MAT 4220 (2008-09) Partial differential equations
Suggested Answer to Assignment 3

Exercise 3.1

1. Solve $u_t = ku_{xx}$; $u(x,0) = e^{-x}$; $u(0,t) = 0$ on the half-line $0 < x < \infty$.
Answer: By the method of odd extension or formula (6), we have

$$\begin{aligned} u(x,t) &= \frac{1}{\sqrt{4\pi kt}} \int_0^\infty [e^{-\frac{(x-y)^2}{4kt}} - e^{-\frac{(x+y)^2}{4kt}}] e^{-y}\, dy \\ &= \frac{1}{\sqrt{4\pi kt}} \int_0^\infty [e^{-\frac{(y+2kt-x)^2}{4kt}+kt-x} - e^{-\frac{(y+2kt+x)^2}{4kt}+kt+x}]\, dy \\ &= \frac{1}{\sqrt{\pi}} e^{kt-x} \int_{\frac{2kt-x}{\sqrt{4kt}}}^\infty e^{-p^2} dp - \frac{1}{\sqrt{\pi}} e^{kt+x} \int_{\frac{2kt+x}{\sqrt{4kt}}}^\infty e^{-q^2} dq \\ &= \frac{1}{2} e^{kt-x} [1 - \mathfrak{E}rf(\frac{2kt-x}{\sqrt{4kt}})] - \frac{1}{2} e^{kt+x} [1 - \mathfrak{E}rf(\frac{2kt+x}{\sqrt{4kt}})], \end{aligned}$$

where $\mathfrak{E}rf(x)$ is defined by

$$\mathfrak{E}rf(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-p^2}\, dp.$$

2. Solve $u_t = ku_{xx}$; $u(x,0) = 0$; $u(0,t) = 1$ on the half-line $0 < x < \infty$.
Answer: Let $v(x,t) = u(x,t) - 1$. Then $v(x,t)$ will satisfy

$$v_t = kv_{xx}, v(x,0) = -1, v(0,t) = 0.$$

Hence

$$\begin{aligned} v(x,t) &= -\frac{1}{\sqrt{4\pi kt}} \int_0^\infty [e^{-\frac{(x-y)^2}{4kt}} - e^{-\frac{(x+y)^2}{4kt}}]\, dy \\ &= -\mathfrak{E}rf(\frac{x}{\sqrt{4kt}}). \end{aligned}$$

Therefore,

$$u(x,t) = v(x,t) + 1 = 1 - \mathfrak{E}rf(\frac{x}{\sqrt{4kt}}). \quad \square$$

3. Derive the solution formula for the half-line Neumann problem $w_t - kw_{xx} = 0$ for $0 < x < \infty, 0 < t < \infty; w_x(0,t) = 0; w(x,0) = \phi(x)$.

Answer: By the method of even reflection, we can translate the original problem for the half-line to the problem for the whole line and then using the formula for the latter to obtain

$$w(x,t) = \frac{1}{\sqrt{4\pi kt}} \int_0^{+\infty} [e^{-(x-y)^2/4kt} + e^{-(x+y)^2/4kt}]\phi(y)dy.$$

For the details, please see your lecture notes of Prof.Wei. $\square$

4. Consider the following problem with a Robin boundary condition:

$$\begin{array}{lll} \text{DE}: \; u_t = ku_{xx} & \text{on the half}-\text{line } 0 < x < \infty \text{ and } 0 < t < \infty & \\ \text{IC}: \; u(x,0) = x & \text{for } t = 0 \text{ and } 0 < x < \infty & (1) \\ \text{BC}: \; u_x(0,t) - 2u(0,t) = 0 & \text{for } x = 0. & \end{array}$$

The purpose of this exercise is to verify the solution formula for (1). Let $f(x) = x$ for $x > 0$, let $f(x) = x + 1$ for $x < 0$, and let

$$v(x,t) = \frac{1}{\sqrt{4\pi kt}} \int_{-\infty}^{\infty} e^{-(x-y)^2/4kt} f(y)\, dy.$$

(a) What PDE and initial condition does $v(x,t)$ satisfy for $-\infty < x < \infty$?

