Solutions to Assignment Five

Exercise6.1

2.By (6.1.(7)) in the book, we have

$$u_{rr} + \frac{2}{r}u_r = k^2 u.$$

Let $u = v/r$, we get

$$u_r = v_r/r - v/r^2,$$

$$u_{rr} = v_{rr}/r - 2v_r/r^2 + 2v/r^3.$$

Hence

$$v_{rr} = k^2 v \Rightarrow v = Ae^{-kr} + Be^{kr}$$

where A, B are constants. Therefore

$$u = A\frac{1}{r}e^{-kr} + B\frac{1}{r}e^{kr},$$

where A, B are constants.

4.From the book, we have known that $-c_1 r^{-1} + c_2$ is a solution, where c_1 and c_2 satisfy the equation:

$$-c_1 a^{-1} + c_2 = A, -c_1 b^{-1} + c_2 = B.$$

Hence

$$u(x,y) = ab\frac{A-B}{b-a}r^{-1} + A + b\frac{B-A}{b-a}, \text{ where } r = \sqrt{x^2+y^2+z^2}$$

is a solution. Therefore it is the unique solution by the Uniqueness Theorem.

5. Firstly we find a solution that depend only on r. As in the book, we have

$$u_{rr} + \frac{1}{r}u_r = 1$$

$$\Rightarrow (ru_r)_r = r$$

$$\Rightarrow u_r = \frac{1}{2}r + \frac{c_1}{r}$$

$$\Rightarrow u = \frac{1}{4}r^2 + c_1 \ln r + c_2$$

Thus by the boundary conditions, we get

$$c_1 = 0,\ \frac{1}{4}a^2 + c_2 = 0.$$

Therefore

$$u(x, y) = \frac{1}{4}r^2 - \frac{1}{4}a^2,\ \text{where } r = \sqrt{x^2 + y^2}$$

is the unique solution by the Uniqueness Theorem.

6. Firstly we find a solution that depend only on r. As in exercise (5), we have

$$u = \frac{1}{4}r^2 + c_1 \ln r + c_2.$$

By the boundary conditions, we get

$$\frac{1}{4}a^2 + c_1 \ln a + c_2 = 0,\ \frac{1}{4}b^2 + c_1 \ln b + c_2 = 0.$$

Hence

$$u(x, y) = \frac{1}{4}(r^2 - a^2) - \frac{b^2 - a^2}{4(\ln b - \ln a)}(\ln r - \ln a),\ \text{where } r = \sqrt{x^2 + y^2}$$

is the unique solution by the Uniqueness Theorem.

7.Firstly we find a solution that depend only on r. As in the book, we have

$$u_{rr} + \frac{2}{r}u_r = 1$$

$$\Rightarrow (r^2 u_r)_r = r^2$$

$$\Rightarrow r^2 u_r = \frac{1}{3}r^3 + c_1$$

$$\Rightarrow u = \frac{1}{6}r^2 + \frac{c_2}{r} + c_2.$$

Thus by the boundary conditions, we get

$$\frac{1}{6}a^2 + \frac{c_2}{a} + c_2 = 0, \frac{1}{6}b^2 + \frac{c_2}{b} + c_2 = 0.$$

Therefore

$$u(x, y) = \frac{1}{6}(r^2 - a^2) + ab\frac{a + b}{6}(\frac{1}{r} - \frac{1}{a}),\ \text{where } r = \sqrt{x^2 + y^2}$$

is the unique solution by the Uniqueness Theorem.

10. Suppose u and v are two solutions. Let $w = u - v$, then w satisfies $\Delta w = 0$ in D, $w = 0$ on bdy D.

Hence by the divergence theorem, we have

$$0 = \int_D w\Delta w dx = -\int_D Dw \cdot Dw ds$$
$$\Rightarrow Dw = 0.$$

Since $w = 0$ on bdy D.we get $w = 0$. Therefore the solution of the Dirichlet problem is unique.

