MAT 4220 (2008-09) Partial differential equations
Suggested Answer to Assignment 4

Exercise 4.1

3. A quantum-mechanical particle on the line with an infinite potential outside the interval $(0, l)$ ("particle in a box") is given by Schrodinger's equation $u_t = iu_{xx}$ on $(0, l)$ with Dirichlet conditions at the ends. Separate variables and use (8) to find its representation as a series.

Answer: Let $u(x, t) = T(t)X(x)$, we get

$$\frac{T'}{iT} = \frac{X''}{X} = -\lambda.$$

Therefore using (8),

$$u(x,t) = \sum_{n=1}^{\infty} A_n e^{-i(\frac{n\pi}{l})^2 t} \sin\frac{n\pi x}{l}. \quad \square$$

4. Consider waves in a resistant medium that satisfy the problem

$$\begin{aligned} u_{tt} = c^2 u_{xx} - r u_t \quad &\text{for } 0 < x < l \\ u = 0 \quad &\text{at both ends} \\ u(x,0) = \phi(x) \quad u_t(x,0) &= \psi(x), \end{aligned}$$

where r is a constant, $0 < r < 2\pi c/l$. Write down the series expansion of the solution.

Answer: Let $u(x, t) = T(t)X(x)$, we get

$$\frac{T'' + rT'}{c^2 T} = \frac{X''}{X} = -\lambda.$$

Hence

$$\lambda_n = (\frac{n\pi}{l})^2, X(x) = \sin\frac{n\pi x}{l}, n = 1, 2, ...$$

Since $0 < r < 2\pi c/l$, we get

$$T_n(t) = A_n \cos(\sqrt{-\Delta_n} t/2) + B_n \sin(\sqrt{-\Delta_n} t/2), n = 1, 2, ...,$$

where $\Delta_n = r^2 - (\frac{2n\pi c}{l})^2$ relative to the equation

$$\lambda^2 + r\lambda + (\frac{n\pi c}{l})^2 = 0.$$

Therefore

$$u(x,t) = \sum_{n=1}^{\infty}[A_n \cos(\sqrt{-\Delta_n}t/2) + B_n \sin(\sqrt{-\Delta_n}t/2)] \sin \frac{n\pi x}{l}. \quad \Box$$

5. Do the same for $2\pi c/l < r < 4\pi c/l$.
Answer: Let $u(x,t) = T(t)X(x)$. As above, we get

$$\frac{T'' + rT'}{c^2T} = \frac{X''}{X} = -\lambda.$$

Hence

$$\lambda_n = (\frac{n\pi}{l})^2, X(x) = \sin\frac{n\pi x}{l}, n = 1, 2, ...$$

For $n = 1$, since $2\pi c/l < r < 4\pi c/l$,

$$T_1(t) = A_1 e^{\lambda_1^+ t} + B_1 e^{\lambda_1^- t},$$

where

$$\lambda_1^{\pm} = \frac{-r \pm \sqrt{r^2 - (\frac{2\pi c}{l})^2}}{2}$$

be the roots of the equation

$$\lambda^2 + r\lambda + (\frac{\pi c}{l})^2 = 0.$$

For $n \geq 2$, as before,

$$T_n(t) = A_n \cos(\sqrt{-\Delta_n}t/2) + B_n \sin(\sqrt{-\Delta_n}t/2), n = 2, 3, ...,$$

where $\Delta_n = r^2 - (\frac{2n\pi c}{l})^2$ relative to the equation

$$\lambda^2 + r\lambda + (\frac{n\pi c}{l})^2 = 0.$$

Therefore

$$u(x,t) = [A_1 e^{\lambda_1^+ t} + B_1 e^{\lambda_1^- t}] \sin\frac{\pi x}{l} + \sum_{n=2}^{\infty}[A_n \cos(\sqrt{-\Delta_n}t/2) + B_n \sin(\sqrt{-\Delta_n}t/2)] \sin \frac{n\pi x}{l}. \quad \Box$$

6. Separate variables for the equation $tu_t = u_{xx} + 2u$ with the boundary conditions $u(0,t) = u(\pi,t) = 0$. Show that there are an infinite number of solutions that satisfy the initial condition $u(x,0) = 0$. So uniqueness is false for this equation!

Answer: Let $u(x,t) = T(t)X(x)$, we get

$$\frac{tT' - 2T}{T} = \frac{X''}{X} = -\lambda.$$

As before we have

$$\lambda_n = n^2, X_n(x) = \sin nx,\ n = 1,2,3,....$$

By the initial condition, we have

$$tT' - 2T = -\lambda T,\ T(0) = 0.$$

Therefore

$$u(x,t) = ct \sin x, \quad \text{for any constant } c,$$

are solutions (You can check them directly). So uniqueness is false for this equation! ☐

Exercise 4.2

1. Solve the diffusion problem $u_t = ku_{xx}$ in $0 < x < l$, with the mixed boundary conditions $u(0,t) = u_x(l,t) = 0$.

Answer: Let $u(x,t) = T(t)X(x)$, we get

$$\frac{T'}{kT} = \frac{X''}{X} = -\lambda.$$

By the boundary condition, we have the following eigenvalue problem

$$-X'' = \lambda X,\ X(0) = X'(l) = 0.$$

Since the eigenvalues be $[(n+\frac{1}{2})\pi]^2/l^2$ and the eigenfunctions be $\sin[(n+\frac{1}{2})\pi x/l]$ for $n = 0,1,2,...$, then we have

$$u(x,t) = \sum_{n=0}^{\infty} A_n e^{-[\frac{(n+\frac{1}{2})\pi}{l}]^2 kt} \sin \frac{(n+\frac{1}{2})\pi x}{l}. \quad \square$$

2. Consider the equation $u_{tt} = c^2 u_{xx}$ for $0 < x < l$, with the boundary conditions $u_x(0,t) = 0, u(l,t) = 0$ (Neumann at the left, Dirichlet at the right).

(a) Show that the eigenfunctions are $\cos[(n+\frac{1}{2})\pi x/l]$.

(b) Write the series expansion for a solution $u(x,t)$.

Answer: (a) This can be proved as before (Please see it in the textbook). Here we give another proof.

