MAT 4220 (2008-09) Partial differential equations
Suggested Answer to Assignment 5

Here I only give the suggested solutions of Exercise 5.3(3),(5.a),(6),(13) and Exercise 5.6(8), because you can find others in "Suggested Answer to Assignment 4". Thanks for your attention.

Exercise 5.3

3. Consider $u_{tt} = c^2 u_{xx}$ for $0 < x < l$, with the boundary conditions $u(0,t) = 0$, $u_x(l,t) = 0$ and the initial conditions $u(x,0) = x$, $u_t(x,0) = 0$. Find the solution explicitly in series form.

Answer: Let $u(x,t) = X(x)T(t)$, then

$$\frac{T''}{c^2 T} = \frac{X''}{X} = -\lambda.$$

By the boundary conditions $X(0) = 0, X'(l) = 0$, we get

$$\lambda_n = \frac{(n+\frac{1}{2})^2\pi^2}{l^2}, X_n(x) = \sin\frac{(n+\frac{1}{2})\pi x}{l}, n = 0, 1, 2, \ldots.$$

Hence

$$T_n(t) = A_n \cos\frac{(n+\frac{1}{2})\pi ct}{l} + B_n \sin\frac{(n+\frac{1}{2})\pi ct}{l}, n = 0, 1, 2, \ldots,$$

and $u(x,t) = \sum_{n=0}^{\infty}\left[A_n \cos\frac{(n+\frac{1}{2})\pi ct}{l} + B_n \sin\frac{(n+\frac{1}{2})\pi ct}{l}\right]\sin\frac{(n+\frac{1}{2})\pi x}{l}$.

Therefore by the initial conditions, we have

$$u(x,t) = \sum_{n=0}^{\infty} A_n \cos\frac{(n+\frac{1}{2})\pi ct}{l}\sin\frac{(n+\frac{1}{2})\pi x}{l},$$

where

$$A_n = \frac{2}{l}\int_0^l x \sin\frac{(n+\frac{1}{2})\pi x}{l}dx = (-1)^n \frac{2l}{(n+\frac{1}{2})^2\pi^2}, n = 0, 1, 2, \ldots. \quad \square$$

5(a). Show that the boundary conditions $u(0,t) = 0$, $u_x(l,t) = 0$ lead to the eigenfunctions $\big(\sin(\pi x/2l), \sin(3\pi x/2l), \sin(5\pi x/2l), \dots\big)$.

Answer: Let $u(x,t) = X(x)T(t)$, then

$$-X''(x) = \lambda X(x),$$

$$X(0) = 0, X'(l) = 0.$$

By Theorem 3 in the textbook, there is no negative eigenvalue. It is easy to check that 0 is not an eigenvalue. Hence there are only positive eigenvalues.

Let $\lambda = \beta^2, \beta > 0$, then

$$X(x) = A\cos\beta x + B\sin\beta x$$

Hence by the boundary conditions,

$$A = 0, \quad B\beta\cos\beta l = 0.$$

Thus

$$\beta_n = \frac{(n+\frac{1}{2})\pi}{l}, n = 0, 1, 2, \dots,$$

and

$$X_n(x) = \sin\frac{(n+\frac{1}{2})\pi x}{l}, n = 0, 1, 2, \dots. \quad \square$$

6. Find the complex eigenvalues of the first-derivative operator d/dx subject to the single boundary condition $X(0) = X(1)$. Are the eigenfunctions orthogonal on the interval $(0,1)$?