11. Integrating the equation $\Delta u = f$, we get

$$\int\int\int_D f dxdydz = \int\int\int_D \Delta u dxdydz = \int\int_{\text{bdy(D)}} \frac{\partial u}{\partial n} dS = \int\int_{\text{bdy(D)}} g dS.$$

Hence there is no solution unless

$$\int\int\int_D f dxdydz = \int\int_{\text{bdy(D)}} g dS.$$

Exercise6.2

1. By the boundary condition, we can guess $u_x(x, y) = x - a$ and $u_y(x.y) = -y + b$. Luckily these also satisfy the equation. Hence

$$u(x, y) = \frac{1}{2}x^2 - \frac{1}{2}y^2 - ax + by + c, \text{ where c is any constant,}$$

are solutions.

(Actually we can prove that they are all solutions by the Hopf maximum principle)

2. Since

$$\int_0^\pi \int_0^\pi (\sin my \sin nz)(\sin m'y \sin n'z) dydz = (\int_0^\pi \sin my \sin m'y dy) \cdot (\int_0^\pi \sin nz \sin n'z dz),$$

so they are orthogonal.

3. We separate variables and use the homogeneous boundary conditions. Then we get:

$$u(x, y) = X(x)Y(y), \ \frac{X''}{X} + \frac{Y''}{Y} = 0$$

$$X(0) = Y'(0) = Y'(\pi) = 0.$$

Hence

$$\lambda_n = n^2, \ Y_n(y) = \cos(ny), \ X_0 = x, X_{n+1} = \sinh[(n+1)x]$$

$$n = 0, 1, 2, ...$$

Therefore,

$$u(x, y) = A_0 x + \sum_{n=1}^{\infty} A_n \sinh(nx)\cos(ny).$$

By the inhomogeneous boundary condition, we get

$$A_0\pi + \sum_{n=1}^{\infty} A_n \sinh(n\pi)\cos(ny) = \frac{1}{2}(1 + \cos 2y).$$

So

$$A_0 = \frac{1}{2\pi}, A_2 = \frac{1}{2\sinh(2\pi)}, A_n = 0, n \neq 0, 2$$

Hence

$$u(x, y) = \frac{x}{2\pi} + \frac{1}{2\sinh(2\pi)}\sinh(2x)\cos(2y).$$

4. Suppose u_1 satisfies

$$\Delta u_1 = 0, \text{ in the square}$$

$$u_1(x, 0) = x, u_1(x, 1) = u_{1,x}(0, y) = u_{1,x}(1, y) = 0$$

and u_2 satisfies

$$\Delta u_2 = 0, \text{ in the square}$$

$$u_2(x, 0) = u_2(x, 1) = u_{2,x}(0, y) = 0, u_{2,x}(1, y) = y^2,$$

then $u = u_1 + u_2$ is a harmonic function which we want to find.

By the method of separate variables, as in the book, we have

$$u_1 = -\frac{A_0}{2}(y - 1) + \sum_{n=1}^{\infty} A_n \cos(n\pi x)(\cosh(n\pi y) - \coth(n\pi)\sinh(n\pi y))$$

where

$$A_0 = 1, A_n = 2\int_0^1 x\cos(n\pi x)dx = \frac{2}{n^2\pi^2}((-1)^n - 1), n = 1, 2, ...$$

$$u_2 = \sum_{n=1}^{\infty} B_n \cosh(n\pi x)\sin(n\pi y),$$

where

$$B_n = \frac{2}{n\pi\sinh(n\pi)}\int_0^1 y^2\sin(n\pi y)dy = \frac{2}{\sinh(n\pi)}\{\frac{(-1)^{n+1}}{n^2\pi^2} + \frac{2}{n^4\pi^4}[(-1)^n - 1]\}$$

Therefore

$$u = -\frac{A_0}{2}(y-1) + \sum_{n=1}^{\infty} A_n\cos(n\pi x)(\cosh(n\pi y) - \coth(n\pi)\sinh(n\pi y))$$

$$+\sum_{n=1}^{\infty} B_n\cosh(n\pi x)\sin(n\pi y),$$

where A_n, B_n be given above.

