Solutions to Assignment Two

Exercise2.1

2. For $\phi(x) = \log(1 + x^2)$ and $\psi(x) = 4 + x$, the solution is

$$u(x,t) = \frac{1}{2}\{\log[1 + (x+ct)^2] + \log[1 + (x-ct)^2]\} + \frac{1}{2c}\int_{x-ct}^{x+ct} 4 + s\ ds$$

$$= \frac{1}{2}\{\log[1 + (x+ct)^2] + \log[1 + (x-ct)^2]\} + 4t + xt$$

5.We have

$$u(x,t) = \frac{1}{2c}\int_{x-ct}^{x+ct} \psi(s)\ ds = \frac{1}{2c}[\text{length } of\ (x-ct,\ x+ct) \cap (-a,\ a)].$$

Then for $t = \frac{a}{2c}$,

$$u(x,t) = \frac{1}{2c}[\text{length } of\ (x - \frac{a}{2},\ x + \frac{a}{2}) \cap (-a,\ a)]$$

$$= \begin{cases} 0, & x > \frac{3a}{2}; \\ \frac{3a}{2} - x, & \frac{3a}{2} \geq x > \frac{a}{2}; \\ a, & \frac{a}{2} \geq x > -\frac{a}{2}; \\ x + \frac{3a}{2}, & -\frac{a}{2} \geq x > -\frac{3a}{2}; \\ 0, & -\frac{3a}{2} \geq x. \end{cases}$$

for $t = \frac{a}{c}$,

$$u(x,t) = \frac{1}{2c}[\text{length } of\ (x - a,\ x + a) \cap (-a,\ a)]$$

$$= \begin{cases} 0, & x > 2a; \\ 2a - x, & 2a \geq x > 0; \\ 2a + x, & 0 \geq x > -2a; \\ 0, & -2a \geq x. \end{cases}$$

for $t = \frac{3a}{2c}$,

$$u(x,t) = \frac{1}{2c}[\text{length } of \ (x - \frac{3a}{2},\ x + \frac{3a}{2}) \cap (-a,\ a)]$$

$$= \begin{cases} 0, & x > \frac{5a}{2}; \\ \frac{5a}{2} - x, & \frac{5a}{2} \geq x > \frac{a}{2}; \\ 2a, & \frac{a}{2} \geq x > -\frac{a}{2}; \\ x + \frac{5a}{2}, & -\frac{a}{2} \geq x > -\frac{5a}{2}; \\ 0, & -\frac{5a}{2} \geq x. \end{cases}$$

for $t = \frac{2a}{c}$,

$$u(x,t) = \frac{1}{2c}[\text{length } of \ (x - 2a,\ x + 2a) \cap (-a,\ a)]$$

$$= \begin{cases} 0, & x > 3a; \\ 3a - x, & 3a \geq x > a; \\ 2a, & a \geq x > -a; \\ x + 3a, & -a \geq x > -3a; \\ 0, & -3a \geq x. \end{cases}$$

for $t = \frac{5a}{c}$,

$$u(x,t) = \frac{1}{2c}[\text{length } of \ (x - 5a,\ x + 5a) \cap (-a,\ a)]$$

$$= \begin{cases} 0, & x > 6a; \\ 6a - x, & 6a \geq x > 4a; \\ 2a, & 4a \geq x > -4a; \\ x + 6a, & -4a \geq x > -6a; \\ 0, & -6a \geq x. \end{cases}$$

Omit the figures

6.

$$\max_x\{u(x,t)\} = \begin{cases} 2ct, & \frac{a}{c} > t \geq 0; \\ 2a, & t \geq \frac{a}{c}. \end{cases}$$

