MAT 4220 (2008-09) Partial differential equations
Suggested Answer to Assignment 2

Exercise 2.1

2. Solve $u_{tt} = c^2 u_{xx}, u(x,0) = \log(1+x^2), u_t(x,0) = 4+x$.

Answer: By the solution formula for the initial-value problem, due to d'Alembert, the solution is

$$u(x,t) = \frac{1}{2}\{\log[1+(x+ct)^2] + \log[1+(x-ct)^2]\} + \frac{1}{2c}\int_{x-ct}^{x+ct}(4+s)\ ds$$

$$= \frac{1}{2}\{\log[1+(x+ct)^2] + \log[1+(x-ct)^2]\} + 4t + xt. \ \Box$$

5. (The hammer blow) Let $\phi(x) \equiv 0$ and $\psi(x) = 1$ for $|x| < a$ and $\psi(x) = 0$ for $|x| \geq 0$. Sketch the string profile (u versus x) at each of the successive instants $t = a/2c, a/c, 3a/2c, 2a/c$, and $5a/c$.

Answer: By the formula,

$$u(x,t) = \frac{1}{2c}\int_{x-ct}^{x+ct}\psi(s)\ ds = \frac{1}{2c}[\text{length of } (x-ct,\ x+ct)\cap(-a,\ a)].$$

Then

$$u(x, a/2c) = \frac{1}{2c}[\text{length of } (x-\frac{a}{2},\ x+\frac{a}{2})\cap(-a,a)]$$

$$= \begin{cases} 0, & x > \frac{3a}{2}; \\ \frac{3a}{2} - x, & \frac{3a}{2} \geq x > \frac{a}{2}; \\ a, & \frac{a}{2} \geq x > -\frac{a}{2}; \\ x + \frac{3a}{2}, & -\frac{a}{2} \geq x > -\frac{3a}{2}; \\ 0, & -\frac{3a}{2} \geq x. \end{cases}$$

$$u(x, a/c) = \frac{1}{2c}[\text{length of } (x-a,\ x+a)\cap(-a,a)]$$

$$= \begin{cases} 0, & x > 2a; \\ 2a - x, & 2a \geq x > 0; \\ 2a + x, & 0 \geq x > -2a; \\ 0, & -2a \geq x. \end{cases}$$

$$u(x, 3a/2c) = \frac{1}{2c}[\text{length of } (x - \frac{3a}{2},\ x + \frac{3a}{2}) \cap (-a, a)]$$

$$= \begin{cases} 0, & x > \frac{5a}{2}; \\ \frac{5a}{2} - x, & \frac{5a}{2} \geq x > \frac{a}{2}; \\ 2a, & \frac{a}{2} \geq x > -\frac{a}{2}; \\ x + \frac{5a}{2}, & -\frac{a}{2} \geq x > -\frac{5a}{2}; \\ 0, & -\frac{5a}{2} \geq x. \end{cases}$$

$$u(x, 2a/c) = \frac{1}{2c}[\text{length } of\ (x - 2a,\ x + 2a) \cap (-a, a)]$$

$$= \begin{cases} 0, & x > 3a; \\ 3a - x, & 3a \geq x > a; \\ 2a, & a \geq x > -a; \\ x + 3a, & -a \geq x > -3a; \\ 0, & -3a \geq x. \end{cases}$$

$$u(x, 5a/c) = \frac{1}{2c}[\text{length of } (x - 5a,\ x + 5a) \cap (-a, a)]$$

$$= \begin{cases} 0, & x > 6a; \\ 6a - x, & 6a \geq x > 4a; \\ 2a, & 4a \geq x > -4a; \\ x + 6a, & -4a \geq x > -6a; \\ 0, & -6a \geq x. \end{cases}$$

Hear we omit the figures. □

6. In Exercise 5, find the greatest displacement, $\max_x u(x, t)$, as a function of t.

Answer:

$$\max_x u(x,t) = \begin{cases} 2ct, & \frac{a}{c} > t \geq 0; \\ 2a, & t \geq \frac{a}{c}. \end{cases} \qquad \square$$

7. If both ϕ and ψ are odd function of x, show that the solution $u(x,t)$ of the wave equation is also odd in x for all t.