Since $X'(0)=0$, by the even expansion $X(-x)=X(x)$ for $-l \le x < 0$, then X satisfies

$$-X'' = \lambda X, X(-l) = X(l) = 0.$$

Hence

$$\lambda_n = [(n+\frac{1}{2})\pi]^2/l^2, X_n(x) = \cos[(n+\frac{1}{2})\pi x/l],\ n = 0,1,2,....$$

(b) As before, using (a) we have

$$u(x,t) = \sum_{n=0}^{\infty}\Big[A_n \cos\frac{(n+\frac{1}{2})\pi ct}{l} + B_n \sin\frac{(n+\frac{1}{2})\pi ct}{l}\Big]\cos\frac{(n+\frac{1}{2})\pi x}{l}. \ \square$$

3. Consider diffusion inside an enclosed circular tube. Let its length (circumference) be $2l$. Let x denote the arc length parameter where $-l \le x \le l$. Then the concentration of the diffusing substance satisfies

$$u_t = k u_{xx} \quad \text{for } -l \le x \le l$$
$$u(-l,t) = u(l,t) \quad \text{and} \quad u_x(-l,t) = u_x(l,t).$$

These are called periodic boundary conditions.

(a) Show that the eigenvalues are $\lambda = (n\pi/l)^2$ for $n = 0,1,2,3,\ldots$.

(b) Show that the concentration is

$$u(x,t) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty}\Big(a_n \cos\frac{n\pi x}{l} + b_n \sin\frac{n\pi x}{l}\Big)e^{-n^2\pi^2 kt/l^2}.$$

Answer: (a) As before, let λ be any complex number and γ be either one of the two square roots of $-\lambda$; the other one is $-\gamma$. Then

$$X(x) = Ce^{\gamma x} + De^{-\gamma x}.$$

The boundary conditions yield

$$Ce^{-\gamma l} + De^{\gamma l} = Ce^{\gamma l} + De^{-\gamma l}, \text{and } C\gamma e^{-\gamma l} - D\gamma e^{\gamma l} = C\gamma e^{\gamma l} - D\gamma e^{-\gamma l}.$$

Hence $e^{2\gamma l} = 1$ and then

$$\gamma = \pm n\pi i/l \quad \text{and} \quad \lambda = -\gamma^2 = (n\pi/l)^2, n = 0, 1, 2, \ldots.$$

(b) As before, by (a) we get

$$u(x,t) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty}\left[a_n \cos\frac{n\pi x}{l} + b_n \sin\frac{n\pi x}{l}\right]e^{-(\frac{n\pi}{l})^2 kt}.$$

Exercise 4.3

1. Find the eigenvalues graphically for the boundary conditions

$$X(0) = 0, \quad X'(l) + aX(l) = 0.$$

Assume that $a \neq 0$.

Answer: Firstly, let's look for the positive eigenvalues $\lambda = \beta^2 > 0$. As usual, the general solution of the ODE is

$$X(x) = C \cos\beta x + D \sin\beta x.$$

By the boundary conditions,

$$C = 0, \text{and } D\beta\cos\beta l + aD\sin\beta l = 0.$$

Hence $\tan\beta l = -\frac{\beta}{a}$. The graph is omitted.

Secondly, let's look for the zero eigenvalue, i.e., $X(x) = Ax + B$, by the boundary conditions, $al + 1 = 0$. Hence $\lambda = 0$ is an eigenvalue if and only if $al + 1 = 0$.

Thirdly, let's look for the negative eigenvalues $\lambda = -\gamma^2 < 0$. As usual, the solution of the ODE is

$$X(x) = C\cosh\gamma x + D\sinh\gamma x.$$

By the boundary conditions,

$$C = 0, \text{and } D\gamma\cosh\gamma l + aD\sinh\gamma l = 0.$$

Hence $\tanh\gamma l = -\frac{\gamma}{a}$ and the graph is also omitted. □

2. Consider the eigenvalue problem with Robin BCs at both ends:

$$-X'' = \lambda X$$
$$X'(0) - a_0X(0) = 0, \quad X'(l) + a_lX(l) = 0.$$

(a) Show that $\lambda = 0$ is an eigenvalue if and only if $a_0 + a_l = -a_0 a_l l$.

(b) Find the eigenfunctions corresponding to the zero eigenvalue. (Hint: First solve the ODE for $X(x)$. The solutions are not sines or cosines.)

Answer: (a) If $\lambda = 0$, then $X(x) = Ax + B$. By the boundary conditions, we get

$$A - a_0 B = 0, \text{and} \quad A + a_l(Al + B) = 0,$$

which is equivalent to

$$a_0 + a_l = -a_0 a_l l.$$

Therefore $\lambda = 0$ is an eigenvalue if and only if $a_0 + a_l = -a_0 a_l l$.

(b) By (a), we have $X(x) = B(a_0 x + 1)$, here B is constant. □

3. Derive the eigenvalue equation (16) for the negative eigenvalues $\lambda = -\gamma^2$ and the formula (17) for the eigenfunctions.

Answer: If $\lambda = -\gamma^2 < 0$, we have

$$X(x) = C \cosh \gamma x + D \sinh \gamma x.$$

Thus

$$X'(x) = C\gamma \sinh \gamma x + D\gamma \cosh \gamma x.$$

Hence by the boundary conditions,

$$D\gamma - a_0 C = 0, \text{and } C\gamma \sinh \gamma l + D\gamma \cosh \gamma l + a_l[C \cosh \gamma l + D \sinh \gamma l] = 0,$$

which implies

$$\tanh \gamma l = -\frac{(a_0 + a_l)\gamma}{\gamma^2 + a_0 a_l}.$$

And the corresponding eigenfunction

$$X(x) = C \cosh \gamma x + \frac{a_0}{\gamma} C \sinh \gamma x,$$

where C is constant. □

4. Consider the Robin eigenvalue problem. If

$$a_0 < 0, a_l < 0 \quad \text{and } -a_0 - a_l < a_0 a_l l,$$

show that there are two negative eigenvalues. This case may be called "substantial absorption at both ends." (Hint: Show that the rational curve $y = (a_0 + a_l)\gamma/(\gamma^2 + a_0 a_l)$ has a single maximum and crosses the line $y = 1$ in two places. Deduce that it crosses the tanh curve in two places.)

