in the future.

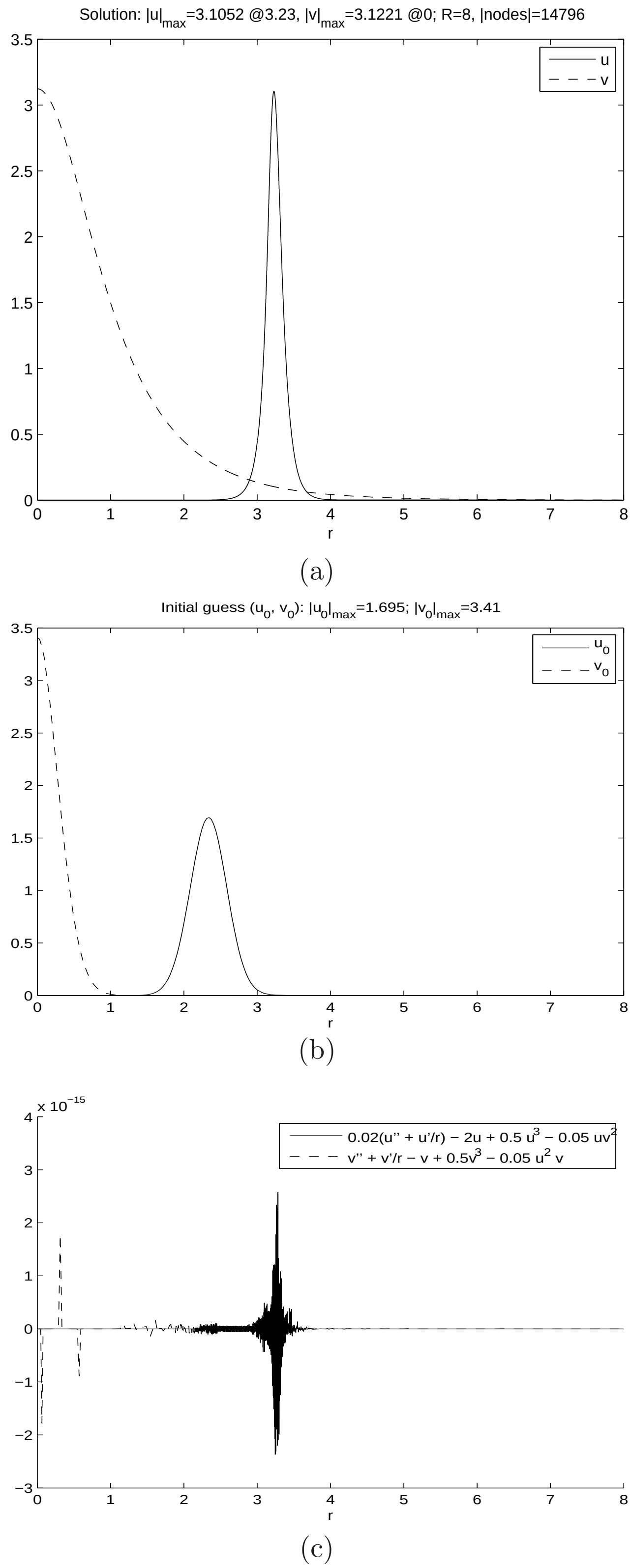


Figure 1: (a) The graph of u_1 and v_1 on $[0, 8]$ with $\varepsilon^2 = 0.02, \lambda_1 = 2, \lambda_2 = 1, \mu_1 = \mu_2 = 0.5, \beta = -0.05$. (b) Initial guess. (c) Computational errors for (u_1, v_1).

