MAT 4220 (2008-09) Partial differential equations
Suggested Answer to Assignment 7

Exercise 7.1

1. Derive the three-dimensional maximum principle from the mean value property.

Answer: First we recall the **maximum principle** as follows:

If D is any solid region, a non-constant harmonic function in D can not take its maximum value inside D, but only on ∂D.

Proof: Suppose there exists one non-constant harmonic function u in D, which attains its maximum M at $x_0 \in D$.

By the mean value property, we have

$$u(x_0) = \frac{1}{|B(x_0,r)|} \iint_{B(x_0,r)} u dS,$$

for all $\overline{B(x_0,r)} \subset D$.

Thus $u(x) = M$ for all $x \in \overline{B(x_0,r)} \subset D$ since u is continuous and M is its maximum.

Now since u is not a constant, there exists $x_1 \in D$ such that $u(x_1) \neq u(x_0)$. Since D is a region, we can find a continuous curve $\gamma(t) \subset D, 0 \leq t \leq 1$ such that $\gamma(0) = x_0, \gamma(1) = x_1$.

Set $E := \{0 \leq t \leq 1 |\ u(\gamma(t)) = M\}$, then E is closed since u and γ are both continuous. Let $t_0 \in E$, then by the above result, $u(x) = M$ for all $x \in \overline{B(\gamma(t_o),r)} \subset D$. So E is open. Hence $E = [0,1]$ since $E \neq \emptyset$ and $[0,1]$ is connected. But this is contradict with $u(x_1) \neq u(x_0)$ and then we complete the proof. $\square$

2. Prove the uniqueness up to constants of the Neumann problem using the energy method.

Answer: Suppose u_1 and u_2 are solutions of the problem. Then $u = u_1 - u_2$ satisfies

$$\Delta u = 0 \text{ in } D, \quad \frac{\partial u}{\partial n} = 0 \text{ on } \partial D.$$

Thus by the Green's first identity for $v = u$, we have

$$\iint_{\partial D} u \cdot \frac{\partial u}{\partial n}\, dS = \iiint_D |\nabla u|^2\, dx + \iiint_D u \Delta u\, dx.$$

Hence

$$\iiint_D |\nabla u|^2 dx = 0.$$

Therefore $\nabla u \equiv 0$ and then $u(x)$ is a constant. $\square$

3. Prove the uniqueness of the Robin problem $\partial u/\partial n + a(\mathbf{x})u(\mathbf{x}) = 0$ provided that $a(\mathbf{x}) > 0$ on the boundary.

Answer: Suppose u_1 and u_2 are solutions of the problem. Then $u = u_1 - u_2$ satisfies

$$\Delta u = 0 \text{ in } D, \quad \frac{\partial u}{\partial n} + au = 0 \text{ on } \partial D.$$

Thus by the Green's first identity for $v = u$,

$$\iint_{\partial D} u \cdot \frac{\partial u}{\partial n} dS = \iiint_D |\nabla u|^2 dx + \iiint_D u\Delta u dx.$$

Hence

$$-\iint_{\partial D} au^2 = \iiint_D |\nabla u|^2 dx.$$

Therefore $\nabla u \equiv 0$ since $a(x) > 0$ and u is continuous. So $u \equiv c$, where c is a constant. Finally, by the Robin boundary conditions, we have $ca(x) = 0$. Therefore $c = 0$ and then $u \equiv 0$. $\square$

4. Generalize the energy method to prove uniqueness for the diffusion equation with Dirichlet boundary conditions in three dimensions.

Answer: Suppose u_1 and u_2 are solutions of the problem. Then $u = u_1 - u_2$ satisfies

$$u_t = k\Delta u \text{ in } D \times (0, \infty), \quad u = 0 \text{ on } \partial D \times (0, \infty), \quad u(x, 0) = 0 \text{ in } D.$$

Thus by the Green's first identity for $v = u$,

$$\iint_{\partial D} u \cdot \frac{\partial u}{\partial n} dS = \iiint_D |\nabla u|^2 dx + \iiint_D u\Delta u dx.$$

Thus by the diffusion equation,

$$\begin{aligned} 0 &= \iiint_D |\nabla u|^2 dx + \frac{1}{k} \iiint_D u u_t dx \\ &= \iiint_D |\nabla u|^2 dx + \frac{1}{2k} \frac{d}{dt} \iiint_D u^2 dx, \end{aligned}$$

which implies $\frac{d}{dt}E(t) \leq 0$, where

$$E(t) := \iiint_D u^2 dx.$$

Since $E(0) = 0$ and $E(t) \geq 0$, we obtain $E(t) \equiv 0$, i.e.,

$$\iiint_D u^2 dx \equiv 0.$$

Hence $u \equiv 0$ and then we complete the proof of uniqueness for the diffusion equation with Dirichlet boundary conditions. $\square$

5. Prove Dirichlet's principle for the Neumann boundary condition. It asserts that among all real-valued functions $w(x)$ on D the quantity

$$E[w] = \frac{1}{2} \iiint_D |\nabla w|^2 \, dx - \iint_{\text{bdy } D} hw \, dS$$

is the smallest for $w = u$, where u is the solution of the Neumann problem

$$-\Delta u = 0 \text{ in } D, \quad \frac{\partial u}{\partial n} = h(x) \text{ on bdy } D.$$

It is required to assume that the average of the given function $h(x)$ is zero (by Exercise 6.1.11).

