On Phase-Separation Model: Asymptotics and Qualitative Properties

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Abstract

In this paper we study bound state solutions of a class of two-component nonlinear elliptic systems with a large parameter tending to infinity. The large parameter giving strong intercomponent repulsion induces phase separation and forms segregated nodal domains divided by an interface. To obtain the profile of bound state solutions near the interface, we prove the uniform Lipschitz continuity of bound state solutions when the spatial dimension is $N = 1$. Furthermore, we show that the limiting nonlinear elliptic system that arises has unbounded solutions with symmetry and monotonicity. These unbounded solutions are useful to derive rigorously the asymptotic expansion of the minimizing energy which is consistent with the hypothesis of [19]. When the spatial dimension is $N = 2$, we establish the De Giorgi type conjecture for the blow-up nonlinear elliptic system under suitable conditions at infinity on bound state solutions. These results naturally lead us to formulate De Giorgi type conjectures for this type of systems in higher dimensions.

1 Introduction

In a binary fluid like a mixture of oil and water, the two components of the fluid may spontaneously separate and form two segregated domains divided by an interface. Such a phenomenon called *phase separation* can be observed as well in cooling binary alloys, glasses and polymer mixtures. The well-known Cahn-Hilliard equation has been proposed as a model to describe the process of phase separation (cf. [9]). It is written in the form:

$$\varphi_t = \Delta \frac{\delta F_\varepsilon}{\delta \varphi} = \Delta \left[\varepsilon^2 \Delta\varphi + (1-\varphi^2)\varphi\right] \qquad \text{for} \qquad x \in \Omega\,, t > 0\,,$$

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with no-flux boundary condition

$$\partial_\nu \varphi = \partial_\nu \left[\varepsilon^2 \Delta\varphi + (1-\varphi^2)\varphi\right] = 0 \quad \text{on} \quad \partial\Omega\,,$$

and mass conservation

$$\frac{1}{|\Omega|}\int_\Omega \varphi(x,t)dx = m.$$

Here $\varphi = \varphi(x,t) \in \mathbb{R}$ is the order parameter, $\Omega \subset \mathbb{R}^N$ is the region occupied by the fluid, ∂_ν is the exterior normal derivative on the boundary $\partial\Omega$ and $\varepsilon > 0$ is a small parameter giving the length of transition regions between the domains. The Cahn-Hilliard free energy functional F_ϵ is defined by

$$F_\varepsilon(u) = \int_\Omega \varepsilon^2 |\nabla u|^2 + \frac{1}{2}(1-u^2)^2 \quad \text{for} \quad u \in H^1(\Omega)\,,$$

(see [14]). Stationary solutions with interfaces of the above equations satisfy

$$\varepsilon^2\Delta\varphi + (1-\varphi^2)\varphi = \lambda_\epsilon \quad \text{in} \quad \Omega\,, \quad \text{and} \quad \partial_\nu\varphi = 0 \quad \text{on} \quad \partial\Omega.$$

It is well-known that as $\epsilon \to 0$ and $\lambda_\epsilon \to 0$, the profile of the solution φ near the interface approaches to a solution of the following Allen-Cahn (AC) equation

$$\Delta\Phi + (1-\Phi^2)\Phi = 0 \quad \text{in} \quad \mathbb{R}^N\,.$$

For AC equation as above, De Giorgi [23] formulated in 1978 the following celebrated conjecture:
Let Φ be a bounded solution of AC equation such that $\partial_{x_N}\Phi > 0$. Then the level sets $\{\Phi = \lambda\}$ are hyperplanes, at least for dimension $N \le 8$. The conjecture has been investigated extensively over the recent years and has been essentially settled by now (see Section 7 for detailed discussions).

Indeed, phase separation is known to occur in a double condensate (cf. [25], [33], [34]). In general, however, phase separation models between two components involve a system of partial differential equations. The aim of this paper is to investigate questions analogous to the previous one in the more general framework. One such system of particular interest arises in a binary mixture of Bose-Einstein condensates with two different hyperfine states denoted by $|1\rangle$ and $|2\rangle$ Due to strong inter-component repulsion, interfaces (so called domain walls) may divide the condensate into segregated domains in the same way as in the mixture of oil and water. A classical model to describe this involves the two component Gross-Pitaevskii (GP) system derived from the following GP functional (cf. [36])

$$E = \frac{1}{2}\int_\Omega \sum_{j=1}^{2}\left(\frac{\hbar^2}{2m}|\nabla\Psi_j|^2 + V_j|\Psi_j|^2 + \frac{1}{2}g_{jj}|\Psi_j|^4\right) + g_{12}|\Psi_1|^2\,|\Psi_2|^2\,dx\,.$$

Here $\hbar$ is Planck constant, m is the atom mass, Ω is the domain for condensate dwelling, V_j's are trapping potentials, and Ψ_j's are wave functions corresponding to states $|j\rangle$'s. Besides, $g_{ij} \sim a_{ij}$, where a_{jj}'s and a_{12} are the intraspecies and interspecies scattering lengths. From the variational principle, the model of double condensates can be written as $i\hbar\partial\Psi_j/\partial t = \delta E/\delta\Psi_j^*$ for $j = 1, 2$, that is,

$$i\hbar\frac{\partial\Psi_j}{\partial t} = -\frac{\hbar^2}{2m}\triangle\Psi_j + V_j\Psi_j + g_{jj}|\Psi_j|^2\Psi_j + g_{12}|\Psi_{3-j}|^2\Psi_j\,, \quad j = 1, 2\,,$$

called the coupled Gross-Pitaevskii (GP) equations (cf. [1] and [36]) giving conservation laws as follows:

$$\int_\Omega |\Psi_j|^2 = N_j\,, \qquad \text{for } t>0\,, j=1,2\,,$$

where N_j's are numbers of atoms.

To study phase separation of double condensates, as is explained in [40], we may switch off trapping potentials V_j's and let $V_j \equiv 0, j=1,2$. Due to Feshbach resonance (cf. [22]), we may further set $g_{12} = \frac{\hbar^2}{2m}\Lambda$ and g_{jj}'s as nonnegative constants, where Λ is a large parameter tending to infinity. Then the condition $g_{12}^2 > g_{11}g_{22}$ for phase separation (cf. [3]) is fulfilled and the GP functional becomes

$$E = \int_\Omega \sum_{j=1}^{2} \left(\frac{\hbar^2}{2m}|\nabla\Psi_j|^2 + \frac{1}{2}g_{jj}|\Psi_j|^4 \right) + \frac{\hbar^2}{2m}\Lambda|\Psi_1|^2\,|\Psi_2|^2\,dx\,.$$

To find standing wave solutions of the coupled GP equations, one sets $\Psi_1(x,t) = e^{-i\epsilon_1 t/\hbar}u(x)$ and $\Psi_2(x,t) = e^{-i\epsilon_2 t/\hbar}v(x)$. Here ϵ_j's are chemical potentials and u, v are the corresponding condensate amplitudes (cf. [17]). Then the coupled GP equations become a class of nonlinear elliptic systems that reads as follows:

$$\begin{cases} -\frac{\hbar^2}{2m}\Delta u + g_{11}u^3 + \frac{\hbar^2}{2m}\Lambda\, v^2 u = \epsilon_1 u & \text{in } \Omega, \\ -\frac{\hbar^2}{2m}\Delta v + g_{22}v^3 + \frac{\hbar^2}{2m}\Lambda\, u^2 v = \epsilon_2 v & \text{in } \Omega, \\ & \\ u, v > 0 \ \text{ in } \ \Omega, & \\ u = v = 0 \ \text{ on } \ \partial\Omega\,. & \end{cases}$$

Due to conservation laws, we may regard ϵ_j's as eigenvalues and u, v as eigenfunctions satisfying normalization conditions

$$\int_\Omega u^2 = N_1\,, \quad \int_\Omega v^2 = N_2\,.$$

By suitable scaling on u, v and spatial variables, the nonlinear elliptic systems with the normalization conditions above can be transformed into

$$-\Delta u + \alpha u^3 + \Lambda v^2 u = \lambda_1 u \qquad \text{in } \Omega, \tag{1.1}$$
$$-\Delta v + \beta v^3 + \Lambda u^2 v = \lambda_2 v \qquad \text{in } \Omega, \tag{1.2}$$
$$u > 0, \quad v > 0 \qquad \text{in } \Omega, \tag{1.3}$$
$$u = 0, \quad v = 0 \qquad \text{on } \partial\Omega\,, \tag{1.4}$$
$$\int_\Omega u^2 = \int_\Omega v^2 = 1\,. \tag{1.5}$$

Hereafter, we assume that Ω is a bounded smooth domain in $\mathbb{R}^N$. Then solutions of (1.1)-(1.5) can be regarded as critical points of the GP functional

$$E_\Lambda(u,v) = \int_\Omega \left(|\nabla u|^2 + |\nabla v|^2\right) + \frac{\alpha}{2}u^4 + \frac{\beta}{2}v^4 + \Lambda u^2 v^2\,, \tag{1.6}$$

on the space $(u, v) \in H_0^1(\Omega) \times H_0^1(\Omega)$ with a constraint given by (1.5). The eigenvalues λ_j's are Lagrange multipliers with respect to (1.5). Both eigenvalues $\lambda_j = \lambda_{j,\Lambda}$'s and eigenfunctions $u = u_\Lambda, v = v_\Lambda$ depend on the parameter Λ. The system of equations (1.1)-(1.5) derived from the GP functional (1.6) is the type of systems we study here.

In this paper, we restrict our attention to solutions (u_Λ, v_Λ) of (1.1)-(1.5) such that the associated eigenvalues $\lambda_{j,\Lambda}$'s are uniformly bounded, that is, we assume here that

$$\sup_{\Lambda>0} \max\{\lambda_{1,\Lambda}\,, \lambda_{2,\Lambda}\} \le C\,, \tag{1.7}$$

where C denotes a positive constant independent of Λ. It is obvious that (1.7) is equivalent to $E_\Lambda(u_\Lambda, v_\Lambda) \le C$. In particular, observe that, the ground state or least energy solution satisfies this condition. Indeed by taking u and v with disjoint support, we derive an upper bound on $\inf E_\Lambda(u, v)$ independent of Λ. More generally, we consider here all bound state solutions that satisfy a boundedness condition on the energy.

Formally, as $\Lambda \to +\infty$ (up to a subsequence), (u_∞, v_∞)-the limit of (u_Λ, v_Λ) satisfies

$$-\Delta u_\infty + \alpha u_\infty^3 = \lambda_{1,\infty} u_\infty \qquad \text{in } \Omega_u\,, \tag{1.8}$$

and

$$-\Delta v_\infty + \beta v_\infty^3 = \lambda_{2,\infty} v_\infty \qquad \text{in } \Omega_v\,, \tag{1.9}$$

where $\Omega_u = \{x \in \Omega : u_\infty(x) > 0\}$ and $\Omega_v = \{x \in \Omega : v_\infty(x) > 0\}$ are positivity domains composed of finitely disjoint domains with positive Lebesgue measure, and each $\lambda_{j,\infty}$ is the limit of $\lambda_{j,\Lambda}$'s as $\Lambda \to \infty$ (up to a subsequence). Effective numerical simulations for (1.8) and (1.9) can be found in [4], [5] and [16]. Several works deal with the convergence of (u_Λ, v_Λ). One may refer to Chang-Lin-Lin-Lin [16] for the pointwise convergence of (u_Λ, v_Λ) away from the interface $\gamma \equiv \{x \in \Omega : u_\infty(x) = v_\infty(x) = 0\}$; Wei-Weth [41] for the uniform equicontinuity of (u_Λ, v_Λ); and Noris-Tavares-Terracini-Verzini [35] for the uniform Hölder continuity of (u_Λ, v_Λ). However, until now, the uniform Lipschitz continuity of the (u_Λ, v_Λ)'s has not yet been obtained. One of the results here is the uniform Lipschitz continuity of the (u_Λ, v_Λ)'s when the spatial dimension is $N = 1$ i.e. $\Omega = (a, b)$ (see Lemma 2.4). For higher dimensions, the problem is still open.

To understand formally the connection between F_ε the Cahn-Hilliard and E_Λ the Gross-Pitaevskii functionals, we set $u = 1 + \rho$, $v = 1 - \rho$ and $\varepsilon = 1/\sqrt{\Lambda}$ a small parameter tending to zero. Then (1.6) becomes

$$E_\Lambda(u, v) = \frac{2}{\varepsilon^2}\left[F_\varepsilon(\rho) + \int_\Omega \frac{\varepsilon^2}{4}\alpha\,(1+\rho)^4 + \frac{\varepsilon^2}{4}\beta\,(1-\rho)^4\right]\,,$$

which is dominated by the Cahn-Hilliard energy F_ε.

One might think that near the interface, the profile of bounded solutions of (1.1)-(1.5) is quite similar to that of bounded solutions of the scalar Allen-Cahn equation. However, this is not the case. As we will see ((1.16) below), the blow up equation is a system, not a scalar equation. One of the main goals of the paper is to study this system.

Here, we completely classify the one-dimensional solution of this system (see Theorem 1.3 below). In particular, we establish the symmetry, monotonicity, uniqueness and nondegeneracy of solutions of (1.16). This leads us to believe that there is an extended De Giorgi conjecture for this new system. When the spatial dimension is $N = 2$, we provide sufficient conditions that give this De Giorgi conjecture for solutions of (1.16) (see Theorem 1.8).

To derive the asymptotic behavior of (u_Λ, v_Λ)'s near the interface $\gamma = \{x \in \Omega : u_\infty(x) = v_\infty(x) = 0\}$, it is sufficient to consider the point $x_\Lambda \in \Omega$ such that $u_\Lambda(x_\Lambda) = v_\Lambda(x_\Lambda) = m_\Lambda \to 0$ and $x_\Lambda \to x_\infty \in \gamma \subset \Omega$ as $\Lambda \to +\infty$ (up to a subsequence). For simplicity, the mention "up to a subsequence" will be understood in the remaining of this paper. When $N = 1$ and $\Omega = (a, b)$, the estimate of m_Λ's is stated as follows:

Theorem 1.1. *Assume that $\Omega = (a, b) \subset \mathbb{R}$, (u_Λ, v_Λ) solves the system (1.1)-(1.5) and (1.7) holds. Let*

$$m_\Lambda = u_\Lambda(x_\Lambda) = v_\Lambda(x_\Lambda) \to 0 \quad \text{as} \quad \Lambda \to +\infty\,, \tag{1.10}$$

and

$$x_\Lambda \to x_\infty \in \Omega \quad \text{as} \quad \Lambda \to +\infty\,. \tag{1.11}$$

Then it holds that

$$m_\Lambda^4 \Lambda \to C_0 \quad \text{as} \quad \Lambda \to +\infty\,, \tag{1.12}$$

where C_0 is a positive constant. On the other hand, if (1.12) holds, then $\Lambda^{1/4} \min(|x_\Lambda - a|, |x_\Lambda - b|) \to +\infty$.