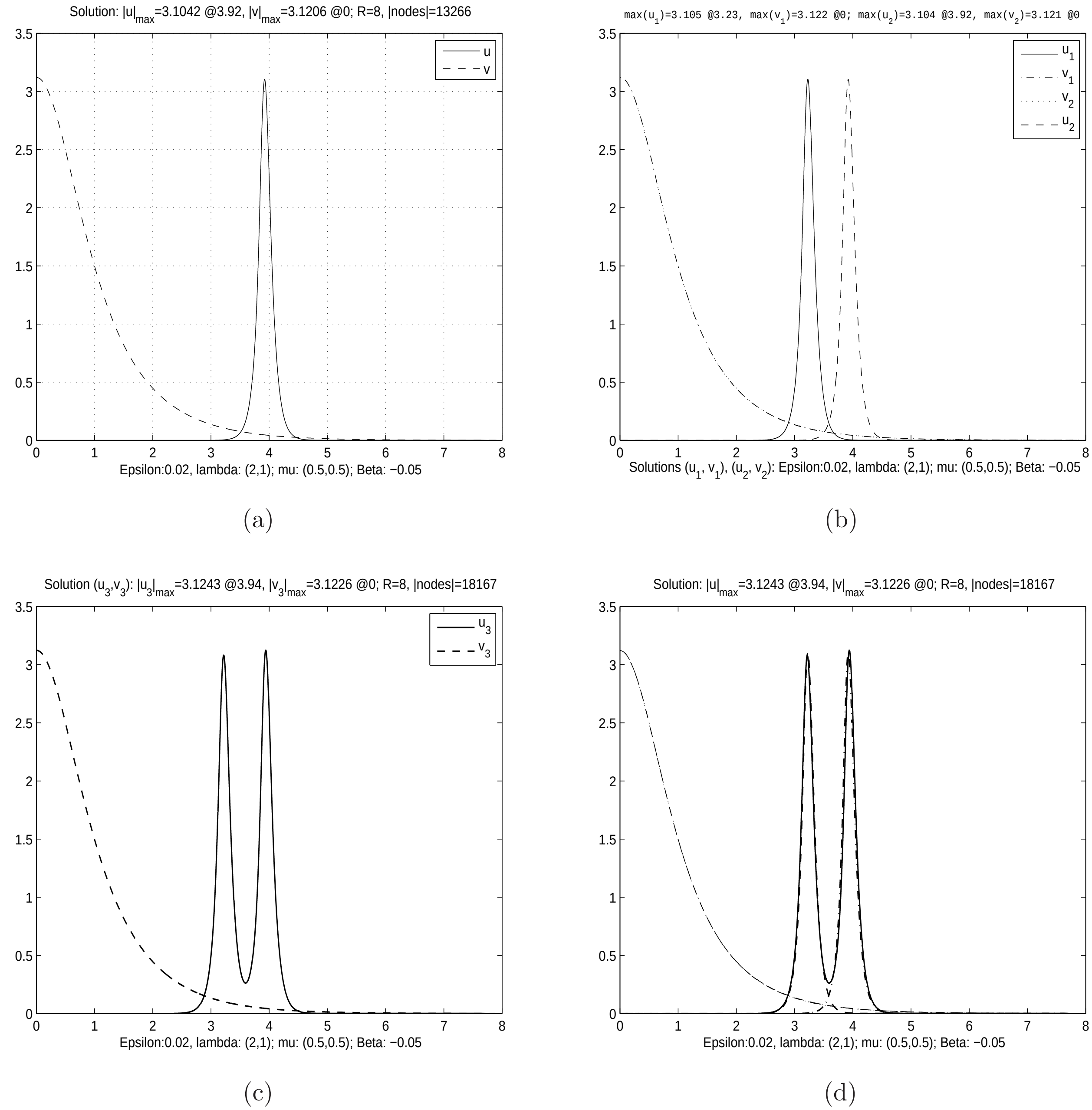