Notice three features of this principle:

(i) There is no constraint at all on the trial functions $w(x)$.

(ii) The function $h(x)$ appears in the energy.

(iii) The functional $E[w]$ does not change if a constant is added to $w(x)$.

(Hint: Follow the method in Section 7.1)

Answer: Suppose $u(x)$ minimizes the energy. Let $v(x)$ be any function and ϵ be any constant, then

$$E[u] \leq E[u+\epsilon v] = E[u]+\epsilon \iiint_D \nabla u \nabla v dx - \epsilon \iint_{\partial D} hv dS + \epsilon^2 \iiint_D |\nabla v|^2 dx.$$

Thus by calculus,

$$\iiint_D \nabla u \nabla v dx - \iint_{\partial D} hv dS = 0$$

for any function v.

Now by Green's first identity and above equality,

$$-\iiint_D \Delta u \cdot v dx + \iint_{\partial D} v \cdot \frac{\partial u}{\partial n} dS - \iint_{\partial D} hv dS = 0.$$

Firstly, we let v vanishes on ∂D and then get $-\Delta u = 0$. Thus we also have

$$\iint_{\partial D} v \cdot \frac{\partial u}{\partial n} dS - \iint_{\partial D} hv dS = 0$$

for any function v. Hence $\frac{\partial u}{\partial n} = h$ on the boundary and then $u(x)$ is a solution of the following Neumann problem

$$-\Delta u = 0 \text{ in } D, \quad \frac{\partial u}{\partial n} = h(x) \text{ on bdy } D.$$

Note that the Neumann problem has a unique solution up to a constant and the functional $E[w]$ does not change if a constant is added to w, thus it is the only function that can minimize the energy up to a constant.

Note that here we assume functions u, v and the domain D are smooth enough, at least under which the Green's first identity can be hold. □

6. Let A and B be two disjoint bounded spatial domains, and let D be their exterior. So bdy $D =$ bdy $A \cup$ bdy B. Consider a harmonic function $u(x)$ in D that tends to zero at infinity, which is constant on bdy A and constant on bdy B, and which satisfies

$$\iint_{\text{bdy } A} \frac{\partial u}{\partial n} dS = Q > 0 \quad \text{and} \quad \iint_{\text{bdy } B} \frac{\partial u}{\partial n} dS = 0.$$

[Interpretation: The harmonic function $u(x)$ is the electrostatic potential of two conductors, A and B; Q is the charge on A, while B is uncharged.]

(a) Show that the solution is unique. (Hint: Use the Hopf maximum principle.)

(b) Show that $u \geq 0$ in D. [Hint: If not, then $u(x)$ has a negative minimum. Use the Hopf principle again.]

(c) Show that $u > 0$ in D.

Answer: (a) Suppose u_1 and u_2 are solutions of the problem. Then $u = u_1 - u_2$ tends to zero at infinity, and is constant on ∂A and constant on ∂B, and which satisfies

$$\iint_{\text{bdy } A} \frac{\partial u}{\partial n} dS = 0 = \iint_{\text{bdy } B} \frac{\partial u}{\partial n} dS.$$

Suppose that $u \neq 0$, then without loss of generality, we assume that there exists $x_0 \in \overline{D}$ such that $u(x_0) > 0$. Since u tends to zero at infinity, so there exists $R \gg 1$ such that $u(x) \leq \frac{u(x_0)}{2}$ if $|x| \geq R$.

Then u is a harmonic function in $D \bigcap B(0,R)$ and $\max\limits_{\partial B(0,R)} u \leq \dfrac{u(x_0)}{2} < u(x_0) \leq \max\limits_{D \bigcap B(0,R)} u$. Thus the maximum is attained on ∂A or ∂B by the Maximum Principle. Without loss of generality, we may assume that u attain the maximum on ∂A and then any point of ∂A is a maximum point since u is constant on ∂A. Hence by the Hopf maximum principle, $\partial u/\partial n > 0$ on ∂A, but this contradicts with the condition

$$\iint_{\text{bdy } A} \frac{\partial u}{\partial n} \, dS = 0.$$

Therefore, $u \equiv 0$ and then the solution is unique.

(b) If not, then $u(x)$ has a negative minimum. As above, by the Minimum Principle, we known that u get its minimum on ∂A or ∂B.

Now we assume that u attains its minimum on ∂A, then $\frac{\partial u}{\partial n} < 0$ by the Hopf principle since u is not a constant. But this is contradict with the fact $\iint_{\partial A} \frac{\partial u}{\partial n} dS = Q > 0$. Surely the same is true for another case. Hence $u \geq 0$ in D.

