

## Solutions to Assignment Six

### Exercise 7.1

#### 1. The maximum principle:

If  $D$  is any solid region, a non-constant harmonic function in  $D$  can not take its maximum value inside  $D$ , but only on  $\partial D$ .

**Proof:** Suppose there exists one non-constant harmonic function  $u$  in  $D$ , which attains its maximum  $M$  at  $x_0 \in D$ .

By the mean value property, we have

$$u(x_0) = \frac{1}{|B(x_0, r)|} \iint_{B(x_0, r)} u dS,$$

for all  $\overline{B(x_0, r)} \subset D$ .

Thus  $u(x) = M$  for all  $x \in \overline{B(x_0, r)} \subset D$  since  $u$  is continuous and  $M$  is its maximum.

Now since  $u$  is not a constant, there exists  $x_1 \in D$  such that  $u(x_1) \neq u(x_0)$ . Since  $D$  is a region, we can find a curve  $\gamma(t) \subset D, 0 \leq t \leq 1$  such that  $\gamma(0) = x_0, \gamma(1) = x_1$ .

Set  $E := \{0 \leq t \leq 1 \mid u(\gamma(t)) = M\}$ , then  $E$  is closed since  $u$  and  $\gamma$  are both continuous. Let  $t_0 \in E$ , then by the above result,  $u(x) = M$  for all  $x \in \overline{B(\gamma(t_0), r)} \subset D$ . So  $E$  is open. Hence  $E = [0, 1]$  since  $E \neq \emptyset$  and  $[0, 1]$  is connected. But this is contradict with  $u(x_1) \neq u(x_0)$ .

2. Suppose  $u_1$  and  $u_2$  are solutions of the problem. Then  $u = u_1 - u_2$  satisfies

$$\Delta u = 0 \text{ in } D, \quad \frac{\partial u}{\partial n} = 0 \text{ on } \partial D.$$

Thus by the Green's first identity for  $v = u$ , we have

$$\iint_{\partial D} u \cdot \frac{\partial u}{\partial n} dS = \iiint_D |\nabla u|^2 dx + \iiint_D u \Delta u dx.$$

Hence

$$\iiint_D |\nabla u|^2 dx = 0.$$

Therefore  $u(x)$  is a constant.

**3.** Suppose  $u_1$  and  $u_2$  are solutions of the problem. Then  $u = u_1 - u_2$  satisfies

$$\Delta u = 0 \text{ in } D, \quad \frac{\partial u}{\partial n} + au = 0 \text{ on } \partial D.$$

Thus by the Green's first identity for  $v = u$ , we have

$$\iint_{\partial D} u \cdot \frac{\partial u}{\partial n} dS = \iiint_D |\nabla u|^2 dx + \iiint_D u \Delta u dx.$$

Hence

$$- \iint_{\partial D} au^2 = \iiint_D |\nabla u|^2 dx.$$

Therefore  $\nabla u = 0$  since  $a(x) > 0$  and  $u$  is continuous. So  $u = c$ , where  $c$  is a constant. Finally, by the Robin boundary conditions, we have  $ca(x) = 0$ . Therefore  $u = 0$ .

**4.** Suppose  $u_1$  and  $u_2$  are solutions of the problem. Then  $u = u_1 - u_2$  satisfies

$$u_t = k\Delta u \text{ in } D \times (0, \infty), \quad u = 0 \text{ on } \partial D \times (0, \infty), \quad u(x, 0) = 0 \text{ in } D.$$

Thus by the Green's first identity for  $v = u$ , we have

$$\iint_{\partial D} u \cdot \frac{\partial u}{\partial n} dS = \iiint_D |\nabla u|^2 dx + \iiint_D u \Delta u dx.$$

Thus

$$0 = \iiint_D |\nabla u|^2 dx + \frac{1}{k} \iiint_D uu_t dx = \iiint_D |\nabla u|^2 dx + \frac{1}{2k} \frac{d}{dt} \iiint_D u^2 dx.$$

Hence  $E(t) := \iiint_D u^2 dx (\geq 0)$  is decrease. Since  $E(0) = 0$ , so  $E(t) \equiv 0$ . Thus we get

$$\iiint_D |\nabla u|^2 dx \equiv 0.$$

Hence  $\nabla u \equiv f(t)$  for some function  $f$ . So  $u(x, t)$  is a constant since  $u_t = k\Delta u = 0$ .