In higher dimension, without loss of generality, we may assume $C_0 = 1$. Let

$$\tilde{u}_\Lambda(y) = \frac{1}{m_\Lambda} u_\Lambda(m_\Lambda y + x_\Lambda)\,, \quad \tilde{v}_\Lambda(y) = \frac{1}{m_\Lambda} v_\Lambda(m_\Lambda y + x_\Lambda)\,, \tag{1.13}$$

for $y \in \Omega_\Lambda \equiv \{y \in \mathbb{R}^N : m_\Lambda y + x_\Lambda \in \Omega\} \to \mathbb{R}^N$ (in general) as $\Lambda \to \infty$. Then $(\tilde{u}_\Lambda, \tilde{v}_\Lambda)$ satisfies

$$-\Delta \tilde{u}_\Lambda + m_\Lambda^4 \alpha \tilde{u}_\Lambda^3 + m_\Lambda^4 \Lambda \tilde{v}_\Lambda^2 \tilde{u}_\Lambda = m_\Lambda^2 \lambda_1 \tilde{u}_\Lambda \text{ in } \Omega_\Lambda, \tag{1.14}$$

$$-\Delta \tilde{v}_\Lambda + m_\Lambda^4 \beta \tilde{v}_\Lambda^3 + m_\Lambda^4 \Lambda \tilde{u}_\Lambda^2 \tilde{v}_\Lambda = m_\Lambda^2 \lambda_1 \tilde{v}_\Lambda \text{ in } \Omega_\Lambda\,. \tag{1.15}$$

In view of (1.12), we expect that in any dimension, the limit of $(\tilde{u}_\Lambda, \tilde{v}_\Lambda)$–$(U, V)$ solves the following blow-up nonlinear elliptic system

$$\Delta U = V^2 U\,, \quad \Delta V = U^2 V\,, \quad U, V \geq 0 \quad \text{in} \quad \mathbb{R}^N\,. \tag{1.16}$$

Here we are only able to establish this fact when the dimension is $N = 1$. This is the statement in the next result.

Theorem 1.2. *Under the same hypotheses as in Theorem 1.1, assume $x_\Lambda \to x_\infty \in \Omega$ as $\Lambda \to +\infty$. Then there exist positive functions $U(y)$, $V(y) \in C^\infty(\mathbb{R})$ such that, as $\Lambda \to \infty$,*

$$\tilde{u}_\Lambda \to U, \quad \tilde{v}_\Lambda \to V \qquad \text{in} \quad C^2_{loc}(\mathbb{R})\,,$$

where (U, V) satisfies

$$\begin{cases} U'' = V^2 U & \text{in} \quad \mathbb{R}\,, \\ V'' = U^2 V & \text{in} \quad \mathbb{R}\,, \end{cases} \tag{1.17}$$

and $U(0) = V(0) = 1$. *Moreover,*

$$U'^2 + V'^2 - U^2V^2 \equiv T_\infty \quad \text{in} \quad \mathbb{R}\,, \tag{1.18}$$

where

$$T_\infty = |\Omega|^{-1} \lim_{\Lambda\to\infty} \frac{1}{2}\left[\lambda_{1,\Lambda} + \lambda_{2,\Lambda} + 3\int_\Omega (u'_\Lambda)^2 + (v'_\Lambda)^2\right] \quad (\text{ up to a subsequence}) \tag{1.19}$$

is a positive constant.

To prove the existence of (U, V) in Theorem 1.2, we need the asymptotic behavior of m_Λ satisfying (1.12) which can be derived from the uniform Lipschitz continuity of (u_Λ, v_Λ). Until now, the uniform Lipschitz continuity of (u_Λ, v_Λ) holds only when the spatial dimension is one (see Lemma 2.4). This is the reason why the result of Theorem 1.2 is only one dimensional.

A more general model is obtained when trapping potentials V_j's are turned on. The system (1.1)-(1.5) then is the form:

$$
\begin{aligned}
&- u'' + P_1(x)u + \alpha u^3 + \Lambda v^2 u = \lambda_1 u && \text{in } (a,b), && (1.20)\\
&- v'' + P_2(x)u + \beta v^3 + \Lambda u^2 v = \lambda_2 v && \text{in } (a,b), && (1.21)\\
&\ \ u > 0, \quad v > 0 && \text{in } (a,b), && (1.22)\\
&\ \ u(a) = u(b) = 0,\ v(a) = v(b) = 0 && && (1.23)\\
&\int_a^b u^2 = \int_a^b v^2 = 1, && && (1.24)
\end{aligned}
$$

where $P_j, j = 1,2$ are $C^1([a,b])$ functions. Assume

$$P_j \geq 0\,, \quad |P_j'| \leq M < \frac{1}{(b-a)^3}\left[\pi^2 + \frac{b-a}{8}(\alpha+\beta)\right] \quad \text{in} \quad (a,b)\,, \quad j = 1,2\,, \tag{1.25}$$

where M is a positive constant independent of Λ. Then Theorem 1.1 and 1.2 also hold for the system (1.20)-(1.24). We refer to Section 3.1 for the details of proofs.

Next, we study the limiting system (1.16) in dimension $N = 1$, that is, (1.17). The existence of an nontrivial solution to (1.17) is given in Lemma 4.1. Using the method of moving planes, we are able to completely classify the one-dimensional solutions of this system (1.16).

Theorem 1.3. *Let* $N = 1$ *and* (U, V) *be an nonnegative solution of (1.17). Then the following properties hold.*

(1) (Symmetry)There exists $x_0 \in \mathbb{R}$ *such that*

$$V(y - x_0) = U(x_0 - y)\,, \quad \text{for} \quad y \in \mathbb{R}\,.$$

(2) (Asymptotic behavior) Either

$$\begin{cases} U(-\infty) = 0,\ U'(-\infty) = 0,\ U' > 0,\ U'(\infty) = \sqrt{T_\infty}\,, \\ V(\infty) = 0,\ V'(\infty) = 0,\ V' < 0,\ V'(-\infty) = -\sqrt{T_\infty}\,, \end{cases}$$

or likewise with U *and* V *interchanged, where* $T_\infty > 0$ *is defined in (1.18).*

(3) *(Nondegeneracy)* (U, V) *is* nondegenerate*, that is, if* (ϕ, ψ) *is a bounded solution of the linearized system*

$$\phi'' = V^2\phi + 2UV\psi\,, \quad \psi'' = U^2\psi + 2UV\phi \quad in \quad \mathbb{R}\,, \tag{1.26}$$

then, it must be the case that $(\phi, \psi) = c(U', V')$ *for some constant* c.

Note that from Theorem 1.3 (2), both U and V are unbounded on $\mathbb{R}$. This is one of the main difficulties in the analysis. In fact, because this is a system, even in dimension one, carrying out the moving planes procedure turns out to be somewhat involved. The question to know whether such a result holds in higher dimension is an open problem.

Remark 1.4. Without loss of generality, we may set $x_0 = 0$ and then Theorem 1.3 (1) gives $U(y) = V(-y)$ for $y \in \mathbb{R}$. Whether or not the solution to (1.17) is unique up to rescaling remains open. Theorem 1.3 (3) shows *local uniqueness*. □

Remark 1.5. Instead of Bose-Einstein condensates, the same system (1.17) also describes a stationary membrane (representing a domain wall) in a static gravitational field of a black hole (cf. [20]). □

Using Theorem 1.1 and 1.3, we can also derive the asymptotic expansion of the minimizing energy as follows:

Theorem 1.6. *Assume* $\Omega = (-1, 1)$ *and* $\alpha = \beta = 0$*. Then the minimizing energy*

$$\mathcal{E}_\Lambda \equiv \min\left\{E_\Lambda(u, v) : (u, v) \in H_0^1(\Omega)^2\,, \int_\Omega u^2 = \int_\Omega v^2 = 1 \;\; and \;\; u(x) = v(-x)\,, \forall x \in (-1, 1)\right\}$$

satisfies

$$2\pi^2 - B_1\Lambda^{-\frac{1}{4}} \leq \mathcal{E}_\Lambda \leq 2\pi^2 - B_2\Lambda^{-\frac{1}{4}}\,, \tag{1.27}$$

as $\Lambda \to \infty$ *(up to a subsequence), where* $B_j, j = 1, 2$ *are positive constants independent of* Λ.

Remark 1.7. In [19], $2\pi^2 - \mathcal{E}_\Lambda$ is assumed to satisfy $2\pi^2 - \mathcal{E}_\Lambda = Q\Lambda^{-\frac{1}{4}} + o(\Lambda^{-\frac{1}{4}})$, where the constant Q can be calculated formally. Here we give a rigorous proof of (1.27) which can be regarded as a partial result of the above assumption. □

As analogue of De Giorgi's conjecture for Allen-Cahn equation, the previous results lead us to state the following conjecture for system (1.16).

Conjecture: *At least up to the dimension* $N = 8$*, under the monotonicity condition*

$$\frac{\partial U}{\partial y_N} > 0, \quad \frac{\partial V}{\partial y_N} < 0, \tag{1.28}$$

a solution (U, V) *of the system (1.16) is necessarily one-dimensional, i.e. there exist* $a \in \mathbb{R}^N$ *such that* $U(y) = U_0(a \cdot y)$ *and* $V(y) = V_0(a \cdot y)$ *for* $y \in \mathbb{R}^N$*, where* $U_0, V_0 : \mathbb{R} \to \mathbb{R}$ *are smooth functions.*

Just note that by Theorem 1.3, the monotonicity condition (1.28) holds in the case of space dimension $N = 1$. When the dimension is $N \geq 2$, Theorem 1.1 and 1.2 are still open. Note that the uniform Lipschitz continuity of the (u_Λ, v_Λ)'s is still open as well in the case of $N \geq 2$.

We now derive further results in dimension $N = 2$. To this end, in addition to the previous ones, we require the following assumptions:

(H0) $x_\Lambda \to x_\infty \in \Omega$ and $m_\Lambda = u_\Lambda(x_\Lambda) = v_\Lambda(x_\Lambda) \to 0$ as $\Lambda \to \infty$.

(H1) $m_\Lambda^4 \Lambda \to C_0 > 0 \quad$ as $\quad \Lambda \to +\infty$, i.e. (1.12) holds.

(H2) $\tilde{u}_\Lambda \to U$, $\tilde{v}_\Lambda \to V$ strongly in $H^1_{loc}(\mathbb{R}^2)$ as $\Lambda \to \infty$, where $(\tilde{u}_\Lambda, \tilde{v}_\Lambda)$ is defined in (1.13) and (U, V) solves (1.16).

(H3) For any large R,

$$\frac{1}{R^4} \int_{B_{2R} \setminus B_R} (U^2 + V^2) \leq C \text{ (independent of } R). \tag{1.29}$$

Under the assumptions (H0)-(H3), we can give an affirmative answer to the above conjecture.

Theorem 1.8. *Let $N = 2$ and assume that conditions (H0)-(H3) hold. Then the solution (U, V) of (1.16) with (1.28) must be one-dimensional.*

Remark 1.9. The remaining question then is to know under which conditions, the previous assumptions, in particular the growth condition (H3), hold. The hypothesis (H3) is satisfied if we can show the following natural growth condition

$$U(x) + V(x) = O(|x|) \quad \text{as} \quad |x| \to \infty. \tag{1.30}$$

Note that one dimensional solutions do satisfy such a growth condition. On the other hand, the hypothesis (H3) is equivalent to the condition of frequency function given by

$$N(m_\Lambda R) \leq 1 + o_\Lambda(1), \quad \text{for any large} \quad R \text{ (independent of } \Lambda), \tag{1.31}$$

where $N(\cdot)$ defined below in (6.4) is the frequency function of (u_Λ, v_Λ)'s, and $o_\Lambda(1)$ is a quantity tending to zero as Λ goes to infinity (see Theorem 6.1). □

The system of equations (1.1)-(1.4) is one particular case of parameter-dependent systems of elliptic equations with k components:

$$\begin{cases} -\Delta u_i = f_i(u_i)u_i - \displaystyle\sum_{\substack{j=1 \\ j \neq i}}^{k} \alpha_{ij} f_{ij}(u_j) u_i & \text{in } \Omega, \\ u_1, \ldots, u_k > 0 & \text{in } \Omega, \\ u_1 = \cdots = u_k = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.32}$$

where $\Omega \subset \mathbb{R}^N$ is a smooth bounded domain, $f_i, f_{ij} : [0, \infty) \to \mathbb{R}$ are continuous locally Lipschitz functions and $\alpha_{ij} > 0$ are parameters for $i, j = 1, \ldots, k$, $j \neq i$. Two special cases of (1.32) have been investigated in the literature. The case $f_{ij}(t) = t$ for $i \neq j$ corresponds to a Lotka-Volterra type system modelling the interaction between biological species in population ecology. In particular, this case has been considered by Dancer and Du [18], Conti, Terracini and Verzini [15] and Caffarelli and Lin [12]. Note that in population dynamics as well, phase separation is known to occur if the repulsion of competition terms is strong enough. Another case where the right hand side $f_i(u_i)u_i - \sum_{\substack{j=1 \\ j\neq i}}^{k} \alpha_{ij} f_{ij}(u_j) u_i$ is replaced by $A(x)\Pi_{i=1}^k u_i^{a_i}$ arises in combustion theory and has been considered recently by Caffarelli and Roquejoffre in [13].

The paper is organized as follows: In Section 2, we give some preliminaries. In Section 3, Theorem 1.1 is proved using blow-up analysis. In Section 4 and 5, we provide the proof of Theorem 1.2, 1.3 and 1.8, respectively. The frequency function and the argument to show the hypothesis (H3) are given in Section 6. Finally, we make a comparison between the Allen-Cahn equation and (1.16), and we propose several open problems in Section 7.
Notations. In this paper, C is denoted as a generic constant which may vary between lines. $U(\pm\infty)$ represents $\lim_{y\to\pm\infty} U(y)$ as usual.

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2 Preliminary

From now on, up to Section 5, we only consider the one dimensional problem. In this section, we will give some basic estimates and relations which are often used later.

Lemma 2.1. *In the interval* $\Omega = (a, b) \subset \mathbb{R}$, *assume that* (u_Λ, v_Λ) *solves system* (1.1)-(1.5) *and that* (1.7) *holds. Then there is a constant* C *independent of* Λ *such that*

$$\|u_\Lambda\|_{C^{1/2}([a,b])} \leq C, \qquad \|v_\Lambda\|_{C^{1/2}([a,b])} \leq C. \tag{2.1}$$

Furthermore, for any $0 < \gamma < \frac{1}{2}$,

$$u_\Lambda \to u_\infty, \quad v_\Lambda \to v_\infty \qquad \text{in } C^\gamma([a, b]), \tag{2.2}$$

with

$$u_\infty v_\infty \equiv 0 \qquad \text{in } (a, b). \tag{2.3}$$

Proof. Testing (1.1) against u'_Λ, we have

$$\int_\Omega |u'_\Lambda|^2 + \alpha \int_\Omega u_\Lambda^4 + \Lambda \int_\Omega u_\Lambda^2 v_\Lambda^2 = \lambda_1.$$

Thus $\|u_\Lambda\|_{H_0^1([a,b])} \leq C$. Thus, (2.1) and (2.2) are standard results of Sobolev Imbedding. Then (2.3) clearly follows from the above equality. □

Remark 2.2. Actually for $\Omega \subset \mathbb{R}^N$, we also know that u_Λ and v_Λ are always uniformly bounded (see [16, Lemma 2.1]). □

In what follows, for simplicity, we write (u, v) and (λ_1, λ_2) rather than (u_Λ, v_Λ) and $(\lambda_{1,\Lambda}, \lambda_{2,\Lambda})$, respectively.