7.We have

$$u(x,t) = \frac{1}{2}[\phi(x+ct) + \phi(x-ct)] + \frac{1}{2c}\int_{x-ct}^{x+ct} \psi(s)\ ds,$$

so

$$\begin{aligned} u(-x,t) &= \frac{1}{2}[\phi(-x+ct) + \phi(-x-ct)] - \frac{1}{2c}\int_{-x-ct}^{-x+ct} \psi(-s)\ ds \\ &= \frac{1}{2}[-\phi(x-ct) - \phi(x+ct)] - \frac{1}{2c}\int_{x-ct}^{x+ct} \psi(s)\ ds \\ &= -\{\frac{1}{2}[\phi(x+ct) + \phi(x-ct)] + \frac{1}{2c}\int_{x-ct}^{x+ct} \psi(s)\ ds\} = -u(x,t). \end{aligned}$$

Thus $u(x,t)$ is also odd in x for all t.

8.(a)Change variables $v = ru$, then

$$v_{tt} = ru_{tt}, \quad v_{rr} = (ru_r + u)_r = ru_{rr} + 2u_r$$

$$\Rightarrow v_{tt} = rc^2(u_{rr} + \frac{2}{r}u_r) = c^2 v_{rr}$$

(b)Using the same skill related to the wave equation (1), we have

$$v(r,t) = f(r+ct) + g(r-ct),$$

where f and g are two arbitrary functions of a single variable. So

$$u = \frac{1}{r}f(r+ct) + \frac{1}{r}g(r-ct).$$

(c)Since $v(r,0) = r\phi(r)$ and $v_t(r,0) = r\psi(r)$ are both odd. We extend v to all of $\mathbb{R}$ by odd reflection. That is, we set

$$\tilde{v}(r,t) := \begin{cases} v(r,t), & r > 0, t \geq 0; \\ 0, & r = 0, t \geq 0; \\ -v(-r,t), & r < 0, t \geq 0. \end{cases}$$

Hence d'Alembert's formula implies

$$\tilde{v}(r,t) = \frac{1}{2}[(r+ct)\phi(r+ct) + (r-ct)\phi(r-ct)] - \frac{1}{2c}\int_{r-ct}^{r+ct} s\psi(s)\, ds.$$

Therefore

$$u(r,t) = \frac{1}{r}v(r,t) = \frac{1}{2r}[(r+ct)\phi(r+ct) + (r-ct)\phi(r-ct)] - \frac{1}{2cr}\int_{r-ct}^{r+ct} s\psi(s)\, ds$$

$$\text{for r} > 0,\ \text{t} > 0$$

9.Using the same skill related to the wave equation (1), let $v = u_x + u_t$

$$\Rightarrow v_x - 4v_t = 0$$

$$\Rightarrow v(x,t) = h(x + \frac{1}{4}t)$$

$$\Rightarrow u_x + u_t = h(x + \frac{1}{4}t)$$

It is easy to check that one solution is

$$u(x,t) = f(x + \frac{1}{4}t), \text{where f}'\text{(s)} = \frac{\text{4h(s)}}{5}.$$

So the general solution is

$$u(x,t) = f(x + \frac{1}{4}t) + g(x-t)$$

Since $x^2 = u(x,0) = f(x) + g(x)$ and $e^x = u_t(x,0) = \frac{1}{4}f'(x) - g'(x)$

$$\Rightarrow f(x) = \frac{4}{5}(x^2 + e^x + a), \quad g(x) = \frac{1}{5}(x^2 - 4e^x - 4a)$$

$$\Rightarrow u(x,t) = \frac{4}{5}[(x + \frac{t}{4})^2 + e^{x+\frac{t}{4}} + a] + \frac{1}{5}[(x-t)^2 - 4e^{x-t} - 4a]$$

$$= \frac{1}{5}[(2x + \frac{t}{2})^2 + (x - t)^2 + 4e^{x+\frac{t}{4}} - 4e^{x-t}]$$

Exercise2.2

1.By the law of conservation of energy, we have

$$E = \frac{1}{2}\int_{-\infty}^{+\infty} (\rho u_t^2 + T u_x^2)\ dx$$

is a constant independent of t.