6.We separate variables and use the homogeneous boundary conditions. Then we get:

$$u(x,y,z) = X(x)Y(y)Z(z),\ \frac{X''}{X} + \frac{Y''}{Y} + \frac{Z''}{Z} = 0$$

$$X'(0) = X'(1) = Y'(0) = Y'(1) = Z'(0) = 0.$$

Hence

$$X_m(x) = \cos(m\pi x), m = 0,1,2,...$$

$$Y_n(y) = \cos(n\pi y), n = 0,1,2,...$$

so that

$$Z'' = (m^2+n^2)\pi^2 Z,\ Z'(0) = 0.$$

Therefore

$$u(x,y,z) = \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} A_{mn}\cos(m\pi x)\cos(n\pi y)\cosh(\sqrt{m^2+n^2}\pi z).$$

Finally, by the inhomogeneous condition, we get

$$g(x,y) = \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} A_{mn}\sqrt{m^2+n^2}\pi\sinh(\sqrt{m^2+n^2}\pi)\cos(m\pi x)\cos(n\pi y).$$

Thus

$$A_{mn} = \frac{4}{\sqrt{m^2+n^2}\pi\sinh(\sqrt{m^2+n^2}\pi)}\int_0^1\int_0^1 g(x,y)\cos(m\pi x)\cos(n\pi y)dxdy.$$

Hence the solutions can be expressed as $u(x, y, z) + c$,where c is any constant, with the coefficients A_{mn}.

(Actually we can prove that they are all solutions by the Hopf maximum principle.)

7. (a) As before, we separate variables and use the homogeneous boundary conditions. Then we get:

$$u(x, y) = X(x)Y(y), \ \frac{X''}{X} + \frac{Y''}{Y} = 0$$

$$X(0) = X(\pi) = 0.$$

Hence

$$X_n(x) = \sin(nx), m = 1, 2, ...$$

so that

$$Y'' = n^2 Y, \ \lim_{y \to \infty} Y(y) = 0.$$

Thus

$$u(x, y) = \sum_{n=1}^{\infty} A_n \sin(nx) e^{-ny}.$$

Finally, by the inhomogeneous condition, we get

$$h(x) = \sum_{n=1}^{\infty} A_n \sin(nx).$$

Hence

$$A_n = \frac{2}{\pi} \int_0^{\pi} h(x) \sin(nx) dx.$$

Therefore

$$u(x, y) = \sum_{n=1}^{\infty} \frac{2}{\pi} (\int_0^{\pi} h(x) \sin(nx) dx) \sin(nx) e^{-ny}.$$

Exercise6.3

1. (a) By the Maximum Principle, we have

$$\max_{\overline{D}} u = \max_{\partial D} u = \max_{\theta} (3 \sin 2\theta + 1) = 4.$$

(b) By the Mean Value property, we have

$$u(0,0) = \frac{1}{2\pi} \int_0^{2\pi} 3\sin 2\theta + 1 d\theta = 1.$$

2.By the formula (10),(11),(12), we have

$$u = \frac{1}{2}A_0 + \sum_{n=1}^{\infty} r^n (A_n \cos n\theta + B_n \sin n\theta),$$

where

$$A_n = \frac{1}{\pi a^n} \int_0^{2\pi} h(\phi) \cos n\phi d\phi$$

$$B_n = \frac{1}{\pi a^n} \int_0^{2\pi} h(\phi) \sin n\phi d\phi.$$

Since $h(\phi) = 1 + 3\sin\phi$, we get

$$A_0 = 2, A_n = 0 \ (n > 0), B_1 = \frac{3}{a}, B_m = 0 \ (m > 1).$$

Hence

$$u(r,\theta) = 1 + \frac{3r}{a} \sin\theta.$$

(You can also solve the equation by Poisson's formula.)