Answer: By the formula,

$$u(x,t) = \frac{1}{2}[\phi(x+ct) + \phi(x-ct)] + \frac{1}{2c}\int_{x-ct}^{x+ct} \psi(s)\,ds,$$

thus

$$u(-x,t) = \frac{1}{2}[\phi(-x+ct) + \phi(-x-ct)] + \frac{1}{2c}\int_{-x-ct}^{-x+ct} \psi(s)\,ds$$

$$= \frac{1}{2}[-\phi(x-ct) - \phi(x+ct)] - \frac{1}{2c}\int_{x-ct}^{x+ct} \psi(s)\,ds$$

$$= -\{\frac{1}{2}[\phi(x+ct) + \phi(x-ct)] + \frac{1}{2c}\int_{x-ct}^{x+ct} \psi(s)\,ds\} = -u(x,t).$$

Thus $u(x,t)$ is also odd in x for all t. $\square$

8. A spherical wave is a solution of the three-dimensional wave equation of the form $u(r,t)$, where r is the distance to the origin (the spherical coordinate). The wave equation takes the form

$$u_{tt} = c^2(u_{rr} + \frac{2}{r}u_r) \quad (\text{“spherical wave equation”}).$$

(a) Change variables $v = ru$ to get the equation for v: $v_{tt} = c^2 v_{rr}$.

(b) Solve for v using (3) and thereby solve the spherical wave equation.

(c) Use (8) to solve it with initial conditions $u(r,0) = \phi(r), u_t(r,0) = \psi(r)$, taking both $\phi(r)$ and $\psi(r)$ to be even functions of r.

Answer: (a) Change variables $v = ru$, then

$$v_{tt} = ru_{tt}, v_{rr} = (ru_r + u)_r = ru_{rr} + 2u_r,$$

which implies

$$v_{tt} = rc^2(u_{rr} + \frac{2}{r}u_r) = c^2 v_{rr}.$$

(b) Using the same skill related to the wave equation (1), we have $v(r,t) = f(r+ct) + g(r-ct)$, where f and g are two arbitrary functions of a single variable. Hence $u = \frac{1}{r}f(r+ct) + \frac{1}{r}g(r-ct)$.

(c) Since $v(r,0) = r\phi(r)$ and $v_t(r,0) = r\psi(r)$ are both odd, we can extend v to all of $\mathbb{R}$ by odd reflection. That is, we set

$$\tilde{v}(r,t) := \begin{cases} v(r,t), & r > 0; \\ 0, & r = 0; \\ -v(-r,t), & r < 0. \end{cases}$$

Hence d'Alembert's formula implies

$$\tilde{v}(r,t) = \frac{1}{2}[(r+ct)\phi(r+ct) + (r-ct)\phi(r-ct)] - \frac{1}{2c}\int_{r-ct}^{r+ct} s\psi(s)\, ds.$$

Therefore for $r > 0$,

$$u(r,t) = \frac{1}{r}v(r,t) = \frac{1}{2r}[(r+ct)\phi(r+ct)+(r-ct)\phi(r-ct)] - \frac{1}{2cr}\int_{r-ct}^{r+ct} s\psi(s)\, ds. \square$$

9. Solve $u_{xx} - 3u_{xt} - 4u_{tt} = 0, u(x,0) = x^2, u_t(x,0) = e^x$.

Answer: Using the same skill related to the wave equation (1), let $v = u_x + u_t$, we must have $v_x - 4v_t = 0$. Thus we have two first-order equations

$$v_x - 4v_t = 0$$

and

$$u_x + u_t = v.$$

As before, we can solve them one at a time and then solve the other one to obtain the general solution is

$$u(x,t) = f(x + \frac{1}{4}t) + g(x - t).$$

The initial condition implies

$$f(x) = \frac{4}{5}(x^2 + e^x + a), g(x) = \frac{1}{5}(x^2 - 4e^x - 4a),$$

where a is a constant. Thus

$$u(x,t) = \frac{4}{5}[(x + \frac{t}{4})^2 + e^{x+\frac{t}{4}} + a] + \frac{1}{5}[(x-t)^2 - 4e^{x-t} - 4a]$$

$$= \frac{1}{5}[(2x + \frac{t}{2})^2 + (x - t)^2 + 4e^{x+\frac{t}{4}} - 4e^{x-t}]. \quad \Box$$

Exercise 2.2

1. Use the energy conservation of the wave equation to prove that the only solution with $\phi \equiv 0$ and $\psi \equiv 0$ is $u \equiv 0$.

Answer: By the law of conservation of energy,

$$E = \frac{1}{2}\int_{-\infty}^{+\infty} (\rho u_t^2 + T u_x^2)\, dx$$

is a constant independent of t.

Since $\phi \equiv 0$ and $\psi \equiv 0$, we have $E \equiv 0$. Thus the first vanishing theorem implies $u_t \equiv 0$ and $u_x \equiv 0$. Thus $u \equiv 0$ since $\phi \equiv 0$. $\Box$

2. For a solution $u(x, t)$ of the wave equation with $\rho = T = c = 1$, the energy density is defined as $e = \frac{1}{2}(u_t^2 + u_x^2)$ and the momentum density as $p = u_t u_x$.