(c) Suppose that there exists $x_0 \in D$ such that $u(x_0) = 0$. Then by (b) and the Strong Minimum Principle, u is a constant in D. But this is contradict with the fact $\iint_{\partial A} \frac{\partial u}{\partial n} dS = Q > 0$. Hence $u > 0$ in D.

Actually we can also prove that $u > 0$ on ∂A and ∂B by the Hopf principle again as in (b). ☐

Extra Problem 1. Consider the following problem

$$\begin{cases} \Delta u = u^3, & \text{in } D, \\ \frac{\partial u}{\partial n} + a(x)u = h, & \text{on } \partial D, \end{cases} \tag{1}$$

where

$$a(x) \geq 0.$$

Show that the solution to (2) (if exists) is unique.

Answer: Suppose u_1 and u_2 are solutions of the problem. Then $u = u_1 - u_2$ satisfies

$$\begin{cases} \Delta u = (u_1^2 + u_1 u_2 + u_2^2)u, & \text{in } D, \\ \frac{\partial u}{\partial n} + a(x)u = 0, & \text{on } \partial D. \end{cases}$$

Using the Green's first identity and the equation of u,

$$-\int_D |\nabla u|^2 \, dx + \int_{\partial D} u \frac{\partial u}{\partial n} \, dx = \int_D u \Delta u \, dx = \int_D (u_1^2 + u_1 u_2 + u_2^2) u^2 \, dx \geq 0.$$

By the boundary condition of u, we have

$$-\int_D |\nabla u|^2 \, dx - \int_{\partial D} a(x) u^2 \, dx \geq 0,$$

which implies $u \equiv 0$ since $a(x) \geq 0$ and thus the solution to (2) (if exists) is unique. □

Extra Problem 2. Consider the following problem

$$\begin{cases} \Delta u - b(x)u = f(x), & \text{in } D, \\ u = h, & \text{on } \partial D, \end{cases} \tag{2}$$

where

$$b(x) \geq 0.$$

Let u be a C^2 function.

(a) Define an energy functional $E[u]$ associated with (2).

(b) Show that u is a solution to (2) if and only if

$$E[w] \geq E[u], \quad \forall w \in C^2, \ w = h \text{ on } \partial D.$$

Answer: (a) The energy functional $E[u]$ associated with (2) is defined by

$$E[u] = \frac{1}{2} \int_D \left[|\nabla u|^2 + b(x) u^2 \right] dx + \int_D f(x) u \, dx.$$

(b) First we show that if u is a solution to (2), then

$$E[w] \geq E[u], \quad \forall w \in C^2, \ w = h \text{ on } \partial D.$$

Let $v = w - u$, then $v = 0$ on ∂D. Thus

$$\begin{aligned} E[w] &= E[u] + \int_D \Big[\nabla u \nabla v + b(x)uv + fv\Big]\,dx + \frac{1}{2}\int_D \Big[|\nabla v|^2 + b(x)v^2\Big]\,dx \\ &= E[u] + +\frac{1}{2}\int_D \Big[|\nabla v|^2 + b(x)v^2\Big]\,dx \ge E[u] \end{aligned}$$

by the Green's first identity and the equation of u.

Next we prove that if $u \in C^2$ minimizes the energy $E[w]$ for all C^2 functions w with $w = h$ on ∂D then u is a solution to (2).

For any function $v \in C^2$ with $v = 0$ on ∂D and any constant ϵ, the function $w = u + \epsilon v \in C^2$ and $w = h$ on ∂D. By direct computation and using the Green's first identity,

$$\begin{aligned} E[w] &= E[u] + \epsilon\int_D \Big[\nabla u \nabla v + b(x)uv + fv\Big]\,dx + \frac{1}{2}\epsilon^2\int_D \Big[|\nabla v|^2 + b(x)v^2\Big]\,dx \\ &= E[u] + \epsilon\int_D \Big[-\Delta u + b(x)u + f\Big]v\,dx + \frac{1}{2}\epsilon^2\int_D \Big[|\nabla v|^2 + b(x)v^2\Big]\,dx. \end{aligned}$$

Since u minimizes the energy $E[w]$, we obtain

$$\int_D \Big[-\Delta u + b(x)u + f\Big]v\,dx = 0$$

for all $v \in C^2$ with $v = 0$ on ∂D. Thus u is a solution to (2). □

Exercise 7.2

1. Derive the representation formula for harmonic functions (7.2.5) in two dimensions.

Answer: As in the dimension three, surely the same is true for any finite dimension. Let D_ϵ be the region D with a ball (of radius ϵ and center x_0) excised. For simplicity let x_0 be the origin and set $r = \sqrt{x^2 + y^2}$.