Answer: It is easily known that the rational curve $y = -\frac{(a_0+a_l)\gamma}{\gamma^2+a_0a_l}$ has a single maximum at $\gamma = \sqrt{a_0a_l}$ and is monotonic in the two intervals $[0, \sqrt{a_0a_l})$ and $[\sqrt{a_0a_l}, \infty)$. Furthermore,

$$\max_{\gamma\in[0,\infty)} y(\gamma) = -\frac{(a_0+a_l)}{2\sqrt{a_0a_l}} > 1, \lim_{\gamma\to\infty} y = 0, \text{and } y'(0) = -\frac{a_0+a_l}{a_0a_l}.$$

Note that $\tanh\gamma l$ is monotonic in $[0,\infty)$,

$$\tanh(\gamma l) < 1 \text{ in } [0,\infty), \lim_{\gamma\to\infty} \tanh\gamma l = 1, \text{ and}(\tanh\gamma l)'|_{\gamma=0} = l > -\frac{a_0+a_l}{a_0a_l}.$$

Therefore the rational curve $y = -\frac{(a_0+a_l)\gamma}{\gamma^2+a_0a_l}$ have two common points with the curve $y = \tanh\gamma l$ and then there are two negative eigenvalues. $\square$

5. In Exercise 4 (substantial absorption at both ends) show graphically that there are an infinite number of positive eigenvalues. Show graphically that they satisfy (11) and (12).

Answer: Since

$$\frac{(a_0+a_l)\beta}{\beta^2 - a_0a_l} = -\frac{[(-a_0)+(-a_l)]\beta}{\beta^2-(-a_0)(-a_l)}$$

and $-a_0 > 0, -a_l > 0, -a_0 - a_l < a_0a_l l$, the graph is similar to the Figure 1 in Section 4.3 in the textbook. $\square$

6. If $a_0 = a_l = a$ in the Robin problem, show that:

(a) There are no negative eigenvalues if $a \geq 0$, there is one if $-2/l \leq a < 0$, and there are two if $a < -2/l$.

(b) Zero is an eigenvalue if and only if $a = 0$ or $a = -2/l$.

Answer: (a) If $a > 0$, this is case 1 and there is only positive eigenvalue. And if $a = 0$, this is the Neumann condition. Hence there are only zero and positive eigenvalues if $a \geq 0$.

By the result of exercise (4), we known that there is one negative eigenvalue if $-2/l \leq a < 0$, and there are two if $a < -2/l$.

(b) By the result of exercise (2), we known that $\lambda = 0$ is an eigenvalue if and only if $a_0 + a_l = -a_0a_l l$, which is equivalent to $a = 0$ or $a = -2/l$. $\square$

7. If $a_0 = a_l = a$, show that as $a \to +\infty$, the eigenvalues tend to the eigenvalues of the Dirichlet problem. That is,

$$\lim_{a\to\infty}\{\lambda_n(a) - [\frac{(n+1)\pi}{l}]^2\} = 0,$$

where $\lambda_n(a)$ is the $(n+1)$st eigenvalue.

Answer: Under the condition of $a_0 = a_l = a$, the eigenvalue equation

$$\tan \beta l = \frac{(a_0 + a_l)\beta}{\beta^2 - a_0 a_l},$$

is changed into the equation

$$\tan \beta l = \frac{2a\beta}{\beta^2 - a^2}.$$

Thus

$$\beta l \longrightarrow n\pi \quad \text{as } a \longrightarrow +\infty.$$

That is,

$$\lim_{a\to\infty} \left\{ \lambda_n(a) - \left[\frac{(n+1)\pi}{l}\right]^2 \right\} = 0. \ \square$$

11. (a) Prove that the (total) energy is conserved for the wave equation with Dirichlet BCs, where the energy is defined to be

$$E = \frac{1}{2}\int_0^l (c^{-2}u_t^2 + u_x^2)\, dx.$$

(Compare this definition with Section 2.2.)

(b) Do the same for the Neumann BCs.

(c) For the Robin BCs, show that

$$E_R = \frac{1}{2}\int_0^l (c^{-2}u_t^2 + u_x^2)\, dx + \frac{1}{2}a_l[u(l,t)]^2 + \frac{1}{2}a_0[u(0,t)]^2$$

is conserved. Thus, while the total energy E_R is still a constant, some of the internal energy is "lost" to the boundary is a_0 and a_l are positive and "gained" from the boundary if a_0 and a_l are negative.

Answer: (a) By the wave equation,

$$\begin{aligned}
\frac{dE}{dt} &= \int_0^l [\frac{1}{c^2}u_t u_{tt} + u_x u_{xt}]\, dx \\
&= \int_0^l [u_t u_{xx} + u_x u_{xt}]\, dx \\
&= (u_t u_x)|_0^l = u_t(l,t)u_x(l,t) - u_t(0,t)u_x(0,t).
\end{aligned}$$

By the Dirichlet boundary conditions $u(l,t) = u(0,t) = 0$, $u_t(l,t) = u_t(0,t) = 0$. Thus $\frac{dE}{dt} \equiv 0$.

(b) The proof is the same as above. Here we omit it.

(c) By the computation in (a) and the Robin boundary conditions, we can get that

$$\frac{dE_R}{dt} = u_t u_x|_0^l + a_l u_t(l,t)u(l,t) + a_0 u_t(0,t)u_x(0,t) \equiv 0. \quad \Box$$

12. Consider the unusual eigenvalue problem

$$\begin{aligned} -v_{xx} &= \lambda v \quad \text{for } 0 < x < l \\ v_x(0) = v_x(l) &= \frac{v(l) - v(0)}{l}. \end{aligned}$$

(a) Show that $\lambda = 0$ is a double eigenvalue.

(b) Get an equation for the positive eigenvalues $\lambda > 0$.

(c) Letting $\gamma = \frac{1}{2}l\sqrt{\lambda}$, reduce the equation in part (b) to the equation

$$\gamma \sin\gamma \cos\gamma = \sin^2\gamma.$$

(d) Use part (c) to find half of the eigenvalues explicitly and half of them graphically.

(e) Assuming that all the eigenvalues are nonnegative, make a list of all the eigenfunctions.

(f) Solve the problem $u_t = ku_{xx}$ for $0 < x < l$, with the BCs given above, and with $u(x,0) = \phi(x)$.

(g) Show that, as $t \to \infty$, $\lim u(x,t) = A + Bx$, assuming that you can take limits term by term.

Answer: (a) Let $\lambda = 0$, we get $-v_{xx} = 0$, which implies $v(x) = Ax + B$. Since $v(x) = Ax + B$ satisfy the boundary conditions for any numbers A and B, so $\lambda = 0$ is a double eigenvalue.

