MAT 4220 (2008-09) Partial differential equations
Suggested Answer to Assignment 6

Exercise 6.1

2. Find the solutions that depend only on r of the equation $u_{xx} + u_{yy} + u_{zz} = k^2 u$, where k is a positive constant.

(Hint: Substitute $u = v/r$.)

Answer: Note that in the spherical coordinates (r, θ, ϕ),

$$\Delta_3 = \frac{\partial^2}{\partial r^2} + \frac{2}{r}\frac{\partial}{\partial r} + \frac{1}{r^2 \sin\theta}\frac{\partial}{\partial\theta}\sin\theta\frac{\partial}{\partial\theta} + \frac{1}{r^2\sin^2\theta}\frac{\partial^2}{\partial\phi^2}.$$

Thus

$$u_{rr} + \frac{2}{r}u_r = \Delta_3 u = k^2 u.$$

Let $u = v/r$, we get

$$u_r = v_r/r - v/r^2, \quad u_{rr} = v_{rr}/r - 2v_r/r^2 + 2v/r^3.$$

Hence by the equation of u,

$$v_{rr} = k^2 v,$$

which implies $v(r) = Ae^{-kr} + Be^{kr}$, where A, B are constants. Therefore

$$u(r) = A\frac{1}{r}e^{-kr} + B\frac{1}{r}e^{kr},$$

where A, B are constants. □

4. Solve $u_{xx} + u_{yy} + u_{zz} = 0$ in the spherical shell $0 < a < r < b$ with the boundary conditions $u = A$ on $r = a$ and $u = B$ on $r = b$, where A and B are constants.

(Hint: Look for a solution depending only on r.)

Answer: We have known that $-c_1 r^{-1} + c_2$ is a solution, where c_1 and c_2 satisfy the equation:

$$-c_1 a^{-1} + c_2 = A, \quad -c_1 b^{-1} + c_2 = B.$$

Hence

$$u(x, y) = ab\frac{A - B}{b - a}r^{-1} + A + b\frac{B - A}{b - a}, \text{ where } r = \sqrt{x^2 + y^2 + z^2},$$

is a solution. Therefore it is the unique solution by the Uniqueness Theorem of the Dirichlet problem for the Laplace's equation. □

5. Solve $u_{xx} + u_{yy} = 1$ in $r < a$ with $u(x, y)$ vanishing on $r = a$.

Answer: Firstly we look for a solution depending only on r. Let $u(r)$, where $r = \sqrt{x^2 + y^2}$, is a solution of $u_{xx} + u_{yy} = 1$, then

$$u_{rr} + \frac{1}{r}u_r = 1.$$

As before, we can obtain

$$u = \frac{1}{4}r^2 + c_1 \ln r + c_2, \text{where } c_1, c_2 \text{ are constants.}$$

Thus by the boundary condition, we have

$$c_1 = 0, \quad \frac{1}{4}a^2 + c_2 = 0.$$

Therefore,

$$u(x, y) = \frac{1}{4}r^2 - \frac{1}{4}a^2,$$

is the unique solution by the Uniqueness Theorem. □

6. Solve $u_{xx} + u_{yy} = 1$ in the annulus $a < r < b$ with $u(x, y)$ vanishing on both parts of the boundary $r = a$ and $r = b$.

Answer: Firstly we find a solution depending only on r. Let $u(r)$, where $r = \sqrt{x^2 + y^2}$, is a solution. As before, we have

$$u = \frac{1}{4}r^2 + c_1 \ln r + c_2, \text{where } c_1, c_2 \text{ are constants.}$$

By the boundary conditions, we get

$$\frac{1}{4}a^2 + c_1 \ln a + c_2 = 0, \quad \frac{1}{4}b^2 + c_1 \ln b + c_2 = 0.$$

Hence

$$u(x, y) = \frac{1}{4}(r^2 - a^2) - \frac{b^2 - a^2}{4(\ln b - \ln a)}(\ln r - \ln a),$$

is the unique solution by the Uniqueness Theorem. □

7. Solve $u_{xx} + u_{yy} + u_{zz} = 1$ in the spherical shell $a < r < b$ with $u(x, y, z)$ vanishing on both the inner and outer boundaries.