Since $\phi \equiv 0$ and $\psi \equiv 0$, we have $E \equiv 0$.

The first vanishing theorem implies

$$u_t \equiv 0, u_x \equiv 0.$$

Thus $u \equiv 0$ since $\phi \equiv 0$.

2.(a)Since

$$\partial e/\partial t = u_t u_{tt} + u_x u_{xt}$$

$$\partial e/\partial x = u_t u_{tx} + u_x u_{xx}$$

$$\partial p/\partial t = u_t u_{xt} + u_{tt} u_x$$

$$\partial p/\partial x = u_t u_{xx} + u_{tx} u_x$$

and

$$u_{tt} = u_{xx}$$

$$u_{xt} = u_{tx},$$

we have

$$\partial e/\partial t = \partial p/\partial x$$

$$\partial e/\partial x = \partial p/\partial t$$

(b)By (a), we have

$$e_{tt} = p_{xt} = p_{tx} = e_{xx},$$

$$p_{tt} = e_{xt} = e_{tx} = p_{xx},$$

so both $e(x,t)$ and $p(x,t)$ also satisfy the wave equation.

3.(a)Let $v(x,t) = u(x-y,t)$, we have

$$v_{tt}(x,t) = u_{tt}(x-y,t) = c^2 u_{xx}(x-y,t) = c^2 v_{xx}(x,t),$$

so it is also a solution.

(b)Let $v(x,t) = u_x$, we have

$$v_{tt}(x,t) = u_{xtt}(x,t) = c^2 u_{xxx}(x,t) = c^2 v_{xx}(x,t),$$

so it is also a solution.

(c)Let $v(x,t) = u(ax,at)$, we have

$$v_{tt}(x,t) = a^2 u_{tt}(ax,at) = a^2 c^2 u_{xx}(ax,at) = c^2 v_{xx}(x,t),$$

so it is also a solution.

5.The energy function is

$$E = \frac{1}{2}\int_{-\infty}^{+\infty} (u_t^2 + c^2 u_x^2)\ dx,$$

so

$$\begin{aligned}
dE/dt &= \frac{1}{2}\int_{-\infty}^{+\infty} (2u_t u_{tt} + 2c^2 u_x u_{xt})\ dx \\
&= \int_{-\infty}^{+\infty} (c^2 u_t u_{xx} - r u_t^2 + c^2 u_x u_{xt})\ dx \\
&= \int_{-\infty}^{+\infty} (c^2 u_t u_{xx} - r u_t^2 - c^2 u_{xx} u_t)\ dx + c^2 u_t u_x|_{-\infty}^{+\infty} \\
&= -r\int_{-\infty}^{+\infty} (u_t^2) \leq 0
\end{aligned}$$

Hence the energy decreases.

More on 2.2

Part(a)

$$\begin{aligned}
\frac{d}{dt}\int_0^l u^2(x,t)dx &= \int_0^l 2uu_t dx = 2k\int_0^l uu_{xx}dx \\
&= 2kuu_x|_0^l - 2k\int_0^l u_x^2 dx = 2k\Big(u(l,t)u_x(l,t) - u(0,t)u_x(0,t)\Big) - 2k\int_0^l u_x^2 dx \\
&= 2k\Big(-\frac{u_x^2(l,t)}{a_l} - \frac{u_x^2(0,t)}{a_0}\Big) - 2k\int_0^l u_x^2 dx \\
&\leq 0.
\end{aligned}$$

Part(b)

Let u_1 and u_2 are two solutions of the above problem. Let $w = u_1 - u_2$, then w satisfies the same equation and the same Robin boundary condition, especially $w(x,0) \equiv 0$.

By Part(a) we know that

$$\int_0^l w^2(x,t)dx \leq \int_0^l w^2(x,0)dx = 0.$$

Thus $w \equiv 0$, i.e. $u_1 \equiv u_2$. Thus the solution of the problem is unique.

