MAT 4220 (2008-09) Partial differential equations
Suggested Answer to Quiz 2

1. Solve the following wave equation:

$$u_{xx} - u_{xt} - 2u_{tt} = 0,$$
$$u(x,0) = x, \quad u_t(x,0) = 1.$$

Answer: Using the same skill related to the wave equation (1), let $v = u_x + u_t$, we must have $v_x - 2v_t = 0$ by the equation of u. Thus we have two first-order equations

$$v_x - 2v_t = 0$$

and

$$u_x + u_t = v.$$

As before, we can solve them one at a time and then solve the other one to obtain the formula of the general solution is

$$u(x,t) = f(x + \frac{1}{2}t) + g(x - t).$$

Substitute the initial condition, we have

$$f(x) = \frac{1}{3}(4x + 2a), \quad \text{and} \quad g(x) = -\frac{1}{3}(x + 2a),$$

where a is a constant. Thus

$$u(x,t) = \frac{1}{3}[4(x + \frac{t}{2}) + 2a] - \frac{1}{3}[(x - t) + 2a] = x + t. \quad \square$$

2. Consider the following diffusion equation

$$u_t = ku_{xx} + bu, \ 0 < x < \pi, t > 0$$
$$u(x,0) = \sin^3 x$$
$$u(0,t) = u(\pi,t) = 0$$

(a) Show that $0 < u(x,t) < (\sin x)e^{bt}$ for all $t > 0$ and $0 < x < \pi$.
(b) Show that $u(x,t) = u(\pi - x, t)$ for all $t > 0$ and $0 \le x \le \pi$.
Hint: Consider the function $v(x,t) = e^{-bt}u(x,t)$.

Answer: (a) Let $v(x,t) = e^{-bt}u(x,t)$, then $v(x,t)$ satisfies

$$\begin{aligned} v_t = kv_{xx},\ 0 < x < \pi, t > 0 \\ v(x,0) = \sin^3 x \\ v(0,t) = v(\pi,t) = 0. \end{aligned} \tag{1}$$

Thus by the Minimum Principle to v or the Maximum Principle to $-v$, $v(x,t) > 0$ for all $t > 0$ and $0 < x < \pi$.

To prove $v(x,t) < \sin x$, we define

$$w(x,t) = v(x,t) - \sin x.$$

Then $w(x,t)$ satisfies

$$\begin{aligned} w_t - kw_{xx} = -k\sin x \leq 0,\ 0 < x < \pi, t > 0 \\ w(x,0) = \sin^3 x - \sin x \\ w(0,t) = w(\pi,t) = 0. \end{aligned}$$

Hence by the generalized Maximum Principle in the assignment 2, More on 2.3, Extra 2, (its proof is mostly the same as the one of Maximum Principle), $w(x,t) < 0$ for all $t > 0$ and $0 < x < \pi$, this implies $v(x,t) < \sin x$ for all $t > 0$ and $0 < x < \pi$.

Combining the above results, we have proved $0 < v(x,t) < \sin x$ for all $t > 0$ and $0 < x < \pi$, this shows that $0 < u(x,t) < (\sin x)e^{bt}$ for all $t > 0$ and $0 < x < \pi$.

(b) It is easy to check that $v(\pi - x, t)$ is also a solution of the problem (1) since $\sin(\pi - x) = \sin x$ and $\sin 0 = \sin \pi = 0$. Thus by the uniqueness for the problem (1), $v(\pi - x, t) = v(x,t)$ for all $t > 0$ and $0 \leq x \leq \pi$, this implies that

$$u(x,t) = u(\pi - x, t)$$

for all $t > 0$ and $0 \leq x \leq \pi$ since $u(x,t) = e^{bt}v(x,t)$. □