Answer: Firstly we look for a solution depending only on $r = \sqrt{x^2+y^2+z^2}$. Let $u(r)$ be a solution, then as before,

$$u_{rr} + \frac{2}{r}u_r = 1,$$

from which we have

$$u = \frac{1}{6}r^2 + \frac{c_1}{r} + c_2, \text{where } c_1, c_2 \text{ are constants.}$$

Thus by the boundary conditions, we get

$$\frac{1}{6}a^2 + \frac{c_1}{a} + c_2 = 0, \frac{1}{6}b^2 + \frac{c_1}{b} + c_2 = 0.$$

Therefore,

$$u(x,y) = \frac{1}{6}(r^2 - a^2) + ab\frac{a+b}{6}(\frac{1}{r} - \frac{1}{a}),$$

is the unique solution by the Uniqueness Theorem. □

10. Prove the uniqueness of the Dirichlet problem $\Delta u = f$ in D, $u = g$ on bdy D by the energy method. That is, after subtracting two solutions $w = u - v$, multiply the Laplace equation for w by w itself and use the divergence theorem.

Answer: Suppose that both u and v are two solutions for the Dirichlet problem. Let $w = u - v$, then w satisfies $\Delta w = 0$ in D, $w = 0$ on bdy D. Thus by the divergence theorem,

$$0 = \int_D w\Delta w dx = -\int_D Dw \cdot Dw ds,$$

which implies $Dw = 0$. Since $w = 0$ on bdy D, we get $w = 0$. Therefore, the solution of the Dirichlet problem is unique. □

11. Show that there is no solution of

$$\Delta u = f \quad \text{in } D, \quad \frac{\partial u}{\partial n} = g \quad \text{on bdy } D$$

in three dimensions, unless

$$\iiint_D f\, dxdydz = \iint_{\text{bdy}(D)} g\, dS.$$

(Hint: Integrate the equation.)

Answer: Integrating the equation $\Delta u = f$ and using the divergence theorem,

$$\iiint_D f\,dxdydz = \iiint_D \Delta u\,dxdydz = \iint_{\text{bdy}(D)} \frac{\partial u}{\partial n}\,dS = \iint_{\text{bdy(D)}} g\,dS.$$

Hence there is no solution unless

$$\iiint_D f\,dxdydz = \iint_{\text{bdy}(D)} g\,dS. \ \square$$

Exercise 6.2

1. Solve $u_{xx} + u_{yy} = 0$ in the rectangle $0 < x < a$, $0 < y < b$ with the following boundary conditions:

$$u_x = -a \quad \text{on } x = 0, \quad u_x = 0 \quad \text{on } x = a$$

$$u_y = b \quad \text{on } y = 0, \quad u_y = 0 \quad \text{on } x = b.$$

(Hint: Note that the necessary condition of Exercise 6.1.11 is satisfied. A shortcut is to guess that the solution might be a quadratic polynomial in x and y.)

Answer: By the boundary conditions, we can guess $u_x(x, y) = x - a$ and $u_y(x.y) = -y + b$. Luckily these also satisfy the equation. Hence

$$u(x, y) = \frac{1}{2}x^2 - \frac{1}{2}y^2 - ax + by + c, \text{ where c is any constant,}$$

are solutions.

Actually we can prove that they are all solutions by the Hopf maximum principle. $\square$

2. Prove that the eigenfunctions $\{\sin my \sin nz\}$ are orthogonal on the square $\{0 < y < \pi, 0 < z < \pi\}$.

Answer: Let $(m, n) \neq (m', n')$, then

$$\begin{aligned} & \int_0^\pi \int_0^\pi (\sin my \sin nz)(\sin m'y \sin n'z)\,dydz \\ = \ & (\int_0^\pi \sin my \sin m'ydy) \cdot (\int_0^\pi \sin nz \sin n'zdz) = 0, \end{aligned}$$

so the eigenfunctions $\{\sin my \sin nz\}$ are orthogonal on the square $\{0 < y < \pi, 0 < z < \pi\}$. $\square$

3. Find the harmonic function $u(x,y)$ in the square $D = \{0 < x < \pi, 0 < y < \pi\}$ with the boundary conditions:

$$u_y = 0 \quad \text{for } y = 0 \text{ and for } y = \pi, \quad u = 0 \text{ for } x = 0 \quad \text{and}$$

$$u = \cos^2 y = \frac{1}{2}(1 + \cos 2y), \quad \text{for } x = \pi.$$

Answer: Let $u(x,y) = X(x)Y(y)$, then

$$\frac{X''}{X} = -\frac{Y''}{Y} = \lambda, \quad X(0) = Y'(0) = Y'(\pi) = 0.$$

Hence

$$\lambda_n = n^2,\ Y_n(y) = \cos(ny),\ X_0 = x, X_{n+1} = \sinh[(n+1)x], n = 0, 1, 2, ...$$

