MAT 4220 (2008-09) Partial differential equations
Suggested Answer to Assignment 1

Exercise 1.1

2. Which of the following operators are linear?

(a) $\mathcal{L}u = u_x + xu + y$; (b) $\mathcal{L}u = u_x + uu_y$;
(c) $\mathcal{L}u = u_x + u_y^2$; (d) $\mathcal{L}u = u_x + u_y + 1$;
(e) $\mathcal{L}u = \sqrt{1+x^2}(\cos y)u_x + u_{yxy} - [\arctan(x/y)]u$.

Answer: (a) and (e). □

3. For each of the following equations, state the order and whether it is nonlinear, linear inhomogeneous, or linear homogeneous; provide reasons.

(a) $u_t - u_{xx} + 1 = 0$; (b) $u_t - u_{xx} + xu = 0$;
(c) $u_t - u_{xxt} + uu_x = 0$; (d) $u_{tt} - u_{xx} + x^2 = 0$;
(e) $iu_t - u_{xx} + u/x = 0$; (f) $u_x(1+u_x^2)^{-1/2} + u_y(1+u_y^2)^{-1/2} = 0$;
(g) $u_x + e^y u_y = 0$; (h) $u_t + u_{xxxx} + \sqrt{1+u} = 0$.

Answer: (a) order 2, linear inhomogeneous; (b) order 2, linear homogeneous; (c) order 3, nonlinear; (d) order 2, linear inhomogeneous; (e) order 2, linear homogeneous; (f) order 1, nonlinear; (g) order 1, linear homogeneous; (h) order 4, nonlinear. □

4. Show that the difference of two solutions of an inhomogeneous linear equation $\mathcal{L}u = g$ with the same g is a solution of the homogeneous equation $\mathcal{L}u = 0$.

Answer: Suppose that $\mathcal{L}u_1 = g$ and $\mathcal{L}u_2 = g$ and let $u = u_1 - u_2$, then $\mathcal{L}u = \mathcal{L}u_1 - \mathcal{L}u_2 = g - g = 0$ since the operator $\mathcal{L}$ is linear. □

11. Verify that $u(x,y) = f(x)g(y)$ is a solution of the PDE $uu_{xy} = u_x u_y$ for all pairs of (differentiable) functions f and g of one variable.

Answer: For $u(x,y) = f(x)g(y)$, at every point (x,y), we have

$$\begin{aligned} u(x,y)u_{xy}(x,y) &= f(x)g(y)f'(x)g'(y) \\ &= f'(x)g(y)f(x)g'(y) \\ &= u_x(x,y)u_y(x,y) \end{aligned}$$

and obtain the desired result. □

12. Verify by direct substitution that

$$u_n(x,y) = \sin nx \sinh ny$$

is a solution of $u_{xx} + u_{yy} = 0$ for every $n > 0$.

Answer: For $u_n(x, y) = \sin nx \sinh ny$, for every $n > 0$, at every point (x, y),

$$u_{xx} + u_{yy} = -n^2 \sin(nx)\sinh(ny) + n^2 \sin(nx)\sinh(ny) = 0,$$

which implies the desired conclusion. $\square$

Exercise 1.2

1. Solve the first-order equation $2u_t + 3u_x = 0$ with the auxiliary condition $u = \sin x$ when $t = 0$.

Answer: Using the characteristic curve method or the coordinate method, we have $u(t, x) = f(3t - 2x)$. Setting $t = 0$ yields the equation $f(-2x) = \sin x$. Letting $w = -2x$ yields $f(w) = -\sin(w/2)$. Therefore, $u(t, x) = \sin(x - 3t/2)$. $\square$

2. Solve the equation $3u_y + u_{xy} = 0$.

Answer: Let $v = u_y$, then $3v + v_x = 0$. Thus we have $v(x, y) = f(y)e^{-3x}$, i.e., $u_y(x, y) = f(y)e^{-3x}$, which implies $u(x, y) = F(y)e^{-3x} + g(x)$, where both F and g are arbitrary (differential) functions. $\square$

3. Solve the linear equation $(1 + x^2)u_x + u_y = 0$. Sketch some of the characteristic curves.

Answer: The characteristic curves satisfy the ODE: $dy/dx = 1/(1+x^2)$, which implies $y = \arctan x + C$. Thus $u(x, y) = f(y - \arctan x)$. For the characteristic curves, please see the Figure 1. $\square$

5. Solve the equation $\sqrt{1 - x^2}u_x + u_y = 0$ with the condition $u(0, y) = y$.

Answer: The characteristic curves satisfy the ODE: $dy/dx = 1/\sqrt{1 - x^2}$, which implies $y = -\arccos x + C$. Thus $u(x, y) = f(y + \arccos x)$. Setting $x = 0$ yields the equation $f(y + \pi/2) = y$. Letting $w = y + \pi/2$ yields $f(w) = w - \pi/2$ and $u(x, y) = y + \arccos x - \pi/2$. $\square$

6. (a) Solve the equation $yu_x + xu_y = 0$ with $u(0, y) = e^{-y^2}$.

