MAT 4220 (2008-09) Partial differential equations
Suggested Answer to Midterm Examination

1. Solve the following wave equation:

$$u_{tt} = c^2 u_{xx} + \cos(ct)\cos x, -\infty < x < +\infty, t > 0,$$

$$u(x,0) = 1, u_t(x,0) = x.$$

Answer: By the d'Alembert's formula,

$$\begin{aligned}
u(x,t) &= \frac{1}{2}[\phi(x+ct)+\phi(x-ct)] + \frac{1}{2c}\int_{x-ct}^{x+ct}\psi(s)\,ds + \frac{1}{2c}\int_0^t\int_{x-c(t-s)}^{x+c(t-s)} f(y,s)\,dyds \\
&= \frac{1}{2}[1+1] + \frac{1}{2c}\int_{x-ct}^{x+ct} s\,ds + \frac{1}{2c}\int_0^t\int_{x-c(t-s)}^{x+c(t-s)} \cos(cs)\cos y\,dyds \\
&= 1 + xt + \frac{1}{2c}t\sin(ct)\cos x. \ \Box
\end{aligned}$$

2. Consider the following wave equation:

$$u_{tt} = 4u_{xx}, 0 < x < 2,$$

$$u(x,0) = x, u_t(x,0) = 1,$$

$$u(0,t) = u(2,t) = 0.$$

Find $u(1,3)$.

Answer: Since they are both homogeneous Dirichlet boundary conditions on line $x = 0$ and $x = 2$, by the method of reflection, we extend the initial dates $\phi(x)$ and $\psi(x)$ to the whole line to be odd with respect to $x = 0$ and $x = 2$. The simplest way to do is to define

$$\phi_{\text{ext}} = \begin{cases} \phi(x), & \text{for } 0 < x < 2; \\ -\phi(-x), & \text{for } -2 < x < 0; \\ \text{extended to be of period 4.} \end{cases}$$

and

$$\psi_{\text{ext}} = \begin{cases} \psi(x), & \text{for } 0 < x < 2; \\ -\psi(-x), & \text{for } -2 < x < 0; \\ \text{extended to be of period 4.} \end{cases}$$

Now let $v(x,t)$ be the solution of the infinite line problem with the extended initial data. Then $u(x,t)$ be the restriction of $v(x,t)$ to the interval $[0,2]$.

Thus $u(x,t)$ is given by the formula

$$u(x,t) = \frac{1}{2}[\phi_{\text{ext}}(x+ct) + \phi_{\text{ext}}(x-ct)] + \frac{1}{2c}\int_{x-ct}^{x+ct} \psi_{\text{ext}}(s)ds$$

Note $1 + 2 \times 3 = 7 = 2 \times 4 - 1$ and $1 - 2 \times 3 = -5 = (-1) \times 4 - 1$. Hence

$$u(1,3) = \frac{1}{2}[-1-1] + 0 = -1. \quad \square$$

3. Solve the following diffusion equation

$$u_t = ku_{xx}, 0 < x < +\infty, t > 0,$$
$$u(x,0) = x, x > 0,$$
$$u_x(0,t) = 0.$$

Answer: Since it is homogeneous Neumann boundary conditions on line $x = 0$, by the method of reflection, we extend the initial data $\phi(x)$ to the whole line to be even with respect to $x = 0$ and define

$$\phi_{\text{even}} = \begin{cases} x, & \text{for } x > 0; \\ -x, & \text{for } x < 0, \end{cases}$$

and then

$$\begin{aligned} u(x,t) &= \int_{-\infty}^{\infty} S(x-y,t)\phi_{\text{even}}(y)\,dy = \int_0^{\infty} \Big[S(x-y,t) + S(x+y,t)\Big] y\,dy \\ &= \frac{1}{\sqrt{4k\pi t}} \int_0^{\infty} \Big[e^{-\frac{(x-y)^2}{4kt}} + e^{-\frac{(x+y)^2}{4kt}}\Big] y\,dy \\ &= \frac{1}{\sqrt{4k\pi t}} \Big[\big(-2kte^{-\frac{(x-y)^2}{4kt}}\big)\big|_0^{+\infty} + x\int_0^{\infty} e^{-\frac{(x-y)^2}{4kt}}\,dy \\ &\qquad + \big(-2kte^{-\frac{(x+y)^2}{4kt}}\big)\big|_0^{+\infty} - x\int_0^{\infty} e^{-\frac{(x+y)^2}{4kt}}\,dy\Big] \\ &= \frac{1}{\sqrt{4k\pi t}}\big(4kte^{-\frac{x^2}{4kt}}\big) + \frac{x}{\sqrt{\pi}}\int_{-\frac{x}{\sqrt{4kt}}}^{+\infty} e^{-p^2}\,dp - \frac{x}{\sqrt{\pi}}\int_{\frac{x}{\sqrt{4kt}}}^{+\infty} e^{-p^2}\,dp \\ &= \sqrt{\frac{4kt}{\pi}} e^{-\frac{x^2}{4kt}} + \frac{2x}{\sqrt{\pi}}\int_0^{\frac{x}{\sqrt{4kt}}} e^{-p^2}\,dp \\ &= \sqrt{\frac{4kt}{\pi}} e^{-\frac{x^2}{4kt}} + x\mathfrak{E}\text{rf}\Big(\frac{x}{\sqrt{4kt}}\Big). \quad \square \end{aligned}$$