3. Since $h(\phi) = \sin^3\phi = 3\sin\phi - 4\sin 3\phi$, we get

$$A_n = 0, B_1 = \frac{3}{a}, B_3 = -\frac{4}{a^3}, B_m = 0 \ (m \neq 1, 3).$$

Hence

$$u(r,\theta) = \frac{3r}{a} \sin\theta - \frac{4r^3}{a^3} \sin 3\theta.$$

4.Let $u \geq 0$ and $\Delta u = 0$ in a unit disk $D = \{(x,y)|x^2 + y^2 \leq 1\}$. Using the Mean-Value Property to prove the following so-called Harnack inequality

$$\frac{1-r}{1+r} u(0,0) \leq u(x,y) \leq \frac{1+r}{1-r} u(0,0)$$

where $r = \sqrt{x^2+y^2} < 1$.

Proof: By the Poisson's formula, we have

$$u(r,\theta) = \frac{1-r^2}{2\pi}\int_0^{2\pi} \frac{h(\phi)}{1-2r\cos(\theta-\phi)+r^2}d\phi,$$

where

$$h(\phi) = u(1,\phi) \geq 0.$$

Since

$$(1-r)^2 \leq 1-2r\cos(\theta-\phi)+r^2 \leq (1+r)^2,$$

we get

$$\frac{1-r^2}{2\pi}\int_0^{2\pi} \frac{h(\phi)}{(1+r)^2}d\phi \leq u(r,\theta) \leq \frac{1-r^2}{2\pi}\int_0^{2\pi} \frac{h(\phi)}{(1-r)^2}d\phi.$$

Now by the Mean-Value property, we have

$$u(0,0) = \frac{1}{2\pi}\int_0^{2\pi} h(\phi)d\phi.$$

Therefore

$$\frac{1-r}{1+r}u(0,0) \leq u(x,y) \leq \frac{1+r}{1-r}u(0,0).$$

5. Suppose that u satisfies $u_{xx}+u_{yy}=0$ for all $(x,y)\in B_1(0)$ except $(x,y)=(0,0)$. Show that if u is a bounded function, the $\lim\limits_{(x,y)\to(0,0)} u(x,y)$ exists and by taking $u(0,0) = \lim\limits_{(x,y)\to(0,0)} u(x,y)$, u is actually smooth in $B_1(0)$.

Hint: Consider the following function: $v_\epsilon = \epsilon\log\frac{1}{r}$.

Proof: Firstly, we consider the following lemma (a uniqueness lemma):

Lemma: Suppose that v satisfies $v_{xx}+v_{yy}=0$ in $D=\{(x,y)|\ 0<x^2+y^2<1\}$ and $v=0$ on $\{(x,y)|\ x^2+y^2=1\}$. Show that $v=0$ if v is bounded.

Proof of the Lemma:

Fix $(x_0,y_0)\in D$. $\forall\epsilon>0$, we consider the harmonic function $v_\epsilon := \epsilon\log\frac{1}{r}$, where $r=\sqrt{x^2+y^2}$. Since $\lim\limits_{r\to0} v_\epsilon = +\infty$ and $v_\epsilon = 0$ on $\{(x,y)|\ x^2+y^2=1\}$, so we can choose r small enough so that $v_\epsilon > \sup v$ (since v is bounded) and $x_0^2+y_0^2 > r^2$.

Hence by Maximum Principle, we get

$$v(x_0, y_0) \leq \epsilon \log \frac{1}{r_0},$$

where $r = \sqrt{x_0^2 + y_0^2}$. Thus $v(x_0, y_0) \leq 0$. Similarly, we can get $v(x_0, y_0) \geq 0$. Therefore $v(x_0, y_0) = 0$. □

Now we use the lemma to prove the statement.

By Poisson's formula, we consider the following harmonic function

$$w(r, \theta) := \frac{1 - r^2}{2\pi} \int_0^{2\pi} \frac{h(\phi)}{1 - 2r\cos(\theta - \phi) + r^2} d\phi, \quad (r < 1)$$

where

$$h(\phi) = u(1, \phi).$$

Here we use polar coordinate.