(b) Let $\lambda = \beta^2 > 0$ and suppose $\beta > 0$, $-v_{xx} = \lambda v$, then $v(x) = C\cos\beta x + D\sin\beta x$. Hence by the boundary conditions, we have

$$D\beta = -C\beta\sin\beta l + D\beta\cos\beta l = \frac{C\cos\beta l + D\sin\beta l - C}{l}.$$

Hence $\lambda = \beta^2$, where β is a root of the following equation

$$\sin(\beta l)(-\sin\beta l + \beta l) = (1 - \cos\beta l)^2.$$

(c) Let $\gamma = \frac{1}{2}l\sqrt{\lambda}$, then γ is a root of the following equation

$$\gamma \sin\gamma \cos\gamma = \sin^2\gamma.$$

(d) By (c), we have $\sin\gamma = 0$ or $\gamma = \tan\gamma$. Then $\lambda_1 = 0$, $\lambda_n = \frac{n^2\pi^2}{l^2}$, for $n = 2, 4, \ldots$, and $\lambda_n = 4\gamma_n^2/l^2$, where $\gamma_n = \tan\gamma_n$, for $n = 3, 5, 7, \ldots$.

(e) By (a) and (d), for $\lambda = 0$, the eigenfunctions are 1 and x; for $\lambda_n = \frac{n^2\pi^2}{l^2}, n = 2, 4, 6, ...$, the eigenfunctions are $\cos(n\pi x/l)$; for $\lambda_n = 4\gamma_n^2/l^2$, where $\gamma_n = \tan\gamma_n, n = 3, 5, 7, \ldots$, the eigenfunctions are

$$l\sqrt{\lambda_n}\cos(\sqrt{\lambda_n}x) - 2\sin(\sqrt{\lambda_n}x).$$

(f) From above, we have

$$u(x,t) = A + Bx + \sum_{n>1 \text{ and odd}} A_n e^{-\lambda_n kt}\Big[l\sqrt{\lambda_n}\cos(\sqrt{\lambda_n}x) - 2\sin(\sqrt{\lambda_n}x)\Big]$$
$$+ \sum_{n>1 \text{ and even}} B_n e^{-\lambda_n kt}\cos(n\pi x/l).$$

(g) By(f), we have $\lim_{t\to\infty} u(x,t) = A + Bx$ since $\lim\limits_{t\to\infty} e^{-\lambda_n kt} = 0$. □

13. Consider a string which is fixed at the end $x = 0$ and is free at the end $x = l$ except that a load (weight) of given mass is attached to the right end.

(a) Show that it satisfies the problem

$$u_{tt} = c^2 u_{xx} \quad \text{for } 0 < x < l$$
$$u(0,t) = 0 \quad u_{tt}(l,t) = -ku_x(l,t)$$

for some constant k.

(b) What is the eigenvalue problem in this case?

(c) Find the equation for the positive eigenvalues and find the eigenfunctions.

Answer: (a) Please see it in the textbook.

(b) Let $u(x,t) = X(x)T(t)$, then

$$T''(t)X(x) = c^2 X''(x)T(t).$$

By the equation and the boundary conditions,

$$\frac{X''(x)}{X(x)} = \frac{T''(t)}{c^2T(t)} = -\lambda,$$

$$X(0) = 0, \text{and} \quad \lambda c^2 X(l) = kX'(l).$$

(c) Let $\lambda = \beta^2$, we obtain eigenfunction

$$X(x) = \sin \beta x,$$

where β satisfies

$$\tan \beta l = \frac{k}{\beta c^2}, \quad k = 1, 2, 3, \dots. \ \square$$

15. Find the equation for the eigenvalues λ of the problem

$$(\kappa(x)X')' + \lambda\rho(x)X = 0 \quad \text{for } 0 < x < l \text{ with } X(0) = X(l) = 0,$$

where $\kappa(x) = \kappa_1^2$ for $x < a$, $\kappa(x) = \kappa_2^2$ for $x > a$, $\rho(x) = \rho_1^2$ for $x < a$, and $\rho(x) = \rho_2^2$ for $x > a$. All these constants are positive and $0 < a < l$.

Answer: Let $\lambda = \beta^2$, then

$$X(x) = A \cos \frac{\beta\rho_1 x}{k_1} + B \sin \frac{\beta\rho_1 x}{k_1}, 0 < x < a;$$

$$X(x) = C \cos \frac{\beta\rho_2 x}{k_2} + D \sin \frac{\beta\rho_2 x}{k_2}, a < x < l.$$

Hence by the boundary conditions,

$$A = 0, \quad C \cos \frac{\beta\rho_2 l}{k_2} + D \sin \frac{\beta\rho_2 l}{k_2} = 0;$$

$$B \sin \frac{\beta\rho_1 a}{k_1} = C \cos \frac{\beta\rho_2 a}{k_2} + D \sin \frac{\beta\rho_2 a}{k_2};$$

$$\text{and } B\frac{\beta\rho_1}{k_1} \cos \frac{\beta\rho_1 a}{k_1} = -C\frac{\beta\rho_2}{k_2} \sin \frac{\beta\rho_2 a}{k_2} + D\frac{\beta\rho_2}{k_2} \cos \frac{\beta\rho_2 a}{k_2}.$$

Therefore $\lambda = \beta^2$, where β is a root of the equation

$$\frac{\rho_1}{k_1} \cot \frac{\beta\rho_1 a}{k_1} + \frac{\rho_2}{k_2} \cot \frac{\beta\rho_2 (l-a)}{k_2} = 0. \ \square$$

16. Find the positive eigenvalues and the corresponding eigenfunctions of the fourth-order operator $+d^4/dx^4$ with the four boundary conditions

$$X(0) = X(l) = X''(0) = X''(l) = 0.$$

Answer: Let $\lambda = \beta^4 > 0$, where $\beta > 0$ and $X(x) = A\cosh(\beta x) + B\sinh(\beta x) + C\cos(\beta x) + D\sin(\beta x)$.

Hence by the boundary conditions,

$$\beta = \frac{n\pi}{l}, \lambda = (\frac{n\pi}{l})^4, X_n(x) = \sin(\frac{n\pi x}{l}), n = 1, 2, \ldots. \ \Box$$

17. Solve the fourth-order eigenvalue problem $X'''' = \lambda X$ in $0 < x < l$, with the four boundary conditions

$$X(0) = X'(0) = X(l) = X'(l) = 0,$$

where $\lambda > 0$. (Hint: First solve the fourth-order ODE.)