(a) Show that $\partial e/\partial t = \partial p/\partial x$ and $\partial p/\partial t = \partial e/\partial x$.

(b) Show that both $e(x, t)$ and $p(x, t)$ also satisfy the wave equation.

Answer: (a) By the chain rule of differential,

$$\partial e/\partial t = u_t u_{tt} + u_x u_{xt}, \partial e/\partial x = u_t u_{tx} + u_x u_{xx}$$

$$\partial p/\partial t = u_t u_{xt} + u_{tt} u_x, \partial p/\partial x = u_t u_{xx} + u_{tx} u_x.$$

Hence

$$\partial e/\partial t = \partial p/\partial x, \text{ and } \partial e/\partial x = \partial p/\partial t$$

since $u_{tt} = u_{xx}, u_{xt} = u_{tx}$.

(b) From the result of (a),

$$e_{tt} = p_{xt} = p_{tx} = e_{xx}, p_{tt} = e_{xt} = e_{tx} = p_{xx}.$$

Thus both $e(x, t)$ and $p(x, t)$ satisfy the wave equation. $\Box$

3. Show that the wave equation has the following invariance properties.

(a) Any translate $u(x - y, t)$, where y is fixed, is also a solution.

(b) Any derivative, say u_x, of a solution is also a solution.

(c) The dilated function $u(ax, at)$ is also a solution.

Answer: (a) Let $v(x, t) = u(x - y, t)$, we have

$$v_{tt}(x, t) = u_{tt}(x - y, t) = c^2 u_{xx}(x - y, t) = c^2 v_{xx}(x, t),$$

so it is also a solution.

(b) Let $v(x,t) = u_x$, we have

$$v_{tt}(x,t) = u_{xtt}(x,t) = c^2 u_{xxx}(x,t) = c^2 v_{xx}(x,t),$$

so it is also a solution.

(c) Let $v(x,t) = u(ax, at)$, we have

$$v_{tt}(x,t) = a^2 u_{tt}(ax, at) = a^2 c^2 u_{xx}(ax, at) = c^2 v_{xx}(x,t),$$

so it is also a solution. □

5. For the damped string, equation (1.3.3), show that the energy decreases.

Answer: The energy function is

$$E = \frac{1}{2}\int_{-\infty}^{+\infty} (u_t^2 + c^2 u_x^2)\, dx.$$

Thus

$$\begin{aligned}
dE/dt &= \frac{1}{2}\int_{-\infty}^{+\infty} (2u_t u_{tt} + 2c^2 u_x u_{xt})\, dx \\
&= \int_{-\infty}^{+\infty} (c^2 u_t u_{xx} - r u_t^2 + c^2 u_x u_{xt})\, dx \\
&= \int_{-\infty}^{+\infty} (c^2 u_t u_{xx} - r u_t^2 - c^2 u_{xx} u_t)\, dx + [c^2 u_t u_x]|_{-\infty}^{+\infty} \\
&= -r \int_{-\infty}^{+\infty} (u_t^2) \leq 0.
\end{aligned}$$

Hence the energy decreases. □

More on 2.2. Consider the diffusion equation with Robin boundary condition

$$\begin{gathered}
u_t - k u_{xx} = 0, \quad 0 < x < l,\ t > 0 \\
u(x,0) = \phi(x) \\
u_x(0,t) - a_0 u(0,t) = 0, u_x(l,t) + a_l u(l,t) = 0
\end{gathered}$$

(a) Show that if $a_0 > 0, a_1 > 0$, then the integral $\int_0^l u^2(x,t)\, dx$ is a decreasing function.

(b) Prove that if $a_0 > 0, a_1 > 0$, then the solution to the above problem is unique.

Answer: (a) By the diffusion equation and the Robin boundary condition,

$$\begin{aligned}
\frac{d}{dt}\int_0^l u^2(x,t)\,dx &= \int_0^l 2uu_t\,dx = 2k\int_0^l uu_{xx}\,dx \\
&= (2kuu_x)|_0^l - 2k\int_0^l u_x^2\,dx \\
&= 2k\Big(u(l,t)u_x(l,t) - u(0,t)u_x(0,t)\Big) - 2k\int_0^l u_x^2\,dx \\
&= 2k\Big(-\frac{u_x^2(l,t)}{a_l} - \frac{u_x^2(0,t)}{a_0}\Big) - 2k\int_0^l u_x^2\,dx \\
&\le 0.
\end{aligned}$$

Thus the integral $\int_0^l u^2(x,t)\,dx$ is a decreasing function.

