Solutions to Quiz One

1. Part(a) The characteristic curves satisfy the ODE

$$\frac{dy}{dx} = \frac{1}{xy}.$$

Solved the ODE,we have

$$x = Ce^{\frac{1}{2}y^2}$$

$$\Rightarrow C = x \cdot e^{-\frac{1}{2}y^2}$$

$$\Rightarrow u(x,y) = f(x \cdot e^{-\frac{1}{2}y^2}).$$

If $y = 0$, then $u_y(x,0) = 0$ by equation for any x. Especially for $x = 1$, but $(e^y)'|_{y=0} = 1$, contradiction. Hence $y \neq 0$. Now we divide into two parts $y > 0$ and $y < 0$.

Using initial data, we get

$$u(x,y) = \begin{cases} e^{\sqrt{2\log\frac{1}{xe^{-y^2/2}}}}, & \text{if} \quad 0 < x < e^{y^2/2}, y > 0 \\ e^{-\sqrt{2\log\frac{1}{xe^{-y^2/2}}}}, & \text{if} \quad 0 < x < e^{y^2/2}, y < 0. \end{cases}$$

The domain of existence is :

$$\Omega := \{(x,y) \in \mathbb{R}^2 | 0 < x < e^{\frac{y^2}{2}}, y \neq 0\}.$$

Part(b)

Let

$$x' = 3x + 4y, \qquad y' = 4x - 3y,$$

then the original equation is changed into

$$25u_{x'} + u = 0.$$

Solve this PDE we get

$$u(x', y') = f(y')e^{-\frac{x'}{25}},$$

thus the solution of the original problem is

$$u(x,y) = f(4x-3y)e^{-\frac{3x+4y}{25}}$$

2.

The corresponding matrix is

$$A = \begin{pmatrix} 1 & -3 \\ -3 & 9 \end{pmatrix}$$

Obviously $\det A = 0$, thus the type is parabolic.

Since the original equation can be rewritten as

$$(\partial x - 3\partial y)^2 u - u_y + u_x = 0,$$

letting

$$\partial\xi = \partial x - 3\partial y, \qquad \partial\eta = -\partial x + \partial y,$$

then the original equation has the following standard form

$$u_{\xi\xi} - u_\eta = 0.$$

At the same time, we can get that

$$\partial x = -\frac{\partial\xi + 3\partial\eta}{2}, \qquad \partial y = -\frac{\partial\xi + \partial\eta}{2}.$$

thus the linear transform is

$$\begin{pmatrix} \xi \\ \eta \end{pmatrix} = \begin{pmatrix} -\frac{1}{2} & -\frac{1}{2} \\ -\frac{3}{2} & -\frac{1}{2} \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}.$$