MAT 4220 (2008-09) Partial differential equations
Suggested Answer to Quiz 4

1. Solve the following PDE:

$$\begin{aligned} u_{xx} + u_{yy} = 0 \quad \text{in } D = (0, \pi) \times (0, \pi), \\ u_y(x, 0) = u(x, \pi) = 0, u(\pi, y) = 0, \\ u(0, y) = \cos^2(y). \end{aligned}$$

Answer: Let $u(x, y) = X(x)Y(y)$, then

$$\frac{X''}{X} = -\frac{Y''}{Y} = \lambda.$$

By the boundary conditions $Y'(0) = 0, Y(\pi) = 0$, we get

$$\lambda_n = (n + \frac{1}{2})^2, Y_n(y) = C_n \cos(n + \frac{1}{2})y, n = 0, 1, 2, \dots.$$

Combining this result and the boundary condition $X(\pi) = 0$, we obtain

$$X_n(x) = A_n \Big[\cosh(n + \frac{1}{2})x - \coth(n + \frac{1}{2})\pi \sinh(n + \frac{1}{2})x \Big], n = 0, 1, 2, \dots.$$

Thus

$$u(x, y) = \sum_{n=0}^{\infty} A_n \Big[\cosh(n + \frac{1}{2})x - \coth(n + \frac{1}{2})\pi \sinh(n + \frac{1}{2})x \Big] \cos(n + \frac{1}{2})y.$$

Hence by the initial conditions $u(0, y) = \cos^2 y$, we have

$$\sum_{n=0}^{\infty} A_n \cos(n + \frac{1}{2})y = \cos^2 y.$$

Therefore,

$$u(x, y) = \sum_{n=0}^{\infty} A_n \Big[\cosh(n + \frac{1}{2})x - \coth(n + \frac{1}{2})\pi \sinh(n + \frac{1}{2})x \Big] \cos(n + \frac{1}{2})y,$$

where

$$A_n = \frac{2}{\pi}\int_0^\pi \cos^2 y \cos(n+\frac{1}{2})y\,dy = \frac{(-1)^n}{\pi}\Big[\frac{1}{2n-3}+\frac{2}{2n+1}+\frac{1}{2n+5}\Big], n = 0, 1, 2, \ldots. \quad \Box$$

2. Solve the following PDE:

$$u_{xx} + u_{yy} = 1 \quad \text{in } D = \{(x, y)|x^2 + y^2 < 4\},$$
$$u(x, y) = x^2 - y^2 \quad \text{on } \partial D = \{(x, y)|x^2 + y^2 = 4\}.$$

Answer: Note that the PDE is linear. Thus we can divide the problem into two subproblems. The first one is the following Poisson's equation with homogeneous Dirichlet boundary condition:

$$v_{xx} + v_{yy} = 1 \quad \text{in } D = \{(x, y)|x^2 + y^2 < 4\},$$
$$v(x, y) = 0 \quad \text{on } \partial D = \{(x, y)|x^2 + y^2 = 4\}.$$

To solve it, let $v(x, y) = v(r)$, $r = \sqrt{x^2 + y^2}$, then $v(r)$ satisfies the following ODE:

$$v'' + \frac{1}{r}v' = 1,\ 0 < r < 2,$$

which implies

$$v(r) = \frac{r^2}{4} + c_1 \ln r + c_2,$$

where c_1, c_2 are constants. By the boundary condition $v(2) = 0$ and v is bound in D, we have $c_1 = 0, c_2 = -1$ and then

$$v(x, y) = v(r) = \frac{r^2}{4} - 1 = \frac{x^2 + y^2}{4} - 1.$$

The other one is the following Laplace's equation with inhomogeneous Dirichlet boundary condition:

$$w_{xx} + w_{yy} = 0 \quad \text{in } D = \{(x, y)|x^2 + y^2 < 4\},$$
$$w(x, y) = x^2 - y^2 \quad \text{on } \partial D = \{(x, y)|x^2 + y^2 = 4\}.$$

To solve it, using the method of separation of variables $\big($note that $x^2 - y^2 = r^2\cos^2\theta - r^2\sin^2\theta = r^2\cos(2\theta)\big)$ or checking directly, we obtain

$$w(x, y) = x^2 - y^2.$$

Therefore, the solution to the original problem is

$$u(x, y) = v(x, y) + w(x, y) = \frac{x^2 + y^2}{4} - 1 + x^2 - y^2 = \frac{5}{4}x^2 - \frac{3}{4}y^2 - 1. \quad \Box$$