Now by the boundary conditions, we have  $u \equiv 0$ . Thus we get uniqueness.

**5.** Suppose  $u(x)$  minimizes the energy. Let  $v(x)$  be any function, then we have

$$E[u] \leq E[u + \epsilon v] = E[u] + \epsilon \iiint_D \nabla u \nabla v dx - \epsilon \iint_{\partial D} hvdS + \epsilon^2 \iiint_D |\nabla v|^2 dx$$

for any constant  $\epsilon$ .

Thus by calculus, we have

$$\iiint_D \nabla u \nabla v dx - \iint_{\partial D} h v dS = 0$$

for any function  $v$ .

Now by Green's first identity, we have

$$- \iiint_D \Delta u \cdot v dx + \iint_{\partial D} v \cdot \frac{\partial u}{\partial n} dS - \iint_{\partial D} h v dS = 0.$$

Firstly, we let  $v$  vanishes on  $\partial D$ , we get  $-\Delta u = 0$ . Thus we also have

$$\iint_{\partial D} v \cdot \frac{\partial u}{\partial n} dS - \iint_{\partial D} h v dS = 0$$

for any function  $v$ . Hence  $\frac{\partial u}{\partial n} = h$ . By uniqueness, it is the only function that can minimize the energy.

Note: we assume functions  $u, v$  and the domain  $D$  are smooth enough, at least under which the Green's first identity can be hold.

6. (a) Suppose  $u_1$  and  $u_2$  are solutions of the problem. Then  $u = u_1 - u_2$  tends to zero at infinity, and is 0 on  $\partial D$ . Suppose that  $u \neq 0$ , then without loss of generality, we assume that  $\exists x_0 \in D$  such that  $u(x_0) > 0$ . Since  $u$  tends to zero at infinity, so there exists  $R \gg 1$  such that  $u(x) \leq \frac{u(x_0)}{2}, \forall x \in B(0, R)$ .

Then  $u$  is a harmonic function in  $D \cap B(0, R)$ , But  $\max_{\partial(D \cap B(0, R))} u \leq \frac{u(x_0)}{2} < u(x_0) \leq \max_{D \cap B(0, R)} u$ , this is contradict with the Maximum Principle. Hence  $u = 0$ .

(b) If not, then  $u(x)$  has a negative minimum. As above, by the Minimum Principle, we known that  $u$  get its minimum on  $\partial A$  or  $\partial B$ .

Now we assume that  $u$  attains its minimum on  $\partial A$ , then  $\frac{\partial u}{\partial n} < 0$  by the Hopf principle since  $u$  is not a constant. But this is contradict with the fact  $\iint_{\partial A} \frac{\partial u}{\partial n} dS = Q > 0$ . Surely the same is true for another case. Hence  $u \geq 0$  in  $D$ .

(c) Suppose that there exists  $x_0 \in D$  such that  $u(x_0) = 0$ . Then by (b) and the Strong Minimum Principle,  $u$  is a constant in  $D$ . But this is contradict with the fact  $\iint_{\partial A} \frac{\partial u}{\partial n} dS = Q > 0$ . Hence  $u > 0$  in  $D$ .

(Actually we can also prove that  $u > 0$  on  $\partial A$  and  $\partial B$  by the Hopf principle again as in (b).)

**Exercise 7.2**

1. (As in the dimension three, surely the same is true for any finite dimension)

Let  $D_\epsilon$  be the region  $D$  with a ball (of radius  $\epsilon$  and center  $x_0$ ) excised. For simplicity let  $x_0$  be the origin,  $r = \sqrt{x^2 + y^2}$ .