Lemma 2.3. *Under the same hypotheses as in Lemma 2.1, there exists a positive constant* T_Λ *and two positive constants* C_1, C_2 *independent of* Λ *such that*

$$u'^2 + v'^2 - \Lambda u^2 v^2 - \frac{\alpha}{2} u^4 - \frac{\beta}{2} v^4 + \lambda_1 u^2 + \lambda_2 v^2 = T_\Lambda \quad \text{in} \quad (a, b)\,, \tag{2.4}$$

and

$$0 < C_1 < T_\Lambda < C_2 < +\infty\,, \tag{2.5}$$

Proof. Multiplying (1.1) on both sides by $2u'$ and (1.2) by $2v'$, we have

$$(u'^2)' - \frac{\alpha}{2}(u^4)' - \Lambda v^2(u^2)' + \lambda_1(u^2)' = 0\,, \quad \text{in} \quad (a,b)\,,$$
$$(v'^2)' - \frac{\beta}{2}(v^4)' - \Lambda u^2(v^2)' + \lambda_2(v^2)' = 0\,, \quad \text{in} \quad (a,b)\,.$$

Adding the two equalities, we get (2.4).

Now we claim $0 < C_1 < T_\Lambda < C_2 < +\infty$. Integrating (2.4) over (a,b), we get

$$\int_a^b u'^2 + \int_a^b v'^2 - \Lambda \int_a^b u^2 v^2 - \frac{\alpha}{2}\int_a^b u^4 - \frac{\beta}{2}\int_a^b v^4 + \lambda_1 + \lambda_2 = T_\Lambda(b-a). \tag{2.6}$$

Using (1.1), (1.2), (1.5) and integrating by parts, we obtain

$$\int_a^b u'^2 + \alpha \int_a^b u^4 + \Lambda \int_a^b u^2 v^2 = \lambda_1\,, \tag{2.7}$$
$$\int_a^b v'^2 + \beta \int_a^b v^4 + \Lambda \int_a^b u^2 v^2 = \lambda_2\,. \tag{2.8}$$

Combining (2.6)-(2.8) yields

$$2\int_a^b u'^2 + 2\int_a^b v'^2 + \Lambda \int_a^b u^2 v^2 + \frac{\alpha}{2}\int_a^b u^4 + \frac{\beta}{2}\int_a^b v^4 = T_\Lambda(b-a).$$

Since from assumption (1.7) we know that λ_1, λ_2 are uniformly bounded with respect to Λ, then it is obvious that $T_\Lambda < C_2 < \infty$. On the other hand, Poincaré's inequality shows that

$$\int_a^b u'^2 \geq C \int_a^b u^2 = C_1 > 0.$$

This gives $T_\Lambda > C_1 > 0$. Here we have used condition (1.5). Therefore, the proof of Lemma 2.3 is complete. □

We now state the uniform Lipschitz continuity of u_Λ and v_Λ.

Lemma 2.4. *Under the same hypotheses as in Lemma 2.1, we have*

$$\|u'\|_{L^\infty} \leq C, \qquad \|v'\|_{L^\infty} \leq C\,,$$

where C is a positive constant independent of Λ.

Proof. By Lemma 2.3 and (1.4), it is easy to check that $u'(a)^2 + v'(a)^2 \leq C$ and $u'(b)^2 + v'(b)^2 \leq C$, which implies that

$$|u'(a)| \leq C, \quad |u'(b)| \leq C, \qquad |v'(a)| \leq C, \quad |v'(b)| \leq C. \tag{2.9}$$

Integrating (1.1) from a to x, we get

$$u'(x) - u'(a) = \alpha \int_a^x u^3 + \Lambda \int_a^x v^2 u - \lambda_1 \int_a^x u\,. \tag{2.10}$$

For $x = b$, by (1.7), (2.1), (2.9) and (2.10), this shows that

$$\Lambda \int_a^b v^2 u \le C. \tag{2.11}$$

Then by (2.1) and (2.11), it follows that $|u'(x) - u'(a)| \le C$. Similarly, integrating (1.1) on (x, b) yields that $|u'(b) - u'(x)| \le C$. Combining these two estimates shows that $|u'| + |v'|$ is uniformly bounded. The proof is thereby complete. $\square$

3 Proof of Theorem 1.1 and 1.2

In this section, we provide the proof of Theorem 1.1. Recall that u_∞ and v_∞ are defined in Lemma 2.1. Owing to condition (1.5), we know that $\int_a^b u_\infty^2 = \int_a^b v_\infty^2 = 1$ which implies that $u_\infty, v_\infty \not\equiv 0$. Hence there exists $x_\Lambda \in \Omega$ such that (1.10) and (1.11) hold.

We start with an a priori bound.

Lemma 3.1. *Under the hypotheses of Theorem 1.1, it holds true that*

$$\limsup_{\Lambda \to \infty} \Lambda\, m_\Lambda^4 < \infty\, .$$

Proof. We argue by contradiction. Suppose that along a sequence $\Lambda_j \to \infty$,

$$m_\Lambda^4 \Lambda \to \infty \qquad \text{as} \qquad \Lambda = \Lambda_j \to \infty\, . \tag{3.1}$$

Then let

$$\tilde{u}(y) = \frac{1}{m_\Lambda} u\left(\frac{y}{m_\Lambda \sqrt{\Lambda}} + x_\Lambda\right), \qquad \tilde{v}(y) = \frac{1}{m_\Lambda} v\left(\frac{y}{m_\Lambda \sqrt{\Lambda}} + x_\Lambda\right),$$

defined for $y \in \tilde{I}_\Lambda$, where $\tilde{I}_\Lambda = \{y \in \mathbb{R} : (a - x_\Lambda) m_\Lambda \sqrt{\Lambda} < y < (b - x_\Lambda) m_\Lambda \sqrt{\Lambda}\}$. Then $(\tilde{u}, \tilde{v})$ solves the system

$$\begin{cases} \tilde{u}'' - \dfrac{\alpha}{\Lambda} \tilde{u}^3 - \tilde{v}^2 \tilde{u} + \dfrac{\lambda_1}{m_\Lambda^2 \Lambda} \tilde{u} = 0 & \text{in} \quad \tilde{I}_\Lambda\, , \\ \tilde{v}'' - \dfrac{\beta}{\Lambda} \tilde{v}^3 - \tilde{u}^2 \tilde{v} + \dfrac{\lambda_2}{m_\Lambda^2 \Lambda} \tilde{v} = 0 & \text{in} \quad \tilde{I}_\Lambda\, . \end{cases}$$

From (2.4) (see Lemma 2.3), we have

$$\tilde{u}'^2 + \tilde{v}'^2 - \tilde{u}^2 \tilde{v}^2 - \frac{\alpha}{2\Lambda} \tilde{u}^4 - \frac{\beta}{2\Lambda} \tilde{v}^4 + \frac{\lambda_1}{m_\Lambda^2 \Lambda} \tilde{u}^2 + \frac{\lambda_2}{m_\Lambda^2 \Lambda} \tilde{v}^2 = \frac{T_\Lambda}{m_\Lambda^4 \Lambda} \quad \text{in} \quad \tilde{I}_\Lambda\, . \tag{3.2}$$

On the other hand, Lemma 2.4 gives

$$\left| \tilde{u}(y) - \frac{1}{m_\Lambda} u(x_\Lambda) \right| \le \frac{C|y|}{m_\Lambda^2 \sqrt{\Lambda}}\, , \quad \left| \tilde{v}(y) - \frac{1}{m_\Lambda} v(x_\Lambda) \right| \le \frac{C|y|}{m_\Lambda^2 \sqrt{\Lambda}} \quad \text{for} \quad y \in \tilde{I}_\Lambda\, . \tag{3.3}$$

Since $m_\Lambda = u_\Lambda(x_\Lambda) = v_\Lambda(x_\Lambda) \to 0$ and $m_\Lambda^4 \Lambda \to \infty$, then these inequalities show that $\tilde{u}$ and $\tilde{v}$ are uniformly bounded and equicontinuous on any compact subinterval of $\tilde{I}_\Lambda$. Owing to (1.11) and (3.1), it is obvious that $\tilde{\Omega}_\Lambda$ tends to the entire real line $\mathbb{R}$ as $\Lambda \to \infty$. Here we have used the fact that $m_\Lambda \to 0$ as $\Lambda \to \infty$. Thus, by (3.1), (3.3) and the Arzela-Ascoli Theorem,

$$\tilde{u} \to 1, \quad \tilde{v} \to 1 \quad \text{in} \quad C_{\text{loc}}(\mathbb{R}), \quad \tilde{u}', \tilde{v}' \to 0 \quad \text{in} \quad L^\infty_{\text{loc}}(\mathbb{R}) \quad \text{as} \quad \Lambda \to \infty, \tag{3.4}$$

Hence (3.2) and (3.4) imply

$$\frac{T_\Lambda}{m_\Lambda^4 \Lambda} \to -1 \quad \text{as} \quad \Lambda \to \infty \quad .$$

However, this contradicts (2.5) and (3.1). Therefore, the proof of Lemma 3.1 is complete. $\square$

We can now prove Theorem 1.1. We argue by contradiction. In view of Lemma 3.1, we assume that $\Lambda\, m_\Lambda^4 \to 0$ as $\Lambda \to \infty$. As in (1.13), let

$$\tilde{u}(y) = \frac{1}{m_\Lambda} u(m_\Lambda y + x_\Lambda), \qquad \tilde{v}(y) = \frac{1}{m_\Lambda} v(m_\Lambda y + x_\Lambda), \tag{3.5}$$

for $y \in I_\Lambda$, where $I_\Lambda = \left(\frac{a - x_\Lambda}{m_\Lambda}, \frac{b - x_\Lambda}{m_\Lambda} \right)$ tends to the entire real line as Λ goes to infinity since we assume $x_\Lambda \to x_\infty \in (a, b)$. As before, $\tilde{u}$ and $\tilde{v}$ satisfy

$$\begin{cases} \tilde{u}'' - m_\Lambda^4 \alpha \tilde{u}^3 - m_\Lambda^4 \Lambda \tilde{v}^2 \tilde{u} + m_\Lambda^2 \lambda_1 \tilde{u} = 0 & \text{in} \quad I_\Lambda, \\ \tilde{v}'' - m_\Lambda^4 \beta \tilde{v}^3 - m_\Lambda^4 \Lambda \tilde{u}^2 \tilde{v} + m_\Lambda^2 \lambda_2 \tilde{v} = 0 & \text{in} \quad I_\Lambda. \end{cases}$$

From Lemma 2.3, we have

$$\tilde{u}'^2 + \tilde{v}'^2 - m_\Lambda^4 \Lambda \tilde{u}^2 \tilde{v}^2 - \frac{\alpha}{2} m_\Lambda^4 \tilde{u}^4 - \frac{\beta}{2} m_\Lambda^4 \tilde{v}^4 + \lambda_1 m_\Lambda^2 \tilde{u}^2 + \lambda_2 m_\Lambda^2 \tilde{v}^2 = T_\Lambda \quad \text{in} \quad I_\Lambda. \tag{3.6}$$

Lemma 2.4 tells us that

$$\tilde{u}(y) = \frac{1}{m_\Lambda} [u(x_\Lambda) + O(1) m_\Lambda y],$$

so $\tilde{u}$ is locally uniformly bounded and so is $\tilde{v}$. By elliptic regularity, we know that $\tilde{u}$ and $\tilde{v}$ are bounded in $C^2_{\text{loc}}(\mathbb{R})$ and thus the Arzelà–Ascoli theorem yields that there exists $U(y)$, $V(y) \in C^2(\mathbb{R})$ such that

$$\tilde{u} \to U, \quad \tilde{v} \to V \qquad \text{in } C^2_{\text{loc}}(\mathbb{R}).$$

Passing to the limit in the associated equations, we see that U, V satisfy the following equations

$$U'' = V'' = 0 \qquad \text{in } \mathbb{R}.$$

Since U, $V \geq 0$, they have to be constants. Furthermore, from (3.6), U and V satisfy

$$U'^2 + V'^2 = T_\infty.$$

Here $T_\Lambda \to T_\infty$ and $T_\infty > 0$. Thus, this is a contradiction with the fact that U, V are constants. Therefore, we know that Λm_Λ^4 converges to some positive constant C_0. This proves (1.12).

Next, we show that $\Lambda^{1/4}\min(|x_\lambda - a|, |x_\lambda - b|) \to +\infty$. To this end, we define

$$\tilde{u}(y) = \Lambda^{1/4} u(\frac{y}{\Lambda^{1/4}} + x_\Lambda), \qquad \tilde{v}(y) = \Lambda^{1/4} v(\frac{y}{\Lambda^{1/4}} + x_\Lambda).$$

Then $\tilde{u}$ and $\tilde{v}$ satisfy

$$\begin{cases} \tilde{u}'' - \dfrac{\alpha}{\Lambda}\tilde{u}^3 - \tilde{v}^2\tilde{u} + \dfrac{\lambda_1}{\sqrt{\Lambda}}\tilde{u} = 0 & \text{in } (a - x_\Lambda, b - x_\Lambda)\sqrt[4]{\Lambda}, \\ \tilde{v}'' - \dfrac{\beta}{\Lambda}\tilde{v}^3 - \tilde{u}^2\tilde{v} + \dfrac{\lambda_2}{\sqrt{\Lambda}}\tilde{v} = 0 & \text{in } (a - x_\Lambda, b - x_\Lambda)\sqrt[4]{\Lambda}. \end{cases}$$

From Lemma 2.3, it follows that

$$\tilde{u}'^2 + \tilde{v}'^2 - \tilde{u}^2\tilde{v}^2 - \frac{\alpha}{2\Lambda}\tilde{u}^4 - \frac{\beta}{2\Lambda}\tilde{v}^4 + \frac{\lambda_1}{\sqrt{\Lambda}}\tilde{u}^2 + \frac{\lambda_2}{\sqrt{\Lambda}}\tilde{v}^2 = T_\Lambda.$$

Without loss of generality, we may assume $\Lambda^{\frac{1}{4}}(a - x_\Lambda) \to -C_1 > -\infty$, where C_1 is a positive constant. A similar argument as above shows that there exist U, $V \in C^\infty([-C_1, \infty))$ such that $\tilde{u} \to U$, $\tilde{v} \to V$ in $C^2_{\text{loc}}([-C_1, \infty))$ and

$$\begin{cases} U'' = V^2 U & \text{in } [-C_1, \infty), \\ V'' = U^2 V & \text{in } [-C_1, \infty), \\ U(0) = V(0) = C_0^{1/4}, \\ U(-C_1) = V(-C_1) = 0. \end{cases}$$

Fatou's Lemma then yields

$$\int_{-C_1}^{\infty} V^2 U \le \liminf_{\Lambda\to\infty} \int_{(a-x_\Lambda)\sqrt[4]{\Lambda}}^{(b-x_\Lambda)\sqrt[4]{\Lambda}} \tilde{v}^2\tilde{u} \le \Lambda \int_a^b v^2 u \le C,$$

where the last inequality is due to (2.11). Thus, $U(\infty) = 0$ or $V(\infty) = 0$. Since U and V are convex on (C_1, ∞), $U(-C_1) = V(-C_1) = 0$ and either $U(\infty) = 0$ or $V(\infty) = 0$, then we know that $U \equiv 0$ or $V \equiv 0$, which contradicts $U(0) = V(0) = C_0^{\frac{1}{4}}$.