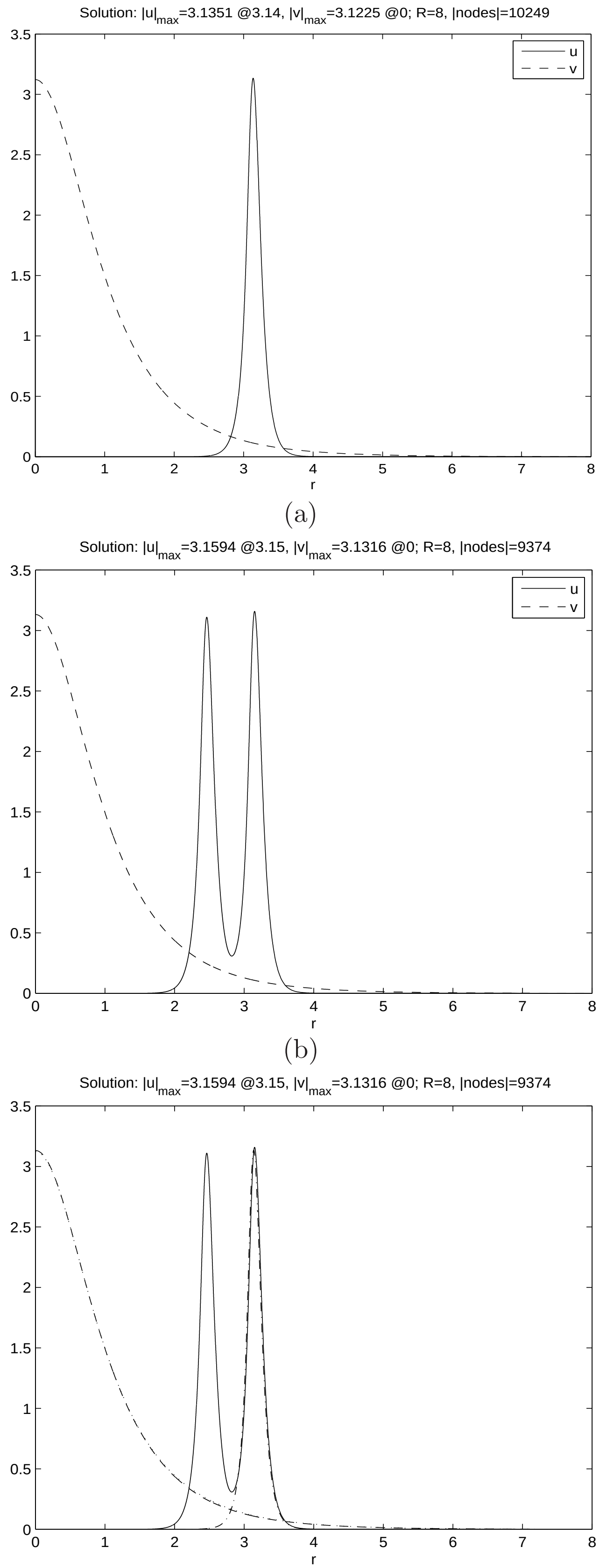