By the Green's second identity, we have

$$\iint_{\partial D_\epsilon} u \cdot \frac{\partial}{\partial n}(\log r) - \frac{\partial u}{\partial n} \cdot \log r \, dS = 0$$

But  $\partial D_\epsilon$  consists of two parts: the original boundary  $\partial D$  and the circle  $\{r = \epsilon\}$ . on the circle,  $\frac{\partial}{\partial n} = -\frac{\partial}{\partial r}$ . Thus the curve integral breaks into two pieces,

$$\iint_{\partial D} [u \cdot \frac{\partial}{\partial n}(\log r) - \frac{\partial u}{\partial n} \cdot \log r] \, dS = \iint_{r=\epsilon} [u \cdot \frac{\partial}{\partial r}(\log r) - \frac{\partial u}{\partial r} \cdot \log r] \, dS.$$

This identity is valid for any small enough  $\epsilon > 0$ .

Now the right side of the identity equals

$$\frac{1}{\epsilon} \iint_{r=\epsilon} u \, dS - \log \epsilon \iint_{r=\epsilon} \frac{\partial u}{\partial r} \, dS = 2\pi \bar{u} - 2\pi \epsilon \log \epsilon \overline{\frac{\partial u}{\partial r}},$$

where  $\bar{u}$  denotes the average value of  $u$  on the circle  $\{r = \epsilon\}$ , and  $\overline{\frac{\partial u}{\partial r}}$  denotes the average value of  $\frac{\partial u}{\partial r}$  on this circle.

As  $\epsilon \rightarrow 0$ ,

$$2\pi \bar{u} - 2\pi \epsilon \log \epsilon \overline{\frac{\partial u}{\partial r}} \rightarrow 2\pi u(0)$$

because  $u$  is continuous and  $\frac{\partial u}{\partial r}$  is bounded. Thus this complete the proof.

2. Let  $D_\epsilon := \{x \mid \epsilon < |x| < R\}$ , where  $R$  is large enough such that  $\phi = 0$  outside  $|x| < R$ .

Using the Green's second identity, let  $r = |x|$ , we have

$$\begin{aligned} \iiint_{D_\epsilon} \frac{1}{|x|} \Delta \phi(x) \, dx &= \iint_{\partial D_\epsilon} \frac{1}{|x|} \cdot \frac{\partial \phi}{\partial n} - \phi \cdot \frac{\partial}{\partial n} \frac{1}{|x|} \, dS \\ &= \iint_{|x|=R} \frac{1}{|x|} \cdot \frac{\partial \phi}{\partial n} - \phi \cdot \frac{\partial}{\partial n} \frac{1}{|x|} \, dS + \iint_{|x|=\epsilon} \frac{1}{|x|} \cdot \frac{\partial \phi}{\partial n} - \phi \cdot \frac{\partial}{\partial n} \frac{1}{|x|} \, dS \end{aligned}$$

$$= - \iint_{|x|=\epsilon} \frac{1}{\epsilon} \cdot \frac{\partial \phi}{\partial r} + \phi \cdot \frac{1}{\epsilon^2} dS = -4\pi \bar{\phi} - 4\pi \epsilon \frac{\overline{\partial \phi}}{\partial r},$$

where  $\bar{\phi}$  denotes the average value of  $u$  on the sphere  $\{|x| = \epsilon\}$ , and  $\frac{\overline{\partial \phi}}{\partial r}$  denotes the average value of  $\frac{\partial u}{\partial r}$  on this sphere.

As  $\epsilon \rightarrow 0$ ,

$$-4\pi \bar{\phi} - 4\pi \epsilon \frac{\overline{\partial \phi}}{\partial r} \rightarrow -4\pi \phi(0)$$

because  $\phi$  is continuous and  $\frac{\partial \phi}{\partial r}$  is bounded. Thus this complete the proof.

**3.** By choosing  $D = B(x_0, R)$  in the representation formula (1), we have

$$\begin{aligned} u(x_0) &= \iint_{\partial B(x_0, R)} \left[ -u(x) \cdot \frac{\partial}{\partial n} \left( \frac{1}{|x - x_0|} \right) + \frac{1}{|x - x_0|} \cdot \frac{\partial u}{\partial n} \right] \frac{dS}{4\pi} \\ &= \iint_{|x-x_0|=R} \left[ \frac{1}{R^2} u(x) + \frac{1}{R} \frac{\partial u}{\partial n} \right] \frac{dS}{4\pi} \\ &= \frac{1}{4\pi R^2} \iint_{|x-x_0|=R} u dS + \frac{1}{4\pi R} \iiint_{|x-x_0|<R} \Delta u dx \\ &= \frac{1}{4\pi R^2} \iint_{|x-x_0|=R} u dS. \end{aligned}$$

This proves the mean value property in three dimensions.