Let $v = u - w$, then v satisfies the conditions of Lemma, thus $v = 0$. Hence $u = w$ for all $(x, y) \in B_1(0)$ except $(x, y) = (0, 0)$. Therefore the $\lim_{(x,y)\to(0,0)} u(x, y)$ exists and $\lim_{(x,y)\to(0,0)} u(x, y) = w(0, 0)$. So by taking $u(0, 0) = \lim_{(x,y)\to(0,0)} u(x, y)$, $u(= w)$ is actually smooth in $B_1(0)$.

Exercise6.4

1. Since the only difference between the formulas of interior and exterior of a disk is that r and a are replaced by r^{-1} and a^{-1}. Therefore by the exercise 6.4.2, we have

$$u(r, \theta) = 1 + \frac{3a}{r} \sin\theta.$$

6. Using the separation-of-variables technique, we have

$$\Theta'' + \lambda\Theta = 0, \ r^2 R'' + rR' - \lambda R = 0.$$

So the homogeneous conditions lead to

$$\Theta'' + \lambda\Theta = 0, \ \Theta(0) = \Theta(\pi) = 0.$$

Hence

$$\lambda_n = n^2, \ \Theta(\theta) = \sin n\theta.$$

As in the book, we also get

$$R(r) = r^n.$$

Thus

$$u(r,\theta) = \sum_{n=1}^{\infty} A_n r^n \sin n\theta.$$

Finally, the inhomogeneous boundary condition requires that

$$\pi \sin\theta - \sin 2\theta = \sum_{n=1}^{\infty} A_n \sin n\theta.$$

So

$$A_1 = \pi, A_2 = -1, A_n = 0, n \neq 1, 2.$$

Therefore

$$u(r,\theta) = \pi r \sin\theta - r^2 \sin 2\theta.$$

9. It is obvious that $u(r,\theta) = \theta$ is a solution. Hence by the uniqueness theorem, $u(r,\theta) = \theta$ is the unique solution.

10. By the example 1 in the book (Please do again), where $\beta = \pi/2, h(\theta) = 1$, we have

$$u(r,\theta) = \sum_{n=1}^{\infty} A_n r^{2n} \sin 2n\theta,$$

where

$$A_n = a^{1-2n} \frac{2}{n\pi} \int_0^{\pi/2} \sin(2n\theta) d\theta = a^{1-2n} \frac{1}{n^2\pi} (1 - (-1)^n).$$

The first two nonzero terms are

$$\frac{2}{a\pi} r^2 \sin 2\theta,\ \frac{2}{9a^5\pi} r^6 \sin 6\theta.$$

11. Multiply u in both sides of equation and integrate by part, we can get that

$$u\frac{\partial u}{\partial n}|_{\partial D} - \int_D |\nabla u|^2 = 0.$$

Using Robin boundary condition we can get that

$$-au^2|_{\partial D} - \int_D |\nabla u|^2 = 0.$$

The only possibility is $\nabla u = 0$ in D and $u = 0$ on ∂D. Hence $u \equiv 0$ in D.

13. Very similar as to the computation of Example 1. Here we only give the result.

For eigenvalue problem we get $\lambda = \left(\frac{n\pi}{\beta-\alpha}\right)^2$ and $\Theta(\theta) = \sin\frac{n\pi\theta}{\beta-\alpha} - \cos\frac{n\pi\theta}{\beta-\alpha}\tan\frac{n\pi\alpha}{\beta-\alpha}$.

Also we have $R(r) = r^{\pm\frac{n\pi\theta}{\beta-\alpha}}$. Thus

$$u(r,\theta) = \sum_{n=1}^{\infty}\left(A_n r^{\frac{n\pi\theta}{\beta-\alpha}} + B_n r^{-\frac{n\pi\theta}{\beta-\alpha}}\right)\left(\sin\frac{n\pi\theta}{\beta-\alpha} - \cos\frac{n\pi\theta}{\beta-\alpha}\tan\frac{n\pi\alpha}{\beta-\alpha}\right).$$

The coefficients are determined by setting $r = a$ and $r = b$.