Figure 2: (a) The graph of u_2 and v_2 on $[0,8]$ with the same numerical parameters as used in Fig. 1(a). (b) Plot two solutions $(u_i, v_i)\,(i = 1,2)$ in Fig. 1(a) and Fig. 2(a) together, where v_1 and v_2 are indistinguishable. (c) The graph of u_3 and v_3 on $[0,8]$ with the same numerical parameters as Fig. 1(a). (d) Plot three solutions $(u_i, v_i)\,(i = 1,2,3)$ in Fig. 1(a) and Fig. 2(a)&(c) together, where v_i's are indistinguishable and $u_3 \approx u_1 + u_2$.



(c)

Figure 3: (a) The graph of u_4 and v_4 (b) The graph of u_5 and v_5 (c) Plot (a)&(b) together, where v_i's are indistinguishable, and the ring profile of u_4 may almost fit the outer ring profile of u_5. Same numerical parameters as those in Fig. 1(a).

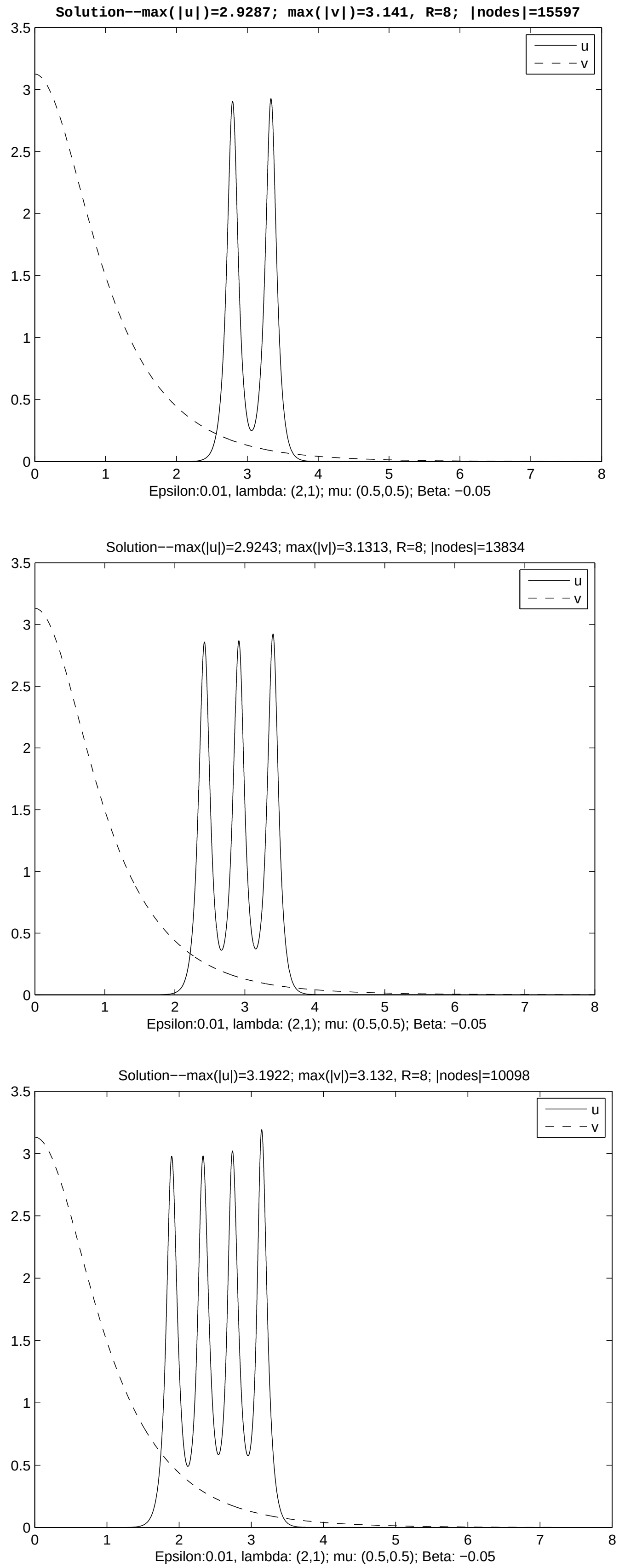


Figure 4: Sketch m-ring profiles on $[0, 8]$ with $\varepsilon^2 = 0.01, \lambda_1 = 2, \lambda_2 = 1, \mu_1 = \mu_2 = 0.5, \beta = -0.05$.