**Extra Problem:** Let  $\phi(x)$  be any  $C^2$  function defined on all of two-dimensional space that vanishes outside some circle. Show that

$$\phi(0) = \iint \log |x| \Delta \phi(x) \frac{dx}{2\pi}.$$

The integration is taken over the region where  $\phi(x)$  is not zero.

**Proof:** Let  $D_\epsilon := \{\epsilon < |x| < R\}$ , where  $R$  is large enough such that  $\phi = 0$  outside  $|x| < R$ .

Using the Green's second identity, let  $r = |x|$ , we have

$$\iint_{D_\epsilon} \log |x| \Delta \phi(x) dx = \int_{\partial D_\epsilon} \log |x| \cdot \frac{\partial \phi}{\partial n} - \phi \cdot \frac{\partial}{\partial n} \log |x| dS$$

$$\begin{aligned}
&= \int_{|x|=R} \log |x| \cdot \frac{\partial \phi}{\partial n} - \phi \cdot \frac{\partial}{\partial n} \log |x| dS + \int_{|x|=\epsilon} \log |x| \cdot \frac{\partial \phi}{\partial n} - \phi \cdot \frac{\partial}{\partial n} \log |x| dS \\
&= - \int_{|x|=\epsilon} \log \epsilon \cdot \frac{\partial \phi}{\partial r} - \phi \cdot \frac{1}{\epsilon} dS = 2\pi \bar{\phi} - 2\pi \epsilon \log \epsilon \frac{\overline{\partial \phi}}{\partial r},
\end{aligned}$$

where  $\bar{\phi}$  denotes the average value of  $u$  on the sphere  $\{|x| = \epsilon\}$ , and  $\frac{\overline{\partial \phi}}{\partial r}$  denotes the average value of  $\frac{\partial u}{\partial r}$  on this sphere.

As  $\epsilon \rightarrow 0$ ,

$$2\pi \bar{\phi} - 2\pi \epsilon \log \epsilon \frac{\overline{\partial \phi}}{\partial r} \rightarrow 2\pi \phi(0)$$

because  $\phi$  is continuous and  $\frac{\partial \phi}{\partial r}$  is bounded. Thus this complete the proof.

### Exercise 7.3

1. Suppose  $G_1$  and  $G_2$  both are the Green's functions for the operator  $-\Delta$  and the domain  $D$  at the point  $x_0 \in D$ .

By (i) and (iii), we know that  $G_1(x) + \frac{1}{4\pi|x-x_0|}$  and  $G_2(x) + \frac{1}{4\pi|x-x_0|}$  both are harmonic function in  $D$ . Thus by (ii) and the uniqueness theorem of harmonic function, we have

$$G_1(x) + \frac{1}{4\pi|x-x_0|} = G_2(x) + \frac{1}{4\pi|x-x_0|}.$$

Therefore the Green's function is unique.

3. In the book,

$$A_\epsilon = \iint_{|x-a|=\epsilon} \left( u \frac{\partial v}{\partial n} - v \frac{\partial u}{\partial n} \right) dS,$$

where  $u(x) = G(x, a)$  and  $v(x) = G(x, b)$ .