(b) In which region of the xy plane is the solution uniquely determined?

Answer: (a) The characteristic curves satisfy the ODE: $dy/dx = x/y$, which implies $y^2 = x^2 + C$ and thus $u(x, y) = f(y^2 - x^2)$. Setting $x = 0$ yields the equation $f(y^2) = e^{-y^2}$. Letting $w = y^2$ yields $f(w) = e^{-w}$ and $u(x, y) = e^{x^2-y^2}$.

(b) Please see the Figure 2. $\square$

7. Solve $au_x + bu_y + cu = 0$.

Answer: Change variables to $x' = ax + by, y' = bx - ay$. By the chain rule,

$$u_x = au_{x'} + bu_{y'}, u_y = bu_{x'} - au_{y'}.$$

We have $(a^2+b^2)u_{x'} + cu = 0$ which implies $u(x',y') = f(y')e^{-cx'/(a^2+b^2)}$ and thus $u(x,y) = f(bx-ay)e^{-c(ax+by)/(a^2+b^2)}$. $\square$

8. Solve $u_x + u_y + u = e^{x+2y}$ with $u(x,0) = 0$.

Answer: Note that $u(x,y) = e^{x+2y}/4$ is a special solution of the inhomogeneous equation, then by the above result, where $a = b = c = 1$ and the result of Exercise 1.1.4, the general solution is

$$u(x,y) = f(x-y)e^{-x/2} + e^{x+2y}/4.$$

Setting $y = 0$ yields the equation $f(x)e^{-x/2} + e^x/4 = 0$ which implies $f(x) = -e^{3x/2}/4$ and thus $u(x,y) = (e^{x+2y} - e^{x-2y})/4$. $\square$

11. Use the coordinate method to solve the equation

$$u_x + 2u_y + (2x-y)u = 2x^2 + 3xy - 2y^2.$$

Answer: Change variables $x' = x+2y, y' = 2x-y$, then the original equation is changed into

$$5u_{x'} + y'u = x'y'.$$

Note that $u(x',y') = x' - \frac{5}{y'}$ is a special solution and $u(x',y') = f(y')e^{-\frac{x'y'}{5}}$ is the general solution of the corresponding homogeneous equation. Hence the general solution of original equation is

$$u(x,y) = f(2x-y)e^{-\frac{(x+2y)(2x-y)}{5}} + x + 2y - \frac{5}{2x-y},$$

where f is an arbitrary (differential) function. $\square$

Extra Problem: Solve the following first order PDEs and state the Domain of Existence.

(1) $u_x + xyu_y = 0,\ u(x,1) = x^2$;

(2) $(1+x^2)u_x - xy^2u_y = 0,\ u(2,y) = e^y$.

Answer: (1) The characteristic curves satisfy the ODE: $\frac{dy}{dx} = xy$. Solved the ODE, we have $y = Ce^{\frac{1}{2}x^2}$, which implies $C = y \cdot e^{-\frac{1}{2}x^2}$ and thus $u(x,y) = f(y \cdot e^{-\frac{1}{2}x^2})$. Setting $y = 1$ yields the equation $x^2 = f(e^{-\frac{1}{2}x^2})$, so that $f(w) = -2\ln w,\ 0 < w \leq 1$. Hence $u(x,y) = -2\ln y + x^2$.

The domain of existence is :

$$\Omega := \{(x,y) \in \mathbb{R}^2 |\ e^{\frac{x^2}{2}} \geq y > 0\}.$$

(2) The characteristic curves satisfy the ODE: $\frac{dy}{dx} = \frac{-xy^2}{1+x^2}$. Solved the ODE, we have $\frac{1}{y} = \frac{1}{2}\ln(1+x^2) + C$ if $y \neq 0$. If $y = 0$ the characteristic curve has

obviously intersected with $x \equiv 2$. If $y \neq 0$, the characteristic curve always has an intersection point with $x \equiv 2$ if and only if $C \neq -\frac{1}{2}\ln 5$. Thus

$$u(x,y) = \begin{cases} f(C) = f(\frac{1}{y} - \frac{1}{2}\ln(1+x^2)), & y \neq 0, \\ 1, & y = 0. \end{cases}$$

According to $u(2,y) = e^y$, we get $f(\frac{1}{y} - \frac{1}{2}\ln 5) = e^y$. Hence $f(w) = e^{1/(\frac{1}{2}\ln 5 + w)}$ and thus $u(x,y) = e^{y/(1+\frac{y}{2}\ln\frac{5}{1+x^2})}$.