Answer: Let $\lambda = \beta^4 > 0$ where $\beta > 0$, and $X(x) = A\cosh\beta x + B\sinh\beta x + C\cos(\beta x) + D\sin(\beta x)$. Hence by the boundary conditions,

$$A + C = 0, B + D = 0,$$

$$A\cosh\beta l + B\sinh\beta l + C\cos(\beta l) + D\sin(\beta l) = 0,$$

$$\text{and } A\sinh\beta l + B\cosh\beta l - C\sin(\beta l) + D\cos(\beta l) = 0.$$

Hence $\lambda = \beta^4$, where β is a root of the equation

$$\cosh\beta l\cos\beta l = 1,$$

and the corresponding eigenfunction is

$$X(x) = (\sinh\beta l - \sin\beta l)(\cosh\beta x - \cos\beta x) - (\cosh\beta l - \cos\beta l)(\sinh\beta x - \sin\beta x). \ \Box$$

18. A tuning fork may be regarded as a pair of vibrating flexible bars with a certain degree of stiffness. Each such bar is clamped at one end and is approximately modeled by the fourth-order PDE $u_{tt} + c^2 u_{xxxx} = 0$. It has initial conditions as for wave equation. Let's say that on the end $x = 0$ it is clamped (fixed), meaning that it satisfies $u(0,t) = u_x(0,t) = 0$. On the other end $x = l$ it is free, meaning that it satisfies $u_{xx}(l,t) = u_{xxxx}(l,t) = 0$. Thus there are a total of four boundary conditions, two at each end.

(a) Separate the time and space variables to get the eigenvalue problem $X'''' = \lambda X$.

(b) Show that zero is not an eigenvalue.

(c) Assuming that all the eigenvalues are positive, write them as $\lambda = \beta^4$ and find the equation for β.

(d) Find the frequencies of vibration.

(e) Compare your answer in part (d) with the overtones of the vibrating string by looking at the ratio β_2^2/β_1^2. Explain why you hear an almost pure tone when you listen to a tuning fork.

Answer: (a) Let $u(x,t) = X(x)T(t)$, then

$$\frac{T''}{-c^2T} = \frac{X''''}{X} = \lambda.$$

(b) Let $\lambda = 0$, then $X(x) = Ax^3 + Bx^2 + Cx + D$. By the boundary conditions, we get $A = B = C = D = 0$. Hence zero is not an eigenvalue.

(c) Similar to the proof of exercise (17), we get β is a root of the equation

$$\cosh \beta l \cos \beta l = -1.$$

(d) The frequencies are $c\beta_n^2$,

$$\beta_1 l = 1.88, \beta_2 l = 4.69, \beta_3 l = 7.85 \ldots.$$

(e) For the bar $\beta_2^2/\beta_1^2 = 6.27$, while for the string the ratio of the first two frequencies if $\beta_2/\beta_1 = 2$. Thus, relative to the fundamental frequency, the first overtone of the bar is higher than the fifth overtone of the string. After a short time the overtones are heavily damped and the fundamental tone dominates. $\square$

Exercise 5.3

7. Show by direct integration that the eigenfunctions associated with the Robin BCs, namely,

$$\phi_n(x) = \cos \beta_n x + \frac{a_0}{\beta_n} \sin \beta_n x \quad \text{where } \lambda_n = \beta_n^2,$$

are mutually orthogonal on $0 \leq x \leq l$,where β_n are the positive roots of (4.3.8).

Answer: For $m \neq n$, since

$$(\beta_j^2 - a_0 a_l) \tan \beta_j l = (a_0 + a_l)\beta_j, j = m, n,$$

$$\begin{aligned}
(\phi_m, \phi_n) &= \int_0^l \phi_m(x)\phi_n(x)dx \\
&= \int_0^l (\cos\beta_m x + \frac{a_0}{\beta_m}\sin\beta_m x)(\cos\beta_n x + \frac{a_0}{\beta_n}\sin\beta_n x)dx \\
&= \int_0^l \Big[\cos\beta_m x\cos\beta_n x + \frac{a_0}{\beta_n}\cos\beta_m x\sin\beta_n x \\
&\quad + \frac{a_0}{\beta_m}\cos\beta_n x\sin\beta_m x + \frac{a_0}{\beta_m}\frac{a_0}{\beta_n}\sin\beta_m x\sin\beta_n x\Big]dx \\
&= \frac{1}{2}\Big[\frac{1}{\beta_m+\beta_n}\sin(\beta_m+\beta_n)l + \frac{1}{\beta_m-\beta_n}\sin(\beta_m-\beta_n)l\Big] \\
&\quad + \frac{a_0}{2\beta_n}\Big[-\frac{1}{\beta_m+\beta_n}(\cos(\beta_m+\beta_n)l-1) + \frac{1}{\beta_m-\beta_n}(\cos(\beta_m-\beta_n)l-1)\Big] \\
&\quad + \frac{a_0}{2\beta_m}\Big[-\frac{1}{\beta_m+\beta_n}(\cos(\beta_m+\beta_n)l-1) - \frac{1}{\beta_m-\beta_n}(\cos(\beta_m-\beta_n)l-1)\Big] \\
&\quad + \frac{a_0^2}{2\beta_m\beta_n}\Big[-\frac{1}{\beta_m+\beta_n}\sin(\beta_m+\beta_n)l + \frac{1}{\beta_m-\beta_n}\sin(\beta_m-\beta_n)l\Big] \\
&= 0. \ \square
\end{aligned}$$

8. Show directly that $(-X_1'X_2 + X_1X_2')|_a^b = 0$ if both X_1 and X_2 satisfy the same Robin boundary condition at $x = a$ and the same Robin boundary condition at $x = b$.

Answer: If

$$X_1'(a) - a_aX_1(a) = X_2'(a) - a_aX_2(a) = 0$$

and

$$X_1'(b) + a_bX_1(b) = X_2'(b) + a_bX_2(b) = 0,$$

then

$$(-X_1'X_2+X_1X_2')|_a^b = -X_1'(b)X_2(b)+X_1(b)X_2'(b)+X_1'(a)X_2(a)-X_1(a)X_2'(a)$$

$$= a_bX_1(b)X_2(b) - X_1(b)a_bX_2(b) + a_aX_1(a)X_2(a) - X_1(a)a_aX_2(a) = 0. \ \square$$

9. Show that the boundary conditions

$$X(b) = \alpha X(a) + \beta X'(a) \quad \text{and} \quad X'(b) = \gamma X(a) + \delta X'(a)$$

on an interval $a \le x \le b$ are symmetric if and only if $\alpha\delta - \beta\gamma = 1$.