Now, Let's computer  $A_\epsilon$ . Firstly note that

$$u(x) = G(x, a) = -\frac{1}{4\pi|x-a|} + H(x, a),$$

where  $H(x, a)$  is a harmonic function throughout the domain  $D$ . Secondly  $v(x) = G(x, b)$  is a harmonic function in  $\{|x-a| < \epsilon\}$ . Thus

$$A_\epsilon = \iint_{|x-a|=\epsilon} \left( \left( -\frac{1}{4\pi|x-a|} + H(x, a) \right) \frac{\partial v}{\partial n} - v \frac{\partial}{\partial n} \left( -\frac{1}{4\pi|x-a|} + H(x, a) \right) \right) dS$$

$$\begin{aligned}
&= \iint_{|x-a|=\epsilon} \left( -\frac{1}{4\pi|x-a|} \frac{\partial v}{\partial n} + v \frac{\partial}{\partial n} \left( \frac{1}{4\pi|x-a|} \right) \right) dS + \iint_{|x-a|=\epsilon} \left( H(x,a) \frac{\partial v}{\partial n} - v \frac{\partial}{\partial n} H(x,a) \right) dS \\
&= v(a) - \iiint_{|x-a|<\epsilon} (H(x,a)\Delta v - v\Delta H(x,a)) dx \\
&= v(a).
\end{aligned}$$

(Here we use the representation formula (7.2.1) and note that  $\frac{\partial}{\partial n} = -\frac{\partial}{\partial r}$ ).

Hence  $\lim_{\epsilon \rightarrow 0} A_\epsilon = v(a)$ .

#### Exercise 7.4

1. By (i), we know that  $G(x)$  is a linear function in  $[0, x_0]$  and  $[x_0, l]$ . Since (ii) and  $G(x)$  is continuous at  $x_0$ , we have

$$G(x) = \begin{cases} kx, & 0 < x \leq x_0; \\ \frac{kx_0}{x_0-l}(x-l), & x_0 < x < l. \end{cases}$$

Hence  $G(x) + \frac{1}{2}|x - x_0|$  is harmonic at  $x_0$  if and only if

$$(G(x) + \frac{1}{2}|x - x_0|)'(x_0^-) = (G(x) + \frac{1}{2}|x - x_0|)'(x_0^+)$$

if and only if

$$k = \frac{l - x_0}{l}.$$

Therefore the one-dimensional Green's function for the interval  $(0, l)$  is

$$G(x) = \begin{cases} \frac{l-x_0}{l}x, & 0 < x \leq x_0; \\ -\frac{x_0}{l}(x-l), & x_0 < x < l. \end{cases}$$

2. Assume that  $h(x, y)$  is a continuous function that vanishes outside  $\{(x, y) \mid x^2 + y^2 \leq r^2\}$  and  $|h(x, y)| \leq M$ . Then by the formulas (3), we have

$$\begin{aligned}
|u(x_0, y_0, z_0)| &\leq \frac{M}{2\pi} \iint_{\{(x,y) \mid x^2+y^2 \leq r^2\}} [(x-x_0)^2 + (y-y_0)^2 + z_0^2]^{-1} dx dy \\
&\leq \frac{Mr^2}{2(\sqrt{x_0^2 + y_0^2 + z_0^2} - r)^2}, \text{ when } \sqrt{x_0^2 + y_0^2 + z_0^2} > r.
\end{aligned}$$

Therefore,  $u$  satisfies the condition at infinity:

$$u(X) \rightarrow 0 \text{ as } |X| \rightarrow \infty.$$

3. From (3), we have

$$\begin{aligned} u(x_0, y_0, z_0) &= \frac{z_0}{2\pi} \iint [(x - x_0)^2 + (y - y_0)^2 + z_0^2]^{-\frac{3}{2}} h(x, y) dx dy \\ &= \frac{z_0}{2\pi} \iint (x^2 + y^2 + z_0^2)^{-\frac{3}{2}} h(x + x_0, y + y_0) dx dy. \end{aligned}$$

Now we change variables such that  $x = z_0 s \cos \theta$ ,  $y = z_0 s \sin \theta$ . Then we have

$$\begin{aligned} u(x_0, y_0, z_0) &= \frac{z_0}{2\pi} \int_0^{2\pi} \int_0^\infty (z_0^2 s^2 + z_0^2)^{-\frac{3}{2}} h(z_0 s \cos \theta + x_0, z_0 s \sin \theta + y_0) z_0^2 s ds d\theta \\ &= \frac{1}{2\pi} \int_0^{2\pi} \int_0^\infty s(s^2 + 1)^{-\frac{3}{2}} h(z_0 s \cos \theta + x_0, z_0 s \sin \theta + y_0) ds d\theta. \end{aligned}$$