By the Green's second identity,

$$\iint_{\partial D_\epsilon} \Big[u \cdot \frac{\partial \log r}{\partial n} - \frac{\partial u}{\partial n} \cdot \log r\Big]\,dS = 0.$$

But ∂D_ϵ consists of two parts: the original boundary ∂D and the circle $\{r = \epsilon\}$. On the circle $\{r = \epsilon\}$, $\frac{\partial}{\partial n} = -\frac{\partial}{\partial r}$. Thus

$$\iint_{\partial D} \Big[u \cdot \frac{\partial \log r}{\partial n} - \frac{\partial u}{\partial n} \cdot \log r\Big]\,dS = \iint_{r=\epsilon} \Big[u \cdot \frac{\partial \log r}{\partial r} - \frac{\partial u}{\partial r} \cdot \log r\Big]\,dS.$$

This identity is valid for any small enough $\epsilon > 0$.

Now the right side of the identity equals

$$\frac{1}{\epsilon}\iint_{r=\epsilon} u\,dS - \log\epsilon \iint_{r=\epsilon} \frac{\partial u}{\partial r}\,dS = 2\pi\overline{u} - 2\pi\epsilon\log\epsilon\overline{\frac{\partial u}{\partial r}},$$

where $\overline{u}$ denotes the average value of u on the circle $\{r = \epsilon\}$, and $\overline{\frac{\partial u}{\partial r}}$ denotes the average value of $\frac{\partial u}{\partial r}$ on this circle.

Note that as $\epsilon \to 0$,

$$2\pi\overline{u} - 2\pi\epsilon\log\epsilon\overline{\frac{\partial u}{\partial r}} \to 2\pi u(0)$$

because u is continuous and $\frac{\partial u}{\partial r}$ is bounded. Thus we obtain the representation formula (7.2.5). $\square$

2. Let $\phi(x)$ be any C^2 function defined on all of three-dimensional space that vanishes outside some sphere. Show that

$$\phi(0) = -\iiint \frac{1}{|x|}\Delta\phi(x)\frac{dx}{4\pi}.$$

The integration is taken over the region where $\phi(x)$ is not zero.

Answer: Let $r = |x|$ and $D_\epsilon := \{x|\ \epsilon < r < R\}$, where R is large enough such that $\phi = 0$ outside $r < R/2$.

Using the Green's second identity,

$$\begin{aligned}
&\iiint_{D_\epsilon} \frac{1}{|x|}\Delta\phi(x)dx \\
= \ & \iint_{\partial D_\epsilon} \left[\frac{1}{|x|}\cdot\frac{\partial\phi}{\partial n} - \phi\cdot\frac{\partial}{\partial n}\frac{1}{|x|}\right] dS \\
= \ & \iint_{|x|=R} \left[\frac{1}{|x|}\cdot\frac{\partial\phi}{\partial n} - \phi\cdot\frac{\partial}{\partial n}\frac{1}{|x|}\right] dS + \iint_{|x|=\epsilon} \left[\frac{1}{|x|}\cdot\frac{\partial\phi}{\partial n} - \phi\cdot\frac{\partial}{\partial n}\frac{1}{|x|}\right] dS \\
= \ & -\iint_{|x|=\epsilon} \left[\frac{1}{\epsilon}\cdot\frac{\partial\phi}{\partial r} + \phi\cdot\frac{1}{\epsilon^2}\right] dS \\
= \ & -4\pi\overline{\phi} - 4\pi\epsilon\overline{\frac{\partial\phi}{\partial r}},
\end{aligned}$$

where $\overline{\phi}$ denotes the average value of ϕ on the sphere $\{r = \epsilon\}$, and $\overline{\frac{\partial\phi}{\partial r}}$ denotes the average value of $\frac{\partial\phi}{\partial r}$ on this sphere.

Because ϕ is continuous and $\frac{\partial\phi}{\partial r}$ is bounded,

$$-4\pi\overline{\phi} - 4\pi\epsilon\overline{\frac{\partial\phi}{\partial r}} \to -4\pi\phi(0) \quad \text{as } \epsilon \to 0.$$

Therefore,

$$\phi(0) = -\iiint\limits_{|x|<R} \frac{1}{|x|}\Delta\phi(x)\frac{dx}{4\pi}. \ \square$$

3. Give yet another derivation of the mean value property in three dimensions by choosing D to be a ball and x_0 its center in the representation formula (1).

Answer: Choosing $D = B(x_0, R)$ in the representation formula (1) and using the divergence theorem,

$$\begin{aligned} u(x_0) &= \iint\limits_{\partial B(x_0,R)} \Big[-u(x)\cdot\frac{\partial}{\partial n}\Big(\frac{1}{|x-x_0|}\Big) + \frac{1}{|x-x_0|}\cdot\frac{\partial u}{\partial n}\Big]\frac{dS}{4\pi} \\ &= \iint\limits_{|x-x_0|=R} \Big[\frac{1}{R^2}u(x) + \frac{1}{R}\frac{\partial u}{\partial n}\Big]\frac{dS}{4\pi} \\ &= \frac{1}{4\pi R^2}\iint\limits_{|x-x_0|=R} u dS + \frac{1}{4\pi R}\iiint\limits_{|x-x_0|<R} \Delta u dx \\ &= \frac{1}{4\pi R^2}\iint\limits_{|x-x_0|=R} u dS. \end{aligned}$$

This proves the mean value property in three dimensions. $\square$

Extra Problem: Formulate and prove Exercise 7.2.2 in two-dimensional case, i.e.,

Let $\phi(x)$ be any C^2 function defined on all of two-dimensional space that vanishes outside some circle. Show that

$$\phi(0) = \iint \log|x|\Delta\phi(x)\frac{dx}{2\pi}.$$

The integration is taken over the region where $\phi(x)$ is not zero.