The domain of existence is:

$$\Omega := \{(x,y) \in \mathbb{R}^2 | y \neq \frac{2}{\ln(1+x^2) - \ln 5}\}. \ \square$$

Exercise 1.3

5. Derive the equation of one-dimensional diffusion in a medium that is moving along the x axis to the right at constant speed V.

Answer: The mass of dye is $M(t) = \int_{x_0}^{x} u(y,t)dy$, so $\frac{\partial M}{\partial t} = \int_{x_0}^{x} u_t(y,t)dy$. Then by the Fick's law,

$$\frac{\partial M}{\partial t} = \text{flow in} - \text{flow out} = V(u(x_0,t) - u(x,t)) + ku_x(x,t) - ku_x(x_0,t).$$

Differentiating with respect to x, we get $u_t = ku_{xx} - Vu_x$. $\square$

6. Consider heat flow in a long circular cylinder where the temperature depends only on t and on the distance r to the axis of the cylinder. Here $r = \sqrt{x^2+y^2}$ is the cylindrical coordinate. From the three-dimensional heat equation derive the equation $u_t = k(u_{rr} + u_r/r)$.

Answer: We have $u(x,y,z,t) = u(\sqrt{x^2+y^2}, t) = u(r,t)$, then by the chain rule,

$$u_x = u_r x/r, \ u_y = u_r y/r, \ u_z = 0,$$

and thus

$$u_{xx} = u_{rr}x^2/r^2 + u_r(r^2-x^2)/r^3, u_{yy} = u_{rr}y^2/r^2 + u_r(r^2-y^2)/r^3, u_{zz} = 0.$$

Therefore, $u_t = k(u_{xx} +_{yy} + u_{zz}) = k(u_{rr} + u_r/r)$. $\square$

7. Same problem as above in a ball except that the temperature depends only on the spherical coordinate $\sqrt{x^2+y^2+z^2}$. Derive the equation $u_t = k(u_{rr} + 2u_r/r)$.

Answer: As above, we also set $r = \sqrt{x^2 + y^2 + z^2}$ and have $u(x, y, z, t) = u(\sqrt{x^2 + y^2 + z^2}, t) = u(r, t)$, then by the chain rule,

$$u_x = u_r x/r, \ u_y = u_r y/r, \ u_z = u_r z/r,$$

and thus

$$\begin{aligned} u_{xx} &= u_{rr}x^2/r^2 + u_r(r^2 - x^2)/r^3, \\ u_{yy} &= u_{rr}y^2/r^2 + u_r(r^2 - y^2)/r^3, \\ u_{zz} &= u_{rr}z^2/r^2 + u_r(r^2 - z^2)/r^3. \end{aligned}$$

Therefore, $u_t = k(u_{xx} +_{yy} + u_{zz}) = k(u_{rr} + 2u_r/r)$. $\square$

8. For the hydrogen atom, if $\int |u|^2 \, dx = 1$ at $t = 0$, show that the same is true at all later times.

Answer: By the rule of derivatives of integrals, we have

$$\frac{d}{dt} \int |u|^2 \, dx = \int \frac{\partial(u\bar{u})}{\partial t} \, dx = \int \bar{u}\frac{\partial u}{\partial t} + u\frac{\partial \bar{u}}{\partial t} \, dx = 0.$$

Hence if$\int |u|^2 \, dx = 1$ at $t = 0$, then $\int |u|^2 \, dx = 1$ at all later times. $\square$

10. If $\mathbf{f}(\mathbf{x})$ is continuous and $|\mathbf{f}(\mathbf{x})| \leq 1/(|\mathbf{x}|^3 + 1)$ for all $\mathbf{x}$, show that

$$\iiint_{\text{all space}} \nabla \cdot \mathbf{f} \, d\mathbf{x} = 0.$$

Answer: By the divergence theorem, we have

$$\begin{aligned} \left| \iiint_{|\mathbf{x}| \leq R} \nabla \cdot \mathbf{f} \, d\mathbf{x} \right| &= \left| \iint_{|\mathbf{x}| = R} \mathbf{f} \cdot \vec{n} \, dS \right| \\ &\leq \iint_{|\mathbf{x}| = R} |\, \mathbf{f} \,| \, dS \\ &\leq \iint_{|\mathbf{x}| = R} 1/|\mathbf{x}|^3 \, dS = 4\pi/R. \end{aligned}$$

Hence

$$\iiint_{\text{all space}} \nabla \cdot \mathbf{f} \, d\mathbf{x} = \lim_{R \to \infty} \iiint_{|\mathbf{x}| \leq R} \nabla \cdot \mathbf{f} \, d\mathbf{x} = 0. \ \square$$

Exercise1.4

1. By trial and error, find a solution of the diffusion equation $u_t = u_{xx}$ with initial condition $u(x, 0) = x^2$.

Answer: Setting $u(x, t) = f(t) + x^2$ yields the equations $f'(t) = 2$ and $f(0) = 0$. Hence $f(t) = 2t$ and $u(x, t) = 2t + x^2$ is a solution of the diffusion equation. □

2. (a) Show that the temperature of a metal rod, insulated at the end $x = 0$, satisfies the boundary condition $\partial u/\partial x = 0$.