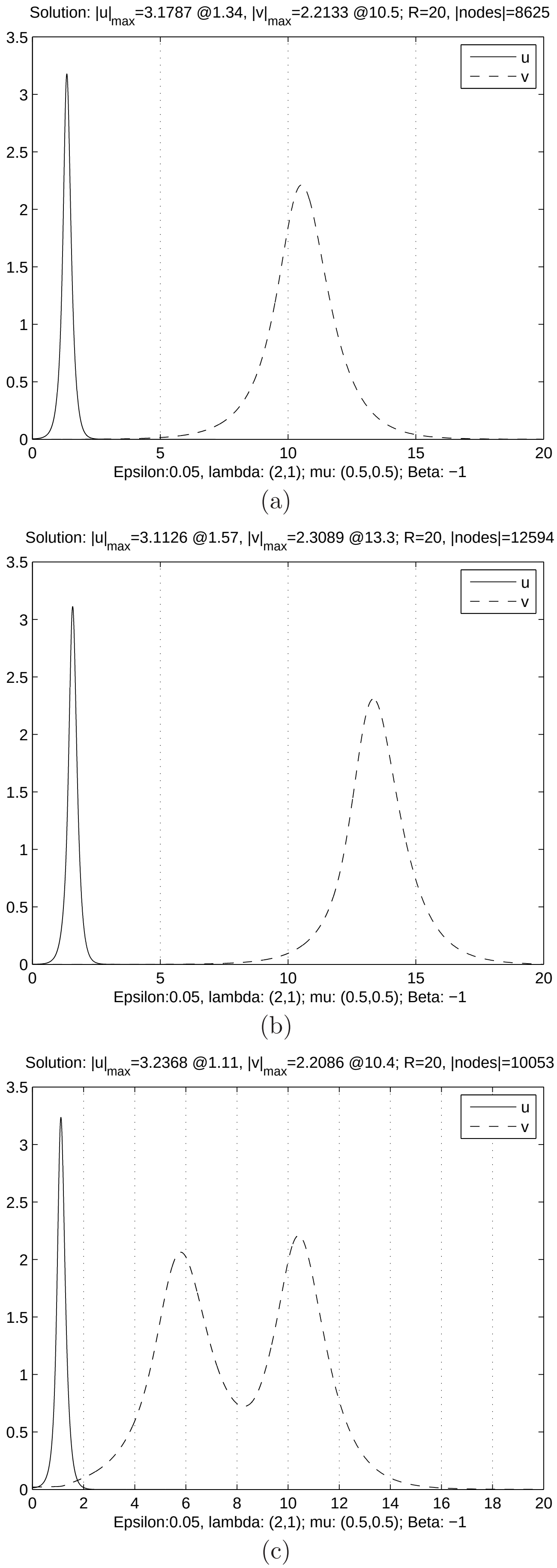


Figure 5: Sketch ring profiles of u and v on $[0, 20]$ with $\varepsilon^2 = 0.05, \lambda_1 = 2, \lambda_2 = 1, \mu_1 = \mu_2 = 0.5, \beta = -1$.

Appendix

Here we give a proof of Lemma 3.1. We assume that $p > 1$ and let $W^{2,p}(\mathbb{R}^2)$ be the usual Sobolev space. Let $\mathcal{V} = W^{2,p}(\mathbb{R}^2) \cap \Sigma_1$ be a normed linear space (in fact, a Banach space) equipped with $W^{2,p}(\mathbb{R}^2)$ norm. For each $f \in L^p(\mathbb{R}^2) \cap \Sigma_1$, there exists a unique $\hat{f} \in \mathcal{V}$ such that

$$\Delta \hat{f} - \lambda_2 \hat{f} = f \text{ in } \mathbb{R}^2. \tag{5.1}$$

Furthermore,

$$\|\hat{f}\|_{W^{2,p}(\mathbb{R}^2)} \le C\|f\|_{L^p(\mathbb{R}^2)}. \tag{5.2}$$

This follows from elliptic regularity. In fact, the solution to (5.1) is given by

$$\hat{f}(x) = -\int_{\mathbb{R}^2} K(|x-y|) f(y) dy \tag{5.3}$$

where $K(r)$ is the unique radial solution of $-\Delta K + \lambda_2 K = \delta_0$. In fact, $K(r)$ is positive and $K(r) \le Cr^{-\frac{1}{2}} e^{-\sqrt{\lambda_2} r}$ for $r > 1$, and for $0 < r \le 1$, $K(r) \le C \log \frac{2}{r}$. See Appendix C of [11] and the book [19].