(b) Let $w = v_x - 2v$. What PDE and initial condition does $w(x,t)$ satisfy for $-\infty < x < \infty$?

(c) Show that $f'(x) - 2f(x)$ is an odd function for $x \neq 0$.

(d) Use Exercise 2.4.11 to show that w is an odd function of x.

(e) Deduce that $v(x,t)$ satisfies (1) for $x > 0$. Assuming uniqueness, deduce that the solution of (1) is given by

$$u(x,t) = \frac{1}{\sqrt{4\pi kt}} \int_{-\infty}^{\infty} e^{-(x-y)^2/4kt} f(y)\, dy.$$

Answer: (a) With the rule for differentiation under an integral sign and the property of source function, $v(x,t)$ satisfies

$$v_t = kv_{xx}, v(x,0) = f(x).$$

(b) By (a), $w(x,t)$ satisfies

$$w_t = kw_{xx}, w(x,0) = f'(x) - 2f(x).$$

(c) By the definition of f,

$$f'(x) - 2f(x) = \begin{cases} 1 - 2x, & x > 0; \\ -1 - 2x, & x < 0. \end{cases}$$

And

$$\begin{aligned} f'(-x) - 2f(-x) &= \begin{cases} -1 + 2x, & x > 0; \\ 1 + 2x, & x < 0. \end{cases} \\ &= -[f'(x) - 2f(x)]. \end{aligned}$$

Hence $f'(x) - 2f(x)$ is an odd function.

(d) Since $w(x, 0)$ is an odd function, using the conclusion in Exercise 2.4.11, w is an odd function of x.

(e) By (a), $v(x, t$ satisfies DE and IC. By (d), $v(x, t$ satisfies BC. Thus we have proved that $v(x, t)$ satisfies (1) for $x > 0$. Hence using the assumption for uniqueness, the solution of (1) is given by

$$u(x, t) = \frac{1}{\sqrt{4\pi kt}} \int_{-\infty}^{\infty} e^{-(x-y)^2/4kt} f(y)\, dy,$$

where

$$f(y) = \begin{cases} y, & y > 0; \\ y + 1, & y < 0. \end{cases} \quad \square$$

Exercise 3.2

1. Solve the Neumann problem for the wave equation on the half-line $0 < x < \infty$.

Answer: By the method of even extension, we have

$$\begin{aligned} v(x, t) &= \frac{1}{2}[\phi_{\text{even}}(x + ct) + \phi_{\text{even}}(x - ct)] + \frac{1}{2c} \int_{x-ct}^{x+ct} \psi_{\text{even}}(y)\, dy \\ &= \begin{cases} \frac{1}{2}[\phi(x + ct) + \phi(x - ct)] + \frac{1}{2c} \int_{x-ct}^{x+ct} \psi(y) dy, \text{for } x \geq ct \geq 0; \\ \frac{1}{2}[\phi(x + ct) + \phi(-x + ct)] + \frac{1}{2c}[\int_0^{-x+ct} \psi(y) dy + \int_0^{x+ct} \psi(y) dy], \text{for } 0 < x < ct. \end{cases} \end{aligned}$$

It is similar for $t < 0$. $\square$

5. Solve $u_{tt} = 4u_{xx}$ for $0 < x < \infty, u(0, t) = 0, u(x, 0) \equiv 1, u_t(x, 0) \equiv 0$ using the reflection method. This solution has a singularity; find its location.

Answer: Using the odd reflection method or formulas (2) and (3), we have

$$u(x,t) = \begin{cases} 1, & x > 2|t|; \\ 0, & x < 2|t|. \end{cases}$$

Hence the singularity is on the lines $x = 2|t|$. $\square$

6. Solve $u_{tt} = c^2 u_{xx}$ in $0 < x < \infty, u(x,0) = 0, u_t(x,0) = V$,

$$u_t(0,t) + a u_x(0,t) = 0,$$

where V, a, and c are positive constants and $a > c$.