Answer: Let $X'(x) = \lambda X(x), \lambda \in \mathbb{C}$, then $X(x) = e^{\lambda x}$. By the boundary condition $X(0) = X(1)$, $e^{\lambda} = 1$. Hence

$$\lambda_n = 2n\pi i, \quad X_n(x) = e^{2n\pi x i}, n \in \mathbb{Z}.$$

Since if $m \neq n$,

$$\int_0^1 X_n(x)\overline{X_m(x)}dx = \int_0^1 e^{2(n-m)\pi x i}dx = 0.$$

Therefore the eigenfunctions are orthogonal on the interval $(0,1)$. $\square$

13. Use Green's first identity to prove Theorem 3. (Hint: Substitute $f(x) = X(x) = g(x)$, a real eigenfunction.)

Answer: Substitute $f(x) = X(x) = g(x)$ in the Green's first identity,

$$\int_a^b X''(x)X(x)\,dx = -\int_a^b (X')^2(x)\,dx + (X'X)|_a^b \leq 0.$$

Since $-X'' = \lambda X$,

$$-\lambda \int_a^b X^2(x)dx \leq 0.$$

Therefore $\lambda \geq 0$ since $X \not\equiv 0$. □

More 1. (a) Consider the following eigenvalue problem

$$\begin{cases} X'' + \lambda X = 0, 0 < x < 1; \\ X(1) = X(0), X'(1) = 5X(0) + X'(0). \end{cases}$$

Show that all eigenvalues are **real**.

(b) Are there **negative eigenvalues**? If yes, find the algebraic equation for negative eigenvalues.

Answer: (a) It is easy to check that the boundary conditions is symmetric (or hermitian), i.e.,

$$(-X'\overline{X} + X\overline{X}')|_0^1 = 0,$$

then by the Theorem 2 in the Section 5.3 (textbook), all eigenvalues are **real**.

(b) Let $\lambda = -\beta^2$, $\beta > 0$, then

$$X(x) = A\sinh\beta x + \cosh\beta x.$$

Hence by the boundary conditions,

$$A\sinh\beta + \cosh\beta = 1, \text{and} \quad A\beta\cosh\beta + \beta\sinh\beta = 5 + A\beta,$$

which implies

$$5\sinh\beta = 2\beta(\cosh\beta - 1), \text{and} \quad A = \frac{1-\cosh\beta}{\sinh\beta}.$$

Note that

$$5\sinh\beta|_{\beta=0} = 2\beta(\cosh\beta - 1)_{\beta=0} = 0,$$

and

$$(5\sinh\beta)'|_{\beta=0} = 5, [2\beta(\cosh\beta - 1)]'|_{\beta=0} = 0,$$

thus $5\sinh\beta > 2\beta(\cosh\beta - 1)$ for β small. On the other hand, if $\beta > 5$, then $2\beta(\cosh\beta - 1) > \beta\cosh\beta > 5\sinh\beta$. Hence the equation $5\sinh\beta = 2\beta(\cosh\beta - 1)$ has solutions.

Note also that

$$[2\beta(\cosh\beta - 1) - 5\sinh\beta]'' = 2\beta\cosh\beta - \sinh\beta > 0, \quad \text{for all } \beta > 0,$$

thus the equation $5\sinh\beta = 2\beta(\cosh\beta - 1)$ has a unique solution and then there is only one negative eigenvalue $\lambda = -\beta^2$, where $\beta > 0$ satisfies $5\sinh\beta = 2\beta(\cosh\beta - 1)$. □

More 2. (a) Solve

$$\begin{cases} u_t - u_{xx} = 0,\ 0 < x < 1, t > 0; \\ u(x,0) = \phi(x),\ 0 < x < 1; \\ u_x(0,t) + u(0,t) = 0, u_x(1,t) + 2u(1,t) = 0. \end{cases}$$

by separation of variables.

(b) Under what conditions on $\phi(x)$, the solution remains bounded as $t \to +\infty$?

Answer: (a) Let $u(x,t) = T(t)X(x)$, we get

$$\frac{T'}{T} = \frac{X''}{X} = -\lambda.$$

By the boundary condition, we have the following eigenvalue problem

$$-X'' = \lambda X,\ X'(0) + X(0) = 0, X'(1) + 2X(1) = 0.$$

It is easy to check that $\lambda = 0$ is not an eigenvalue. For negative eigenvalue, let $\lambda = -\beta^2, \beta > 0$, then $X(x) = A\sinh\beta x + B\cosh\beta x$, and thus

$$A\beta + B = 0, \quad A\beta\cosh\beta + B\beta\sinh\beta + 2(A\sinh\beta + B\cosh\beta) = 0,$$