Therefore,

$$u(x,y) = A_0 x + \sum_{n=1}^{\infty} A_n \sinh(nx)\cos(ny).$$

By the inhomogeneous boundary condition, we get

$$A_0\pi + \sum_{n=1}^{\infty} A_n \sinh(n\pi)\cos(ny) = \frac{1}{2}(1 + \cos 2y),$$

which implies

$$A_0 = \frac{1}{2\pi}, A_2 = \frac{1}{2\sinh(2\pi)}, A_n = 0,\ \text{if } n \neq 0, 2$$

Hence

$$u(x,y) = \frac{x}{2\pi} + \frac{1}{2\sinh(2\pi)}\sinh(2x)\cos(2y). \ \square$$

4. Find the harmonic function in the square $\{0 < x < 1, 0 < y < 1\}$ with the boundary conditions $u(x,0) = x$, $u(x,1) = 0$, $u_x(0,y) = 0$, $u_x(1,y) = y^2$.

Answer: Let u_1 satisfies

$$\Delta u_1 = 0,\ \text{in the square } \{0 < x < 1, 0 < y < 1\},$$

$$u_1(x,0) = x, u_1(x,1) = u_{1,x}(0,y) = u_{1,x}(1,y) = 0,$$

and u_2 satisfies

$$\Delta u_2 = 0, \text{ in the square } \{0 < x < 1, 0 < y < 1\},$$

$$u_2(x,0) = u_2(x,1) = u_{2,x}(0,y) = 0, u_{2,x}(1,y) = y^2,$$

then $u = u_1 + u_2$ is a harmonic function which we want to find.

By the method of separate variables,

$$u_1 = -\frac{A_0}{2}(y-1) + \sum_{n=1}^{\infty} A_n \cos(n\pi x)\Big[\cosh(n\pi y) - \coth(n\pi)\sinh(n\pi y)\Big],$$

where

$$A_0 = 1, A_n = 2\int_0^1 x\cos(n\pi x)dx = \frac{2}{n^2\pi^2}\big[(-1)^n - 1\big], n = 1, 2, \ldots.$$

And

$$u_2 = \sum_{n=1}^{\infty} B_n \cosh(n\pi x)\sin(n\pi y),$$

where

$$\begin{aligned} B_n &= \frac{2}{n\pi\sinh(n\pi)}\int_0^1 y^2\sin(n\pi y)\,dy \\ &= \frac{2}{\sinh(n\pi)}\Big\{\frac{(-1)^{n+1}}{n^2\pi^2} + \frac{2}{n^4\pi^4}\big[(-1)^n - 1\big]\Big\}, n = 1, 2, \ldots. \end{aligned}$$

Therefore,

$$u = -\frac{A_0}{2}(y-1) + \sum_{n=1}^{\infty} A_n \cos(n\pi x)(\cosh(n\pi y) - \coth(n\pi)\sinh(n\pi y))$$

$$+\sum_{n=1}^{\infty} B_n \cosh(n\pi x)\sin(n\pi y),$$

where A_n, B_n be given above. □

6. Solve the following Neumann problem in the cube $\{0 < x < 1, 0 < y < 1, 0 < z < 1\}$: $\Delta u = 0$ with $u_z(x,y,1) = g(x,y)$ and homogeneous Neumann conditions on the other five faces, where $g(x, y)$ is an arbitrary function with zero average.

Answer: Let $u(x, y, z) = X(x)Y(y)Z(z)$, then

$$\frac{X''}{X} + \frac{Y''}{Y} + \frac{Z''}{Z} = 0, \quad X'(0) = X'(1) = Y'(0) = Y'(1) = Z'(0) = 0.$$

Hence

$$X_m(x) = \cos(m\pi x), m = 0, 1, 2, \ldots, Y_n(y) = \cos(n\pi y), n = 0, 1, 2, \ldots,$$

and

$$Z'' = (m^2 + n^2)\pi^2 Z, \; Z'(0) = 0.$$

Therefore,

$$u(x, y, z) = \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} A_{mn} \cos(m\pi x) \cos(n\pi y) \cosh(\sqrt{m^2 + n^2}\pi z).$$