Exercise 2.3

1.$u(x,t) = 1 - x^2 - 2kt$, then

$$u(0,0) = \max_{0\leq x\leq 1,\ 0\leq t\leq T} u(x,t) = 0$$

$$u(1,T) = \min_{0\leq x\leq 1,\ 0\leq t\leq T} u(x,t) = -2kT.$$

2.(a)By the definition of Maximum, $M(T)$ increase(i.e., nondecreasing);

(b)By the definition of minimum, $M(T)$ decrease(i.e., nondecreasing).

3.(a) Use the strong minimum principle;

(b)Use the minimum principle or Let the maximum occur at the point $X(t)$, so that $\mu(t) = u(X(t),t)$, Differentiate $\mu(t)$, assuming that $X(t)$ is differentiable, we have

$$\dot{\mu} = u_x(X(t),t)\dot{X(t)} + u_t(X(t),t) \leq 0.$$

Hence $\mu(t)$ decrease;

(c)Omit the figure.

4.(a)Use the strong minimum principle;

(b)Since both $u(x,t)$ and $u(1-x,t)$ are the solution of

$$u_t = u_{xx},\ 0 < x < 1,\ 0 < t < \infty$$

$$u(0,t) = u(1,t) = 0, \text{and u(x, 0) = 4x(1 - x)},$$

the uniqueness theorem implies that $u(x,t) = u(1-x,t)$;

(c)We have

$$d/dt \int_0^1 u^2\ dx = \int_0^1 2uu_t\ dx = 2\int_0^1 uu_{xx}\ dx = -2\int_0^1 u_x^2\ dx.$$

Since $u(x,t) > 0$ for all $t > 0$ and $0 < x < 1$, so u_x is not zero function.

Hence

$$d/dt \int_0^1 u^2 \, dx < 0.$$

Therefore $\int_0^1 u^2 \, dx$ is a strictly decreasing function of t.

6.Set $w = u - v$ and use the maximum principle.

7.(a)Set $w = u - v$ and use the generalized Maximum Principle(which will be proved in Extra(2)(a)):

If $u_t - u_{xx} \leq 0$ in $\mathrm{R} = [0, \mathrm{l}] \times [0, \mathrm{T}]$, then

$$\max_R u(x,t) = \max_{x=0, x=l, t=0} u(x,t).$$

(b)Let $u(x,t) = (1 - e^{-t}) \sin x$, then $u(0,t) = 0, u(\pi,t) = 0, u(x,0) = 0, u_t - u_{xx} = \sin x$. Hence part(a) show that $v(x,t) \geq (1 - e^{-t}) \sin x$.

8.We have

$$d/dt \int_0^l u^2(x,t) \, dx = \int_0^l 2uu_t \, dx = 2k \int_0^l uu_{xx} \, dx = 2kuu_x|_0^l - 2k \int_0^l u_x^2 \, dx$$

$$= 2k[u(l,t)u_x(l,t) - u(0,t)u_x(0,t)] - 2k \int_0^l u_x^2 \, dx$$

$$= 2k[-\frac{u_x^2(l,t)}{a_l} - \frac{u_x^2(0,t)}{a_0}] - 2k \int_0^l u_x^2 \, dx \leq 0,$$

so $\int_0^l u^2(x,t) \, dx$ decrease.

Extra**1**. **Note**: In(1),the conclusion "$0 < u(x,t) < 1$" has changed to "$0 < u(x,t) < e^{at}$". If the conclusion does not change, the condition of a must limited to $a \leq \pi^2 k$.

(1)Let $v(x,t) = e^{-at} u(x,t)$,then

$$v_t = v_{xx}$$

$$v(0,t) = v(1,t) = 0$$

$$v(x,0) = \sin(\pi x)$$

Using the strong maximum principle, we have $0 < v(x,t) < 1$, therefore

$$0 < u(x,t) < e^{at}$$

for all $t > 0$ and $0 < x < 1$.