Answer: For $j = 1, 2$, suppose that

$$X_j(b) = \alpha X_j(a) + \beta X'_j(a)$$

and

$$X'_j(b) = \gamma X_j(a) + \delta X'_j(a).$$

Then

$$\begin{aligned}
(X'_1X_2 - X_1X'_2)|_a^b &= X'_1(b)X_2(b) - X_1(b)X'_2(b) - X'_1(a)X_2(a) + X_1(a)X'_2(a) \\
&= \big[\gamma X_1(a) + \delta X'_1(a)\big]\big[\alpha X_2(a) + \beta X'_2(a)\big] \\
&\quad -\big[\alpha X_1(a) + \beta X'_1(a)\big]\big[\gamma X_2(a) + \delta X'_2(a)\big] \\
&\quad -X'_1(a)X_2(a) + X_1(a)X'_2(a) \\
&= (\alpha\delta - \beta\gamma - 1)X'_1(a)X_2(a) + (1 + \beta\gamma - \alpha\delta)X_1(a)X'_2(a) \\
&= (\alpha\delta - \beta\gamma - 1)(X_1X_2)'|_{x=a}.
\end{aligned}$$

Therefore the boundary conditions are symmetric if and only if $\alpha\delta - \beta\gamma = 1$. □

10. (The Gram-Schmidt orthogonalization procedure) If $X_1, X_2, \ldots$ is any sequence (finite or infinite) of linearly independent vectors in any vector space with an inner product, it can be replaced by a sequence of linear combinations that are mutually orthogonal. The idea is that at each step one subtracts off the components parallel to the previous vectors. The procedure is as follows. First, we let $Z_1 = X_1/\|X_1\|$. Second, we define

$$Y_2 = X_2 - (X_2, Z_1)Z_1 \quad \text{and} \quad Z_2 = \frac{Y_2}{\|Y_2\|}.$$

Third, we define

$$Y_3 = X_3 - (X_3, Z_2)Z_2 - (X_3, Z_1)Z_1 \quad \text{and} \quad Z_3 = \frac{Y_3}{\|Y_3\|},$$

and so on.

(a) Show that all the vectors $Z_1, Z_2, Z_3, \ldots$ are orthogonal to each other.

(b) Apply the procedure to the pair of functions $\cos x + \cos 2x$ and $3\cos x - 4\cos 2x$ in the interval $(0, \pi)$ to get an orthogonal pair.

Answer: (a) This can be proved by Mathematics induction. Firstly, since

$$(Y_2, Z_1) = (X_2, Z_1) - (X_2, Z_1)(Z_1, Z_1) = 0,$$

we get $(Z_2, Z_1) = 0$.

Secondly, suppose that $Z_1, , Z_2, ..., Z_n$ are orthogonal to each other, then we have

$$(Y_{n+1}, Z_j) = (X_{n+1}, Z_j) - (X_{n+1}, Z_j)(Z_j, Z_j) = 0, j = 1, 2, ...n.$$

Hence Z_{n+1} is orthogonal to $Z_1, Z_2, ..., Z_n$.

Therefore $Z_1, Z_2, \dots$ are orthogonal to each other.

(b) Let $X_1 = \cos x + \cos 2x$ and $X_2 = 3\cos x - 4\cos 2x$, applying the procedure above,

$$Z_1 = \frac{X_1}{\|X_1\|} = \frac{\cos x + \cos 2x}{\sqrt{\pi}},$$

$$\text{and} \quad Y_2 = X_2 - (X_2, Z_1)Z_1 = \frac{7}{2}(\cos x - \cos 2x).$$

Hence

$$Z_2 = \frac{Y_2}{\|Y_2\|} = \frac{\cos x - \cos 2x}{\sqrt{\pi}}. \quad \square$$

11. (a) Show that the condition $f(x)f'(x)|_a^b \leq 0$ is valid for any function $f(x)$ that satisfies Dirichlet, Neumann, or periodic boundary conditions.

(b) Show that it is also valid for Robin BCs provided that the constants a_0 and a_l are positive.

Answer: (a) For the Dirichlet boundary condition,

$$f(x)f'(x)|_a^b = f(b)f'(b) - f(a)f'(a) = 0;$$

For the Neumann boundary condition,

$$f(x)f'(x)|_a^b = f(b)f'(b) - f(a)f'(a) = 0;$$

For the periodic boundary condition,

$$f(x)f'(x)|_a^b = f(b)f'(b) - f(a)f'(a) = 0.$$

(b) For the Robin boundary condition,we have

$$f(x)f'(x)|_0^l = f(l)f'(l) - f(0)f'(0) = -a_l f^2(l) - a_0 f^2(0) \leq 0,$$

provided that the constants a_0 and a_l are positive. $\square$

12. Prove Green's first identity: For every pair of functions $f(x), g(x)$ on (a, b),

$$\int_a^b f''(x)g(x)\,dx = -\int_a^b f'(x)g'(x)\,dx + f'g|_a^b.$$

Answer: By the divergence theorem,

$$f'g|_a^b = \int_a^b (f'(x)g(x))'dx = \int_a^b f''(x)g(x) + f'(x)g'(x)dx,$$

which implies

$$\int_a^b f''(x)g(x)dx = -\int_a^b f'(x)g'(x)dx + f'g|_a^b. \ \square$$

Exercise 5.4

1. $\sum_{n=0}^{\infty}(-1)^n x^{2n}$ is a geometric series.
(a) Does it converge pointwise in the interval $-1 < x < 1$?
(b) Does it converge uniformly in the interval $-1 < x < 1$?
(c) Does it converge in the L^2 sense in the interval $-1 < x < 1$?
(Hint: You can compute its partial sums explicitly.)
Answer: Firstly, the partial sum is given by

$$S_n = \frac{1-(-1)^n x^{2n}}{1+x^2}.$$

(a) Obviously for any x_0 fixed, $S_n \to \frac{1}{1+x_0^2}$. Thus it converges to $\frac{1}{1+x^2}$ pointwise.
(b) Let $x_n = 1 - \frac{1}{n}$, then $x^{2n} \to e^{-2}$. Thus it doesn't converge uniformly.
(c) It will converge to $S(x) \equiv \frac{1}{1+x^2}$ in the L^2 sense since

$$\begin{aligned}\int_{-1}^1 |S_n - S|^2\,dx &= \int_{-1}^1 \frac{x^{4n}}{(1+x^2)^2}\,dx \\ &\le \int_{-1}^1 x^{4n}\,dx \\ &\le \frac{2}{4n+1} \to 0 \quad \text{as } n \to \infty. \ \square\end{aligned}$$

3. Let γ_n be a sequence of constants tending to ∞. Let $f_n(x)$ be the sequence of functions defined as follows: $f_n(\frac{1}{2}) = 0, f_n(x) = \gamma_n$ in the interval

$[\frac{1}{2}-\frac{1}{n},\frac{1}{2})$, let $f_n(x)=-\gamma_n$ in the interval $(\frac{1}{2},\frac{1}{2}+\frac{1}{n}]$, and let $f_n(x)=0$ elsewhere. Show that:

(a) $f_n(x)\to 0$ pointwise.