Finally, by the inhomogeneous boundary condition, we get

$$g(x, y) = \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} A_{mn} \sqrt{m^2 + n^2}\pi \sinh(\sqrt{m^2 + n^2}\pi) \cos(m\pi x) \cos(n\pi y),$$

which implies

$$A_{mn} = \frac{4}{\sqrt{m^2 + n^2}\pi \sinh(\sqrt{m^2 + n^2}\pi)} \int_0^1 \int_0^1 g(x, y) \cos(m\pi x) \cos(n\pi y) dx dy.$$

Therefore, the solutions can be expressed as $u(x, y, z) + c$, where c is any constant, with the coefficients A_{mn}. Actually we can prove that they are all solutions by the Hopf maximum principle. $\square$

7(a). Find the harmonic function in the semi-infinite strip $\{0 \le x \le \pi, 0 \le y < \infty\}$ that satisfies the "boundary conditions":

$$u(0, y) = u(\pi, y) = 0, u(x, 0) = h(x), \lim_{y \to \infty} u(x, y) = 0.$$

Answer: Let $u(x, y) = X(x)Y(y)$, then

$$\frac{X''}{X} + \frac{Y''}{Y} = 0, \quad X(0) = X(\pi) = 0.$$

Hence

$$X_n(x) = \sin(nx), n = 1, 2, \ldots, \text{ and } Y'' = n^2 Y, \lim_{y\to\infty} Y(y) = 0.$$

Thus

$$u(x, y) = \sum_{n=1}^{\infty} A_n \sin(nx) e^{-ny}.$$

Finally, by the inhomogeneous condition,

$$h(x) = \sum_{n=1}^{\infty} A_n \sin(nx),$$

which implies

$$A_n = \frac{2}{\pi} \int_0^{\pi} h(x) \sin(nx) dx.$$

Therefore,

$$u(x, y) = \sum_{n=1}^{\infty} \frac{2}{\pi} \Big(\int_0^{\pi} h(x) \sin(nx)\, dx \Big) \sin(nx) e^{-ny}. \quad \square$$

Exercise 6.3

1. Suppose that u is a harmonic function in the disk $D = \{r < 2\}$ and that $u = 3\sin 2\theta + 1$ for $r = 2$. Without finding the solution, answer the following questions.

(a) Find the maximum value of u in $\overline{D}$.

(b) Calculate the value of u at the origin.

Answer: (a) By the Maximum Principle,

$$\max_{\overline{D}} u = \max_{\partial D} u = \max_{\theta} (3\sin 2\theta + 1) = 4.$$

(b) By the Mean Value property,

$$u(0, 0) = \frac{1}{2\pi} \int_0^{2\pi} (3\sin 2\theta + 1) d\theta = 1. \quad \square$$

2. Solve $u_{xx} + u_{yy} = 0$ in the disk $\{r < a\}$ with the boundary condition

$$u = 1 + 3\sin\theta \quad \text{on } r = a.$$

Answer: By the formula (10)-(12) in the textbook,

$$u = \frac{1}{2}A_0 + \sum_{n=1}^{\infty} r^n (A_n \cos n\theta + B_n \sin n\theta),$$

where

$$A_n = \frac{1}{\pi a^n}\int_0^{2\pi} h(\theta)\cos n\theta d\theta,\ B_n = \frac{1}{\pi a^n}\int_0^{2\pi} h(\theta)\sin n\theta d\theta.$$

Since $h(\theta) = 1 + 3\sin\theta$, we get

$$A_0 = 2, A_n = 0 \text{ (for } n > 0), B_1 = \frac{3}{a}, B_m = 0 \text{ (for } m > 1).$$

Hence

$$u(r,\theta) = 1 + \frac{3r}{a}\sin\theta.$$

You can also solve the equation by Poisson's formula. □

3. Same for the boundary condition $u = \sin^3\theta$.

(Hint: Use the identity $\sin^3\theta = 3\sin\theta - 4\sin 3\theta$.)