Answer: Let $r = |x|$ and $D_\epsilon := \{x|\ \epsilon < r < R\}$, where R is large enough such that $\phi = 0$ outside $r < R/2$.

Using the Green's second identity,

$$\begin{aligned}
&\iint_{D_\epsilon} \log|x|\Delta\phi(x)dx \\
=\ & \int_{\partial D_\epsilon} \left[\log|x|\cdot\frac{\partial\phi}{\partial n} - \phi\cdot\frac{\partial}{\partial n}\log|x|\right] dS \\
=\ & \int_{|x|=R} \left[\log|x|\cdot\frac{\partial\phi}{\partial n} - \phi\cdot\frac{\partial}{\partial n}\log|x|\right] dS + \int_{|x|=\epsilon} \left[\log|x|\cdot\frac{\partial\phi}{\partial n} - \phi\cdot\frac{\partial}{\partial n}\log|x|\right] dS \\
=\ & -\int_{|x|=\epsilon} \left[\log\epsilon\cdot\frac{\partial\phi}{\partial r} - \phi\cdot\frac{1}{\epsilon}\right] dS \\
=\ & 2\pi\overline{\phi} - 2\pi\epsilon\log\epsilon\overline{\frac{\partial\phi}{\partial r}},
\end{aligned}$$

where $\overline{\phi}$ denotes the average value of ϕ on the sphere $\{|x| = \epsilon\}$, and $\overline{\frac{\partial\phi}{\partial r}}$ denotes the average value of $\frac{\partial\phi}{\partial r}$ on this sphere.

Because ϕ is continuous and $\frac{\partial\phi}{\partial r}$ is bounded,

$$2\pi\overline{\phi} - 2\pi\epsilon\log\epsilon\overline{\frac{\partial\phi}{\partial r}} \to 2\pi\phi(0) \quad \text{as } \epsilon \to 0,$$

and then we complete the proof.

Exercise 7.3

1. Show that the Green's function is unique. (Hint: Take the difference of two of them.)

Answer: Suppose G_1 and G_2 both are the Green's functions for the operator $-\Delta$ and the domain D at the point $x_0 \in D$.

By (i) and (iii), we known that $G_1(x) + \frac{1}{4\pi|x-x_0|}$ and $G_2(x) + \frac{1}{4\pi|x-x_0|}$ both are harmonic functions in D. Thus by (ii) and the uniqueness theorem of harmonic function, we have

$$G_1(x) + \frac{1}{4\pi|x-x_0|} = G_2(x) + \frac{1}{4\pi|x-x_0|}.$$

Therefore, the Green's function is unique. □

3. Verify the limit of A_ϵ as claimed in the proof of the symmetry of the Green's function.

Answer: In the textbook,

$$A_\epsilon = \iint_{|x-a|=\epsilon} (u\frac{\partial v}{\partial n} - v\frac{\partial u}{\partial n})dS,$$

where $u(x) = G(x,a)$ and $v(x) = G(x,b)$.

Now, Let's computer A_ϵ. Firstly note that

$$u(x) = G(x,a) = -\frac{1}{4\pi|x-a|} + H(x,a),$$

where $H(x,a)$ is a harmonic function throughout the domain D. Secondly $v(x) = G(x,b)$ is a harmonic function in $\{|x-a| < \epsilon\}$. Thus

$$\begin{aligned}
A_\epsilon &= \iint_{|x-a|=\epsilon} \Big[(-\frac{1}{4\pi|x-a|} + H(x,a))\frac{\partial v}{\partial n} - v\frac{\partial}{\partial n}(-\frac{1}{4\pi|x-a|} + H(x,a))\Big]\, dS \\
&= \iint_{|x-a|=\epsilon} \Big[-\frac{1}{4\pi|x-a|}\frac{\partial v}{\partial n} + v\frac{\partial}{\partial n}(\frac{1}{4\pi|x-a|})\Big]\, dS \\
&\quad + \iint_{|x-a|=\epsilon} \Big[H(x,a)\frac{\partial v}{\partial n} - v\frac{\partial}{\partial n}H(x,a)\Big]\, dS \\
&= v(a) - \iiint_{|x-a|<\epsilon} \Big[H(x,a)\Delta v - v\Delta H(x,a)\Big] dx = v(a).
\end{aligned}$$

Here we use the representation formula (7.2.1) and note that $\frac{\partial}{\partial n} = -\frac{\partial}{\partial r}$. Hence $\lim\limits_{\epsilon\to 0} A_\epsilon = v(a) = G(a,b)$. $\square$

Exercise 7.4

1. Find the one-dimensional Green's function for the interval $(0,l)$. The three properties defining it can be restated as follows.