Answer: By the method of reflection and the Robin boundary condition, we should make the following extension:

$$\phi_{ext}(x) \equiv 0, \quad \text{for any} \;\; x$$

and

$$\psi_{ext}(x) = \begin{cases} V, & \text{if} \quad x > 0, \\ \frac{a+c}{a-c} V, & \text{if} \quad x < 0. \end{cases}$$

Then we get

$$u_{ext} = \frac{1}{2c} \int_{x-ct}^{x+ct} \psi_{ext}(y)\, dy.$$

Its restriction on $x > 0$ is our solution. That is,

$$u(x,t) = \begin{cases} tV, & \text{if} \quad x > ct > 0, \\ \frac{at-x}{a-c} V, & \text{if} \quad ct > x > 0. \end{cases} \quad \square$$

9. (a) Find $u(\frac{2}{3}, 2)$ if $u_{tt} = u_{xx}$ in $0 < x < 1, u(x,0) = x^2(1-x), u_t(x,0) = (1-x)^2, u(0,t) = u(1,t) = 0$.

(b) Find $u(\frac{1}{4}, \frac{7}{2})$.

Answer: (a) Using the reflection method, we have

$$\begin{aligned} u(2/3, 2) &= 1/2[\phi(2/3) + \phi(2/3)] + 1/2 \int_{2/3}^{2/3} \psi(y)\, dy \\ &= (2/3)^2(1 - 2/3) = 4/27. \end{aligned}$$

(b) Using the reflection method, we have

$$\begin{aligned} u(1/4,7/2) &= 1/2[-\phi(1/4)+\phi(3/4)]+1/2\int_{3/4}^{1/4}\psi(y)\,dy \\ &= 1/2[-(1/4)^2(1-1/4)+(3/4)^2(1-3/4)]+1/2\int_{3/4}^{1/4}(1-y)^2dy \\ &= -1/48. \quad \square \end{aligned}$$

10. Solve $u_{tt}=9u_{xx}$ in $0<x<\pi/2, u(x,0)=\cos x, u_t(x,0)=0, u_x(0,t)=u(\pi/2,t)=0$.

Answer: Since it is Neumann boundary condition on line $x=0$ and it is Dirichlet boundary condition on line $x=\pi/2$, by the method of reflection, we extend the initial data $\phi(x)$ and $\psi(x)$ to the whole line to be even with respect to $x=0$ and to be odd with respect to $x=\pi/2$.The simplest way to do is to define

$$\phi_{\text{ext}} = \begin{cases} \phi(x), & \text{for } 0<x<\pi/2; \\ \phi(-x), & \text{for } -\pi/2<x<0; \\ -\phi(\pi-x), & \text{for } \pi/2<x<3\pi/2; \\ \text{extended to be of period } 2\pi. & \end{cases}$$

and $\psi_{\text{ext}}\equiv 0$. Now let $v(x,t)$ be the solution of the infinite line problem with the extended initial data. Then $u(x,t)$ be the restriction of $v(x,t)$ to the interval $(0,\pi/2)$.

Thus $u(x,t)$ is given by the formula

$$u(x,t)=\frac{1}{2}[\phi_{\text{ext}}(x+ct)+\phi_{\text{ext}}(x-ct)]+\frac{1}{2c}\int_{x-ct}^{x+ct}\psi_{\text{ext}}(s)ds$$

Note $\phi(x)=\cos x$ and $\psi(x)=0$ are our extended functions.Hence

$$u(x,t)=\frac{1}{2}[\cos(x+3t)+\cos(x-3t)]=\cos x\cos(3t). \quad \square$$

Exercise 3.3

1. Solve the inhomogeneous diffusion equation on the half-line with Dirichlet boundary condition:

$$\begin{aligned} u_t-ku_{xx}=f(x,t) \quad (0<x<\infty, 0<t<\infty) \\ u(0,t)=0, \quad u(x,0)=\phi(x) \end{aligned}$$

using the method of reflection.