(b) The convergence is not uniform.

(c) $f_n(x)\to 0$ in the L^2 sense if $\gamma_n=n^{1/3}$.

(d) $f_n(x)$ does not converge in the L^2 sense if $\gamma_n=n$.

Answer: (a) For any fixed point x_0, W.L.O.G., we assume $x_0<\frac{1}{2}$. Then there is N_0 such that for $n>N_0$,

$$x_0<\frac{1}{2}-\frac{1}{n},$$

which implies that $f_n(x_0)\equiv 0$. Thus $f_n(x)\to 0$ pointwise.

(b) Let $x_n=\frac{1}{2}-\frac{1}{n}$, then $f_n(x_n)=-\gamma_n\to-\infty$, which implies that the convergence is not uniform.

(c) First, by direct computation, we can get

$$\begin{aligned}\int f_n^2(x)dx &= \int_{\frac{1}{2}-\frac{1}{n}}^{\frac{1}{2}}\gamma_n^2dx+\int_{\frac{1}{2}}^{\frac{1}{2}+\frac{1}{n}}\gamma_n^2dx\\ &= \gamma_n^2\times\frac{2}{n}.\end{aligned}$$

$$\text{For}\gamma_n=n^{\frac{1}{3}},\quad\int f_n^2(x)dx=2\times n^{-\frac{1}{3}}\to 0\quad\text{as}\quad n\to\infty.$$

(d) By the computation in (c), for $\gamma_n=n$,

$$\int f_n^2(x)dx=2n\to\infty\quad\text{as}\quad n\to\infty.\ \square$$

4. Let

$$g_n(x)=\begin{cases}1\text{ in the interval }[\frac{1}{4}-\frac{1}{n^2},\frac{1}{4}+\frac{1}{n^2}), & \text{for odd } n;\\ 1\text{ in the interval }[\frac{3}{4}-\frac{1}{n^2},\frac{3}{4}+\frac{1}{n^2}), & \text{for odd } n;\\ y+1, & \text{for all other } x.\end{cases}$$

Show that $g_n(x)\to 0$ in the L^2 sense but that $g_n(x)$ does not tend to zero in the pointwise sense.

Answer: For odd n,

$$\int_{\frac{1}{4}-\frac{1}{n^2}}^{\frac{1}{4}+\frac{1}{n^2}} 1^2 dx = \frac{2}{n^2} \to 0.$$

For even n,

$$\int_{\frac{3}{4}-\frac{1}{n^2}}^{\frac{3}{4}+\frac{1}{n^2}} 1^2 dx = \frac{2}{n^2} \to 0.$$

Thus for any n,

$$\|g_n(x)\|_{L^2}^2 = \frac{2}{n^2} \to 0 \quad \text{as} \quad n \to \infty. \ \square$$

7. Let

$$\phi(x) = \begin{cases} -1-x, & -1 < x < 0; \\ +1-x, & 0 < x < 1. \end{cases}$$

(a) Find the full Fourier series of $\phi(x)$ in the interval $(-1, 1)$.
(b) Find the first three nonzero terms explicitly.
(c) Does it converge in the mean square sense?
(d) Does it converge pointwise?
(e) Does it converge uniformly?

Answer: (a) Obviously $\phi(x)$ is odd. Thus its full Fourier series is just the Sine Fourier series, i.e.,

$$\sum_{n=1}^{\infty} B_n \sin n\pi x,$$

where B_n satisfies

$$B_n = \int_{-1}^{1} \phi(x) \sin n\pi x dx = \frac{2}{n\pi}.$$

(b) By (a), the first three nonzero terms are

$$\frac{2}{\pi} \sin \pi x, \frac{1}{\pi} \sin 2\pi x, \frac{2}{3\pi} \sin 3\pi x.$$

(c) Since

$$\int_{-1}^{1} |\phi(x)|^2 dx = 2 \int_{0}^{1} (1-x)^2 dx \le 2,$$

it converges in the mean square sense according to Corollary 7.

(d) According to Theorem 4, it converges pointwise.

(e) No. Since

$$\sum_{n=1}^{\infty} B_n \sin(n\pi \frac{1}{2n}) = \sum_{n=1}^{\infty} \frac{2}{n\pi} = \infty. \ \Box$$

9. Let $f(x)$ be a function on $(-l, l)$ that has a continuous derivative and satisfies the periodic BCs. Let a_n and b_n be the Fourier coefficients of $f(x)$, and let a'_n and b'_n be the Fourier coefficients of its derivative $f'(x)$. Show that

$$a'_n = \frac{n\pi b_n}{l} \quad \text{and} \quad b'_n = \frac{-n\pi a_n}{l} \quad \text{for } n \neq 0.$$

(Hint: Write the formulas for a'_n and b'_n and integrate by parts.) This means that the Fourier series of $f'(x)$ is what you'd obtain as if you differentiated term by term. It does not mean that the differentiated series converges.

Answer: By the definition,

$$\begin{aligned} a'_n &= \frac{1}{l}\int_{-l}^{l} f'(x)\cos\frac{n\pi x}{l}dx \\ &= \frac{1}{l}\cos\frac{n\pi x}{l} f(x)|_{-l}^{l} + \frac{n\pi}{l^2}\int_{-l}^{l} f(x)\sin\frac{n\pi x}{l}dx \\ &= \frac{n\pi b_n}{l}. \end{aligned}$$

Similarly, we can get $b'_n = -\frac{n\pi a_n}{l}$. $\Box$

10. Deduce from Exercise 9 that there is a constant k so that

$$|a_n| + |b_n| \leq \frac{k}{n} \quad \text{for all } n.$$

Answer: Since $f'(x)$ is bounded, we can obtain that $|a'_n| \leq 2\max|f'|$. Thus using the result of exercise 9,

$$|b_n| \leq \frac{2l\max|f'|}{n\pi}.$$

Let $k = 4l\max|f'|/\pi$, then $|b_n| \leq \frac{k}{2n}$ for all n. Similarly we can get that $|a_n| \leq \frac{k}{2n}$ for all n and then complete the proof. $\Box$

Exercise 5.6

1. (a) Solve as a series the equation $u_t = u_{xx}$ in $(0,1)$ with $u_x(0,t) = 0$, $u(1,t) = 1$, and $u(x,0) = x^2$. Compute the first two coefficients explicitly.