Finally we prove Theorem 1.2. Recall that $\tilde{u}$ and $\tilde{v}$ satisfy

$$\begin{cases} \tilde{u}'' - \dfrac{\alpha}{\Lambda}\tilde{u}^3 - \tilde{v}^2\tilde{u} + \dfrac{\lambda_1}{\sqrt{\Lambda}}\tilde{u} = 0 & \text{in } (a - x_\Lambda, b - x_\Lambda)\Lambda^{1/4}, \\ \tilde{v}'' - \dfrac{\beta}{\Lambda}\tilde{v}^3 - \tilde{u}^2\tilde{v} + \dfrac{\lambda_2}{\sqrt{\Lambda}}\tilde{v} = 0 & \text{in } (a - x_\Lambda, b - x_\Lambda)\Lambda^{1/4}. \end{cases}$$

From Lemma 2.3, we have

$$\tilde{u}'^2 + \tilde{v}'^2 - \tilde{u}^2\tilde{v}^2 - \frac{\alpha}{2\Lambda}\tilde{u}^4 - \frac{\beta}{2\Lambda}\tilde{v}^4 + \frac{\lambda_1}{\sqrt{\Lambda}}\tilde{u}^2 + \frac{\lambda_2}{\sqrt{\Lambda}}\tilde{v}^2 = T_\Lambda.$$

By similar arguments as to the one we have already used, we pass to the limit in the above two equations, say $\tilde{u} \to U$ and $\tilde{v} \to V$. The Maximum Principle yields that $U > 0$ and $V > 0$ and this completes the proof.

3.1 Special trapping potential case

We now observe that the previous arguments of Theorems 1.1 and 1.2 can be extended to the more general system (1.20)-(1.24). First, for Lemma 2.1, it is easy to check that

$$\|u\|_{L^\infty}\,, \quad \|v\|_{L^\infty} \leq c_1\,, \tag{3.7}$$

where c_1 is a positive constant independent of Λ. Since $\int_a^b u^2 = \int_a^b v^2 = 1$, and $u, v \in H_0^1((a,b))$, it is obvious that

$$\int_a^b u'^2\,, \quad \int_a^b v'^2 \geq \pi^2\,(b-a)^{-2} > 0\,, \tag{3.8}$$

and by Cauchy-Schwartz inequality,

$$\int_a^b u^4 \geq \frac{1}{b-a}\left(\int_a^b u^2\right)^2 = (b-a)^{-1}\,, \quad \int_a^b v^4 \geq \frac{1}{b-a}\left(\int_a^b v^2\right)^2 = (b-a)^{-1}\,. \tag{3.9}$$

Regarding the analogue of (2.4) in Lemma 2.3, we have

$$\begin{aligned} & u'^2 + v'^2 - P_1 u^2 - P_2 v^2 - \Lambda u^2 v^2 - \frac{\alpha}{2}u^4 - \frac{\beta}{2}v^4 + \lambda_1 u^2 + \lambda_2 v^2 \\ = \;& u'^2(a) + v'^2(a) - \int_a^x P_1' u^2 - \int_a^x P_2' v^2 \equiv \tilde{T}_\Lambda(x) \qquad \text{for} \;\; x \in (a,b)\,. \end{aligned} \tag{3.10}$$

Now we want to show that

$$C_1 \leq \tilde{T}_\Lambda \leq C_2 \quad \text{ for } \quad x \in (a,b)\,, \tag{3.11}$$

where C_j's are positive constant independent of Λ. As before, we know that

$$\int_\Omega u'^2 + \int_\Omega P_1\, u^2 + \alpha \int_\Omega u^4 + \Lambda \int_\Omega u^2 v^2 = \lambda_1\,, \tag{3.12}$$

$$\int_\Omega v'^2 + \int_\Omega P_2\, v^2 + \beta \int_\Omega v^4 + \Lambda \int_\Omega u^2 v^2 = \lambda_2\,. \tag{3.13}$$

Adding these relations, we obtain

$$\int_a^b 2\left(u'^2 + v'^2\right) + \int_a^b \frac{\alpha}{2}\,u^4 + \frac{\beta}{2}\,v^4 + \Lambda \int_a^b u^2 v^2 + \int_a^b \int_a^x P_1' u^2 + P_2' v^2 = [u'^2(a) + v'^2(a)](b-a)\,. \tag{3.14}$$

Thus, by (1.25), (3.8), (3.9), (3.10) and (3.14), we get

$$\tilde{T}_\Lambda \geq 4\pi^2\,(b-a)^{-3} + \frac{1}{2}\,(\alpha+\beta)\,(b-a)^{-2} - 4M > 0 \quad \text{ in } \quad (a,b)\,. \tag{3.15}$$

Here we have used (1.25) and the fact that

$$\int_a^x |P_1'|\,u^2 \leq M\,, \quad \int_a^x |P_2'|\,v^2 \leq M\,,$$

for $x \in (a, b)$. On the other hand, (3.10) and (3.14) imply that

$$\tilde{T}_\Lambda \leq 4M + 2(\lambda_1 + \lambda_2)(b - a)^{-1} . \tag{3.16}$$

Combining (1.7), (3.15), (3.16), we thus obtain (3.11).

By (3.10) and (3.14), it is easy to check that $u'(a)$ and $v'(a)$ are uniformly bounded. Hence as in Lemma 2.4, we get

$$|u'| \leq C \qquad \text{and} \qquad |v'| \leq C\,, \quad \forall x \in (a, b)\,. \tag{3.17}$$

Therefore, the blow-up argument in Section 3 is also applicable here and we get the same result as in Theorem 1.1 and 1.2.

4 Proof of Theorem 1.3 and 1.6

In this section, we completely characterize the entire solution of the one-dimensional system:

$$U'' = UV^2, V'' = VU^2,\ U, V \geq 0 \ \text{ in } \mathbb{R} \tag{4.1}$$

and prove Theorem 1.3.

First we shall prove the existence of solutions to (4.1).

Lemma 4.1. *There exists an entire solution to (4.1) with the property that $U(x) = V(-x)$.*

Proof. First we solve (4.1) in a bounded interval $[0, R]$

$$\begin{cases} U'' = UV^2, -R < x < R, U(-R) = 0, U(R) = R, \\ V'' = VU^2, -R < x < R, V(-R) = R, V(R) = 0, \\ U(-x) = V(x), -R < x < R. \end{cases}$$

We denote such solution as (U_R, V_R). The existence of (U_R, V_R) follows from direct minimization of the energy functional $\frac{1}{2}\int_{-R}^{R}((U')^2 + (V')^2 + U^2V^2)dx$ over the space $\{U \in H^1(-R, R), V \in H^1(-R, R), U(-R) = V(R) = 0, U(R) = V(-R) = R, U(-x) = V(x), U, V \geq 0\}$. By maximum Principle, $U, V > 0$ in $(-R, R)$. Since $U_R'' \geq 0$ and $U_R(-R) = 0$, we deduce that $U_R'(x) > 0$ for $x \in (-R, R)$. Similarly, $V_R'(x) < 0$ for $x \in (-R, R)$. Thus U_R and V_R only meet at the origin. So $U_R(x) - V_R(x) > 0$ for $x \in (0, R)$. This implies that $(U_R - V_R)''(x) \leq 0$ in $(0, R)$. By comparison principle, $U_R - V_R \geq x$ for $x \in (0, R)$. Denoting $x^+ = \max(x, 0)$, we see that $V'' \geq (x^+)^2 V$ in $(-R, R)$. Let V_0 be the unique solution of $V_0'' - (x^+)^2 V_0 = 0, V_0(+\infty) = 0, V_0(-\infty) = -2x + o(1)$ as $x \to -\infty$. Then for R large, we have $V_R(x) \leq V_0(x)$. By symmetry assumption we also have $U_R(x) \leq V_0(-x)$. Letting $R \to +\infty$ and noting that $U_R(x) \geq x^+$, we see that (U_R, V_R) approaches a (nontrivial) solution to (4.1) in $(0, +\infty)$ with the property that $U(x) = V(-x)$. □

Our second lemma shows that U and V must be monotone.

Lemma 4.2. *Let (U, V) satisfy (4.1) with $(U, V) \neq (0, 0)$. Then either $U' > 0$ and $V' < 0$, or $U' < 0$ and $V' > 0$ on $\mathbb{R}$. Furthermore, there exists a constant C such that $|U'| + |V'| \leq C$.*

Proof. Clearly, the quantity $(U^{'})^2 + (V^{'})^2 - U^2V^2$ is a constant, i.e.,

$$(U^{'})^2 + (V^{'})^2 - U^2V^2 \equiv T_\infty \quad \text{on} \quad \mathbb{R}\,, \tag{4.2}$$

for some constant T_∞. Since $U^{''} \geq 0$, either $U^{'} > 0$,or $U^{'} < 0$ on $\mathbb{R}$, or there exists x_1 such that $U^{'}(x) > 0$ for $x > x_1$, $U^{'}(x) < 0$ for $x < x_1$. Similar statement holds for V. Let us first assume that there exists x_0 such that $U^{'} > 0, V^{'} > 0$ for $x > x_0$. Then $U(\infty) = V(\infty)$ and for x large enough, say $x \geq x_1$, the equations of U and V show that $(U+V)^{''} \geq 2(U^2+V^2) \geq (U+V)^2$. If we set $\varphi = U+V > 0$, it satisfies $\varphi^{''} \geq \varphi^2$ for $x \geq x_1$. Since φ is defined in (x_1, ∞), this implies $\varphi(x) \to 0$ as $x \to +\infty$ which is clearly a contradiction. This argument actually shows that all cases are excluded except the one when $U^{'}$ and $V^{'}$ do not change sign and $U^{'}V^{'} < 0$ in $\mathbb{R}$. Furthermore, it is easily seen using (4.2) and the equations that $|U^{'}|$ and $|V^{'}|$ are bounded. □

Let us assume that $U^{'} > 0$ and $V^{'} < 0$. By a slight shift of the origin and rescaling, we may assume that

$$U(0) = V(0) = 1. \tag{4.3}$$

It is then straightforward to derive the asymptotic behavior of U and V.

Proposition 4.3. *Let (U, V) be a solution of (4.1)-(4.3) such that $U^{'} > 0$ and $V^{'} < 0$. Then we have*

$$V^2U \to 0, \quad U^2V \to 0 \qquad \text{as } y \to \pm\infty. \tag{4.4}$$

Furthermore, the following holds true.

$$\begin{cases} U(-\infty) = 0, \ U'(-\infty) = 0, \ U'(\infty) = \sqrt{T_\infty}, \\ V(\infty) = 0, \ V'(\infty) = 0, \ \ V'(-\infty) = -\sqrt{T_\infty}, \end{cases} \tag{4.5}$$

Proof. We only treat the limits as $x \to +\infty$ since the ones at $-\infty$ follow from the change of unknowns $\tilde{U}(x) = V(-x), \tilde{V}(x) = U(-x)$. Since $U^{''}, V^{''} > 0$, $U^{'}$ and $V^{'}$ have limits as $x \to +\infty$. Obviously, $V^{''}(+\infty) = 0$. Assume by the way of contradiction that $U^{'}(\infty) = \infty$. Then, by (4.2), $UV \to +\infty$ as $x \to +\infty$. This implies that $V^{''} = U^2V \to +\infty$ as $x \to +\infty$ which is impossible. Hence by (4.2), $U^{'}$ and UV have limits as $x \to +\infty$. Thus $\lim_{x\to+\infty} UV^2 = 0$. Since U has at most linear growth at ∞ and $-V^{''} + V \leq 0$ for large x with $V(\infty) = 0, V > 0$, we see that $\lim_{x\to+\infty} UV = 0$ and $\lim_{x\to+\infty} U^2V = 0$. Therefore, by (4.2), we get $U^{'}(\infty) = \sqrt{T_\infty}$ and complete the proof of (4.5). □

Lemma 4.4. *Let U and V be solutions of (4.5) such that $U^{'} > 0$. Then U has two asymptotic lines $y = 0$ at $-\infty$, and*

$$y = \sqrt{T_\infty}x + b_1 \qquad \text{for some} \quad b_1 \in \mathbb{R}$$

at ∞ with $T_\infty > 0$. Similarly V has two asymptotic lines, $y = 0$ at ∞, and at $-\infty$

$$y = -\sqrt{T_\infty}x + b_2 \qquad \text{for some } b_2 \in \mathbb{R}.$$

Remark 4.5. Actually, it will be seen below that necessarily $b_1 = b_2$ in the above lemma (see Corollary 4.7). □

Proof. We will just prove that U has two asymptotic lines. We only need to consider the situation at ∞. Since $U^{''} > 0$, $U^{'} > 0$ implies that $T_\infty = (U^{'}(\infty))^2 > 0$. Since $U'(\infty) = \sqrt{T_\infty}$, it is not difficult to get

$$(\sqrt{T_\infty} - \epsilon)x \leq U(x) \leq (\sqrt{T_\infty} + \epsilon)x \qquad \text{for } x \in (M, \infty) \text{ and } M \text{ large.}$$

It is easily seen that one can define function V_1 such that

$$\begin{cases} V_1'' = (\sqrt{T_\infty} - \epsilon)^2 x^2 V_1 & \text{in } (M, \infty), \\ V_1 > 0, \\ V_1(M) = V(M),\ V_1(\infty) = 0. \end{cases}$$

Then, for any $\rho, 0 < \rho < \frac{\sqrt{T_\infty} - \epsilon}{2}$, there exists a constant $C > 0$ and $M > 0$ such that $V_1(y) \leq Ce^{-\rho y^2}$ for all $y \in (M, \infty)$. Furthermore we have, in the considered region,

$$(V - V_1)'' - U^2(V - V_1) = [U^2 - (\sqrt{T_\infty} - \epsilon)^2 x^2]V_1 \geq 0,$$

which implies, by Maximum Principle,

$$V(x) \leq Ce^{-\frac{\rho y^2}{2}}, \qquad \forall y \in (M, \infty). \tag{4.6}$$

From the equation on V, it follows that $V^{''} \leq Cy^2 e^{-\rho y^2}$ for $y \geq M$. Therefore, by integrating, we get $0 \geq V^{'}(y) \geq -Ce^{-\rho y^2}$ for $y \geq M$. Hence $(U')^2 - T_\infty \geq -(V')^2 \geq Ce^{-\rho y^2}$ for $y \geq M$. This implies that $U(y) - \sqrt{T_\infty} y \geq -C$ for $y \geq M$. Letting $b_1 := \inf_{\mathbb{R}}\{U - \sqrt{T_\infty} y\}$, in view of the convexity of U, we see that $z = \sqrt{T_\infty} y + b_1$ is an asymptotic line of U at $+\infty$. □

Proposition 4.6. *Let (U, V) be a solution of (4.1)-(4.3).Then we have*

$$V(y) = U(-y).$$

Proof. We assume here that $U^{'} > 0$ hence (4.5) holds. Also without loss of generality, we may assume that $b_1 \geq b_2$ where b_1 and b_2 are the $y-$intercept of the asymptotic lines in Lemma 4.4. Indeed, if needed we substitute (U, V) by $(V(-x), U(-x))$ which also satisfies (4.5) and this substitution interchanges b_1 and b_2.

To prove Proposition 4.6, we will use the method of moving planes which actually reduces here to a "moving point". For $\lambda > 0$, define $I_\lambda = \{x; x > \lambda\} = (\lambda, \infty)$ and for $x \in I_\lambda$, set

$$U_\lambda(x) := U(2\lambda - x), \quad V_\lambda(x) := V(2\lambda - x)$$

and

$$w_1 := U - V_\lambda, \quad w_2 := U_\lambda - V.$$

Note that U_λ and V_λ also satisfy $U_\lambda^{''} = U_\lambda V_\lambda^2$, $V_\lambda^{''} = V_\lambda U_\lambda^2$. Therefore, w_1 and w_2 satisfy the following equation

$$-w_1^{''} + U_\lambda^2 w_1 = p_\lambda w_2 \tag{4.7}$$

$$-w_2^{''} + U^2 w_2 = q_\lambda w_1 \tag{4.8}$$

where $p_\lambda = U(V+U_\lambda)$ and $q_\lambda = U_\lambda(U+V_\lambda)$ so that $p_\lambda, q_\lambda > 0$. Note that $q_\lambda(x) = p_\lambda(2\lambda - x)$ as was clear from the symmetry of the problem.