(i) It solves $G''(x) = 0$ for $x \neq x_0$ ("harmonic").

(ii) $G(0) = G(l) = 0$.

(iii) $G(x)$ is continuous at x_0 and $G(x) + \frac{1}{2}|x-x_0|$ is harmonic at x_0.

Answer: By (i), we known that $G(x)$ is a linear function in $[0,x_0]$ and $[x_0,l]$. Since (ii) and $G(x)$ is continuous at x_0, we have

$$G(x) = \begin{cases} kx, & 0 < x \le x_0; \\ \frac{kx_0}{x_0-l}(x-l), & x_0 < x < l. \end{cases}$$

Hence $G(x) + \frac{1}{2}|x-x_0|$ is harmonic at x_0 if and only if

$$\big(G(x) + \frac{1}{2}|x-x_0|\big)'|(x_0^-) = \big(G(x) + \frac{1}{2}|x-x_0|\big)'|(x_0^+)$$

if and only if

$$k = \frac{l-x_0}{l}.$$

Therefore the one-dimensional Green's function for the interval $(0, l)$ at the point $x_0 \in (0, l)$ is

$$G(x, x_0) = \begin{cases} \frac{l-x_0}{l}x, & 0 < x \leq x_0; \\ -\frac{x_0}{l}(x-l), & x_0 < x < l. \end{cases} \quad \square$$

2. Verify directly from (3) or (4) that the solution of the half-space problem satisfies the condition at infinity:

$$u(\mathbf{x}) \to 0 \quad \text{as } |\mathbf{x}| \to \infty.$$

Assume that $h(x, y)$ is a continuous function that vanished outside some circle.

Answer: Assume that $h(x, y)$ is a continuous function that vanished outside $\{(x, y)|\ x^2 + y^2 \leq R^2\}$ and $|h(x, y)| \leq M$. Then by the formulas (3), we have

$$|u(x_0, y_0, z_0)| \leq \frac{M}{2\pi} \iint\limits_{\{(x,y)|\ x^2+y^2 \leq R^2\}} [(x-x_0)^2 + (y-y_0)^2 + z_0^2]^{-1} dxdy$$

$$\leq \frac{MR^2}{2(\sqrt{x_0^2 + y_0^2 + z_0^2} - R)^2}, \text{when } \sqrt{x_0^2 + y_0^2 + z_0^2} > R.$$

Therefore, u satisfies the condition at infinity:

$$u(\mathbf{x}) \to 0 \quad \text{as } |\mathbf{x}| \to \infty. \ \square$$

3. Show directly from (3) that the boundary condition is satisfied: $u(x, y, z) \to h(x, y)$ as $z \to 0$.

[Hint: Change variables $s^2 = [(x-x_0)^2 + (y-y_0)^2]/z_0^2$ and use the fact that $\int_0^\infty s(s^2+1)^{-3/2}\, ds = 1$.]

Answer: From (3), we have

$$\begin{aligned} u(x_0, y_0, z_0) &= \frac{z_0}{2\pi} \iint [(x-x_0)^2 + (y-y_0)^2 + z_0^2]^{-\frac{3}{2}} h(x, y) dxdy \\ &= \frac{z_0}{2\pi} \iint (x^2 + y^2 + z_0^2)^{-\frac{3}{2}} h(x + x_0, y + y_0) dxdy. \end{aligned}$$

Now we change variables such that $x = z_0 s\cos\theta,\ y = z_0 s \sin\theta$. Then

$$\begin{aligned} u(x_0,y_0,z_0) &= \frac{z_0}{2\pi}\int_0^{2\pi}\int_0^{\infty}(z_0^2s^2+z_0^2)^{-\frac{3}{2}}h(z_0 s\cos\theta+x_0, z_0 s\sin\theta+y_0)z_0^2 s ds d\theta \\ &= \frac{1}{2\pi}\int_0^{2\pi}\int_0^{\infty} s(s^2+1)^{-\frac{3}{2}}h(z_0 s\cos\theta+x_0, z_0 s\sin\theta+y_0)ds d\theta. \end{aligned}$$

Since

$$\int_0^{\infty} s(s^2+1)^{-\frac{3}{2}}ds = -(s^2+1)^{-\frac{3}{2}}|_0^{\infty} = 1,$$

so

$$\lim_{z_0\to 0} u(x_0,y_0,z_0) = h(x_0,y_0).$$

if the limit can be taken in the integration, for example, when $h(x,y)$ is bounded. □

5. Notice that the function xy is harmonic in the half-plane $\{y > 0\}$ and vanishes on the boundary line $\{y = 0\}$. The function 0 has the same properties. Does this mean that the solution is not unique? Explain.