Answer: Using the method of reflection and the formula (2) in Section 3.3, we have

$$\begin{aligned} u(x,t) &= \int_{-\infty}^{\infty} S(x-y,t)\phi_{\text{odd}}(y)dy + \int_0^t \int_{-\infty}^{\infty} S(x-y,t-s)f_{\text{odd}}(y,s)dyds \\ &= \int_0^{\infty} [S(x-y,t) - S(x+y,t)]\phi(y)dy \\ &\quad + \int_0^t \int_0^{\infty} [S(x-y,t-s) - S(x+y,t-s)]f(y,s)dyds, \end{aligned}$$

where $f_{\text{odd}}(y,s)$ is the odd extension of $f(y,s)$ in the variable y, and

$$S(x,t) = \frac{1}{2\sqrt{\pi k t}} e^{-\frac{x^2}{4kt}}, \ t > 0. \ \square$$

3. Solve the inhomogeneous Neumann diffusion problem on the half-line

$$\begin{aligned} w_t - k w_{xx} = 0 \quad &\text{for } 0 < x < \infty, 0 < t < \infty, \\ w_x(0,t) = h(t), \quad &w(x,0) = \phi(x) \end{aligned}$$

by the subtraction method indicated in the text.

Answer: Let $W(x,t) = w(x,t) - xh(t)$. Then $W(x,t)$ will satisfy

$$\begin{aligned} W_t - k W_{xx} = -xh'(t) \quad &\text{for } 0 < x < \infty, 0 < t < \infty, \\ W_x(0,t) = 0, \quad &w(x,0) = \phi(x) - xh(0). \end{aligned}$$

Using the method of reflection and the formula (2) in Section 3.3, we have

$$\begin{aligned} W(x,t) &= \int_{-\infty}^{\infty} S(x-y,t)\phi_{\text{even}}(y)dy + \int_0^t \int_{-\infty}^{\infty} S(x-y,t-s)f_{\text{even}}(y,s)dyds \\ &= \int_0^{\infty} [S(x-y,t) + S(x+y,t)][\phi(y) - xh(0)]dy \\ &\quad + \int_0^t \int_0^{\infty} [S(x-y,t-s) + S(x+y,t-s)][-yh'(s)]dyds, \end{aligned}$$

and thus $w(x,t) = W(x,t) + xh(t)$, where $f_{\text{even}}(y,s)$ is the even extension of $f(y,s)$ in the variable y, and

$$S(x,t) = \frac{1}{2\sqrt{\pi k t}} e^{-\frac{x^2}{4kt}}, \ t > 0. \ \square$$

Exercise 3.4

1. Solve $u_{tt} = c^2 u_{xx} + xt, u(x,0) = 0, u_t(x,0) = 0$.
Answer: By the Theorem 1 in Section 3.4, we have

$$\begin{aligned} u(x,t) &= \frac{1}{2c} \iint_{\Delta} ysdyds \\ &= \frac{1}{2c} \int_0^t \int_{x-c(t-s)}^{x+c(t-s)} ysdyds \\ &= \frac{xt^3}{6}. \quad \square \end{aligned}$$

2. Solve $u_{tt} = c^2 u_{xx} + e^{ax}, u(x,0) = 0, u_t(x,0) = 0$.
Answer: By the Theorem 1 in Section 3.4, we have

$$\begin{aligned} u(x,t) &= \frac{1}{2c} \iint_{\Delta} e^{ay} dyds \\ &= \frac{1}{2c} \int_0^t \int_{x-c(t-s)}^{x+c(t-s)} e^{ay} dyds \\ &= \begin{cases} \frac{e^{ax}}{a^2c^2}\left(\frac{e^{act}+e^{-act}}{2} - 1\right), & \text{for } a \neq 0; \\ \frac{1}{2}t^2, & \text{for } a = 0. \end{cases} \quad \square \end{aligned}$$

3. Solve $u_{tt} = c^2 u_{xx} + \cos x, u(x,0) = \sin x, u_t(x,0) = 1 + x$.
Answer: By the Theorem 1 in Section 3.4, we have

$$\begin{aligned} u(x,t) &= 1/2[\sin(x+ct) + \sin(x-ct)] + \frac{1}{2c}\int_{x-ct}^{x+ct}(1+s)ds + \frac{1}{2c}\iint_{\Delta} \cos y dyds \\ &= \sin x \cos(ct) + (x+1)t + \frac{1}{c^2}\cos x[1 - \cos(ct)]. \quad \square \end{aligned}$$