(b) Let u_1 and u_2 are two solutions of the above problem. Let $w = u_1 - u_2$, then w satisfies the same equation and the same Robin boundary condition, especially $w(x,0) \equiv 0$.

By Part(a) we know that

$$\int_0^l w^2(x,t)\,dx \le \int_0^l w^2(x,0)\,dx = 0.$$

Thus $w \equiv 0$, i.e., $u_1 \equiv u_2$. Hence the solution of the problem is unique. □

Exercise 2.3

1. Consider the solution $1 - x^2 - 2kt$ of the diffusion equation. Find the locations of its maximum and its minimum in the closed rectangle $\{0 \le x \le 1, 0 \le t \le T\}$.

Answer: Let $u(x,t) = 1 - x^2 - 2kt$, then

$$u(0,0) = \max_{0\le x\le 1,\ 0\le t\le T} u(x,t) = 0$$

$$u(1,T) = \min_{0\le x\le 1,\ 0\le t\le T} u(x,t) = -2kT. \quad \square$$

2. Consider a solution of the diffusion equation $u_t = u_{xx}$ in $\{0 \le x \le l, 0 \le t < \infty\}$.

(a) Let $M(T)$ = the maximum of $u(x,t)$ in the rectangle $\{0 \le x \le l, 0 \le t \le T\}$. Does $M(T)$ increase or decrease as a function of T?

(b) Let $m(T)$ = the minimum of the $u(x,t)$ in the rectangle $\{0 \le x \le l, 0 \le t \le T\}$. Does $m(T)$ increase or decrease as a function of T?

Answer: (a) By the definition of maximum, $M(T)$ increase (i.e., nondecreasing).

(b) By the definition of minimum, $m(T)$ decrease (i.e., nondecreasing). $\square$

3. Consider the diffusion equation $u_t = u_{xx}$ in the interval $(0,1)$ with $u(0,t) = u(1,t) = 0$ and $u(x,0) = 1 - x^2$. Note that this initial function does not satisfy the boundary condition at the left end, but that the solution will satisfy it for all $t > 0$.

(a) Show that $u(x,t) > 0$ at all interior points $0 < x < 1, 0 < t < \infty$.

(b) For each $t > 0$, let $\mu(t)$ = the maximum of $u(x,t)$ over $0 \le x \le 1$. Show that $\mu(t)$ is a deceasing (i.e., nondecreasing) function of t.

(c) Draw a rough sketch of what you think the solution looks like (u versus x) at a few times. (If you have appropriate software available, compute it.)

Answer: (a) Use the strong minimum principle.

(b) Use the minimum principle or Let the maximum occur at the point $X(t)$, so that $\mu(t) = u(X(t),t)$. Differentiate $\mu(t)$, assuming that $X(t)$ is differentiable, we have

$$\dot{\mu}(t) = u_x(X(t),t)\dot{X}(t) + u_t(X(t),t) = u_{xx}(X(t),t) \le 0.$$

Hence $\mu(t)$ decrease.

(c) Hear we omit the figure. $\square$

4. Consider the diffusion equation $u_t = u_{xx}$ in $\{0 < x < 1, 0 < t < \infty\}$ with $u(0,t) = u(1,t) = 0$ and $u(x,0) = 4x(1-x)$.

(a) Show that $0 < u(x,t) < 1$ for all $t > 0$ and $0 < x < 1$.

(b) Show that $u(x,t) = u(1-x,t)$ for all $t \ge 0$ and $0 \le x \le 1$.

(c) Use the energy method to show that $\int_0^1 u^2\,dx$ is a strictly decreasing function of t.

Answer: (a) Use the strong minimum principle.

(b) Since both $u(x,t)$ and $u(1-x,t)$ are the solution of

$$u_t = u_{xx},\ 0 < x < 1,\ 0 < t < \infty$$

$$u(0,t) = u(1,t) = 0, \text{and } u(x,0) = 4x(1-x),$$

the uniqueness theorem implies that $u(x,t) = u(1-x,t)$.

(c) By the equation

$$\frac{d}{dt}\int_0^1 u^2\,dx = \int_0^1 2uu_t\;dx = 2\int_0^1 uu_{xx}\,dx = -2\int_0^1 u_x^2\,dx.$$

Since $u(x,t) > 0$ for all $t > 0$ and $0 < x < 1$, so u_x is not zero function. Hence $\frac{d}{dt}\int_0^1 u^2\,dx < 0$. Therefore $\int_0^1 u^2\,dx$ is a strictly decreasing function of t. □

6. Prove the comparison principle for the diffusion equation: If u and v are two solutions, and if $u \le v$ for $t = 0$, for $x = 0$, and for $x = l$, then $u \le v$ for $0 \le t < \infty, 0 \le x \le l$.