Answer: Here one **Suggested explain** is given.

Since the half-plane $\{y > 0\}$ is not bounded, this only means that the solution is not unique under the (uncompleted) boundary condition or it is not a well-posed problem. But generally it will be a well-posed problem if we add another boundary condition, such as

$$u(\mathbf{x}) \to 0 \ \text{ as } |\mathbf{x}| \to \infty. \ \Box$$

6. (a) Find the Green's function for the half-plane $\{(x,y) :\ y > 0\}$.

(b) Use it to solve the Dirichlet problem in the half-plane with boundary values $h(x)$.

(c) Calculate the solution with $u(x,0) = 1$.

Answer: (a) Using the method of reflection, as in the dimension three, we have

$$G(X,X_0) = \frac{1}{2\pi}\log|X-X_0| - \frac{1}{2\pi}\log|X-X_0^*|,$$

where $X = (x,y), X_0 = (x_0,y_0), X_0^* = (x_0,-y_0)$.

(b) Since

$$\begin{aligned} -\frac{\partial G}{\partial y} &= \frac{y+y_0}{2\pi\big[(x-x_0)^2+(y+y_0)^2\big]} - \frac{y-y_0}{2\pi\big[(x-x_0)^2+(y-y_0)^2\big]} \\ &= \frac{y_0}{\pi\big[(x-x_0)^2+y_0^2\big]} \end{aligned}$$

on $y = 0$, Therefore, the solution is given by

$$u(x_0, y_0) = \frac{y_0}{\pi} \int \frac{h(x)}{(x - x_0)^2 + y_0^2}\, dx$$

(c) By (b), we have

$$u(x_0, y_0) = \frac{y_0}{\pi} \int \frac{1}{(x - x_0)^2 + y_0^2}\, dx = \frac{1}{\pi} \int \frac{1}{x^2 + 1}\, dx = 1. \quad \square$$

7. (a) If $u(x, y) = f(x/y)$ is a harmonic function, solve the ODE satisfied by f.

(b) Show that $\partial u/\partial r \equiv 0$, where $r = \sqrt{x^2 + y^2}$ as usual.

(c) Suppose that $v(x, y)$ is any harmonic function in $\{y > 0\}$ such that $\partial v/\partial r \equiv 0$. Show that $v(x, y)$ is a function of the quotient x/y.

(d) Find the boundary values $\lim_{y \to 0} u(x, y) = h(x)$.

(e) Show that your answer to parts (c) and (d) agrees with the general formula from Exercise 6.

Answer: (a) If $u(x, y) = f(\frac{x}{y})$, then

$$u_x = \frac{1}{y} f'(\frac{x}{y}), u_y = -\frac{x}{y^2} f'(\frac{x}{y}),$$

Thus

$$u_{xx} + u_{yy} = \frac{1}{y^2} f''(\frac{x}{y}) + \frac{x^2}{y^4} f''(\frac{x}{y}) + 2\frac{x}{y^3} f'(\frac{x}{y}).$$

Hence the ODE satisfied by f is

$$(1 + t^2) f''(t) + 2t f'(t) = 0.$$

Therefore,

$$f(t) = \int_0^t \frac{a}{1 + s^2} ds + b = a \arctan t + b,$$

where a, b are constants.

(b)

$$\frac{\partial u}{\partial r} = \frac{\partial u}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial r} = \frac{1}{y} f'(\frac{x}{y})\frac{x}{r} - \frac{x}{y^2} f'(\frac{x}{y})\frac{y}{r} = 0.$$

(c) Using the polar coordinate,

$$v(r, \theta) = c\theta + d,$$

where c, d are constants. Hence in the (x, y)-coordinate, we have

$$v(x, y) = c \arctan \frac{y}{x} + d.$$

(d) By (a),

$$u(x, y) = f(\frac{x}{y}) = a \arctan \frac{x}{y} + b.$$

Thus

$$h(x) = \lim_{y \to 0} u(x, y) = \begin{cases} \frac{\pi}{2}a + b, & x > 0; \\ -\frac{\pi}{2}a + b, & x < 0; \\ b, & x = 0. \end{cases}$$

(e) By (d), we see that $h(x)$ is not continuous unless u is a constant. This agrees with the condition that $h(x)$ is continuous in Exercise 6. □

9. Find the Green's function for the tilted half-space $\{(x, y, z) : \ ax + by + cz > 0\}$.

(Hint: Either do it from scratch by reflecting across the titled plane, or change variables in the double integral (3) using a linear transformation.)