(2)Let $v(x,t) = e^{-at}u(x,t)$, then both $v(x,t)$ and $v(1-x,t)$ are solutions of the diffusion equation $v_t = v_{xx}$ with the boundary conditions:

$$v(0,t) = v(1,t) = 0$$

$$v(x,0) = \sin(2x).$$

Hence the uniqueness theorem implies that $v(x,t) = v(1-x,t)$. Therefore

$$u(x,t) = u(1-x,t)$$

for all $t \geq 0$ and $0 \leq x \leq 1$.

Another Proof:

Let $v(x,t) = e^{-at}u(x,t)$,then

$$v_t = v_{xx}$$

$$v(0,t) = v(1,t) = 0$$

$$v(x,0) = \sin(2x)$$

Using the method of separation of variables, we set$v(x,t) = X(x)T(t)$, then

$$X\dot{T} - k\ddot{X}T = 0$$

$$\Rightarrow \frac{\ddot{X}}{X} = -\lambda = \frac{\dot{T}}{kT}$$

$$\Rightarrow \ddot{X} + \lambda X = 0,\ \dot{T} = -\lambda kT.$$

Hence by the boundary conditions

$$X(0) = X(1) = 0,\ X(x)T(0) = v(x,0) = \sin(\pi x),$$

we have

$$v(x,t) = e^{-\pi^2 kt}\sin(\pi x)$$

$$\Rightarrow u(x,t) = e^{at-\pi^2 kt}\sin(\pi x)$$

Therefore $0 < u(x,t) < e^{at}$ for all $t > 0$ and $0 < x < 1$, and $u(x,t) = u(1-x,t)$ for all $t \geq 0$ and $0 \leq x \leq 1$.

Extra**2**.(a)Let ϵ be a positive constant and let

$$v(x,t) = u(x,t) + \epsilon x^2.$$

This function v satisfies

$$v_t - kv_{xx} = u_t - k(u + \epsilon x^2)_{xx} = u_t - ku_{xx} - 2\epsilon k \leq -2\epsilon k < 0,$$

which is the "diffusion in equality." Now suppose that $v(x,t)$ attains its maximum at an interior point or on the top edge $(x_0,\ t_0)$.That is, $0 < x_0 < 1,\ 0 < t_0 \leq T$. By ordinary calculus, we know that

$$v_x(x_0,t_0) = 0,\ v_{xx}(x_0,t_0) \leq 0$$

and

$$v_t(x_0,t_0) \geq 0$$

This contradicts the diffusion inequality. So

$$v(x,t) \leq M + \epsilon l^2$$

$$\Rightarrow u(x,t) \leq M + \epsilon(l^2 - x^2),$$

$$\Rightarrow u(x,t) \leq M$$

where $M := \max_{x=0,x=l,t=0} u(x,t)$.

Therefore

$$\max_R u(x,t) = \max_{x=0,x=l,t=0} u(x,t).$$

(b)Consider

$$u(x,t) = v(x,t) - t \max_{-\infty<x<\infty, 0<t<T} f(x,t),$$

then

$$u_t - ku_{xx} = - \max_{-\infty<x<\infty, 0<t<T} f(x,t) + f(x,t) \le 0,$$

$$u(x,0) = 0.$$

Hence part(a) or the proof of (a) implies that $u(x,t) \le 0$.Therefore

$$v(x,t) \le T \max_{-\infty<x<\infty, 0<t<T} f(x,t).$$

Exercise 2.4

1.By the general formula, we have

$$u(x,t) = \frac{1}{\sqrt{4\pi kt}} \int_{-l}^{l} e^{-(x-y)^2/4kt} \, dy$$

$$= \frac{1}{\sqrt{\pi}} \int_{-l-x}^{l-x} e^{-p^2} \, dp$$

$$= 1/2\{\mathfrak{E}\mathrm{rf}[(\mathrm{x}+\mathrm{l})/(2\sqrt{\mathrm{kt}})] - \mathfrak{E}\mathrm{rf}[(\mathrm{x}-\mathrm{l})/(2\sqrt{\mathrm{kt}})]\}.$$

