Solutions to Quiz 3

1. Using separation of variables we get the eigenvalue problem

$$X'' + \lambda X = 0, \quad X'(0) = X(\pi) = 0.$$

Solving this eigenvalue problem we get

$$\lambda_n = (n - \frac{1}{2})^2, \qquad X_n = \cos(n - \frac{1}{2})x.$$

Solving

$$T'' = -\lambda_n c^2 T$$

we obtain

$$T_n = A_n \cos(n - \frac{1}{2})ct + B_n \sin(n - \frac{1}{2})ct.$$

Thus

$$u(x,t) = \sum_{n=1}^{\infty} \left(A_n \cos(n - \frac{1}{2})ct + B_n \sin(n - \frac{1}{2})ct \right) \cos(n - \frac{1}{2})x.$$

Using initial condition, we get that

$$A_1 = 1, \quad A_n = 0 \quad \text{for} \quad n = 2, 3, \ldots$$

and

$$B_2 = \frac{4}{3c}, \quad B_n = 0 \quad \text{for} \quad n \neq 2.$$

Hence we get the solution

$$u(x,t) = \cos\frac{ct}{2} \cos\frac{x}{2} + \frac{4}{3c} \sin\frac{3ct}{2} \cos\frac{3x}{2}.$$

2. Using the boundary conditions we get that

$$al + 1 = 0 \qquad \text{for} \quad \lambda = 0,$$

which is impossible for $a \to +\infty$ since $l > 0$ is a fixed number.

$$(-\frac{\lambda}{a^2} + 1) \tanh\sqrt{-\lambda} l + \frac{2\sqrt{-\lambda}}{a} = 0, \quad \text{for} \quad \lambda < 0.$$

Since $\tanh\sqrt{-\lambda}l > 0$, as $a \to +\infty$ we get contradiction.

$$(-\frac{\lambda}{a^2}+1)\sin\sqrt{\lambda}l + \frac{2\sqrt{\lambda}}{a}\cos\sqrt{\lambda}l = 0, \quad \text{for} \quad \lambda > 0.$$

Taking $a \to +\infty$, we get that

$$\sin\sqrt{\lambda}l \to 0.$$

Thus

$$\lambda_n - (\frac{(n+1)\pi}{l})^2 \to 0.$$