We first claim: for λ sufficiently large, w_1 and w_2 are positive in I_λ.

In fact, from the asymptotic behavior (4.5), we know that

$$U(x) \geq \sqrt{T_\infty}x + b_1, V(x) \leq -\sqrt{T_\infty}x^- + K,$$

for all $x \in \mathbb{R}$, where $x^- = \max(-x, 0)$ and K is a constant. Therefore, $w_1(x) \geq \sqrt{T_\infty}(x + (2\Lambda - x)^- + b_1 - K)$ for all $x \in I_\lambda$. Clearly, if λ is sufficiently large, say $\lambda \geq \lambda_0$, we get $w_1(x) > 0$ for all $x \in I_\lambda$. From the equation (4.8), we see that if $\lambda \geq \lambda_0$,

$$\begin{cases} -w_2'' + U^2 w_2 > 0 \text{ in } I_\lambda, \\ w_2(\lambda) = U(\lambda) - V(\lambda) > 0, \ w_2(\infty) = 0. \end{cases} \tag{4.9}$$

If w_2 were not positive, then it would reach a minimum at some $\bar{x} \in I_\lambda$ such that $w_2(\bar{x}) = \min_{I_\lambda} w_2 < 0$ with $w_2''(\bar{x}) \geq 0$ and that obviously contradicts the equation (4.9). Hence $w_1, w_2 > 0$ in I_λ for $\lambda \geq \lambda_0$. The claim is thus proved.

Next, let

$$\lambda^* := \inf\{\lambda > 0; w_1^\mu(x) > 0 \text{ in } I_\mu \text{ for all } \mu \geq \lambda\}.$$

Here we write w_1^μ, w_2^μ to emphasize the dependence of w_1 and w_2 on the parameter μ. By equation (4.8) above and by the Maximum Principle as already used, $w_2 > 0$ in I_λ for all $\lambda > \lambda^*$. Since $U(0) = V(0) = 1$, $\lambda^* \geq 0$.

By continuity, we know that $w_1^{\lambda^*}, w_2^{\lambda^*} \geq 0$ in I_{λ^*}. We claim that $\lambda^* = 0$ and, to this end, argue by contradiction, and assume that $\lambda^* > 0$. Then since $w_1 = w_1^{\lambda^*} \geq 0, w_2 = w_2^{\lambda^*} \geq 0$ and $w_1(\lambda^*), w_2(\lambda^*) > 0$, from the equation (4.7) and (4.8) and the strong maximum principle, we know that $w_1, w_2 > 0$ in I_{λ^*}.

From the asymptotic behvaior, it follows that, as $x \to +\infty$,

$$w_1(x) - \left[\sqrt{T_\infty}x + b_1 + \sqrt{T_\infty}(2\lambda^* - b_2) - b_2\right] > 0.$$

Hence $w_1^{\lambda^*}(\infty) = 2\sqrt{T_\infty}\lambda^* + b_1 - b_2 > 0$ since we have chosen $b_1 \geq b_2$.

The same asymptotic behavior shows that for all λ close to λ^* and below it, say $0 < \lambda^* - \eta \leq \lambda \leq \lambda^*$, with $\eta > 0$ sufficiently small, one can choose A sufficiently large and find $\nu > 0$ such that

$$w_1^\lambda(x) \geq \nu > 0 \text{ for all } \ x \geq A \text{ and } \lambda^* - \eta \leq \lambda \leq \lambda^*.$$

Now, we know that $\min_{x \in [\lambda^*, A]} w_1^{\lambda^*}(x) > 0$. then by continuity, we get

$$\min_{[\lambda, A]} w_1^\lambda(x) > 0 \text{ for all } \lambda^* - \eta \leq \lambda \leq \lambda^*.$$

Together with the previous inequality this yields $w_1^\lambda(x) \geq 0$ for all λ such that $\lambda^* - \eta \leq \lambda \leq \lambda^*$. We thus get a contradiction to the minimality of λ^*. Therefore $\lambda^* = 0$.

We can now conclude. Going all the way to $\lambda^* = 0$, we get $U(x) \geq V(-x)$ for $x \geq 0$. Now $w_2^0 \geq 0$ implies $U(-x) \geq V(x), \forall x \geq 0$. Putting these together we get

$$U(x) \geq V(-x) \ \forall x \in \mathbb{R}.$$

Since $U(0) = V(0) = 1, U(x) - V(-x)$ reaches a minimum at $x = 0$ implying $U'(0) = -V'(0)$. This shows that $w_1^0(0) = 0, (w_1^0)'(0) = U'(0) + V'(0) = 0$. Then, the Hopf Lemma applied to $w_1^0 \geq 0$ as a solution of (4.7) in $\mathbb{R}^+$ shows that $w_1^0 \equiv 0$, that is $U(x) \equiv V(-x)$. The proof of Proposition 1.1 is thereby complete.

□

The above proposition implies the following consequence.

Corollary 4.7. *In Lemma 4.4 it holds that*

$$b_1 = b_2.$$

Proof of Theorem 1.3. Part (1) and (2) of the theorem follows Proposition 4.3 and Proposition 4.6.

It remains to prove (3) of Theorem 1.3. Let (ϕ, ψ) be a solution of (1.26) and $|\phi| + |\psi| \leq 1$. Let

$$\phi = U'\bar{\phi}, \psi = V'\bar{\psi}.$$

Then it is easy to see that $\bar{\phi}$ satisfies

$$U'\bar{\phi}'' + 2U''\bar{\phi}' = 2UVV'(\bar{\psi} - \bar{\phi}) \tag{4.10}$$

and $\bar{\psi}$ satisfies

$$V'\bar{\psi}'' + 2V''\bar{\psi}' = 2UVU'(\bar{\phi} - \bar{\psi}) \tag{4.11}$$

Multiplying (4.10) by $U'\bar{\phi}$ and integrating over (a, b), we have

$$(U')^2\bar{\phi}'\bar{\phi}|_a^b = \int_a^b (U')^2(\bar{\phi}')^2 + \int_a^b 2UVU'V'(\bar{\psi} - \bar{\phi})\bar{\phi} \tag{4.12}$$

Similarly we also have

$$(V')^2\bar{\psi}'\bar{\psi}|_a^b = \int_a^b (V')^2(\bar{\psi}')^2 + \int_a^b 2UVU'V'(\bar{\phi} - \bar{\psi})\bar{\psi} \tag{4.13}$$

Adding (4.12) and (4.13), we obtain

$$(U')^2\bar{\phi}'\bar{\phi}|_a^b + (V')^2\bar{\psi}'\bar{\psi}|_a^b = \int_a^b (V')^2(\bar{\psi}')^2 + \int_a^b (U')^2(\bar{\phi}')^2 - \int_a^b 2UVU'V'(\bar{\psi} - \bar{\phi})^2 \tag{4.14}$$

We calculate

$$(U')^2\bar{\phi}'\bar{\phi} = \phi\phi' - \frac{U''}{U'}\phi^2.$$

From the equation for $\phi(x)$, we see that ϕ is exponentially decaying as $x \to -\infty$. In fact, by the asymptotic behavior of U and V as $x \to -\infty$, we infer that ϕ'' is exponentially decaying as $x \to -\infty$ and hence $\lim_{x\to-\infty} \phi'$ exists and must be zero since ϕ is bounded. By the equation again, ϕ' is exponentially decaying as $x \to -\infty$ and so the $\lim_{x\to-\infty} \phi(x) = \phi(-\infty)$ exists. But this limit must be zero since $\phi'' = V^2\phi + 2UV\psi$. Because $\phi'(x)$ is exponentially decaying as $x \to -\infty$ and $\phi(-\infty) = 0$, we conclude that ϕ is exponentially decaying as $x \to -\infty$. Similarly as $x \to +\infty$, ϕ'' is exponentially decaying and hence $\phi'(+\infty)$ exists and must be zero. This yields also that ϕ' is exponentially decaying when $x \to +\infty$. Thus we have

$$(U')^2\bar{\phi}'\bar{\phi}|_{-\infty}^{+\infty} = \lim_{b\to+\infty}(-\frac{U''}{U'}\phi^2) = 0 \tag{4.15}$$

Similarly, we have

$$(V')^2\bar{\psi}'\bar{\psi} = \psi\psi' - \frac{V''}{V'}\psi^2$$

$$(V')^2\bar{\psi}'\bar{\psi}|_{-\infty}^{+\infty} = \lim_{a\to+\infty}(\frac{V''}{V'}\psi^2) = 0. \tag{4.16}$$

Combining (4.14), (4.15) and (4.16), we deduce that

$$\bar{\phi} = C_1, \bar{\psi} = C_2, \bar{\phi} = \bar{\psi} \tag{4.17}$$

Thus $(\phi, \psi) = c(U', V')$.

$\square$

4.1 Proof of Theorem 1.6.

Assume $\Omega = (-1,1)$ and $\alpha = \beta = 0$. Then (1.6) becomes now $E_\Lambda(u,v) = \int_{-1}^1 (u'^2 + v'^2) + \Lambda\int_{-1}^1 u^2v^2$. Here we want to estimate the minimizing energy

$$\mathcal{E}_\Lambda \equiv \min\left\{E_\Lambda(u,v) : (u,v) \in H_0^1(-1,1)^2\,, \int_{-1}^1 u^2 = \int_{-1}^1 v^2 = 1\,, u(x) = v(-x)\,, \forall x \in (-1,1)\right\}. \tag{4.18}$$

Firstly, we want to prove the lower bound estimate of $\mathcal{E}_\Lambda$. (I state the proof of lower bound estimate firstly, and then use (4.25)-(4.27) to show (4.41) to avoid disconcerting readers as the notes suggest) Let (u_Λ, v_Λ) be the minimizer of (4.18). For simplicity, we may set $(u,v) = (u_\Lambda, v_\Lambda)$ which satisfies

$$\begin{cases} u'' = \epsilon^{-4}v^2u - \lambda u & \text{in } (-1,1)\,, \\ v'' = \epsilon^{-4}u^2v - \lambda v & \text{in } (-1,1)\,, \\ u = v = 0 & \text{at } \pm 1\,, \end{cases} \tag{4.19}$$

and

$$u(x) = v(-x) \quad \text{for} \quad x \in (-1,1)\,, \tag{4.20}$$

where $\epsilon = \Lambda^{-\frac{1}{4}}$. Here we have invoked the symmetry of the problem to use a single parameter λ as the Lagrangian-multipliers for both constraints. Moreover, both u and v are positive functions on $(-1,1)$. By the same arguments of Theorem 1.2, Proposition 2.1 and 2.2 of [16], it is easy to check that as $\epsilon \to 0$ i.e. $\Lambda \to \infty$ (up to a subsequence),

$$\lambda \to \pi^2\,, \tag{4.21}$$

$$u(x) \to u_0(x) = \begin{cases} \sqrt{2}\sin(\pi x) & \text{for } x \in (0,1]\,, \\ 0 & \text{for } x \in [-1,0]\,, \end{cases} \tag{4.22}$$

and

$$v(x) \to v_0(x) = \begin{cases} -\sqrt{2}\,\sin(\pi\, x) & \text{for } \ x \in [-1,0)\,, \\ 0 & \text{for } \ x \in [0,1]\,, \end{cases} \tag{4.23}$$

for $x \in (-1,1)$.

Without loss of generality, by (4.20), (4.22), (4.23), Theorem 1.1 and 1.2, we may assume

$$u(0) = v(0) = \epsilon, \quad u'(0) > 0 \quad \text{and} \quad v'(0) < 0\,. \tag{4.24}$$

By (4.19), it is easy to check that

$$(v'u - u'v)' = \epsilon^{-4}\left(u^2 - v^2\right) uv \quad \text{in } \ (-1,1)\,, \tag{4.25}$$

and

$$\left(u^2 - v^2\right)'' + 2\lambda\left(u^2 - v^2\right) = 2\left(u'^2 - v'^2\right) \quad \text{in} \quad (-1,1)\,. \tag{4.26}$$

Then we have

Proposition 4.8. *There does not exist an interval $(\xi,\eta) \subset (0,1)$ such that $u \geq v \geq 0$ in (ξ,η), $u' < 0, v' > 0$ at ξ and $u' > 0, v' < 0$ at η.*

Proof. We may prove this by contradiction. Suppose there exists an interval $(\xi,\eta) \subset (0,1)$ such that $u \geq v \geq 0$ in (ξ,η), $u' < 0, v' > 0$ at ξ and $u' > 0, v' < 0$ at η. Then integrating (4.25) from ξ to η, we may get the contradiction and complete the proof. □

Proposition 4.9. *There does not exist $0 < \rho < 1$ such that $u(\rho) = v(\rho) > 0$ and $u > v > 0$ on $(0,\rho)$.*

Proof. We may also prove this by contradiction. Suppose there exists $0 < \rho < 1$ such that $u(\rho) = v(\rho) > 0$ and $u > v > 0$ on $(0,\rho)$. Let $\phi(x) = u^2(x) - v^2(x)$ for $x \in (-1,1)$. Then by (4.26) and Lemma 2.4, we have

$$|\phi'' + 2\lambda\,\phi| \leq K_0 \quad \text{in} \quad (-1,1)\,, \tag{4.27}$$

where K_0 is a positive constant independent of ϵ. Note that $\phi(0) = \phi(\rho) = \phi(1) = 0$. Hence (4.21) and (4.27) imply $\rho \in (\gamma_1,\gamma_2) \subset (0,1)$, where γ_j's are positive constant independent of ϵ. This may contradict to (4.22) and (4.23). Therefore, we may complete the proof. □

Due to (4.19), $u'' \geq 0$ if and only if $v \geq \epsilon^2\sqrt{\lambda}$. Similarly, $v'' \geq 0$ if and only if $u \geq \epsilon^2\sqrt{\lambda}$. Hence by (4.24), Proposition 4.8 and 4.9, we obtain $v'(x) \leq 0$ for $x \in [0,1)$. Consequently, $0 < v(x) \leq v(0) = \epsilon$ for $x \in [0,1]$ and

$$\int_0^1 v^2(x)\,dx \leq \epsilon^2\,. \tag{4.28}$$

Similarly, we may have $0 < u(x) \leq u(0) = \epsilon$ for $x \in [-1,0]$ and

$$\int_{-1}^0 u^2(x)\,dx \leq \epsilon^2\,. \tag{4.29}$$

Thus (4.28), (4.29) and the constraint $\int_{-1}^{1} u^2 = \int_{-1}^{1} v^2 = 1$ give

$$\int_0^1 u^2(x)\,dx\,,\quad \int_{-1}^0 v^2(x)\,dx \geq 1-\epsilon^2\,. \tag{4.30}$$

On the other hand,

$$\int_0^1 u'^2 \geq \mathcal{E}_{u,\epsilon} \int_0^1 u^2\,, \tag{4.31}$$

and

$$\mathcal{E}_{u,\epsilon} = \inf\left\{\int_0^1 w'^2 : w \in H^1(0,1)\,, w(0)=\epsilon\,, w(1)=0\,, \int_0^1 w^2 = 1\right\} \geq \pi^2 - C_0\epsilon\,, \tag{4.32}$$

where C_0 is a positive constant independent of ϵ. Then (4.30)-(4.32) give

$$\int_0^1 u'^2 \geq \pi^2 - C_1\epsilon\,, \tag{4.33}$$

where C_1 is a positive constant independent of ϵ. Similarly,

$$\int_{-1}^0 v'^2 \geq \pi^2 - C_2\epsilon\,, \tag{4.34}$$

where C_2 is a positive constant independent of ϵ. Combining (4.33) and (4.34), we may have the lower bound estimate of $\mathcal{E}_\Lambda$ given as follows:

$$\mathcal{E}_\Lambda \geq 2\pi^2 - C_3\,\Lambda^{-\frac{1}{4}}\,, \tag{4.35}$$

where C_3 is a positive constant independent of Λ. Here we have used the fact that $\epsilon = \Lambda^{-\frac{1}{4}}$.