Answer: Here the method of reflection is used. Let $X_0 = (x_0, y_0, z_0)$ such that $ax_0 + by_0 + cz_0 > 0$, then its reflection point about the hyperplane $\{ax + by + cz = 0\}$ $X_0' = (x_0', y_0', z_0')$, where

$$x_0' = x_0 - \frac{2a}{a^2 + b^2 + c^2}(ax_0 + by_0 + cz_0),$$

$$y_0' = y_0 - \frac{2b}{a^2 + b^2 + c^2}(ax_0 + by_0 + cz_0),$$

$$z_0' = z_0 - \frac{2c}{a^2 + b^2 + c^2}(ax_0 + by_0 + cz_0).$$

Thus the Green's function for the tilted half-space $\{(x, y, z) : \ ax + by + cz > 0\}$ is given by

$$G(X, X_0) = -\frac{1}{4\pi|X - X_0|} + \frac{1}{4\pi|X - X_0'|}. \quad \Box$$

10. Verify the formula (11) for $G(\mathbf{x}, \mathbf{0})$, the Green's function with its second argument at the center of the sphere.

Answer: In case $\mathbf{x}_0 = 0$, the formula for the Green's function is

$$G(\mathbf{x}, 0) = -\frac{1}{4\pi|\mathbf{x}|} + \frac{1}{4\pi a},$$

since $-\frac{1}{4\pi|\mathbf{x}|} + \frac{1}{4\pi a} \in C^2$ and is harmonic in $B_a(\mathbf{0})$, except at the point $\mathbf{x} = \mathbf{0}$; $\big[-\frac{1}{4\pi|\mathbf{x}|} + \frac{1}{4\pi a}\big]\big|_{|x|=a} = 0$; and $\big[-\frac{1}{4\pi|\mathbf{x}|} + \frac{1}{4\pi a}\big] + \frac{1}{4\pi|\mathbf{x}|}$ is harmonic at $\mathbf{0}$. $\square$

13. Find the Green's function for the half-ball $D = \{x^2 + y^2 + z^2 < a^2,\ z > 0\}$.

(Hint: The easiest method is to use the solution for the whole ball and reflect it across the plane.)

Answer: Let $X_0 \in D$, we have known that

$$G(X, X_0) = -\frac{1}{4\pi|X - X_0|} + \frac{1}{4\pi|r_0 X/a - aX_0/r_0|}$$

is the Green's function at X_0 for the whole ball.

Let $X_0' = (x_0, y_0, -z_0)$, then

$$G(X, X_0') = -\frac{1}{4\pi|X - X_0'|} + \frac{1}{4\pi|r_0 X/a - aX_0'/r_0|}$$

is the Green's function at X_0' for the whole ball. Note that $G(X, X_0) = G(X, X_0')$ on ∂D and $G(X, X_0')$ is a harmonic function in D. Hence $G(X, X_0) - G(X, X_0')$ is the Green's function for D by the uniqueness of the Green's function. $\square$

15. (a) Show that if $v(x, y)$ is harmonic, so is $u(x, y) = v(x^2 - y^2, 2xy)$.

(b) Show that the transformation $(x, y) \to (x^2 - y^2, 2xy)$ maps the first quadrant onto the half-plane $\{y > 0\}$.

(Hint: Use polar coordinates.)

Answer: (a) If $v(x, y)$ is harmonic and $u(x, y) = v(x^2 - y^2, 2xy)$, then

$$u_x = 2xv_x + 2yv_y, \quad u_y = -2yv_x + 2xv_y.$$

Thus

$$\begin{aligned} u_{xx} + u_{yy} &= 2v_x + 4x^2 v_{xx} + 4xyv_{xy} + 4xyv_{yx} + 4y^2 v_{yy} \\ &\quad -2v_x + 4y^2 v_{xx} - 4xyv_{xy} - 4xyv_{yx} + 4x^2 v_{yy} \\ &= 4(x^2 + y^2)(v_{xx} + v_{yy}) = 0. \end{aligned}$$

(b) Using the polar coordinates, the transformation is

$$(r\cos\theta, r\sin\theta) \mapsto (r^2\cos 2\theta, r^2\sin 2\theta).$$

Therefore, it maps the first quadrant onto the half-plane $\{y > 0\}$. Note that the transformation is one-one which also maps the x-positive-axis to x-positive-axis and y-positive-axis to x-negative-axis. $\square$

17(a). Find the Green's function for the quadrant

$$Q = \{(x, y) : \ x > 0, y > 0\}.$$

(Hint: Either use the method of reflection or reduce to the half-plane problem by the transformation in Exercise 15.)

Answer: Here the method of reflection is used and you can also use the result of Exercise 15 to find the Green's function for the quadrant Q. Let $X_0 = (x_0, y_0) \in Q$, $X_0^y = (-x_0, y_0)$, $X_0^x = (x_0, -y_0)$ and $X_0^0 = (-x_0, -y_0)$. Then it is easily to check that

$$G(X, X_0) = -\frac{1}{4\pi}\frac{1}{|X - X_0|} + \frac{1}{4\pi}\frac{1}{|X - X_0^y|} + \frac{1}{4\pi}\frac{1}{|X - X_0^x|} - \frac{1}{4\pi}\frac{1}{|X - X_0^0|}$$

is the Green's function at X_0 for the quadrant Q. $\square$