Answer: As before, since

$$h(\theta) = \sin^3\theta = 3\sin\theta - 4\sin 3\theta,$$

we get

$$A_n = 0 \text{ (for any } n), B_1 = \frac{3}{a}, B_3 = -\frac{4}{a^3}, B_m = 0 \text{ (for } m \neq 1, 3).$$

Hence

$$u(r,\theta) = \frac{3r}{a}\sin\theta - \frac{4r^3}{a^3}\sin 3\theta. \quad \square$$

Problem 4. Let $u \geq 0$ and $\Delta u = 0$ in a unit disk $D = \{(x,y)|x^2+y^2 \leq 1\}$. Using the Mean-Value Property to prove the following so-called Harnack inequality

$$\frac{1-r}{1+r}u(0,0) \leq u(x,y) \leq \frac{1+r}{1-r}u(0,0),$$

where $r = \sqrt{x^2+y^2} < 1$.

Answer: By the Poisson's formula,

$$u(r,\theta) = \frac{1-r^2}{2\pi}\int_0^{2\pi}\frac{h(\phi)}{1-2r\cos(\theta-\phi)+r^2}d\phi,$$

where $h(\phi) = u(1, \phi)$.

Note that $h(\phi) \geq 0$ and

$$(1-r)^2 \leq 1 - 2r\cos(\theta - \phi) + r^2 \leq (1+r)^2,$$

we get

$$\frac{1-r^2}{2\pi(1+r)^2} \int_0^{2\pi} h(\phi) d\phi \leq u(r,\theta) \leq \frac{1-r^2}{2\pi(1-r)^2} \int_0^{2\pi} h(\phi) d\phi.$$

Now by the Mean-Value property,

$$u(0,0) = \frac{1}{2\pi} \int_0^{2\pi} h(\phi) d\phi.$$

Therefore,

$$\frac{1-r}{1+r} u(0,0) \leq u(x,y) \leq \frac{1+r}{1-r} u(0,0). \ \Box$$

Problem 5. Suppose that u satisfies $u_{xx} + u_{yy} = 0$ for all $(x,y) \in B_1(0)$ except $(x,y) = (0,0)$. Show that if u is a bounded function, the $\lim_{(x,y)\to(0,0)} u(x,y)$ exists and by taking $u(0,0) = \lim_{(x,y)\to(0,0)} u(x,y)$, u is actually smooth in $B_1(0)$.

(Hint: Consider the following function: $v_\epsilon = \epsilon \log \frac{1}{r}$.)

Answer: Firstly, we prove the following uniqueness lemma.

Lemma: Suppose that v satisfies $v_{xx} + v_{yy} = 0$ in $D = \{(x,y)|\ 0 < x^2 + y^2 < 1\}$ and $v = 0$ on $\{(x,y)|\ x^2 + y^2 = 1\}$. Show that $v = 0$ if v is bounded.

Proof of the Lemma:

Fix $(x_0, y_0) \in D$. For any $\epsilon > 0$, we consider the harmonic function

$$v_\epsilon(r) := \epsilon \log \frac{1}{r}, \quad \text{where } r = \sqrt{x^2 + y^2}.$$

Since $\lim_{r\to 0} v_\epsilon = +\infty$ and $v_\epsilon = 0$ on $\{(x,y)|\ x^2 + y^2 = 1\}$, so we can choose r small enough so that $v_\epsilon(r) > \sup v$ (since v is bounded) and $x_0^2 + y_0^2 > r^2$. Hence by Maximum Principle on the domain $\{(x,y)|\ r \leq x^2 + y^2 = 1\}$, we get

$$v(x_0, y_0) \leq \epsilon \log \frac{1}{r_0},$$

where $r_0 = \sqrt{x_0^2 + y_0^2}$. Letting $\epsilon \to 0$, then $v(x_0, y_0) \le 0$. Similarly, we can get $v(x_0, y_0) \ge 0$. Therefore, $v(x_0, y_0) = 0$ and we complete the proof of the lemma. □

Now we can use the lemma to prove the conclusion in the problem. Let w be the harmonic function given by the following Poisson's formula,

$$w(r, \theta) := \frac{1 - r^2}{2\pi} \int_0^{2\pi} \frac{h(\phi)}{1 - 2r\cos(\theta - \phi) + r^2} d\phi, \quad (r < 1)$$

where $h(\phi) = u(1, \phi)$. Here we use polar coordinate (r, ϕ).