About the upper bound estimate of $\mathcal{E}_\Lambda$, it is easy to get a upper bound of $\mathcal{E}_\Lambda$ independent of Λ. Indeed, taking test functions

$$u_0(x) = \begin{cases} \sqrt{2}\sin(\pi x) & \text{if} \quad 0 \leq x \leq 1\,, \\ 0 & \text{if} \quad -1 \leq x < 0\,, \end{cases}$$

and $v_0(x) = u_0(-x)$ for $x \in [-1,1]$, we get $\mathcal{E}_\Lambda \leq E_\Lambda(u_0, v_0) = 2\pi^2$. However, Theorem 1.6 involves a more precise upper bound on $\mathcal{E}_\Lambda$ describing how $\mathcal{E}_\Lambda$ approaches to $2\pi^2$ as $\Lambda \to \infty$. The idea of the more precise test functions we construct below is an approximation of u_0 (and v_0) involving the solution (U,V) of the system (1.17).

We use the results of Section 1 to construct the comparison function and obtain the upper bound of $\mathcal{E}_\Lambda$. By Theorem 1.2, there exists the associated limit (U,V) solving (1.17) and satisfying (1.18). Since $\Omega = (-1,1)$ and $\alpha = \beta = 0$, then by (1.19) and (4.21)-(4.23), we may obtain $T_\infty = 2\pi^2$. Consequently,

$$U'' = V^2 U\,, \qquad V'' = U^2 V \quad \text{in} \quad \mathbb{R}\,, \tag{4.36}$$

and

$$U'^2 + V'^2 - U^2V^2 = 2\pi^2 \qquad \text{in} \quad \mathbb{R}\,. \tag{4.37}$$

By Theorem 1.3 and Lemma 4.4, we have $U(y) = V(-y)$ for $y \in \mathbb{R}$, $U(R) = \sqrt{2}\pi R + B + O(e^{-CR})$ and $U'(R) = \sqrt{2}\pi + O(e^{-CR})$ as $R \to \infty$, where $B \in \mathbb{R}$ is a constant. Integrating (4.36) and (4.37) on $[-R, R]$, we use integration by part to get

$$\int_{-R}^{R} U^2V^2 = \frac{2\sqrt{2}\pi}{3}B + O(e^{-CR}), \quad \int_{-R}^{R} U'^2 = \int_{-R}^{R} V'^2 = 2\pi^2 R + \frac{\sqrt{2}\pi}{3}B + O(e^{-CR}), \quad (4.38)$$

for $R > 0$ sufficiently large. Now we define the comparison function of $\mathcal{E}_\Lambda$ as follows:
Let

$$\breve{u}_\epsilon(x) = \begin{cases} \chi(x)\epsilon U(\frac{x}{\epsilon}) & \text{for } -1 \le x \le \epsilon^{\frac{2}{3}}, \\ \sqrt{\frac{2}{1+\delta_\epsilon}} \sin\left[\frac{\pi}{1+\delta_\epsilon}(x+\delta_\epsilon)\right] & \text{for } \epsilon^{\frac{2}{3}} \le x \le 1, \end{cases}$$

where $\epsilon = \Lambda^{-\frac{1}{4}}$ and $\chi(x)$ is a smooth cut-off function such that $\chi \equiv 1$ if $x \ge -\frac{1}{2}$ and $\chi \equiv 0$ if $x \le -1$. Besides, by Taylor expansion, there exists $\delta_\epsilon = \frac{B}{\sqrt{2}\pi}\epsilon + \frac{3B}{2\sqrt{2}\pi}\epsilon^{\frac{5}{3}} + O(\epsilon^2)$ such that

$$\sqrt{\frac{2}{1+\delta_\epsilon}} \sin\left[\frac{\pi}{1+\delta_\epsilon}(\epsilon^{\frac{2}{3}}+\delta_\epsilon)\right] = \epsilon U(\epsilon^{-\frac{1}{3}}) = \sqrt{2}\pi\epsilon^{\frac{2}{3}} + B\epsilon + O(e^{-C\epsilon^{-\frac{1}{3}}}).$$

Then it is easy to check that

$$\begin{aligned} \int_{-1}^{1} \breve{u}_\epsilon^2 &= \int_{-1}^{\epsilon^{\frac{2}{3}}} \chi^2\epsilon^2 U^2(\frac{x}{\epsilon}) + \int_{\epsilon^{\frac{2}{3}}}^{1} \frac{2}{1+\delta_\epsilon} \sin^2\left[\frac{\pi}{1+\delta_\epsilon}(x+\delta_\epsilon)\right] \\ &= 1 + O(\epsilon^2). \end{aligned}$$

Let $\hat{u}_\epsilon = \|\breve{u}_\epsilon\|_{L^2(-1,1)}^{-1}\breve{u}_\epsilon$. Then $\hat{u}_\epsilon \in H_0^1(-1,1)$ and $\|\hat{u}_\epsilon\|_{L^2(-1,1)} = 1$. Similarly, we may define $\hat{v}_\epsilon$ and have $\hat{v}_\epsilon(x) = \hat{u}_\epsilon(-x)$ for $x \in (-1,1)$. Hence by (4.38), we have $\int_{-1}^{1} \breve{u}_\epsilon'^2 = \pi^2 - \frac{5\sqrt{2}}{3}B\pi\epsilon + O(\epsilon^2)$, which gives that

$$\int_{-1}^{1} \hat{u}_\epsilon'^2 = \int_{-1}^{1} \hat{v}_\epsilon'^2 = \pi^2 - \frac{5\sqrt{2}}{3}B\pi\epsilon + O(\epsilon^2) \quad (4.39)$$

because of $\|\breve{u}_\epsilon\|_{L^2(-1,1)}^2 = 1 + O(\epsilon^2)$. On the other hand, we may also obtain that

$$\frac{1}{\epsilon^4}\int_{-\epsilon^{\frac{2}{3}}}^{\epsilon^{\frac{2}{3}}} \breve{u}_\epsilon^2\breve{v}_\epsilon^2 = \epsilon\int_{-\epsilon^{-\frac{1}{3}}}^{\epsilon^{-\frac{1}{3}}} U^2(y)V^2(y)\mathrm{d}y = \epsilon\int_{-\infty}^{\infty} U^2V^2 + O(e^{-C\epsilon^{-\frac{1}{3}}}),$$

$$\frac{1}{\epsilon^4}\int_{-1}^{-\epsilon^{\frac{2}{3}}} \breve{u}_\epsilon^2\breve{v}_\epsilon^2 \le \frac{C}{\epsilon^4}\int_{-1}^{-\epsilon^{\frac{2}{3}}} \breve{u}_\epsilon^2 = \frac{C}{\epsilon}\int_{-\frac{1}{\epsilon}}^{-\epsilon^{-\frac{1}{3}}} U^2(y)\mathrm{d}y = O(e^{-C\epsilon^{-\frac{1}{3}}}),$$

and similarly

$$\frac{1}{\epsilon^4}\int_{\epsilon^{\frac{2}{3}}}^{1} \breve{u}_\epsilon^2\breve{v}_\epsilon^2 = O(e^{-C\epsilon^{-\frac{1}{3}}}).$$

Thus

$$\frac{1}{\epsilon^4}\int_{-1}^{1}\hat{u}_\epsilon^2\hat{v}_\epsilon^2 = \epsilon\int_{-\infty}^{\infty}U^2V^2 + O(\epsilon^3) = \frac{2\sqrt{2}}{3}B\pi\epsilon + O(\epsilon^3). \tag{4.40}$$

Therefore, by (4.39) and (4.40), we get

$$E(\hat{u}_\epsilon, \hat{v}_\epsilon) = 2\pi^2 - \frac{8\sqrt{2}}{3}B\pi\epsilon + O(\epsilon^2),$$

which implies the following upper bound estimate

$$\mathcal{E}_\Lambda \leq 2\pi^2 - \frac{8\sqrt{2}}{3}B\pi\Lambda^{-\frac{1}{4}} + O(\Lambda^{-\frac{1}{2}}). \tag{4.41}$$

Therefore, combining (4.35) and (4.41), we may complete the proof of Theorem 1.6.

5 De Giorgi type result

In this section we will study the high-dimensional case of the equation of U and V.

The proof of Theorem 1.8 is analogous to that in [28]. First we give the following non-degenerate result.

Proposition 5.1. *Suppose that U, V satisfy (1.16) and (1.28). Assume that ϕ, ψ satisfy*

$$\frac{1}{R^2}\int_{B_{2R}\setminus B_R}(\phi^2+\psi^2) \leq C \qquad \textit{for large } R, \tag{5.1}$$

and

$$\begin{cases}\Delta\phi = V^2\phi + 2UV\psi & \textit{in } \mathbb{R}^N,\\ \Delta\psi = 2UV\phi + U^2\psi & \textit{in } \mathbb{R}^N.\end{cases}$$

Then for some constant $C \in \mathbb{R}$,

$$(\phi,\psi) = C(\partial_N U, \partial_N V).$$

Remark: The proof of Proposition (5.1) also gives the proof of Part II of Theorem 1.2.

Proof. It is more convenient to designate $\sigma_1(y) = \partial_N U(y)$ and $\sigma_2(y) = \partial_N V(y)$. Define $\tilde{\phi}$ and $\tilde{\psi}$ such that

$$\phi = \sigma_1\tilde{\phi}, \qquad \psi = \sigma_2\tilde{\psi}.$$

Then the following equalities hold

$$\operatorname{div}(\sigma_1^2\nabla\tilde{\phi}) + 2UV\sigma_1\sigma_2(\tilde{\phi}-\tilde{\psi}) = 0, \tag{5.2}$$

$$\operatorname{div}(\sigma_2^2\nabla\tilde{\psi}) - 2UV\sigma_1\sigma_2(\tilde{\phi}-\tilde{\psi}) = 0. \tag{5.3}$$

Let $\chi_R(y) = \chi(|y|/R)$ be a cut-off function where $\chi(s) = 1$ for $0 < s < 1$, $\chi(s) = 0$ for $s > 2$. Testing (5.2) against $\tilde{\phi}\chi_R^2$, we have

$$\int_{\mathbb{R}^N}\chi_R^2\sigma_1^2|\nabla\tilde{\phi}|^2 + \int_{\mathbb{R}^N}2\sigma_1^2\tilde{\phi}\chi_R\nabla\tilde{\phi}\nabla\chi_R - \int_{\mathbb{R}^N}2UV\sigma_1\sigma_2(\tilde{\phi}-\tilde{\psi})\tilde{\phi}\chi_R^2 = 0. \tag{5.4}$$

Similarly, we also have

$$\int_{\mathbb{R}^N} \chi_R^2 \sigma_2^2 |\nabla \tilde{\psi}|^2 + \int_{\mathbb{R}^N} 2\sigma_2^2 \tilde{\psi} \chi_R \nabla \tilde{\psi} \nabla \chi_R + \int_{\mathbb{R}^N} 2UV\sigma_1\sigma_2(\tilde{\phi} - \tilde{\psi})\tilde{\psi}\chi_R^2 = 0. \tag{5.5}$$

Adding (5.4) and (5.5) may give

$$\begin{aligned}
&\int_{\mathbb{R}^N} \chi_R^2 \sigma_1^2 |\nabla \tilde{\phi}|^2 + \int_{\mathbb{R}^N} \chi_R^2 \sigma_2^2 |\nabla \tilde{\psi}|^2 \\
= &-\int_{\mathbb{R}^N} 2\sigma_1^2 \tilde{\phi} \chi_R \nabla \tilde{\phi} \nabla \chi_R - \int_{\mathbb{R}^N} 2\sigma_2^2 \tilde{\psi} \chi_R \nabla \tilde{\psi} \nabla \chi_R + \int_{\mathbb{R}^N} 2UV\sigma_1\sigma_2(\tilde{\phi} - \tilde{\psi})^2\chi_R^2 \\
\leq &-\int_{B_{2R}\setminus B_R} 2\sigma_1^2 \tilde{\phi} \chi_R \nabla \tilde{\phi} \nabla \chi_R - \int_{B_{2R}\setminus B_R} 2\sigma_2^2 \tilde{\psi} \chi_R \nabla \tilde{\psi} \nabla \chi_R\,.
\end{aligned} \tag{5.6}$$

Here we have used (1.28) and the fact that $\int_{\mathbb{R}^N} 2UV\sigma_1\sigma_2(\tilde{\phi} - \tilde{\psi})^2\chi_R^2$ is non-positive. Moreover, by Hölder's inequality, the right hand side term of (5.6) can be dominated by

$$\begin{aligned}
&\left|\int_{B_{2R}\setminus B_R} 2\sigma_1^2 \tilde{\phi} \chi_R \nabla \tilde{\phi} \nabla \chi_R\right| + \left|\int_{B_{2R}\setminus B_R} 2\sigma_2^2 \tilde{\psi} \chi_R \nabla \tilde{\psi} \nabla \chi_R\right| \\
\leq\ & 2\left(\int_{B_{2R}\setminus B_R} \chi_R^2\sigma_1^2|\nabla\tilde{\phi}|^2\right)^{\frac{1}{2}} \left(\int_{B_{2R}\setminus B_R} \sigma_1^2\tilde{\phi}^2|\nabla\chi_R|^2\right)^{\frac{1}{2}} \\
&\qquad + 2\left(\int_{B_{2R}\setminus B_R} \chi_R^2\sigma_2^2|\nabla\tilde{\psi}|^2\right)^{\frac{1}{2}} \left(\int_{B_{2R}\setminus B_R} \sigma_2^2\tilde{\psi}^2|\nabla\chi_R|^2\right)^{\frac{1}{2}} \\
\leq & 2\left(\int_{B_{2R}\setminus B_R} \chi_R^2\sigma_1^2|\nabla\tilde{\phi}|^2 + \int_{B_{2R}\setminus B_R} \chi_R^2\sigma_2^2|\nabla\tilde{\psi}|^2\right)^{\frac{1}{2}} \\
&\qquad \cdot \left(\int_{B_{2R}\setminus B_R} \sigma_1^2\tilde{\phi}^2|\nabla\chi_R|^2 + \int_{B_{2R}\setminus B_R} \sigma_2^2\tilde{\psi}^2|\nabla\chi_R|^2\right)^{\frac{1}{2}}.
\end{aligned} \tag{5.7}$$

The last inequality comes from $|A \cdot B| \leq |A||B|$ for $A, B \in \mathbb{R}^2$. Thus by (5.6) and (5.7), we may obtain

$$\int_{\mathbb{R}^N} \chi_R^2\sigma_1^2|\nabla\tilde{\phi}|^2 + \int_{\mathbb{R}^N} \chi_R^2\sigma_2^2|\nabla\tilde{\psi}|^2 \leq 4\left(\int_{B_{2R}\setminus B_R} \sigma_1^2\tilde{\phi}^2|\nabla\chi_R|^2 + \int_{B_{2R}\setminus B_R} \sigma_2^2\tilde{\psi}^2|\nabla\chi_R|^2\right). \tag{5.8}$$