Answer: Let $w = u - v$ and then use the maximum principle. □

7. (a) More generally, if $u_t - ku_{xx} = f, v_t - kv_{xx} = g, f \le g$, and $u \le v$ at $x = 0, x = l$ and $t = 0$, prove that $u \le v$ for $0 \le x \le l, 0 \le t$.

(b) If $v_t - v_{xx} \ge \sin x$ for $0 \le x \le \pi, 0 < t < \infty$, and if $v(0,t) \ge v(\pi,t) \ge 0$ and $v(x,0) \ge \sin x$, use part (a) to show that $v(x,t) \ge (1 - e^{-t})\sin x$.

Answer: (a) Let $w = u - v$ and use the generalized Maximum Principle (which will be proved in Extra 2 (a)):

If $w_t - w_{xx} \le 0$ in $R = [0,l] \times [0,T]$, then

$$\max_R w(x,t) = \max_{x=0,\,x=l,\,t=0} w(x,t).$$

(b) Let $u(x,t) = (1 - e^{-t})\sin x$, then $u(0,t) = 0, u(\pi,t) = 0, u(x,0) = 0$ and $u_t - u_{xx} = \sin x$. Hence the result of part(a) shows that $v(x,t) \ge (1 - e^{-t})\sin x$. □

8. Consider the diffusion equation on $(0,l)$ with the Robin boundary condition $u_x(0,t) - a_0u(0,t) = 0$ and $u_x(l,t) + a_lu(l,t) = 0$. If $a_0 > 0$ and $a_l > 0$, use the energy method to show that the endpoints contribute to the decrease of $\int_0^l u^2(x,t)\,dx$. (This is interpreted to mean that part of the "energy" is lost at the boundary, so we call the boundary conditions "radiating" or "dissipative.")

Answer: Please see the answer of **More on 2.2 (a)**. □

More on 2.3.

Extra 1. Consider the diffusion equation $u_t = ku_{xx} + au$ in $\{0 < x < 1, 0 < t < \infty\}$ with $u(0,t) = u(1,t) = 0$ and $u(x,0) = \sin(\pi x)$, where $k > 0$, a are real numbers.

(1) Show that $0 < u(x,t) < e^{at}$ for all $t > 0$ and $0 < x < 1$.

(2) Show that $u(x,t) = u(1-x,t)$ for all $t \ge 0$ and $0 \le x \le 1$.

Answer: Note: In (1), the conclusion "$0 < u(x,t) < 1$" has changed to "$0 < u(x,t) < e^{at}$". If the conclusion does not change, the real number a in the equation must limited to $a \leq \pi^2 k$.

(1) Let $v(x,t) = e^{-at}u(x,t)$, then v will satisfies

$$\begin{aligned} v_t &= v_{xx}, \\ v(0,t) &= v(1,t) = 0, \\ v(x,0) &= \sin(\pi x). \end{aligned}$$

Using the strong maximum principle, we have $0 < v(x,t) < 1$, therefore $0 < u(x,t) < e^{at}$ for all $t > 0$ and $0 < x < 1$.

(2) Let $v(x,t) = e^{-at}u(x,t)$, then both $v(x,t)$ and $v(1-x,t)$ are solutions of the diffusion equation $v_t = v_{xx}$ with the boundary conditions:

$$v(0,t) = v(1,t) = 0, \quad v(x,0) = \sin(2x).$$

Hence the uniqueness theorem implies that $v(x,t) = v(1-x,t)$. Therefore $u(x,t) = u(1-x,t)$ for all $t \geq 0$ and $0 \leq x \leq 1$.

Another Proof:

Let $v(x,t) = e^{-at}u(x,t)$,then v satisfies

$$\begin{aligned} v_t &= v_{xx}, \\ v(0,t) &= v(1,t) = 0, \\ v(x,0) &= \sin(2x). \end{aligned}$$

Using the method of separation of variables, $v(x,t) = e^{-\pi^2 kt}\sin(\pi x)$ and then $u(x,t) = e^{at-\pi^2 kt}\sin(\pi x)$. Therefore $0 < u(x,t) < e^{at}$ for all $t > 0$ and $0 < x < 1$, and $u(x,t) = u(1-x,t)$ for all $t \geq 0$ and $0 \leq x \leq 1$. □

Extra 2. (a) Prove the following generalized Maximum Principle:

If $u_t - ku_{xx} \leq 0$ in $R = [0,l] \times [0,T]$, then

$$\max_R u(x,t) = \max_{\partial R} u(x,t).$$

Hint: follow the proof of Maximum Principle.