which implies

$$\tanh\beta = \frac{\beta}{2 - \beta^2}.$$

It is easy to prove that the above equation has a unique positive solution, which we denote by β_0. For positive eigenvalues, let $\lambda = \beta^2, \beta > 0$, then $X(x) = A\sin\beta x + B\cos\beta x$, and thus

$$A\beta + B = 0, \quad A\beta\cos\beta - B\beta\sin\beta + 2(A\sin\beta + B\cos\beta) = 0,$$

which implies

$$\tan\beta = \frac{\beta}{2+\beta^2}.$$

It is easy to show that the above equation has a countable infinite number of positive solutions, which we denote by β_n, $n = 1, 2, \ldots$. Hence

$$u(x,t) = A_0(\sinh\beta_0 x - \beta_0\cosh\beta_0 x)e^{\beta_0^2 t} + \sum_{n=1}^{\infty} A_n(\sin\beta_n x - \beta_n\cos\beta_n x)e^{-\beta_n^2 t},$$

where the constants A_0, A_n are given by

$$\phi(x) = A_0(\sinh\beta_0 x - \beta_0\cosh\beta_0 x) + \sum_{n=1}^{\infty} A_n(\sin\beta_n x - \beta_n\cos\beta_n x).$$

(b) By the above result, the solution remains bounded as $t \to +\infty$ if and only if $A_0 = 0$. □

Exercise 5.6

8. Solve $u_t = ku_{xx}$ in $(0, l)$, with $u(0,t) = 0$, $u(l,t) = At$, $u(x,0) = 0$, where A is a constant.

Answer: (Expansion Method)

Let

$$u(x,t) = \sum_{n=1}^{\infty} u_n(t)\sin\frac{n\pi x}{l},$$

$$\frac{\partial u}{\partial t}(x,t) = \sum_{n=1}^{\infty} v_n(t)\sin\frac{n\pi x}{l},$$

$$\text{and}\quad \frac{\partial^2 u}{\partial x^2}(x,t) = \sum_{n=1}^{\infty} w_n(t)\sin\frac{n\pi x}{l}.$$

Then

$$v_n(t) = \frac{2}{l}\int_0^l \frac{\partial u}{\partial t}\sin\frac{n\pi x}{l}dx = \frac{du_n}{dt},$$

and

$$\begin{aligned} w_n(t) &= \frac{2}{l}\int_0^l \frac{\partial^2 u}{\partial x^2}\sin\frac{n\pi x}{l}dx \\ &= -\frac{2}{l}\int_0^l (\frac{n\pi}{l})^2 u(x,t)\sin\frac{n\pi x}{l}dx + \frac{2}{l}(u_x\sin\frac{n\pi x}{l} - \frac{n\pi}{l}u\cos\frac{n\pi x}{l})|_0^l \\ &= -\lambda_n u_n(t) - 2n\pi l^{-2}(-1)^n At, \end{aligned}$$

where $\lambda_n = (n\pi/l)^2$. Here we used the Green's second identity and the boundary conditions.

Hence by the PDE $u_t = ku_{xx}$ and the initial condition $u(x,0) = 0$,

$$u_n(0) = 0, \text{and} \quad \frac{du_n}{dt} = k\Big[-\lambda_n u_n(t) - 2n\pi l^{-2}(-1)^n At\Big].$$

Hence

$$u_n(t) = (-1)^{n+1} 2n\pi l^{-2} A\Big[\frac{t}{\lambda_n} - \frac{1}{\lambda_n^2 k} + \frac{e^{-\lambda_n kt}}{\lambda_n^2 k}\Big].$$

Therefore

$$u(x,t) = \sum_{n=1}^{\infty} (-1)^{n+1} 2n\pi l^{-2} A\Big[\frac{t}{\lambda_n} - \frac{1}{\lambda_n^2 k} + \frac{e^{-\lambda_n kt}}{\lambda_n^2 k}\Big] \sin\frac{n\pi x}{l},$$

where $\lambda_n = (n\pi/l)^2$. $\square$