Since

$$\int_0^\infty s(s^2 + 1)^{-\frac{3}{2}} ds = -(s^2 + 1)^{-\frac{3}{2}} \Big|_0^\infty = 1,$$

so

$$\lim_{z_0 \rightarrow 0} u(x_0, y_0, z_0) = h(x_0, y_0).$$

**5. Suggested explain:** Since the half-plane  $\{y > 0\}$  is not bounded, this only means that the solution is not unique under the (uncompleted) boundary condition or it is not a well-posed problem. But generally it will be a well-posed problem if we add another boundary condition, such as

$$u(X) \rightarrow 0 \text{ as } |X| \rightarrow \infty.$$

6. (a) As in the dimension three, we have

$$G(X, X_0) = \frac{1}{2\pi} \log |X - X_0| - \frac{1}{2\pi} \log |X - X_0^*|,$$

where  $X = (x, y)$ ,  $X_0 = (x_0, y_0)$ ,  $X_0^* = (x_0, -y_0)$ .

(b) Since

$$\begin{aligned} -\frac{\partial G}{\partial y} &= \frac{y + y_0}{2\pi((x - x_0)^2 + (y + y_0)^2)} - \frac{y - y_0}{2\pi((x - x_0)^2 + (y - y_0)^2)} \\ &= \frac{y_0}{\pi((x - x_0)^2 + y_0^2)} \end{aligned}$$

on  $y = 0$ , Therefore, the solution is

$$u(x_0, y_0) = \frac{y_0}{\pi} \int \frac{h(x)}{(x - x_0)^2 + y_0^2} dx$$

(c) By (b), we have

$$u(x_0, y_0) = \frac{y_0}{\pi} \int \frac{1}{(x - x_0)^2 + y_0^2} dx = 1.$$

7. (a) If  $u(x, y) = f(\frac{x}{y})$ , then

$$u_x = \frac{1}{y} f'(\frac{x}{y}), u_y = -\frac{x}{y^2} f'(\frac{x}{y}),$$

Thus

$$u_{xx} + u_{yy} = \frac{1}{y^2} f''(\frac{x}{y}) + \frac{x^2}{y^4} f''(\frac{x}{y}) + 2\frac{x}{y^3} f'(\frac{x}{y}).$$

Hence the ODE satisfied by  $f$  is

$$(1 + t^2)f''(t) + 2tf'(t) = 0.$$

Therefore,

$$f(t) = \int_0^t \frac{a}{1 + s^2} ds + b = a \arctan t + b,$$

where  $a, b$  are constants.

(b) Since

$$u(x, y) = f(\frac{x}{y}) = a \arctan \frac{x}{y} + b,$$

so

$$\frac{\partial u}{\partial r} = \frac{\partial u}{\partial x} \frac{\partial x}{\partial r} + \frac{\partial u}{\partial y} \frac{\partial y}{\partial r} = \frac{a}{1 + (x/y)^2} \frac{1}{y} \frac{x}{r} + \frac{a}{1 + (x/y)^2} \frac{-x}{y^2} \frac{y}{r} = 0.$$

(c) Using the polar coordinate, we have

$$v(r, \theta) = c\theta + d,$$

where  $c, d$  are constants. Hence in the  $(x, y)$ -coordinate, we have

$$v(x, y) = c \arctan \frac{y}{x} + d.$$

(d) By (a), we have

$$u(x, y) = f\left(\frac{x}{y}\right) = a \arctan \frac{x}{y} + b.$$

Thus

$$h(x) = \lim_{y \rightarrow 0} u(x, y) = \begin{cases} \frac{\pi}{2}a + b, & x > 0; \\ -\frac{\pi}{2}a + b, & x < 0; \\ b, & x = 0. \end{cases}$$

(e) By (d), we see that  $h(x)$  is not continuous unless  $u$  is a constant. This agrees with the condition that  $h(x)$  is continuous in Exercise 6.