(b) Show that if $v(x,t)$ satisfies

$$\begin{gathered} v_t = kv_{xx} + f(x,t), -\infty < x < +\infty, 0 < t < T \\ v(x,0) = 0 \end{gathered}$$

then

$$v(x,y) \le T \max_{-\infty<x<\infty, 0<t<T} f(x,t).$$

Hint: Consider

$$u(x,t) = v(x,t) - t \max_{-\infty<x<\infty, 0<t<T} f(x,t)$$

and then use (a).

Answer: (a) For any positive constant ϵ, let $v(x,t) = u(x,t) + \epsilon x^2$, then v satisfies

$$v_t - kv_{xx} = u_t - k(u + \epsilon x^2)_{xx} = u_t - ku_{xx} - 2\epsilon k \le -2\epsilon k < 0,$$

which is the "diffusion inequality". Now suppose that $v(x,t)$ attains its maximum at an interior point or on the top edge (x_0, t_0). That is, $0 < x_0 < 1,\ 0 < t_0 \le T$. By ordinary calculus, we know that

$$v_x(x_0,t_0) = 0,\ v_{xx}(x_0,t_0) \le 0, \text{ and } v_t(x_0,t_0) \ge 0.$$

This contradicts the diffusion inequality. Thus $v(x,t) \le M + \epsilon l^2$, where $M := \max_{x=0,\, x=l,\, t=0} u(x,t)$, which implies

$$u(x,t) \le M + \epsilon(l^2 - x^2)$$

and then $u(x,t) \le M$ since ϵ can be choose small enough. Therefore

$$\max_R u(x,t) = \max_{x=0, x=l, t=0} u(x,t).$$

(b) Consider $u(x,t) = v(x,t) - t \max_{-\infty<x<\infty, 0<t<T} f(x,t)$, then u satisfies

$$u_t - ku_{xx} = - \max_{-\infty<x<\infty, 0<t<T} f(x,t) + f(x,t) \le 0, \quad u(x,0) = 0.$$

Hence the result of (a) implies that $u(x,t) \le 0$. Therefore

$$v(x,t) \le T \max_{-\infty<x<\infty, 0<t<T} f(x,t). \ \square$$

Exercise 2.4

1. Solve the diffusion equation with the initial condition

$$\phi(x) = 1 \quad \text{for } |x| < l \quad \text{and } \phi(x) = 0 \quad \text{for } |x| > l.$$

Write your answer in terms of $\mathfrak{E}\text{rf}(x)$.

Answer: By the general formula,

$$\begin{aligned} u(x,t) &= \frac{1}{\sqrt{4\pi kt}} \int_{-l}^{l} e^{-(x-y)^2/4kt}\, dy \\ &= \frac{1}{\sqrt{\pi}} \int_{(-l-x)/(2\sqrt{kt})}^{(l-x)/(2\sqrt{kt})} e^{-p^2}\, dp \\ &= 1/2\{\mathfrak{E}\text{rf}[(x+l)/(2\sqrt{kt})] - \mathfrak{E}\text{rf}[(x-l)/(2\sqrt{kt})]\}. \ \square \end{aligned}$$

2. Do the same for $\phi(x) = 1$ for $x > 0$ and $\phi(x) = 3$ for $x < 0$.

Answer: By the general formula,

$$\begin{aligned} u(x,t) &= \frac{1}{\sqrt{4\pi kt}} \int_{0}^{\infty} e^{-(x-y)^2/4kt}\, dy + \frac{1}{\sqrt{4\pi kt}} \int_{-\infty}^{0} 3e^{-(x-y)^2/4kt}\, dy \\ &= 1/2 + 1/2\mathfrak{E}\text{rf}[x/(2\sqrt{kt})] + 3/2 - 3/2\mathfrak{E}\text{rf}[x/(2\sqrt{kt})] \\ &= 2 - \mathfrak{E}\text{rf}[x/(2\sqrt{kt})]. \ \square \end{aligned}$$

4. Solve the diffusion equation if $\phi(x) = e^{-x}$ for $x > 0$ and $\phi(x) = 0$ for $x < 0$.

Answer: By the general formula,

$$\begin{aligned} u(x,t) &= \frac{1}{\sqrt{4\pi kt}} \int_{0}^{\infty} e^{-(x-y)^2/4kt} e^{-y}\, dy \\ &= \frac{1}{\sqrt{4\pi kt}} \int_{0}^{\infty} e^{-\frac{(y+2kt-x)^2}{4kt}+kt-x}\, dy \\ &= e^{kt-x} \int_{\frac{2kt-x}{2\sqrt{kt}}}^{\infty} e^{-p^2}\, dp/\sqrt{\pi} \\ &= \frac{1}{2}e^{kt-x} - \frac{e^{kt-x}}{2}\mathfrak{E}\text{rf}(\frac{2kt-x}{2\sqrt{kt}}). \ \square \end{aligned}$$

