MAT 4220 (2008-09) Partial differential equations
Suggested Answer to Quiz 3

1. Solve the following wave equation:

$$\begin{aligned} u_{tt} &= c^2 u_{xx}, 0 < x < l, \\ u(x,0) = 0, \quad u_t(x,0) &= x, \\ u(0,t) = 0, u_x(l,t) &= 0. \end{aligned}$$

Answer: Let $u(x,t) = X(x)T(t)$, then

$$\frac{T''}{c^2 T} = \frac{X''}{X} = -\lambda.$$

By the boundary conditions $X(0) = 0, X'(l) = 0$, we get

$$\lambda_n = \frac{(n+\frac{1}{2})^2\pi^2}{l^2}, X_n(x) = C_n \sin\frac{(n+\frac{1}{2})\pi x}{l}, n = 0, 1, 2, \dots.$$

Hence

$$T_n(t) = A_n \cos\frac{(n+\frac{1}{2})\pi ct}{l} + B_n \sin\frac{(n+\frac{1}{2})\pi ct}{l}, n = 0, 1, 2, \dots,$$

$$\text{and } u(x,t) = \sum_{n=0}^{\infty}\left[A_n \cos\frac{(n+\frac{1}{2})\pi ct}{l} + B_n \sin\frac{(n+\frac{1}{2})\pi ct}{l}\right]\sin\frac{(n+\frac{1}{2})\pi x}{l}.$$

Therefore by the initial conditions $u(x,0) = 0$ and $u_t(x,0) = x$, we have

$$\sum_{n=0}^{\infty} A_n \sin\frac{(n+\frac{1}{2})\pi x}{l} = 0,$$

and

$$\sum_{n=0}^{\infty}\left[B_n\frac{(n+\frac{1}{2})\pi c}{l}\right]\sin\frac{(n+\frac{1}{2})\pi x}{l} = x.$$

Therefore,

$$u(x,t) = \sum_{n=0}^{\infty} B_n \sin\frac{(n+\frac{1}{2})\pi ct}{l}\sin\frac{(n+\frac{1}{2})\pi x}{l},$$

where

$$B_n = \frac{l}{(n+\frac{1}{2})\pi c} \frac{2}{l} \int_0^l x \sin \frac{(n+\frac{1}{2})\pi x}{l^2} dx = (-1)^n \frac{2l^2}{(n+\frac{1}{2})^3 \pi^3 c}, n = 0, 1, 2, \ldots. \ \square$$

2. Consider the following eigenvalue problem

$$X'' + \lambda X = 0, 0 < x < 1,$$
$$X(1) = X(0), X'(1) = 5X(0) + X'(0).$$

Show that all eigenvalues are **real**.

Answer: The first way is to check that the boundary conditions is symmetric (or hermitian), i.e.,

$$(-X'\overline{X} + X\overline{X}')|_0^1 = 0,$$

and then by the Theorem 2 in the Section 5.3 (see the textbook or Prof.Wei's Lecture notes), all eigenvalues are **real**.

Next we use the method in the proof of the theorem 2 to prove the result.

Let λ be an eigenvalue, possibly complex, and $X(x)$ be its eigenfunction, also possibly complex. Then

$$-X'' = \lambda X \quad \text{and } X(1) = X(0), X'(1) = 5X(0) + X'(0).$$

Taking the complex conjugate of these equations, we obtain

$$-\overline{X}'' = \overline{\lambda}\,\overline{X} \quad \text{and } \overline{X}(1) = \overline{X}(0), \overline{X}'(1) = 5\overline{X}(0) + \overline{X}'(0).$$

Now using the Green's second identity with the functions X and $\overline{X}$,

$$\begin{aligned}(\lambda - \overline{\lambda}) \int_0^1 |X|^2 \, dx &= \int_0^1 (-X''\overline{X} + X\overline{X}'') \, dx = (-X'\overline{X} + X\overline{X}')|_0^1 \\ &= -\big[5X(0) + X'(0)\big]\overline{X}(0) + X(0)\big[5\overline{X}(0) + \overline{X}'(0)\big] \\ &\quad + X'(0)\overline{X}(0) - X(0)\overline{X}'(0) = 0.\end{aligned}$$

Note that $\int_0^1 |X|^2 \, dx > 0$, thus $\lambda - \overline{\lambda} = 0$, which means exactly that λ is real. $\square$