Let $v = u - w$, then v satisfies the conditions of Lemma, thus $v = 0$. Hence $u = w$ for all $(x, y) \in B_1(0)$ except $(x, y) = (0, 0)$. Therefore the $\lim\limits_{(x,y)\to(0,0)} u(x, y)$ exists and $\lim\limits_{(x,y)\to(0,0)} u(x, y) = w(0, 0)$. Therefore, after taking $u(0, 0) = \lim\limits_{(x,y)\to(0,0)} u(x, y)$, u is actually smooth in $B_1(0)$ since $u = w$ and w is a harmonic function. □

Exercise 6.4

1. Solve $u_{xx} + u_{yy} = 0$ in the exterior $\{r > a\}$ of a disk, with the boundary condition $u = 1 + 3\sin\theta$ on $r = a$, and the condition at infinity that u be bounded as $r \to \infty$.

Answer: Since the only difference between the formulas of harmonic function in the interior and exterior of a disk is that r and a are replaced by r^{-1} and a^{-1}. Therefore by the result in the exercise 6.4.2,

$$u(r, \theta) = 1 + \frac{3a}{r} \sin\theta. \ \square$$

6. Find the harmonic function u in the semidisk $\{r < 1, 0 < \theta < \pi\}$ with u vanishing on the diameter $(\theta = 0, \pi)$ and

$$u = \pi \sin\theta - \sin 2\theta \quad \text{on } r = 1.$$

Answer: Using the separation of variables technique, we have

$$\Theta'' + \lambda\Theta = 0, \ r^2R'' + rR' - \lambda R = 0.$$

So the homogeneous conditions lead to

$$\Theta'' + \lambda\Theta = 0, \ \Theta(0) = \Theta(\pi) = 0.$$

Hence

$$\lambda_n = n^2, \ \Theta(\theta) = \sin n\theta, n = 1, 2, \ldots,$$

and then

$$R_n(r) = r^n, n = 1, 2, \ldots.$$

Thus

$$u(r, \theta) = \sum_{n=1}^{\infty} A_n r^n \sin n\theta.$$

Finally, the inhomogeneous boundary condition requires that

$$\pi \sin\theta - \sin 2\theta = \sum_{n=1}^{\infty} A_n \sin n\theta,$$

which implies

$$A_1 = \pi, A_2 = -1, A_n = 0, \text{ for } n \neq 1, 2.$$

Therefore,

$$u(r, \theta) = \pi r \sin\theta - r^2 \sin 2\theta. \ \square$$

9. Solve $u_{xx} + u_{yy} = 0$ in the wedge $r < a$, $0 < \theta < \beta$ with the BCs

$$u = \theta \quad \text{on } r = a, \quad u = 0 \text{ on } \theta = 0, \quad \text{and} \quad u = \beta \text{ on } \theta = \beta.$$

(Hint: Look for a solution independent of r.)

Answer: It is obvious that $u(r, \theta) = \theta$ is a solution. Hence by the uniqueness theorem, $u(r, \theta) = \theta$ is the unique solution. $\square$

10. Solve $u_{xx} + u_{yy} = 0$ in the quarter-disk $\{x^2 + y^2 < a^2, x > 0, y > 0\}$ with the following BCs:

$$u = 0 \quad \text{on } x = 0 \text{ and on } y = 0 \quad \text{and} \quad \frac{\partial u}{\partial r} = 1 \quad \text{on } r = a.$$

Write the answer as an infinite series and write the first two nonzero terms explicitly.

Answer: By the example 1 in the textbook Section 6.4 (Please do it again) and letting $\beta = \pi/2, h(\theta) = 1$, we have

$$u(r, \theta) = \sum_{n=1}^{\infty} A_n r^{2n} \sin 2n\theta,$$

where

$$A_n = a^{1-2n} \frac{2}{n\pi} \int_0^{\pi/2} \sin(2n\theta) d\theta = a^{1-2n} \frac{1}{n^2\pi} \big[1 - (-1)^n\big].$$

The first two nonzero terms are

$$\frac{2}{a\pi} r^2 \sin 2\theta, \ \frac{2}{9a^5\pi} r^6 \sin 6\theta. \ \square$$

11. Prove the uniqueness of the Robin problem

$$\Delta u = 0 \quad \text{in } D \quad \frac{\partial u}{\partial n} + au = 0 \quad \text{on bdy } D,$$

where D is any domain in three dimensions and where a is a positive constant.