4. Compute the energy of the following wave equation

$$u_{tt} = c^2 u_{xx}, -\infty < x < +\infty,$$

$$u(x,0) = \begin{cases} \sin x, \text{ when } |x| \le \pi, \\ 0, \text{ when } |x| > \pi. \end{cases}$$

$$u_t(x,0) = \begin{cases} 1, \text{ when } |x| \le a, \\ 0, \text{ when } |x| > a. \end{cases}$$

Answer: Firstly, we prove the law of conservation of energy, i.e., $E(t) = E(0)$, for all t, where

$$E(t) = \frac{1}{2} \int_{-\infty}^{+\infty} (u_t^2 + c^2 u_x^2)\, dx.$$

By the d'Alembert's formula and the initial condition,

$$u(x,t) = 0 \quad \text{if } |x| > c|t| + \max\{\pi, a\}.$$

Thus

$$\int_{-\infty}^{+\infty} c^2 (u_t u_{xx} + u_x u_{xt})\, dx = (c^2 u_t u_x)|_{-\infty}^{+\infty} = 0,$$

which implies

$$\int_{-\infty}^{+\infty} (u_t u_{tt} + c^2 u_x u_{xt})\, dx = 0,$$

by the wave equation. By the direct computation, this shows that

$$\frac{dE}{dt} = 0 \quad \text{for all } t,$$

and then the law of conservation of energy is proved.

Therefore,

$$\begin{aligned} E(t) = E(0) &= \frac{1}{2} \int_{-\infty}^{+\infty} [u_t^2(0,x) + c^2 u_x^2(0,x)]\, dx \\ &= \frac{1}{2}(\int_{-a}^{a} 1^2\, dx + c^2 \int_{-\pi}^{\pi} \cos^2 x\, dx) \\ &= a + \frac{c^2}{2}\pi. \ \square \end{aligned}$$

5. Consider the following diffusion equation with periodic boundary condition

$$u_t = ku_{xx}, 0 < x < l, 0 < t < T,$$

$$u(x, 0) = \phi(x),$$

$$u(0, t) = u(l, t), u_x(0, t) = u_x(l, t).$$

Prove the following maximum principle:

$$(*) \quad \max_R u(x, t) = \max_{0 \le x \le l} \phi(x)$$

where $R = [0, l] \times [0, T]$.

Hint: Consider $u = v + \epsilon t$. If v attains its maximum on $\{x = 0, t > 0\}$, then $v_x(0, t) = 0$.

Answer: Let ϵ be a positive constant and let $v(x, t) = u(x, t) - \epsilon t$. Then the function v satisfies

$$v_t - kv_{xx} = u_t - \epsilon - ku_{xx} = -\epsilon < 0,$$

which is the "diffusion inequality".

Now suppose that $v(x, t)$ attains its maximum at an interior point (x_0, t_0), i.e., $0 < x_0 < l$, $0 < t_0 < T$. By ordinary calculus, $v_t(x_0, t_0) = 0$ and $v_{xx}(x_0, t_0) \le 0$. This contradicts the diffusion inequality. So there cann't be an interior maximum.

Next suppose that $v(x, t)$ attains a maximum at a point (x_0, t_0) on the top edge, i.e., $0 < x_0 < l$, $t_0 = T$. By ordinary calculus, $v_t(x_0, t_0) \ge 0$ and $v_{xx}(x_0, t_0) \le 0$. This also contradicts the diffusion inequality. So there cann't be an maximum point on the top edge.

Finally, suppose that $v(x, t)$ attains a maximum at a point (x_0, t_0) on the left or right boundary, i.e., $x_0 = 0, \text{or } l$ and $0 < t_0 \le T$. Without loss of generality, we assume that $x_0 = 0$. Then by ordinary calculus, $v_t(x_0, t_0) \ge 0$ and $v_x(x_0, t_0) \le 0$. On the other hand, by the first periodic boundary condition, $v(x, t)$ also attains a maximum at (l, t_0). Then by ordinary calculus, $v_x(l, t_0) \ge 0$. Using the second periodic boundary condition, we obtain $v_x(x_0, t_0) = v_x(l, t_0)$ and then $v_x(x_0, t_0) = 0$. Thus by the ordinary calculus, $v_{xx}(x_0, t_0) \le 0$. Combinate this with $v_t(x_0, t_0) \ge 0$ give a contradiction to the diffusion inequality. So there cann't be an maximum point on the left and right boundaries.

Therefore, $v(x,t)$ attains its maximum on the bottom and then

$$\max_R v(x,t) = \max_{0\le x\le l,\, t=0} v(x,t) = \max_{0\le x\le l} \phi(x).$$

Thus

$$\max_{0\le x\le l} \phi(x) \le \max_R u(x,t) \le \max_R v(x,t) + \epsilon T \le \max_{0\le x\le l} \phi(x) + \epsilon T,$$

and then

$$\max_R u(x,t) = \max_{0\le x\le l} \phi(x). \quad \Box$$

6. Show that Maximum Principle doesnot hold for the wave equation

$$u_{tt} = c^2 u_{xx}.$$

Answer: To show that Maximum Principle does not hold for the wave equation, we give a counterexample as follows.

Let $u(x,t) = -x^2 - (t-1)^2$ be the unique solution of the wave equation with boundary conditions:

$$\begin{gathered} u_{tt} = u_{xx}, \text{ for } -1 < x < 1, 0 < t < \infty, \\ u(x,0) = -x^2 - 1, \; u_t(x,0) = 2, \\ u(-1,t) = u(1,t) = -t^2 + 2t - 2. \end{gathered}$$

But $u(x,t)$ attains its maximum 0 at $(0,1)$. $\Box$