Solutions to Midterm Examination

1. Using formula we can get that

$$u(x,t) = \int_{-\infty}^{\infty} \frac{1}{\sqrt{4k\pi t}} e^{-\frac{(x-y)^2}{4kt}} y dy + \int_0^t \int_{-\infty}^{\infty} \frac{1}{\sqrt{4k\pi(t-s)}} e^{-\frac{(x-y)^2}{4k(t-s)}} ysdyds.$$

Compute directly we get

$$\int_{-\infty}^{\infty} \frac{1}{\sqrt{4k\pi t}} e^{-\frac{(x-y)^2}{4kt}} y dy = \int_{-\infty}^{\infty} \frac{1}{\sqrt{4k\pi t}} e^{-\frac{(x-y)^2}{4kt}} x dy = x,$$

since

$$\int_{-\infty}^{\infty} \frac{1}{\sqrt{4k\pi t}} e^{-\frac{(x-y)^2}{4kt}} (y-x) dy = 0.$$

Similarly we can obtain

$$\int_0^t \int_{-\infty}^{\infty} \frac{1}{\sqrt{4k\pi(t-s)}} e^{-\frac{(x-y)^2}{4k(t-s)}} ysdyds = \frac{xt^2}{2}.$$

Thus the solution $u(x,t) = x + \frac{xt^2}{2}$.

2. Part(a).

(1) Hyperbolic.

(2) Let $v = u_x$, then v satisfies

$$v_y + 4v = 0.$$

Solve it we can get $v = f(x)e^{-4y}$ for any arbitrary smooth function $f(x)$. Thus

$$u_x = f(x)e^{-4y}$$

which leads to $u = F(x)e^{-4y} + G(y)$ for any smooth functions $F(x), G(y)$. According to the boundary condition we get

$$F(x)e^{-4x} + G(x) = x, \qquad F'(x)e^{-4x} = 1,$$

from which we get

$$F(x) = \frac{1}{4}e^{4x} + c, \qquad G(x) = x - \frac{1}{4} - ce^{-4x}.$$

Hence

$$u(x,y) = \frac{1}{4}e^{4x-4y} + y - \frac{1}{4}.$$

Part(b). Using formula we get

$$u(\frac{1}{2},2) = \frac{1}{2}(\phi_{ext}(4\frac{1}{2}) + \phi_{ext}(-3\frac{1}{2})) + \frac{1}{4}\int_{-3\frac{1}{2}}^{4\frac{1}{2}} \psi_{ext}(s)ds = \frac{1}{2}(\phi(\frac{1}{2}) + \phi(\frac{1}{2})) + 0 = 1.$$

3. Let

$$u_n(x,t) = \frac{1}{n}e^{-nkt}\sin\sqrt{n}x,$$

then u_n satisfies the same equation with $\phi_n(x) = \frac{1}{n}\sin\sqrt{n}x$ which goes to zero as $n \to \infty$. But u_n doesn't tend to zero since $t < 0$.

4. Let u_1 and u_2 be two solutions of the equation. Then $u = u_1 - u_2$ satisfies the homogeneous equation with homogeneous initial data and the same boundary conditions. Let

$$E(t) = \frac{1}{2}\int_0^l u^2(x,t)dx,$$

then

$$\frac{dE}{dt} = kuu_x|_0^l - k\int_0^l u^2dx = -k\int_0^l u^2dx \leq 0.$$

Obviously $E(0) = 0$, $E(t) \equiv 0$ for any t. Thus $u(x,t) \equiv 0$ and we get the uniqueness.

5. One Proof.

If not, we assume $M_1 < M$. Let

$$v(x,t) = u(x,t) - \frac{M - M_1}{2T}t.$$

Then $v(x,t)$ satisfies

$$v_t - kv_{xx} = -\frac{M - M_1}{2T} < 0.$$

Using the same argument as the proof of maximum principle in the book we know that the maximum point (x_0, t_0) of $v(x,t)$ must lie in $x = 0, x = l, t = 0$.

If $x_0 = 0, t_0 \neq 0$ then $v_t|_{(x_0,t_0)} = 0, v_x|_{(x_0,t_0)} = 0$. On the other hand $v_{xx} > 0$ by $v_t - kv_{xx} = -\frac{M-M_1}{2T} < 0$ and continuity. Hence (x_0, t_0) can't be the maximum point.

By the same argument we know that $x_0 = l, t_0 \neq 0$ can't happen.

Obviously,

$$\max_R v(x,t) \geq M - \frac{M - M_1}{2} > M_1,$$

which shows that $t_0 = 0$ is impossible. All the above leads to contradiction. We complete the proof.

Another Proof.

If not, we assume $M_1 < M$. Let

$$v(x,t) = u(x,t) - \frac{M - M_1}{2}\varepsilon kt - \frac{M - M_1}{8}(x - \frac{l}{2})^2\varepsilon.$$

Then $v(x,t)$ satisfies

$$v_t - v_{xx} = -\frac{\varepsilon k(M - M_1)}{4} < 0.$$

Using the same argument as the proof of maximum principle in the book we know that the maximum point (x_0, t_0) of $v(x,t)$ must lie in $x = 0, x = l, t = 0$.

If $x_0 = 0, t_0 \neq 0$ then $v_x|_{(x_0,t_0)} \leq 0$. On the other hand $v_x = u_x - \frac{M-M_1}{4}(x - \frac{l}{2})\varepsilon$. Especially $v_x|_{(0,t_0)} = 0 + \frac{(M-M_1)\varepsilon}{8} > 0$, contradiction.

By the same argument we know that $x_0 = l, t_0 \neq 0$ can't happen. Thus

$$\max_R v(x,t) = \max_{0 \leq x \leq l} v(x,0) \leq M_1.$$

That is to say

$$\max_R u(x,t) - \frac{M - M_1}{2}\varepsilon kT - \frac{M - M_1}{8}(\frac{l}{2})^2\varepsilon \leq M_1.$$

Let $\varepsilon \to 0$, we get

$$M = \max_R u(x,t) \leq M_1$$

which leads to contradiction. We complete the proof.

The third proof.

According to the Neumann boundary condition, $\phi(x)$ can be extended by the following

$$\phi_{ext}(x) = \begin{cases} \phi(x), & 0 < x < l \\ \phi(-x), & -l < x < 0 \\ \text{peridic extention.} \end{cases}$$

Obviously

$$\max_{-\infty<y<\infty} \phi_{ext}(y) = \max_{0<y<l} \phi(y).$$

Then using formula we get

$$\begin{aligned} u(x,t) &= \int_{-\infty}^{\infty} S(x-y,t)\phi_{ext}(y)dy \\ &\leq \int_{-\infty}^{\infty} S(x-y,t)\Big(\max_{0<y<l} \phi_{ext}(y)\Big)dy \\ &= \max_{0<y<l} \phi(y) = M_1, \end{aligned}$$

which leads to $M \leq M_1$. Obviously $M_1 \leq M$. We complete the proof.