By (5.1) and $|\nabla\chi_R| \leq \frac{C}{R}$, we have

$$\int_{B_{2R}\setminus B_R} (\sigma_1^2\tilde{\phi}^2|\nabla\chi_R|^2 + \sigma_2^2\tilde{\psi}^2|\nabla\chi_R|^2) \leq C, \tag{5.9}$$

which yields that

$$\int_{B_R} \sigma_1^2|\nabla\tilde{\phi}|^2 + \int_{B_R} \sigma_2^2|\nabla\tilde{\psi}|^2 \leq C.$$

Here we have used (5.8) and (5.9). Then it is obvious that $\int_{\mathbb{R}^N} \sigma_1^2|\nabla\tilde{\phi}|^2 + \int_{\mathbb{R}^N} \sigma_2^2|\nabla\tilde{\psi}|^2 < \infty$, so

$$\int_{B_{2R}\setminus B_R} \sigma_1^2|\nabla\tilde{\phi}|^2 + \int_{B_{2R}\setminus B_R} \sigma_2^2|\nabla\tilde{\psi}|^2 \to 0 \qquad \text{as } R \to \infty. \tag{5.10}$$

Using (5.6)–(5.10) again, we finally get that

$$\int_{\mathbb{R}^N} \chi_R^2 \sigma_1^2 |\nabla \tilde{\phi}|^2 + \int_{\mathbb{R}^N} \chi_R^2 \sigma_2^2 |\nabla \tilde{\psi}|^2 \to 0 \qquad \text{as } R \to \infty.$$

Thus $\nabla \tilde{\phi} \equiv \nabla \tilde{\psi} \equiv 0$ in $\mathbb{R}^N$ i.e. $\tilde{\phi} \equiv C_1$, $\tilde{\psi} \equiv C_2$. Therefore, by (5.2) and (5.3), it is easy to check that $C_1 = C_2$ and we may complete the proof of Proposition 5.1. □

Proof of Theorem 1.8. Testing the first equation in (1.16) against $\chi_R^2 U$, we have

$$\int_{\mathbb{R}^N} \chi_R^2 |\nabla U|^2 + \int_{\mathbb{R}^N} 2\chi_R U \nabla \chi_R \nabla U = -\int_{\mathbb{R}^N} \chi_R^2 U^2 V^2 \le 0,$$

where χ_R is defined in the previous proof. Thus Hölder's inequality yields that

$$\int_{B_{2R}} \chi_R^2 |\nabla U|^2 \le 4 \int_{B_{2R} \setminus B_R} |\nabla \chi_R|^2 U^2 \,. \tag{5.11}$$

Similarly we also have

$$\int_{B_{2R}} \chi_R^2 |\nabla V|^2 \le 4 \int_{B_{2R} \setminus B_R} |\nabla \chi_R|^2 V^2 \,. \tag{5.12}$$

Then by (5.11) and (5.12), we obtain

$$\begin{aligned} \frac{1}{R^2} \int_{B_{2R} \setminus B_R} |\nabla U|^2 + |\nabla V|^2 &\le \frac{1}{R^2} \int_{B_{4R}} \chi_{2R}^2 (|\nabla U|^2 + |\nabla V|^2) \\ &\le \frac{C}{R^2} \int_{B_{4R} \setminus B_{2R}} |\nabla \chi_{2R}|^2 (U^2 + V^2) \\ &\le \frac{C}{R^4} \int_{B_{4R} \setminus B_{2R}} (U^2 + V^2) \le C \,, \end{aligned} \tag{5.13}$$

where the last inequality may come from the assumption (H3) i.e. (1.29).

Let $\phi = \nabla U \cdot \nu$ and $\psi = \nabla V \cdot \nu$ where $\nu \in \mathbb{R}^2$ such that $\nabla U(0) \cdot \nu = 0$. Then Proposition 5.1 and (5.13) may imply $(\phi, \psi) = C(\partial_N U, \partial_N V)$, where C is a constant. Hence by $\nabla U(0) \cdot \nu = 0$ and(1.28), $C = 0$ i.e. $\phi \equiv 0$ and $\psi \equiv 0$. Therefore, $U(y) = U_0(a \cdot y)$, $V(y) = V_0(a \cdot y)$ for $y \in \mathbb{R}^2$ and we may complete the proof of Theorem 1.8, where $U_0, V_0 : \mathbb{R} \to \mathbb{R}$ and $a \perp \nu$. □

Furthermore, we can get the stability result of U and V, that is,

$$\int_{\mathbb{R}^N} |\nabla \phi|^2 + |\nabla \psi|^2 + \int_{\mathbb{R}^N} V^2 \phi^2 + U^2 \psi^2 + 4UV\phi\psi \ge 0,$$

for any compactly supported smooth functions ϕ, ψ.

Proposition 5.2. *The solution U, V of (1.16) is stable for any dimension.*

Proof. We use the method in [27]. Note that

$$\Delta \partial_N U = V^2 \partial_N U + 2UV \partial_N V, \tag{5.14}$$

$$\Delta \partial_N V = 2UV \partial_N U + U^2 \partial_N V. \tag{5.15}$$

Let ϕ, ψ be two compactly supported smooth functions in $\mathbb{R}^N$. Multiplying (5.14) with $\frac{\phi^2}{\partial_N U}$ and integrating by parts, we have

$$\int_{\mathbb{R}^N} \frac{2\phi\nabla\phi\nabla(\partial_N U)}{\partial_N U} - \int_{\mathbb{R}^N} \frac{\phi^2|\nabla\partial_N U|^2}{(\partial_N U)^2} + \int_{\mathbb{R}^N} V^2\phi^2 + \int_{\mathbb{R}^N} 2UV\frac{\partial_N V}{\partial_N U}\phi^2 = 0.$$

Using Young's inequality, we then obtain that

$$\int_{\mathbb{R}^N} |\nabla\phi|^2 + \int_{\mathbb{R}^N} V^2\phi^2 + \int_{\mathbb{R}^N} 2UV\frac{\partial_N V}{\partial_N U}\phi^2 \geq 0.$$

Dealing similarly with (5.15) and $\frac{\psi^2}{\partial_N V}$, and we get

$$\int_{\mathbb{R}^N} |\nabla\psi|^2 + \int_{\mathbb{R}^N} U^2\psi^2 + \int_{\mathbb{R}^N} 2UV\frac{\partial_N U}{\partial_N V}\psi^2 \geq 0.$$

Thus an addition of the above two inequalities says that

$$\int_{\mathbb{R}^N} |\nabla\phi|^2 + |\nabla\psi|^2 + \int_{\mathbb{R}^N} V^2\phi^2 + U^2\psi^2 + 2UV\left(\frac{\partial_N V}{\partial_N U}\phi^2 + \frac{\partial_N U}{\partial_N V}\psi^2\right) \geq 0.$$

Since $UV \geq 0$ and $\partial_N U \partial_N V < 0$, we finally get

$$\int_{\mathbb{R}^N} |\nabla\phi|^2 + |\nabla\psi|^2 + \int_{\mathbb{R}^N} V^2\phi^2 + U^2\psi^2 + 4UV\phi\psi \geq 0.$$

The proof is concluded. $\square$

6 Frequency function

In this section, we show how to use (1.31) the condition of the frequency function of (u_Λ, v_Λ)'s to prove (1.29) when the spatial dimension $N = 2$. For notation convenience, we may denote u_Λ, v_Λ and λ_j^Λ's by u, v and λ_j's, respectively. To define the frequency function, we may introduce the following quantities:

$$H(r) = \int_{\partial B_r} (u^2 + v^2) dS_x, \tag{6.1}$$

$$I\,(r) = \int_{B_r} \left\{|\nabla u|^2 + |\nabla v|^2 + \bar{P}_\Lambda(u, v)\right\} dx, \tag{6.2}$$

where $B_r \subset \Omega$ is the two-dimensional ball with center at x_Λ and radius r, and $\bar{P}_\Lambda$ is set as

$$\bar{P}_\Lambda(u, v) = \alpha\, u^4 + \beta\, v^4 + 2\Lambda\, u^2\, v^2 - \left(\lambda_1 u^2 + \lambda_2 v^2\right)\,. \tag{6.3}$$

Now we define the frequency function N of (u, v) as follows:

$$N(r) = \frac{rI\,(r)}{H\,(r)}\,, \tag{6.4}$$

for $r > 0$ such that $B_r \subset \Omega$. By (1.13), it is easy to check that

$$N(m_\Lambda r) = \tilde{N}(r) \equiv \frac{r\tilde{I}\,(r)}{\tilde{H}\,(r)}\,, \tag{6.5}$$

for $r > 0$, where

$$\tilde{H}(r) = \int_{\partial B_r} \left(\tilde{u}^2 + \tilde{v}^2\right) dS_y\,, \tag{6.6}$$

$$\tilde{I}\,(r) = \int_{B_r} \left\{|\nabla\tilde{u}|^2 + |\nabla\tilde{v}|^2 + \tilde{P}_\Lambda(\tilde{u},\tilde{v})\right\} dy, \tag{6.7}$$

and

$$\tilde{P}_\Lambda(\tilde{u},\tilde{v}) = m_\Lambda^4\,\alpha\,\tilde{u}^4 + m_\Lambda^4\,\beta\,\tilde{v}^4 + 2m_\Lambda^4\,\Lambda\,\tilde{u}^2\,\tilde{v}^2 - \left(m_\Lambda^2\,\lambda_1\tilde{u}^2 + m_\Lambda^2\,\lambda_2\tilde{v}^2\right)\,. \tag{6.8}$$

Hereafter, we denote $\tilde{u}, \tilde{v}$ as $\tilde{u}_\Lambda, \tilde{v}_\Lambda$ for notation convenience. Besides, we also use the same notation B_r to denote a two-dimensional ball with radius r and center at the origin.

Due to the hypothesis (H0) (i.e. $x_\Lambda \to x_\infty \in \Omega$ as $\Lambda \to \infty$), it is obvious that

$$m_\Lambda^{-1}\,dist(x_\Lambda, \partial\Omega) \to \infty \qquad \text{as } \ \Lambda \to \infty\,, \tag{6.9}$$

and then the frequency function $\tilde{N}(r)$ is well-defined for $r > 0$ and Λ sufficiently large. As for (1.12) of [21], we may obtain

$$\tilde{H}' = \frac{1}{r}\,\tilde{H} + 2\int_{\partial B_r} \left(\tilde{u}\,\tilde{u}_\rho + \tilde{v}\,\tilde{v}_\rho\right)\,dS_y\,, \quad \forall\, r > 0\,, \tag{6.10}$$

where $\rho = |y|$. Here we have used the fact that the spatial dimension $N = 2$. By (1.14) and (1.15), it is obvious that

$$\frac{1}{2}\Delta(\tilde{u}^2) \;=\; |\nabla\tilde{u}|^2 + m_\Lambda^4\alpha\tilde{u}^4 + m_\Lambda^4\Lambda\tilde{u}^2\tilde{v}^2 - m_\Lambda^2\lambda_1\tilde{u}^2 \quad \text{ in } \ B_r\,, \tag{6.11}$$

$$\frac{1}{2}\Delta(\tilde{v}^2) \;=\; |\nabla\tilde{v}|^2 + m_\Lambda^4\beta\tilde{v}^4 + m_\Lambda^4\Lambda\tilde{u}^2\tilde{v}^2 - m_\Lambda^2\lambda_2\tilde{v}^2 \quad \text{ in } \ B_r\,. \tag{6.12}$$

Hence by (6.7), (6.11), (6.12) and the divergence theorem, we have

$$\tilde{I}(r) = \frac{1}{2}\int_{B_r} \Delta\left(\tilde{u}^2 + \tilde{v}^2\right) dy = \frac{1}{2}\int_{\partial B_r} \partial_\rho\left(\tilde{u}^2 + \tilde{v}^2\right)\,dy\,. \tag{6.13}$$

Moreover, by (6.13), (6.10) can be transformed into

$$\tilde{H}' = \frac{1}{r}\,\tilde{H} + 2\tilde{I}\,, \quad \forall\, r > 0\,, \tag{6.14}$$

i.e.

$$\frac{d}{dr}\left(\log\frac{\tilde{H}}{r}\right) = \frac{2}{r}\,\tilde{N}(r)\,, \quad \forall\, r > 0\,. \tag{6.15}$$

Integrating (6.15) from R_0 to l, we have

$$\tilde{H}(l) = \frac{1}{R_0}\,\tilde{H}(R_0)\,l\,e^{\int_{R_0}^{l}\frac{2}{r}\,\tilde{N}(r)\,dr}\,, \qquad \text{for} \qquad l > R_0\,, \tag{6.16}$$

where $R_0 > 1$ is a positive constant. Using integration by part, it is obvious that

$$\int_{R_0}^{l} \frac{1}{r}\,\tilde{N}\,dr = \tilde{N}(l)\log l - \tilde{N}(R_0)\log R_0 - \int_{R_0}^{l} (\log r)\,\tilde{N}'(r)\,dr\,. \tag{6.17}$$

By (6.5), it is easy to check that

$$\tilde{N}' = \left(\frac{\tilde{I}'}{\tilde{I}} + \frac{1}{r} - \frac{\tilde{H}'}{\tilde{H}}\right)\tilde{N}\,. \tag{6.18}$$

Consequently, by (6.14), (6.18) becomes

$$\tilde{N}' = \left(\frac{\tilde{I}'}{\tilde{I}} - \frac{2\tilde{I}}{\tilde{H}}\right)\tilde{N}\,. \tag{6.19}$$

