MAT 4220 (2008-09) Partial differential equations
Suggested Answer to Quiz 1

1. (a) Solve the linear equation $y^2u_x + xu_y = 0$ with $u(x,1) = e^x$. In which region of the xy-plane is the solution uniquely determined?

(b) Find a general formula for the equation $4u_x + 3u_y = 4x + 3y$.

Answer: (a) The characteristic curves satisfy the ODE: $dy/dx = \frac{x}{y^2}$, which implies $\frac{1}{3}y^3 = \frac{1}{2}x^2 + C$. Thus

$$u(x,y) = f(\frac{1}{3}y^3 - \frac{1}{2}x^2).$$

Setting $y = 1$ yields the equation $f(\frac{1}{3} - \frac{1}{2}x^2) = e^x$. Letting $w = \frac{1}{3} - \frac{1}{2}x^2$ yields $f(w) = e^{\pm\sqrt{\frac{2}{3}-2w}}$, $w \leq \frac{1}{3}$. Note also that $e^{-x} \neq e^x$ if $x \neq 0$, so $u(x,y)$ is not well defined in the region

$$\{(x,y) \in \mathbb{R}^2 | x = 0, y < 1\},$$

and then

$$u(x,y) = \begin{cases} e^{\sqrt{x^2-\frac{2}{3}y^3+\frac{2}{3}}}, & x^2 - \frac{2}{3}y^3 + \frac{2}{3} \geq 0 \quad \text{and } x > 0; \\ e^{-\sqrt{x^2-\frac{2}{3}y^3+\frac{2}{3}}}, & x^2 - \frac{2}{3}y^3 + \frac{2}{3} \geq 0 \quad \text{and } x < 0. \end{cases}$$

Thereby the domain of the solution uniquely determined is :

$$\Omega := \{(x,y) \in \mathbb{R}^2 | x^2 - \frac{2}{3}y^3 + \frac{2}{3} \geq 0\} \setminus \{(x,y) \in \mathbb{R}^2 | x = 0, y < 1\}.$$

(b) Change variables to $x' = 4x + 3y, y' = 3x - 4y$. By the chain rule,

$$u_x = 4u_{x'} + 3u_{y'}, u_y = 3u_{x'} - 4u_{y'}.$$

We have $25u_{x'} = x'$ which implies $u(x',y') = \frac{1}{50}x'^2 + f(y')$ and thus

$$u(x,y) = \frac{1}{50}(4x + 3y)^2 + f(3x - 4y),$$

where f is an arbitrary (differential) function. □

2. What is the type of the following equation

$$u_{xx} - 10u_{xy} + 25u_{yy} - 2u_y + u_x = 0?$$

Find a linear transformation so that it becomes one of the following standard forms:

$$u_{\xi\xi} - u_{\eta\eta} = 0, u_\eta - u_{\xi\xi} = 0, u_{\xi\xi} + u_{\eta\eta} = 0.$$

Answer: The corresponding matrix is:

$$A = \begin{pmatrix} 1 & -5 \\ -5 & 25 \end{pmatrix}.$$

Since $\det(A) = (-5)^2 - 25 = 0$, the type of the equation is parabolic.

Note that the original equation can be written as

$$(\partial_x - 5\partial_y)^2 u - (-\partial_x + 2\partial_y)u = 0.$$

Let

$$\partial_\xi = \partial_x - 5\partial_y, \partial_\eta = -\partial_x + 2\partial_y.$$

Then

$$\partial_x = -\frac{2}{3}\partial_\xi - \frac{5}{3}\partial_\eta, \partial_\eta = -\frac{1}{3}\partial_\xi - \frac{1}{3}\partial_\eta.$$

Hence by the following transformation

$$\xi = -\frac{2}{3}x - \frac{1}{3}y, \eta = -\frac{5}{3}x - \frac{1}{3}y,$$

the original equation becomes the standard form: $u_\eta - u_{\xi\xi} = 0$. □