(b) What is the equilibrium state (the term that does not tend to zero)?

Answer: (Using the method of shifting the data)

Let $v(x,t) := u(x,t) - 1$, then v solves

$$v_t = v_{xx}, v_x(0,t) = v(1,t) = 0, \text{and } v(x,0) = x^2 - 1.$$

By the method of separation of variables, we have

$$v(x,t) = \sum_{n=0}^{\infty} A_n e^{-(n+\frac{1}{2})^2\pi^2 t} \cos\left[(n+\frac{1}{2})\pi x\right], A_n = (-1)^{n+1}4(n+\frac{1}{2})^{-3}\pi^{-3}.$$

Hence

$$u(x,t) = 1 + \sum_{n=0}^{\infty} A_n e^{-(n+\frac{1}{2})^2\pi^2 t} \cos[(n+\frac{1}{2})\pi x], A_n = (-1)^{n+1}4(n+\frac{1}{2})^{-3}\pi^{-3}.$$

(b) 1. □

2. For problem (1), complete the calculation of the series in case $j(t) = 0$ and $h(t) = e^t$.

Answer: In case $j(t) = 0$ and $h(t) = e^t$, by (10) and the initial condition $u_n(0) = 0$,

$$u_n(t) = \frac{2n\pi k}{(\lambda_n k + 1)l^2}(e^t - e^{-\lambda_n k t}).$$

Therefore

$$u(x,t) = \sum_{n=1}^{\infty} \frac{2n\pi k}{(\lambda_n k + 1)l^2}(e^t - e^{-\lambda_n k t}) \sin\frac{n\pi x}{l}. \quad \square$$

4. Solve $u_{tt} = c^2 u_{xx} + k$ for $0 < x < l$, with the boundary conditions $u(0,t) = 0, u_x(l,t) = 0$ and the initial condition $u(x,0) = 0, u_t(x,0) = V$. Here k and V are constants.

Answer: It is easy to check $c^{-2}kx(l - \frac{1}{2}x)$ solves

$$v_{tt} = c^2 v_{xx} + k, \quad v(0,t) = v_x(l,t) = 0.$$

Using the method of shifting the data, we have

$$u(x,t) = c^{-2}kx(l-\frac{1}{2}x)+\sum_{n=0}^{\infty}\left(A_n \cos\frac{(n+\frac{1}{2})\pi ct}{l}+B_n \sin\frac{(n+\frac{1}{2})\pi ct}{l}\right)\sin\left[(n+\frac{1}{2})\pi x/l\right],$$

where

$$A_n = -\frac{2}{l}\int_0^l c^{-2}kx(l-\frac{1}{2}x)\sin\big[(n+\frac{1}{2})\pi x/l\big]dx,$$

$$\text{and}\quad B_n = \frac{2Vl}{(n+\frac{1}{2})^2\pi^2 c}, \quad n = 0,1,2,\ldots. \ \square$$

5. Solve $u_{tt} = c^2 u_{xx} + e^t \sin 5x$ for $0 < x < \pi$, with $u(0,t) = u(\pi,t) = 0$ and the initial conditions $u(x,0) = 0, u_t(x,0) = \sin 3x$.

Answer: It is easy to check $\frac{e^t}{1+25c^2}\sin 5x$ solves

$$v_{tt} = c^2 v_{xx} + e^t \sin 5x, \text{and} \quad v(0,t) = v(\pi,t) = 0.$$

Using the method of shifting the data, we have

$$u(x,t) = \frac{1}{1+25c^2}e^t \sin 5x + \sum_{n=1}^{\infty}\big(A_n \cos(nct) + B_n \sin(nct)\big)\sin(nx),$$

where

$$A_n = -\frac{2}{\pi}\int_0^\pi \frac{1}{1+25c^2}\sin 5x \sin(nx)dx,$$

$$\text{and}\quad B_n = \frac{2}{nc\pi}\int_0^\pi \big[\sin 3x - \frac{1}{1+25c^2}\sin 5x\big]\sin(nx)dx. \ \square$$

6. Solve $u_{tt} = c^2 u_{xx} + g(x)\sin\omega t$ for $0 < x < l$, with $u = 0$ at both ends and $u = u_t = 0$ when $t = 0$. For which values of ω can resonance occur? (Resonance means growth in time.)

Answer: Let

$$u(x,t) = \sum_{n=1}^{\infty} u_n(t)\sin\frac{n\pi x}{l},$$

by the Expansion Method or (12) in the book,

$$\frac{d^2 u_n}{dt^2} + c^2\lambda_n u_n(t) = f_n(t)$$

with the initial conditions

$$u_n(0) = 0, \ u_n'(0) = 0,$$

where

$$f_n(t) = \frac{2}{l}\int_0^l g(x)\sin\omega t \sin\frac{n\pi x}{l}dx = g_n \sin\omega t.$$

Hence when $\omega = \frac{n\pi c}{l}$, resonance can occur. $\square$

9. Use the method of subtraction to solve $u_{tt} = 9u_{xx}$ for $0 \le x \le 1 = l$, with $u(0,t) = h, u(1,t) = k$, where h and k are given constants, and $u(x,0) = 0, u_t(x,0) = 0$.

Answer: To use the method of subtraction, we firstly find a solution of

$$v_{tt} = 9v_{xx}, \text{and} \quad v(0,t) = h, \; v(1,t) = k.$$

Obviously $v(x,t) = h + (k-h)x$ is a solution.

By the method of subtraction, we get $u(x,t) - v(x,t)$ satisfying

$$w_{tt} = 9w_{xx}, \text{and} \quad w(0,t) = 0, \; w(1,t) = 0,$$

with the initial conditions

$$w(x,0) = -[h + (k-h)x], w_t(x,0) = 0.$$

Hence by the method of separation of variables, we have

$$u(x,t) - v(x,t) = \sum_{n=1}^{\infty} \frac{2}{n\pi}\big[(-1)^n k + h\big] \cos 3n\pi t \sin n\pi x.$$

Therefore

$$u(x,t) = h + (k-h)x + \sum_{n=1}^{\infty} \frac{2}{n\pi}\big[(-1)^n k + h\big] \cos 3n\pi t \sin n\pi x. \; \square$$