By (6.7), it is obvious that

$$\begin{aligned}
\tilde{I}'(r) &= \int_{\partial B_r} \{|\nabla\tilde{u}|^2 + |\nabla\tilde{v}|^2 + \tilde{P}_\Lambda\} dS_y \\
&= \int_{\partial B_r} \{(|\nabla\tilde{u}|^2 + |\nabla\tilde{v}|^2)\sum_{s=1}^{2}\frac{y^s}{r}\nu_s + \tilde{P}_\Lambda\} dS_y,
\end{aligned}$$

where $\nu_s = y^s/r$ is the s component of outer normal vector $\nu = \frac{y}{r}$. Using integration by parts, we may get

$$\begin{aligned}
\tilde{I}'(r) &= \frac{1}{r}\int_{B_r}\sum_{k,s=1}^{2}\left[\frac{\partial}{\partial y^s}(\tilde{u}_k^2 + \tilde{v}_k^2)\right] y^s dy + \frac{2}{r}\int_{B_r}(|\nabla\tilde{u}|^2 + |\nabla\tilde{v}|^2)\,dy + \int_{\partial B_r}\tilde{P}_\Lambda dS_y \\
&= \frac{2}{r}\int_{B_r}\sum_{k,s=1}^{2}(\tilde{u}_k\tilde{u}_{ks} + \tilde{v}_k\tilde{v}_{ks})y^s dy + \frac{2}{r}\int_{B_r}(|\nabla\tilde{u}|^2 + |\nabla\tilde{v}|^2)dy + \int_{\partial B_r}\tilde{P}_\Lambda\,dS_y\,,
\end{aligned}$$

where $\tilde{u}_k$, $\tilde{v}_k$, $\tilde{u}_{ks}$ and $\tilde{v}_{ks}$ denote $\partial_{y_k}\tilde{u}$, $\partial_{y_k}\tilde{v}$, $\partial_{y_k}\partial_{y_s}\tilde{u}$ and $\partial_{y_k}\partial_{y_s}\tilde{v}$, respectively. Moreover,

$$\begin{aligned}
&\frac{2}{r}\int_{B_r}\sum_{k,s=1}^{2}(\tilde{u}_k\tilde{u}_{ks} + \tilde{v}_k\tilde{v}_{ks})y^s dy \\
= &\frac{2}{r}\int_{\partial B_r}\sum_{k,s=1}^{2}(\tilde{u}_k\tilde{u}_s + \tilde{v}_k\tilde{v}_s)y^s\cdot\nu_k dS_y - \frac{2}{r}\int_{B_r}\sum_{k,s=1}^{2}(\tilde{u}_{kk}\tilde{u}_s + \tilde{v}_{kk}\tilde{v}_s)y^s dy \\
&-\frac{2}{r}\int_{B_r}\sum_{k,s=1}^{2}(\tilde{u}_k\tilde{u}_s + \tilde{v}_k\tilde{v}_s)\delta_{ks}dy\,,
\end{aligned}$$

where $\delta_{ks} = \begin{cases} 1 & \text{if } k = s \\ 0 & \text{if } k \neq s. \end{cases}$ Therefore,

$$\tilde{I}'(r) = \int_{\partial B_r}\tilde{P}_\Lambda dS_y + 2\int_{\partial B_r}\left(\tilde{u}_\rho^2 + \tilde{v}_\rho^2\right)dS_y - \frac{2}{r}\int_{B_r}\sum_{s=1}^{2}(y^s\tilde{u}_s\Delta\tilde{u} + y^s\tilde{v}_s\Delta\tilde{v})dy \tag{6.20}$$

where $\tilde{u}_\rho = \partial_\rho \tilde{u}$ and $\tilde{v}_\rho = \partial_\rho \tilde{v}$. By (1.14) and (1.15) and (6.8), it is easy to check that

$$-\frac{2}{r}\int_{B_r}\sum_{s=1}^{2}(y^s\tilde{u}_s\Delta\tilde{u}+y^s\tilde{v}_s\Delta\tilde{v})dy=\frac{-1}{r}\int_{B_r}y\cdot\nabla\hat{P}_\Lambda dy\,, \tag{6.21}$$

where

$$\hat{P}_\Lambda(\tilde{u},\tilde{v})=m_\Lambda^4\frac{\alpha}{2}\tilde{u}^4+m_\Lambda^4\frac{\beta}{2}\tilde{v}^4+m_\Lambda^4\Lambda\tilde{u}^2\tilde{v}^2-m_\Lambda^2(\lambda_1\tilde{u}^2+\lambda_2\tilde{v}^2)\,. \tag{6.22}$$

Using integration by parts,

$$\begin{aligned}\frac{-1}{r}\int_{B_r}y\cdot\nabla\hat{P}_\Lambda dy &= -\frac{1}{r}\int_{\partial B_r}(y\cdot\nu)\,\hat{P}_\Lambda dS_y+\frac{2}{r}\int_{B_r}\hat{P}_\Lambda dy\\ &= -\int_{\partial B_r}\hat{P}_\Lambda dS_y+\frac{2}{r}\int_{B_r}\hat{P}_\Lambda dy.\end{aligned}$$

Then (6.20) can be rewritten as

$$\tilde{I}'(r)=\int_{B_r}\frac{2}{r}\hat{P}_\Lambda dy+2\int_{\partial B_r}\left(\tilde{u}_\rho^2+\tilde{v}_\rho^2\right)dS_y+\int_{\partial B_r}\left(\tilde{P}_\Lambda-\hat{P}_\Lambda\right)dS_y\,. \tag{6.23}$$

Hence by (6.8), (6.22) and (6.23), we obtain

$$\tilde{I}'(r)\geq 2\int_{\partial B_r}\left(\tilde{u}_\rho^2+\tilde{v}_\rho^2\right)dS_y-m_\Lambda^2\frac{2\lambda}{r}\int_{B_r}(\tilde{u}^2+\tilde{v}^2)dy, \tag{6.24}$$

where $\lambda=\sup_{\Lambda>0}\{\lambda_1,\lambda_2\}$. From (6.13), (6.19) and (6.24), we obtain

$$\begin{aligned}\tilde{N}'(r) &= \left(\frac{\tilde{I}'}{\tilde{I}}-\frac{2\tilde{I}}{\tilde{H}}\right)\tilde{N}(r) \qquad (6.25)\\ &\geq \left(\frac{2\int_{\partial B_r}\left(\tilde{u}_\rho^2+\tilde{v}_\rho^2\right)dS_y}{\tilde{I}(r)}-\frac{2m_\Lambda^2\lambda\int_{B_r}(\tilde{u}^2+\tilde{v}^2)dy}{r\tilde{I}(r)}-\frac{\int_{\partial B_r}\frac{\partial}{\partial\rho}(\tilde{u}^2+\tilde{v}^2)dS_y}{\tilde{H}(r)}\right)\tilde{N}(r)\\ &\geq -\frac{2m_\Lambda^2\lambda\int_{B_r}(\tilde{u}^2+\tilde{v}^2)dy}{\tilde{H}(r)}\,.\end{aligned}$$

Here we have used (6.5) and the Schwartz inequality. Note that by (H1), (H2) and (6.5)-(6.8), it is obvious that

$$\tilde{N}(r)\to\tilde{N}_\infty(r)\equiv\frac{\int_{B_r}|\nabla U|^2+|\nabla V|^2+2C_0\,U^2V^2\,dy}{\int_{B_r}U^2+V^2\,dy}>0\quad\text{as}\quad\Lambda\to\infty\,, \tag{6.26}$$

for $r>0$.

Now we want to claim that

$$\begin{cases} r\int_{\partial B_r}\tilde{u}^2\,dS_y\geq\int_{B_r}\tilde{u}^2dy\,,\\ r\int_{\partial B_r}\tilde{v}^2\,dS_y\geq\int_{B_r}\tilde{v}^2\,dy\,,\end{cases} \tag{6.27}$$

i.e.

$$\int_{B_r} (\tilde{u}^2 + \tilde{v}^2) dy \le r\tilde{H}(r)\,, \tag{6.28}$$

for $r > 0$ and Λ sufficiently large. Using integration by parts, it is easy to check that

$$\int_{B_r} (r^2 - |y|^2)\, \Delta(\tilde{u}^2)\, dy = 2r \int_{\partial B_r} \tilde{u}^2\, dS_y - 4 \int_{B_r} \tilde{u}^2\, dy \quad \forall r > 0\,. \tag{6.29}$$

and

$$\int_{B_r} (r^2 - |y|^2)\, \Delta(\tilde{v}^2)\, dy = 2r \int_{\partial B_r} \tilde{v}^2\, dS_y - 4 \int_{B_r} \tilde{v}^2\, dy \quad \forall r > 0\,. \tag{6.30}$$

Put (6.11) and (6.12) into (6.29) and (6.30). Then we have

$$r \int_{\partial B_r} \tilde{u}^2\, dS_y \ge \int_{B_r} \left[2 - m_\Lambda^2 \lambda_1 (r^2 - |y|^2)\right] \tilde{u}^2\, dy \ge \int_{B_r} \tilde{u}^2\, dy\,, \tag{6.31}$$

and

$$r \int_{\partial B_r} \tilde{v}^2\, dS_y \ge \int_{B_r} \left[2 - m_\Lambda^2 \lambda_1 (r^2 - |y|^2)\right] \tilde{v}^2\, dy \ge \int_{B_r} \tilde{v}^2\, dy\,, \tag{6.32}$$

for $r > 0$ and Λ sufficiently large. Here we have used the hypothesis that m_Λ tends to zero as Λ goes to infinity. Therefore, we complete the proof of (6.27) and (6.28).

Combining (6.25) and (6.28), it is obvious that

$$\tilde{N}'(r) \ge -2m_\Lambda^2 \lambda\, r\,, \tag{6.33}$$

for $r > 0$ and Λ sufficiently large. Then (6.16), (6.17) and (6.33) may give

$$\int_R^{2R} \tilde{H}(l)\, dl = K_0 \int_R^{2R} l^{1+2\tilde{N}(l)}\, dl + o_\Lambda(1) \quad \text{for} \quad R > R_0\,, \tag{6.34}$$

where K_0 is a positive constant independent of Λ and $o_\Lambda(1)$ is a small quantity tending to zero as Λ goes to infinity. Hence the condition (1.29) is equivalent to

$$\limsup_{\Lambda\to\infty} \int_R^{2R} l^{1+2\tilde{N}(l)}\, dl \le CR^4\,, \tag{6.35}$$

for R sufficiently large. On the other hand, by (6.26) and (6.33), we have

$$\begin{aligned}
\tilde{N}_\infty(R+h) - \tilde{N}_\infty(R) &= \lim_{\Lambda\to\infty} \tilde{N}(R+h) - \tilde{N}(R) \\
&= \lim_{\Lambda\to\infty} \int_R^{R+h} \tilde{N}'(r)\, dr \\
&\ge \lim_{\Lambda\to\infty} \int_R^{R+h} -2m_\Lambda^2 \lambda\, r\, dr \\
&= -\lim_{\Lambda\to\infty} m_\Lambda^2 \lambda (2Rh + h^2) = 0\,,
\end{aligned}$$

i.e.

$$\tilde{N}_\infty(R+h) \ge \tilde{N}_\infty(R) \quad \text{for} \quad R, h > 0\,. \tag{6.36}$$

Suppose $\tilde{N}_\infty(R_1) > 1 + \delta$ for some $R_1 > R_0$, where δ is any positive constant. Then (6.36) implies

$$\tilde{N}_\infty(R) \geq 1 + \delta \quad \text{for} \quad R \geq R_1\,, \tag{6.37}$$

which may fail the condition (6.35) i.e. (1.29). Therefore, we have proved that

Theorem 6.1. *Assume (H0)-(H2) hold. Then the hypothesis (H3) i.e. (1.29) is equivalent to*

$$\tilde{N}_\infty(R) \leq 1 \quad \textit{for} \quad R \textit{ sufficiently large} \tag{6.38}$$

i.e. $\tilde{N}(R) \leq 1 + o_\Lambda(1)$, *where* $o_\Lambda(1)$ *is a small quantity tending to zero as* Λ *goes to infinity.*

7 Comparison with Allen-Cahn equation and some open problems

In this section, we compare the system (1.16) and the Allen-Cahn equation

$$\Delta u + u - u^3 = 0 \quad \text{in } \mathbb{R}^N\,. \tag{7.1}$$

Equation (7.1) arises in the gradient theory of phase transitions by Allen-Cahn, in connection with the energy functional in bounded domains Ω

$$J_\epsilon(u) = \frac{\epsilon}{2}\int_\Omega |\nabla u|^2 + \frac{1}{4\epsilon}\int_\Omega (1-u^2)^2, \quad \epsilon > 0 \tag{7.2}$$

whose Euler-Lagrange equation corresponds precisely to a ϵ-scaling of equation (7.1) in the expanding domain $\epsilon^{-1}\Omega$. The theory of Γ-convergence developed in the 70s and 80s, showed a deep connection between this problem and the theory of minimal surfaces, see Modica, Mortola, Kohn, Sternberg, [30, 31, 32, 39]. In fact, it is known that for a family u_ϵ of local minimizers of J_ϵ with uniformly bounded energy must converge, up to subsequences, in L^1-sense to a function of the form $\chi_E - \chi_{E^c}$ where χ denotes characteristic function, and ∂E has minimal perimeter. Thus the interface between the stable *phases* $u = 1$ and $u = -1$, represented by the sets $[u_\epsilon = \lambda]$ with $|\lambda| < 1$ approach a minimal hypersurface, see Caffarelli and Córdoba [10, 11], and also Röger and Tonegawa [37], for stronger convergence and uniform regularity results on these level surfaces.

E. De Giorgi [23] formulated in 1978 the following celebrated conjecture concerning entire solutions of equation (7.1).

Let u be a bounded solution of equation (7.1) *such that* $\frac{\partial u}{\partial x_N} > 0$. *Then u is one-dimensional,* **at least** *for dimension $N \leq 8$.*

Equivalently, u depends on just one Euclidean variable so that it must have the form

$$u(x) = \tanh\left(\frac{x\cdot a - b}{\sqrt{2}}\right), \tag{7.3}$$

for some $b \in \mathbb{R}$ and some a with $|a| = 1$ and $a_N > 0$, where the function $w(t) = \tanh\left(t/\sqrt{2}\right)$ is the unique solution of the one-dimensional problem,

$$w'' + (1-w^2)w = 0, \quad w(0) = 0 \quad w(\pm\infty) = \pm 1\,.$$

Great advance in De Giorgi conjecture has been achieved in recent years, having been fully established in dimensions $N = 2$ by Ghoussoub and Gui [29] and for $N = 3$ by Ambrosio and Cabré [2]. Savin [38] established its validity for $4 \le N \le 8$ under the following mild additional assumption

$$\lim_{x_N \to \pm\infty} u(x^{'}, x_N) = \pm 1 \tag{7.4}$$

Recently, del Pino, Kowalczyk and Wei [24] has disproved De Giorgi's conjecture in dimension $N \ge 9$ by constructing a bounded solution of equation (7.1) which is monotone in one direction whose level sets are not hyperplanes. The basis of their construction is a minimal graph different from a hyperplane built by Bombieri, de Giorgi and Giusti [8].

Condition (7.4) is related to the so-called Gibbons' Conjecture:

Gibbons' Conjecture: Let u be a bounded solution of equation (7.1) *satisfying*

$$\lim_{x_N \to \pm\infty} u(x^{'}, x_N) = \pm 1, \text{ uniformly in } x'. \tag{7.5}$$

Then the level sets $\{u = \lambda\}$ are all hyperplanes.

Gibbons' Conjecture has been proved in all dimensions with different methods by Caffarelli and Córdoba [11], Farina [26], Barlow, Bass and Gui [6], and Berestycki, Hamel, and Monneau [7]. In references [11, 6], it is proven that the conjecture is true for any solution that has one level set which is a globally Lipschitz graph.

Our problem (1.1)-(1.5) produces the limiting system (1.16). We have initiated study on (1.1)-(1.5) and (1.16), but there are still many questions remains. The major difficulty is that we have a system of equations instead of a single equation. In the following, we list some of the emerging questions:

1. Is there a "Γ-Convergence" theory for (1.1)-(1.5)?

2. The key estimate (1.12) in higher dimensions is still missing. It will require some extra techniques.

3. For Allen-Cahn equation (7.1), u is bounded between -1 and $+1$. On the other hand, the system (1.16) has unbounded one-dimensional solutions. Is there a growth estimate for (1.16)? We believe that the following growth should hold

$$U(x) + V(x) = O(|x|) \tag{7.6}$$

4. The De Giorgi type result for (1.16) is completely open, except in dimension two. What is the underlying geometry? We tend to believe that minimal surface is the underlying geometry.

5. Similar to the Gibbons's Conjecture, we also have the following conjecture

Conjecture: Let (U, V) be a solution of system (1.16) satisfying

$$\lim_{x_N \to -\infty} U(x^{'}, x_N) = 0, \quad \lim_{x_N \to +\infty} U(x^{'}, x_N) = +\infty, \text{ uniformly in } x', \tag{7.7}$$

$$\lim_{x_N \to -\infty} V(x^{'}, x_N) = +\infty, \quad \lim_{x_N \to +\infty} V(x^{'}, x_N) = 0, \text{ uniformly in } x'. \tag{7.8}$$

Then (U, V) are one-dimensional. This conjecture is completely open.

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