4. Show that the solution of the inhomogeneous wave equation

$$u_{tt} = c^2 u_{xx} + f, \quad u(x,0) = \phi(x), \quad u_t(x,0) = \psi(x),$$

is the sum of three terms, one each for f, ϕ, and ψ.
Answer: Let u_1 be the solution of the wave equation

$$u_{tt} = c^2 u_{xx} + f, \quad u(x,0) = 0, \quad u_t(x,0) = 0,$$

u_2 be the solution of the wave equation

$$u_{tt} = c^2 u_{xx}, \quad u(x,0) = \phi(x), \quad u_t(x,0) = 0,$$

and u_3 be the solution of the wave equation

$$u_{tt} = c^2 u_{xx}, \quad u(x,0) = 0, \quad u_t(x,0) = \psi(x).$$

Then $u = u_1 + u_2 + u_3$ is the unique solution for the original problem since the equation and conditions are linear and the uniqueness of the wave equation. Note that u_1, u_2, u_3 are terms each for f, ϕ, and ψ. Hence the solution of the original problem can be written in the sum of three terms, one each for f, ϕ, and ψ. □

11. Show by direct substitution that $u(x,y) = h(t - x/c)$ for $x < ct$ and $u(x,t) = 0$ for $x \geq ct$ solves the wave equation on the half-line $(0,\infty)$ with zero initial data and boundary condition $u(0,t) = h(t)$.

Answer: By the chain rule, it is easily to check the conclusion. □

12. Derive the solution of the fully inhomogeneous wave equation on the half-line

$$\begin{aligned} v_{tt} - c^2 v_{xx} = f(x,t) \quad \text{in } 0 < x < \infty \\ v(x,0) = \phi(x), \quad v_t(x,0) = \psi(x) \\ v(0,t) = h(t), \end{aligned}$$

by means of the method using Green's theorem.

Answer:

$$u(x,t) = \begin{cases} \frac{1}{2}[\phi(x+ct) + \phi(x-ct)] + \frac{1}{2c}\int_{x-ct}^{x+ct} \psi + \frac{1}{2c}\iint\limits_{\triangle} f, & \text{if } x > ct > 0, \\ \frac{1}{2}[\phi(x+ct) - \phi(ct-x)] + \frac{1}{2c}\int_{ct-x}^{x+ct} \psi + h(t - \frac{x}{c}) + \frac{1}{2c}\iint\limits_{\triangle} f, & \text{if } 0 < x < ct. \end{cases}$$

Please see the details on the Page 76 in the textbook. □

13. Solve $u_{tt} = c^2 u_{xx}$ for $0 < x < \infty$, $u(0,t) = t^2, u(x,0) = x, u_t(x,0) = 0$.

Answer: By the formula in the answer of Excrcise 3.12 (the above one) with $f \equiv 0, \phi(x) = x, \psi \equiv 0, h(t) = t^2$, for $x \geq ct$, we have

$$\begin{aligned} u(x,t) &= \frac{1}{2}[\phi(x+ct) + \phi(x-ct)] + \frac{1}{2c}\int_{x-ct}^{x+ct} \psi(y)dy + \frac{1}{2c}\iint\limits_{D} f \\ &= \frac{1}{2}[(x+ct) + (x-ct)] \\ &= x. \end{aligned}$$

For $x < ct$, we have

$$\begin{aligned} u(x,t) &= \frac{1}{2}[\phi(x+ct) - \phi(-x+ct)] + h(t-\frac{x}{c}) \\ &= \frac{1}{2}[(x+ct)+(x-ct)] + (t-\frac{x}{c})^2 \\ &= x + (t-\frac{x}{c})^2. \end{aligned}$$

Hint of another proof: Let $v(x,t) = u(x,t) - t^2$ and use the method of reflection. □

Exercise 3.5

1. Prove that if ϕ is any piecewise continuous function, then

$$\frac{1}{\sqrt{4\pi}} \int_0^{\pm\infty} e^{-p^2/4} \phi(x + \sqrt{kt}p)\, dp \to \pm\frac{1}{2}\phi(x\pm) \quad \text{as } t \searrow 0.$$