Answer: Multiplying u in the both sides of equation and using the divergence theorem,

$$\int_{\partial D} u \frac{\partial u}{\partial n} - \int_D |\nabla u|^2 = 0.$$

Using Robin boundary condition,

$$-a \int_{\partial D} u^2 - \int_D |\nabla u|^2 = 0,$$

which implies $\nabla u = 0$ in D and $u = 0$ on ∂D and then $u \equiv 0$ in D since $a > 0$. $\square$

13. Solve $u_{xx} + u_{yy} = 0$ in the region $\{\alpha < \theta < \beta, a < r < b\}$ with the boundary conditions $u = 0$ on the two sides $\theta = \alpha$ and $\theta = \beta$, $u = g(\theta)$ on the arc $r = a$, and $u = h(\theta)$ on the arc $r = b$.

Answer: It is similar to the Example 1 in Section 6.4 in the textbook. Here we only give the result and leave the details to you.

For the eigenvalue problem of $\Theta(\theta)$, we get

$$\lambda_n = \left(\frac{n\pi}{\beta - \alpha}\right)^2$$

and

$$\Theta_n(\theta) = \sin \frac{n\pi\theta}{\beta - \alpha} - \cos \frac{n\pi\theta}{\beta - \alpha} \tan \frac{n\pi\alpha}{\beta - \alpha}, \quad n = 1, 2, \ldots.$$

For the eigenvalue problem of $R(r)$, we get

$$R_n(r) = A_n r^{\frac{n\pi\theta}{\beta-\alpha}} + B_n r^{-\frac{n\pi\theta}{\beta-\alpha}}, \quad n = 1, 2, \ldots.$$

Therefore,

$$u(r, \theta) = \sum_{n=1}^{\infty} \left(A_n r^{\frac{n\pi\theta}{\beta-\alpha}} + B_n r^{-\frac{n\pi\theta}{\beta-\alpha}}\right)\left(\sin \frac{n\pi\theta}{\beta - \alpha} - \cos \frac{n\pi\theta}{\beta - \alpha} \tan \frac{n\pi\alpha}{\beta - \alpha}\right).$$

The coefficients A_n and B_n are determined by setting $r = a$ and $r = b$. □

Problem 6. Using the method of separation of variables to solve the following problem

$$\begin{cases} u_{rr} + \frac{1}{r}u_r + \frac{1}{r^2}u_{\theta\theta} = 0, & \text{in } D = \{(r,\theta) | 1 < r < 2\}; \\ u_r(1,\theta) + u(1,\theta) = \sin^3\theta, & \text{for } r = 1; \\ u(2,\theta) = 1, & \text{for } r = 2. \end{cases}$$

Answer: Using the method of separation of variables, by the example 2 in the textbook Section 6.4 (Please do it again),

$$u(r,\theta) = \frac{1}{2}(C_0 + D_0 \log r) + \sum_{n=1}^{\infty} \Big[(C_n r^n + D_n r^{-n}) \cos n\theta + (A_n r^n + B_n r^{-n}) \sin n\theta\Big].$$

By the boundary conditions, we obtain

$$\begin{aligned} 3\sin\theta - 4\sin 3\theta &= \sin^3\theta \\ &= u_r(1,\theta) + u(1,\theta) \\ &= \frac{1}{2}(C_0 + D_0) + \sum_{n=1}^{\infty} \Big[\Big((n+1)C_n - (n-1)D_n\Big) \cos n\theta \\ &\quad + \Big((n+1)A_n - (n-1)B_n\Big) \sin n\theta\Big], \end{aligned}$$

and

$$\begin{aligned} 1 &= u(2,\theta) \\ &= \frac{1}{2}(C_0 + D_0 \log 2) + \sum_{n=1}^{\infty} \Big[(C_n 2^n + D_n 2^{-n}) \cos n\theta + (A_n 2^n + B_n 2^{-n}) \sin n\theta\Big], \end{aligned}$$

which implies

$$C_0 = \frac{2}{1 - \log 2}, D_0 = -\frac{2}{1 - \log 2}, A_1 = \frac{3}{2}, B_1 = -6, A_3 = -\frac{1}{33}, B_3 = \frac{64}{33},$$

and

$$A_n = B_n = 0 \text{ (for } n \neq 1, 3) \text{ and } C_n = D_n = 0 \text{ (for } n = 1, 2, \dots).$$

Therefore,

$$u(r,\theta) = \frac{1 - \log r}{1 - \log 2} + (\frac{3}{2}r - 6r^{-1}) \sin\theta + (-\frac{1}{33}r^3 + \frac{64}{33}r^{-3}) \sin 3\theta. \quad \square$$