6. Compute $\int_0^\infty e^{-x^2}\, dx$.

Answer: Note that

$$\begin{aligned}\int_0^\infty e^{-x^2}\,dx \cdot \int_0^\infty e^{-y^2}\,dy &= \int_0^\infty \int_0^\infty e^{-(x^2+y^2)}\,dxdy \\ &= \frac{\pi}{2}\int_0^\infty e^{-r^2} r\,dr \\ &= \frac{\pi}{4}(-e^{-r^2})|_0^\infty \\ &= \frac{\pi}{4}.\end{aligned}$$

Hence

$$\int_0^\infty e^{-x^2}\,dx = \frac{\sqrt{\pi}}{2}. \quad \square$$

7. Use Exercise 6 to show that $\int_{-\infty}^\infty e^{-p^2}\,dp = \sqrt{\pi}$. Then substitute $p = x/\sqrt{4kt}$ to show that

$$\int_{-\infty}^\infty S(x,t)\,dx = 1.$$

Answer: Firstly,

$$\begin{aligned}\int_{-\infty}^\infty e^{-p^2}\,dp &= \int_0^\infty e^{-p^2}\,dp + \int_{-\infty}^0 e^{-p^2}\,dp \\ &= 2\int_0^\infty e^{-p^2}\,dp \\ &= 2\frac{\sqrt{\pi}}{2} = \sqrt{\pi}.\end{aligned}$$

Then

$$\int_{-\infty}^\infty S(x,t)\,dx = \int_{-\infty}^\infty \frac{1}{\sqrt{4\pi kt}} e^{-(x-y)^2/4kt}\,dx = \frac{1}{\sqrt{\pi}}\int_{-\infty}^\infty e^{-p^2}\,dp = 1. \quad \square$$

8. Show that for any fixed $\delta > 0$ (no matter how small),

$$\max_{\delta \le |x| < \infty} S(x,t) \to 0 \quad \text{as } t \to 0.$$

[This means that the tail of $S(x,t)$ is "uniformly small".]

Answer: By the definition of $S(x,t)$,

$$\max_{\delta \le x < \infty} S(x,t) = \frac{1}{\sqrt{4\pi kt}} e^{-\delta^2/4kt},$$

so

$$\lim_{t \to 0} \max_{\delta \le x < \infty} S(x,t) = \lim_{t \to 0} \frac{1}{\sqrt{4\pi kt}} e^{-\delta^2/4kt} = \lim_{x \to +\infty} \frac{\sqrt{x}}{\sqrt{4\pi k}} e^{-x\delta^2/4k} = 0. \quad \square$$

11. (a) Consider the diffusion equation on the whole line with the usual initial condition $u(x,0) = \phi(x)$. If $\phi(x)$ is an odd function, show that the solution $u(x,t)$ is also an odd function of x.

(b) Show that the same is true if "odd" is replaced by "even".

(c) Show that the analogous statements are true for the wave equation.

Answer: (a) Since both $u(x,t)$ and $-u(-x,t)$ are solutions, by the uniqueness, we have $u(x,t) = -u(-x,t)$.

(b) Similar to (a).

(c) Similar to (a). $\square$

14. Let $\phi(x)$ be a continuous function such that $|\phi(x)| \le Ce^{ax^2}$. Show that formula (8) for the solution of the diffusion equation makes sense for $0 < t < 1/(4ak)$, but not necessarily for larger t.

Answer: Since

$$|e^{-(x-y)^2/4kt}\phi(y)| \le Ce^{-(x-y)^2/4kt + ay^2} = Ce^{(a - \frac{1}{4kt})y^2 + \frac{x}{2kt}y - \frac{x^2}{4kt}},$$

so that formula

$$u(x,t) = \frac{1}{\sqrt{4\pi kt}} \int_{-\infty}^{\infty} e^{-(x-y)^2/4kt}\phi(y)\, dy$$

makes sense for $0 < t < 1/(4ak)$, but not necessarily for larger t, for example, $\phi(x) = e^{ax^2}$. $\square$

15. Prove the uniqueness of the diffusion problem with Neumann boundary conditions:

$$u_t - ku_{xx} = f(x,t) \quad \text{for } 0 < x < l, t > 0 \quad u(x,0) = \phi(x)$$
$$u_x(0,t) = g(t) \quad u_x(l,t) = h(t)$$

by the energy method.