Answer: Since

$$\frac{1}{\sqrt{4\pi}} \int_0^{\infty} e^{-p^2/4} dp = 1/2,$$

we obtain

$$\begin{aligned} & |\frac{1}{\sqrt{4\pi}} \int_0^{\infty} e^{-p^2/4} \phi(x + \sqrt{kt}p) dp - \frac{1}{2}\phi(x+)| \\ \le & \frac{1}{\sqrt{4\pi}} \int_0^{\infty} e^{-p^2/4} |\phi(x + \sqrt{kt}p) - \phi(x+)| dp \\ = & \frac{1}{\sqrt{4\pi}} \int_{p_0}^{\infty} e^{-p^2/4} |\phi(x + \sqrt{kt}p) - \phi(x+)| dp \\ & + \frac{1}{\sqrt{4\pi}} \int_0^{p_0} e^{-p^2/4} |\phi(x + \sqrt{kt}p) - \phi(x+)| dp. \end{aligned}$$

Let $\epsilon > 0$. The first part is

$$\begin{aligned} & \frac{1}{\sqrt{4\pi}} \int_{p_0}^{\infty} e^{-p^2/4} |\phi(x + \sqrt{kt}p) - \phi(x+)| dp \\ \le & \frac{1}{\sqrt{4\pi}} \cdot 2\max|\phi| \cdot \int_{p_0}^{\infty} e^{-p^2/4} dp < \frac{\epsilon}{2} \end{aligned}$$

by choosing p_0 sufficiently large, since the integral $\int_0^\infty e^{-p^2/4} dp$ converges.

Fix p_0, let $\delta > 0$ be so small that

$$|\phi(x+\sqrt{kt}p) - \phi(x+)| < \epsilon$$

for all $p \le p_0, t < \delta$. Then the second part is

$$\begin{aligned} & \frac{1}{\sqrt{4\pi}} \int_0^{p_0} e^{-p^2/4} |\phi(x+\sqrt{kt}p) - \phi(x+)| dp \\ \le & \ (\frac{1}{\sqrt{4\pi}} \int_0^{\infty} e^{-p^2/4} dp) \cdot \epsilon = \frac{\epsilon}{2}. \end{aligned}$$

Hence

$$\frac{1}{\sqrt{4\pi}} \int_0^{\infty} e^{-p^2/4} \phi(x+\sqrt{kt}p) dp \to \frac{1}{2}\phi(x+) \text{ as } t \searrow 0.$$

Similarly we can prove that

$$\frac{1}{\sqrt{4\pi}} \int_0^{-\infty} e^{-p^2/4} \phi(x+\sqrt{kt}p) dp \to -\frac{1}{2}\phi(x-) \text{ as } t \searrow 0. \ \square$$

2. Use Exercise 1 to prove Theorem 2.

Answer: The proof of the infinitely differentiable is the same as before. So we only need to prove that

$$\lim_{t \searrow 0} u(x,t) = \frac{1}{2}[\phi(x+) + \phi(x-)].$$

By the result of Exercise 1, we have

$$\begin{aligned} u(x,t) &= \frac{1}{\sqrt{4\pi}} \int_{-\infty}^{\infty} e^{-p^2/4} \phi(x - p\sqrt{kt}) dp \\ &= \frac{1}{\sqrt{4\pi}} \int_0^{\infty} e^{-p^2/4} \phi(x - p\sqrt{kt}) dp + \frac{1}{\sqrt{4\pi}} \int_{-\infty}^{0} e^{-p^2/4} \phi(x - p\sqrt{kt}) dp \\ &= -\frac{1}{\sqrt{4\pi}} \int_0^{-\infty} e^{-p^2/4} \phi(x + p\sqrt{kt}) dp + \frac{1}{\sqrt{4\pi}} \int_0^{\infty} e^{-p^2/4} \phi(x + p\sqrt{kt}) dp \\ &\to \frac{1}{2}[\phi(x-) + \phi(x+)] \quad \text{as } t \searrow 0. \ \square \end{aligned}$$