2.By the general formula, we have

$$u(x,t) = \frac{1}{\sqrt{4\pi kt}} \int_{0}^{\infty} e^{-(x-y)^2/4kt} \, dy + \frac{1}{\sqrt{4\pi kt}} \int_{-\infty}^{0} 3e^{-(x-y)^2/4kt} \, dy$$

$$= 1/2 + 1/2\mathfrak{E}\mathrm{rf}[\mathrm{x}/(2\sqrt{\mathrm{kt}})] + 3/2 - 3/2\mathfrak{E}\mathrm{rf}[\mathrm{x}/(2\sqrt{\mathrm{kt}})]$$

$$= 2 - \mathfrak{E}\mathrm{rf}[\mathrm{x}/(2\sqrt{\mathrm{kt}})].$$

4.By the general formula, we have

$$u(x,t) = \frac{1}{\sqrt{4\pi kt}} \int_{0}^{\infty} e^{-(x-y)^2/4kt} e^{-y} \, dy$$

$$= \frac{1}{\sqrt{4\pi kt}} \int_{0}^{\infty} e^{-\frac{(y+2kt-x)^2}{4kt}+kt-x} \, dy$$

$$= e^{kt-x} \int_{\frac{2kt-x}{2\sqrt{kt}}}^{\infty} e^{-p^2} \, dp/\sqrt{\pi} = \frac{1}{2}e^{kt-x} - \frac{e^{kt-x}}{2}\mathfrak{E}\mathrm{rf}(\frac{2\mathrm{kt}-\mathrm{x}}{2\sqrt{\mathrm{kt}}}).$$

6.Since

$$\int_0^\infty e^{-x^2}\ dx \cdot \int_0^\infty e^{-y^2}\ dy = \int_0^\infty \int_0^\infty e^{-(x^2+y^2)}\ dxdy$$
$$= \frac{\pi}{2}\int_0^\infty e^{-r^2} r\ dr = \frac{\pi}{4}(-e^{-r^2})|_0^\infty = \frac{\pi}{4},$$

Hence

$$\int_0^\infty e^{-x^2}\ dx = \frac{\sqrt{\pi}}{2}.$$

7.We have

$$\int_{-\infty}^\infty e^{-p^2}\ dp = \int_0^\infty e^{-p^2}\ dp + \int_{-\infty}^0 e^{-p^2}\ dp$$
$$= 2\int_0^\infty e^{-p^2}\ dp = 2\frac{\sqrt{\pi}}{2} = \sqrt{\pi}.$$

So

$$\int_{-\infty}^\infty S(x,t)\ dx = \int_{-\infty}^\infty \frac{1}{\sqrt{4\pi kt}} e^{-(x-y)^2/4kt}\ dx = \frac{1}{\sqrt{\pi}}\int_{-\infty}^\infty e^{-p^2}\ dp = 1$$

8.We have

$$\max_{\delta \le x < \infty} S(x,t) = \frac{1}{\sqrt{4\pi kt}} e^{-\delta^2/4kt},$$

so

$$\lim_{t\to 0}\max_{\delta\le x<\infty} S(x,t) = \lim_{t\to 0}\frac{1}{\sqrt{4\pi kt}} e^{-\delta^2/4kt} = \lim_{x\to+\infty}\frac{\sqrt{x}}{\sqrt{4\pi k}} e^{-x\delta^2/4k} = 0$$

11.(a)Since both $u(x,t)$ and $-u(-x,t)$ are solutions, by the uniqueness, we have $u(x,t) = -u(-x,t)$;

(b)Since both $u(x,t)$ and $-u(-x,t)$ are solutions, by the uniqueness, we have $u(x,t) = u(-x,t)$;

(c)similar.