Answer: Support that both u and v are solution of the diffusion problem with the same Neumann boundary condition. Let $w(x,t) = u(x,t) - v(x,t)$, then w satisfies

$$w_t = kw_{xx}, \quad w(x,0) = w_x(0,t) = w_x(l,t) = 0.$$

Thus by the integration by part and the Neumann boundary condition,

$$\frac{d}{dt}\int_0^l 1/2[w(x,t)]^2\,dx = -k\int_0^l [w_x(x,t)]^2\,dx \le 0.$$

Hence by the initial condition,

$$\int_0^l 1/2[w(x,t)]^2\ dx \le \int_0^l 1/2[w(x,0)]^2\,dx = 0$$

Therefore, $w = 0$ and thus $u = v$ for all $t > 0$. $\square$

16. Solve the diffusion equation with constant dissipation:

$$u_t - ku_{xx} + bu = 0 \quad \text{for } -\infty < x < \infty \quad \text{with } u(x,0) = \phi(x),$$

where $b > 0$ is a constant.

Answer: Let $v(x,t) = e^{bt}u(x,t)$, then v satisfies

$$v_t - kv_{xx} = 0,\ v(x,0) = u(x,0) = \phi(x).$$

By the general formula,

$$v(x,t) = \frac{1}{\sqrt{4\pi kt}}\int_{-\infty}^{\infty} e^{-(x-y)^2/4kt}\phi(y)\,dy,$$

which implies

$$u(x,t) = \frac{e^{-bt}}{\sqrt{4\pi kt}}\int_{-\infty}^{\infty} e^{-(x-y)^2/4kt}\phi(y)\,dy.\ \square$$

18. Solve the heat equation with convection:

$$u_t - ku_{xx} + Vu_x = 0 \quad \text{for } -\infty < x < \infty \quad \text{with } u(x,0) = \phi(x),$$

where V is a constant.

Answer: Let $v(x,t) = u(x+Vt,t)$, then v satisfies

$$v_t - kv_{xx} = 0,\ v(x,0) = u(x,0) = \phi(x).$$

By the general formula,

$$v(x,t) = \frac{1}{\sqrt{4\pi kt}} \int_{-\infty}^{\infty} e^{-(x-y)^2/4kt} \phi(y)\, dy,$$

which implies

$$u(x,t) = \frac{1}{\sqrt{4\pi kt}} \int_{-\infty}^{\infty} e^{-(x-Vt-y)^2/4kt} \phi(y)\, dy. \ \Box$$

19. (a) Show that $S_2(x,y,t) = S(x,t)S(y,t)$ satisfies the diffusion equation $S_t = k(S_{xx} + S_{yy})$.

(b) Deduce that $S_2(x,y,t)$ is the source function for two-dimensional diffusions.

Answer: (a) By the chain rule,

$$S_{2t}(x,y,t) = S_t(x,t)S(y,t) + S(x,t)S_t(y,t),$$

and

$$S_{2xx}(x,y,t) = S_{xx}(x,t)S(y,t), S_{2yy}(x,y,t) = S(x,t)S_{yy}(y,t).$$

Hence

$$\begin{aligned} S_{2t}(x,y,t) &= S_t(x,t)S(y,t) + S(x,t)S_t(y,t) \\ &= kS_{xx}(x,t)S(y,t) + kS(x,t)S_{yy}(y,t) \\ &= k(S_{2xx} + S_{2yy}). \end{aligned}$$

(b) By the definition of $S(x,t)$,

$$\begin{aligned} S_2(x,y,t) &= S(x,t)S(y,t) = \frac{1}{\sqrt{4\pi kt}} e^{-x^2/4kt} \frac{1}{\sqrt{4\pi kt}} e^{-y^2/4kt} \\ &= \frac{1}{4\pi kt} e^{-\frac{x^2+y^2}{4kt}}. \ \Box \end{aligned}$$

Exercise 2.5

1. Show that there is no maximum principle for the wave equation.

Answer: Let $u(x,t) = -x^2 - (t-1)^2$ be the unique solution of the wave equation with boundary conditions:

$$u_{tt} = u_{xx}, \text{ for } -1 < x < 1, 0 < t < \infty,$$
$$u(x,0) = -x^2 - 1, \; u_t(x,0) = 2,$$
$$u(-1,t) = u(1,t) = -t^2 + 2t - 2.$$

But $u(x,t)$ attains its maximum 0 at $(0,1)$. $\square$