**9.** We have known that the Green's function for the half-space is

$$G'(X', X'_0) = -\frac{1}{4\pi} \sqrt{(x' - x'_0)^2 + (y' - y'_0)^2 + (z' - z'_0)^2} \\ + \frac{1}{4\pi} \sqrt{(x' - x'_0)^2 + (y' - y'_0)^2 + (z' + z'_0)^2}.$$

Now suppose that  $c \neq 0$  and make the linear transformation:  $x' = x, y' = y, z' = ax + by + cz$ , then the Green's function for the tilted half-space  $\{(x, y, z) | ax + by + cz > 0\}$  is

$$G(X, X_0) = -\frac{1}{4\pi} \sqrt{(x - x_0)^2 + (y - y_0)^2 + (ax + by + cz - ax_0 - by_0 - cz_0)^2} \\ + \frac{1}{4\pi} \sqrt{(x - x_0)^2 + (y - y_0)^2 + (ax + by + cz + ax_0 + by_0 + cz_0)^2}$$

since the linear transformation present the harmonic property and the uniqueness of the Green's function. It is similar for cases  $a \neq 0$  and  $b \neq 0$ .

**10.** In case  $X_0 = 0$ , the formula for the Green's function is

$$G(X, 0) = -\frac{1}{4\pi|X|} + \frac{1}{4\pi a}.$$

It is easy to check (i),(ii) and (iii).

**13.** We have known that

$$G(X, X_0) = -\frac{1}{4\pi|X - X_0|} + \frac{1}{4\pi|r_0X/a - aX_0/r_0|}$$

is the Green's function for the whole ball.

Let  $X'_0 = (x_0, y_0, -z_0)$ , then

$$G(X, X'_0) = -\frac{1}{4\pi|X - X'_0|} + \frac{1}{4\pi|r_0X/a - aX'_0/r_0|}$$

is also the Green's function for the whole ball. Note that  $G(X, X_0) = G(X, X'_0)$  on  $\partial D$  and  $G(X, X'_0)$  is a harmonic function in  $D$ . Hence  $G(X, X_0) - G(X, X'_0)$  is the Green's function for  $D$  by the uniqueness of the Green's function.

**15.** (a) If  $v(x, y)$  is harmonic and  $u(x, y) = v(x^2 - y^2, 2xy)$ , then

$$u_x = 2xv_x + 2yv_y, u_y = -2yv_x + 2xv_y.$$

Thus

$$\begin{aligned} u_{xx} + u_{yy} &= 2v_x + 4x^2v_{xx} + 4xyv_{xy} + 4xyv_{yx} + 4y^2v_{yy} - 2v_x + 4y^2v_{xx} - 4xyv_{xy} - 4xyv_{yx} + 4x^2v_{yy} \\ &= 4(x^2 + y^2)(v_{xx} + v_{yy}) = 0. \end{aligned}$$

(b) Using the polar coordinates, the transformation is

$$(r \cos \theta, r \sin \theta) \mapsto (r^2 \cos 2\theta, r^2 \sin 2\theta).$$

Therefore, it maps the first quadrant onto the half-plane  $\{y > 0\}$ .

(Note that the transformation is one-one which also maps the x-positive-axis to x-positive-axis and y-positive-axis to x-negative-axis. )

**17.**(a) We have known that

$$G'(X, X_0) = -\frac{1}{4\pi|X - X_0|} + \frac{1}{4\pi|X - X_0^*|}$$

is the Green's function for the half-plane. Now by the Exercise 15, we find the Green's function for the first quadrant is

$$G(X, X_0) = G'(X', X'_0) = -\frac{1}{4\pi|X' - X'_0|} + \frac{1}{4\pi|X' - (X'_0)^*|}$$

$$= -\frac{1}{4\pi\sqrt{(x^2 + x_0^2 - y^2 - y_0^2)^2 + 4(xy - x_0y_0)^2}} + \frac{1}{4\pi\sqrt{(x^2 + x_0^2 - y^2 - y_0^2)^2 + 4(xy + x_0y_0)^2}}$$

where  $X = (x, y)$ ,  $X_0 = (x_0, y_0)$ ,  $X' = (x^2 - y^2, 2xy)$ ,  $X'_0 = (x_0^2 - y_0^2, 2x_0y_0)$ , since the Green's function is unique.