14.Since

$$|\ e^{-(x-y)^2/4kt}\phi(y)| \le Ce^{-(x-y)^2/4kt+ay^2} = Ce^{(a-\frac{1}{4kt})y^2+\frac{x}{2kt}y-\frac{x^2}{4kt}},$$

so that formula

$$u(x,t) = \frac{1}{\sqrt{4\pi kt}}\int_{-\infty}^\infty e^{-(x-y)^2/4kt}\phi(y)\ dy$$

makes sense for $0 < t < 1/(4ak)$, but not necessarily for larger t.

15.Support that $u,\ v$ are solution of the diffusion problem with Neumann boundary condition. Let$w(x,t) = u(x,t) - v(x,t)$, then

$$w_t = kw_{xx}$$

$$w(x,0) = w_x(0,t) = w_x(l,t) = 0,$$

and

$$d/dt \int_0^l 1/2[w(x,t)]^2\ dx = -k \int_0^l [w_x(x,t)]^2\ dx \leq 0.$$

So

$$\int_0^l 1/2[w(x,t)]^2\ dx \leq \int_0^l 1/2[w(x,0)]^2\ dx = 0$$

Hence $w = 0$ and $u = v$ for all $t > 0$.

16.Let $v(x,t) = e^{bt}u(x,t)$, then

$$v_t - kv_{xx} = 0,\ v(x,0) = u(x,0) = \phi(x)$$

$$\Rightarrow v(x,t) = \frac{1}{\sqrt{4\pi kt}} \int_{-\infty}^{\infty} e^{-(x-y)^2/4kt}\phi(y)\ dy$$

$$\Rightarrow u(x,t) = \frac{e^{-bt}}{\sqrt{4\pi kt}} \int_{-\infty}^{\infty} e^{-(x-y)^2/4kt}\phi(y)\ dy$$

18.Let $v(x,t) = u(x+Vt,t)$, then

$$v_t - kv_{xx} = 0,\ v(x,0) = u(x,0) = \phi(x)$$

$$\Rightarrow v(x,t) = \frac{1}{\sqrt{4\pi kt}} \int_{-\infty}^{\infty} e^{-(x-y)^2/4kt}\phi(y)\ dy$$

$$\Rightarrow u(x,t) = \frac{1}{\sqrt{4\pi kt}} \int_{-\infty}^{\infty} e^{-(x-Vt-y)^2/4kt}\phi(y)\ dy$$

19.(a)$S_2(x,y,t) = S(x,t)S(y,t)$,so

$$S_{2t}(x,y,t) = S_t(x,t)S(y,t) + S(x,t)S_t(y,t)$$

$$S_{2xx}(x,y,t) = S_{xx}(x,t)S(y,t)$$

$$S_{2yy}(x,y,t) = S(x,t)S_{yy}(y,t).$$

Hence

$$S_{2t}(x,y,t) = S_t(x,t)S(y,t) + S(x,t)S_t(y,t)$$

$$= kS_{xx}(x,t)S(y,t) + kS(x,t)S_{yy}(y,t) = k(S_{2xx} + S_{2yy});$$

(b)We have

$$S_2(x,y,t) = S(x,t)S(y,t) =$$

$$\frac{1}{\sqrt{4\pi kt}}e^{-x^2/4kt}\frac{1}{\sqrt{4\pi kt}}e^{-y^2/4kt} = \frac{1}{4\pi kt}e^{-\frac{x^2+y^2}{4kt}}$$

Exercise 2.5

1.Let $u(x,t) = -x^2 - (t-1)^2$ be the unique solution of the wave equation with boundary conditions:

$$u_{tt} = u_{xx}, \text{ for} -1 < \text{x} < 1, 0 < \text{t}$$

$$u(x,0) = -x^2 - 1, \; u_t(x,0) = 2, \; u(-1,t) = u(1,t) = -t^2 + 2t - 2.$$

But $u(x,t)$ attains its maximum at $(0,1)$.