Solutions to Quiz Four

1. Part(a). According to mean-value property we get

$$u(0,0) = \frac{1}{2\pi}\int_0^{2\pi} 3 + \cos^2(7\theta)d\theta = \frac{7}{2}.$$

By Maximum Principle we get that

$$\max_{\bar{D}} u = \max_{\partial D} u = 4$$

and by Minimum Principle

$$\min_{\bar{D}} u = \min_{\partial D} u = 3.$$

Part(b). By the Poisson's formula, we have

$$u(r,\theta) = \frac{1-r^2}{2\pi}\int_0^{2\pi} \frac{h(\phi)}{1 - 2r\cos(\theta-\phi) + r^2} d\phi,$$

where

$$h(\phi) = u(1,\phi) \geq 0.$$

Since

$$(1-r)^2 \leq 1 - 2r\cos(\theta-\phi) + r^2 \leq (1+r)^2,$$

we get

$$\frac{1-r^2}{2\pi}\int_0^{2\pi} \frac{h(\phi)}{(1+r)^2} d\phi \leq u(r,\theta) \leq \frac{1-r^2}{2\pi}\int_0^{2\pi} \frac{h(\phi)}{(1-r)^2} d\phi.$$

Now by the Mean-Value property, we have

$$u(0,0) = \frac{1}{2\pi}\int_0^{2\pi} h(\phi)d\phi.$$

Therefore

$$\frac{1-r}{1+r}u(0,0) \leq u(x,y) \leq \frac{1+r}{1-r}u(0,0).$$

2. Part(a)

$$E(u) = \frac{1}{2}\int_D \left(|\nabla u|^2 + b(x)u^2\right)dx + \int_D f(x)u dx.$$

Part(b). Let $v = w - u$, then

$$v = 0 \quad \text{on} \quad \partial D.$$

By Green's first identity we can get that

$$\int_D v\Delta u + \nabla v \cdot \nabla u = 0.$$

Hence

$$\begin{aligned} E(w) - E(u) =& \frac{1}{2}\int_D (|\nabla v|^2 + b(x)v^2) + \int_D \big(\nabla v \cdot \nabla u + b(x)uv + f(x)v\big)dx \\ =& \frac{1}{2}\int_D (|\nabla v|^2 + b(x)v^2) + \int_D \big(-v\Delta u + b(x)uv + f(x)v\big)dx \\ =& \frac{1}{2}\int_D (|\nabla v|^2 + b(x)v^2) + \int_D \big(-\Delta u + b(x)u + f(x)\big)vdx \\ =& \frac{1}{2}\int_D (|\nabla v|^2 + b(x)v^2) \geq 0 \end{aligned}$$

since $b(x) \geq 0$.