Now for each $\phi \in \mathcal{V}$, we define the map

$$T[\phi] = (-\Delta + \lambda_2)^{-1} (3\mu_2 \omega_2^2 \phi)\,. \tag{5.4}$$

By (5.3), the map T can be written as

$$T[\phi] = \int_{\mathbb{R}^2} K(|x-y|) \left(3\mu_2 \omega_2^2 \phi\right) dy\,. \tag{5.5}$$

Notice that ω_2 is radially symmetric, $T[\phi] \in \Sigma_1$ if $\phi \in \Sigma_1$. Due to Sobolev inequality and $p > 1$, we have

$$\|\phi\|_{L^\infty(\mathbb{R}^2)} \le C\|\phi\|_{W^{2,p}(\mathbb{R}^2)}\,, \quad \forall\, \phi \in \mathcal{V}\,. \tag{5.6}$$

Consequently,

$$|\omega_2^2 \phi(y)| \le C\|\phi\|_{W^{2,p}(\mathbb{R}^2)} e^{-\sqrt{\lambda_2}|y|} \quad \text{for } \; y \in \mathbb{R}^2\,,\; \phi \in \mathcal{V}\,. \tag{5.7}$$

Here we have used the fact that $\omega_2^2(y)$ decays to zero exponentially as $|y|$ goes to infinity. By (5.5) and (5.7), it is easy to obtain

$$|T[\phi](x)| \le C\, \|\phi\|_{W^{2,p}(\mathbb{R}^2)} \int_{\mathbb{R}^2} K(|z|) e^{-\sqrt{\lambda_2}|x-z|}\, dz\,, \quad \text{for } \; x \in \mathbb{R}^2\,,\; \phi \in \mathcal{V}\,. \tag{5.8}$$

Similarly,

$$|\partial_{x_j}^k\, T[\phi](x)| \le C\, \|\phi\|_{W^{2,p}(\mathbb{R}^2)} \int_{\mathbb{R}^2} |\partial_{z_j}^k\, K(|z|)|\, e^{-\sqrt{\lambda_2}|x-z|}\, dz\,, \tag{5.9}$$

for $x = (x_1, x_2) \in \mathbb{R}^2$, $j, k = 1, 2$ and $\phi \in \mathcal{V}$. Note that $\partial_{z_j}^k\, K(|z|)$'s decay to zero exponentially as $|z|$ goes to infinity. Thus by (5.8), (5.9) and Arzela-Ascoli Theorem, the map T is a compact operator from $\mathcal{V}$ to $\mathcal{V}$.

By Theorem 5.3 of [9], we have the following Fredholm alternatives: either (i) the homogeneous equation

$$\phi - T[\phi] = 0 \tag{5.10}$$

has a nontrivial solution $\phi \in \mathcal{V}$, or (ii) for each $h \in \mathcal{V}$, the equation

$$\phi - T[\phi] = h \tag{5.11}$$

has a uniquely determined solution $\phi \in \mathcal{V}$. Furthermore, in case (ii), the operator $(I - T)^{-1}$ is bounded.

Now we want to claim that (i) is impossible by contradiction. Suppose (i) holds. Then there exists a nontrivial solution to

$$\Delta \phi - \lambda_2 \phi + 3\mu_2 \omega_2^2 \phi = 0\,, \quad \phi \in W^{2,p}(\mathbb{R}^2) \cap \Sigma_1\,. \tag{5.12}$$

Due to $p > 1$ and (5.6), ϕ is bounded. By Lemma C of [20], we have $\phi = \sum_{j=1}^{2} c_j \frac{\partial \omega_2}{\partial y_j}$. Since $\phi \in \Sigma_1$, we conclude that $c_1 = c_2 = 0$ i.e. $\phi \equiv 0$. This may give a contradiction to (i). Hence by Fredholm alternative, (ii) holds and we have

$$\|\phi\|_{W^{2,p}(\mathbb{R}^2)} \leq C\|h\|_{W^{2,p}(\mathbb{R}^2)}. \tag{5.13}$$

Set $h = \hat{f}$. Then (5.2) and (5.13) imply

$$\|\phi\|_{W^{2,p}(\mathbb{R}^2)} \leq C\|\hat{f}\|_{W^{2,p}(\mathbb{R}^2)} \leq C\|f\|_{L^p(\mathbb{R}^2)}. \tag{5.14}$$

Note that the equation $\phi - T[\phi] = \hat{f}$ is equivalent to $\Delta\phi - \lambda_2\phi + 3\mu_2\omega_2^2\phi = f$. Therefore, we may complete the proof of Lemma 3.1. □

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