(b) Do the same for the diffusion of gas along a tube that is closed off at the end $x = 0$.

(c) Show that the three-dimensional version of (a) (insulated solid) or (b) (impermeable container) leads to the boundary condition $\partial u/\partial n = 0$.

Answer: (a) NO heat flows across the boundary, by the Fourier's law, we have $\partial u/\partial x = 0$;

(b) NO gas flows across the boundary, by the Fick's law, we have $\partial u/\partial x = 0$;

(c) NO heat or gas flows across the boundary, by the Fourier's or Fick's law, we have $\partial u/\partial n = 0$. □

Exercise1.5

1. Consider the problem

$$\frac{d^2u}{dx^2} + u = 0$$
$$u(0) = 0 \text{ and } u(L) = 0,$$

consisting of an ODE and a pair of boundary conditions. Clearly, the function $u(x) = 0$ is a solution. Is this solution unique, or not? Does the answer depend on L?

Answer: The generate solution of the ODE is $u(x) = C_1 \cos x + C_2 \sin x$, thus the solution(s) of the problem be

$$u(x) = C_1 \cos x + C_2 \sin x,$$

where C_1 and C_2 satisfy

$$C_1 = 0 \text{ and } C_1 \cos(L) + C_2 \sin(L) = 0,$$

i.e., $C_1 = 0$ and $C_2 \sin(L) = 0$. Therefore, the solution is unique if only if L is not an integer multiple of π. □

2. Consider the problem

$$u''(x) + u'(x) = f(x)$$
$$u'(0) = u(0) = \tfrac{1}{2}[u'(l) + u(l)],$$

with $f(x)$ a given function.

(a) Is the solution unique? Explain.

(b) Does a solution necessarily exist, or is there a condition that $f(x)$ must satisfy for existence? Explain.

Answer: (a) If there is no solution, then the problem is unique. But if there exists a solution u_0, the problem is not unique since $u = u_0 + C(e^{-x} - 2)$ is also a solution for any constant C.

(b) A solution does not necessary exist since there is a necessarily condition that $f(x)$ must satisfy for existence:

$$\int_0^l f(x)\,dx = \int_0^l [u''(x) + u'(x)]\,dx = u'(l) + u(l) - u'(0) - u(0) = 0. \ \square$$

3. Solve the boundary problem $u'' = 0$ for $0 < x < 1$ with $u'(0) + ku(0) = 0$ and $u'(1) \pm ku(1) = 0$. Do the $+$ and $-$ cases separately. What is special about the case $k = 2$?

Answer: The generate solution of the ODE is $u(x) = ax + b$, where a and b are constants. Hence when we do with the $+$ case , the solution(s) of the boundary problem be 0 if $k \neq 0$ and b if $k = 0$. When we do with the $-$ case, the solution(s) of the boundary problem be 0 if $k \neq 0, 2$ and b if $k = 0$ and $-2bx + b$ if $k = 2$.

If $k = 2$, the boundary problem is unique for the $+$ case, but it is not unique for the $-$ case. $\square$

4. Consider the Neumann problem

$$\Delta u = f(x, y, z) \quad \text{in } D$$
$$\frac{\partial u}{\partial n} = 0 \quad \text{on bdy } D$$

(a) What can we surely add to any solution to get another solution? So we don't have uniqueness.

(b) Use the divergence theorem and the PDE to show that

$$\iiint_D f(x, y, z)\,dxdydz = 0$$

is a necessary condition for the Neumann problem to have a solution.

(c) Can you give a physical interpretation of part (a) and/or (b) for either heat flow or diffusion?

Answer: (a) Adding a constant $C(\neq 0)$ to a solution will give another solution, so we don't have uniqueness if there is a solution;

(b) Integrating $f(x, y, z)$ on D and using the divergence theorem, we obtain

$$\begin{aligned} \iiint\limits_D f(x,y,z)\,dxdydz &= \iiint\limits_D \Delta u\,dxdydz \\ &= \iiint\limits_D \nabla\cdot\nabla u\,dxdydz \\ &= \iint\limits_{bdyD} \nabla u\cdot\vec{n}dS = 0; \end{aligned}$$

(c) The temperature of the object $u(x, y, z) \equiv C$ reaches a steady (or equilibrium) state. □

Exercise1.6

1. What are the types of the following equations?

(a) $u_{xx} - u_{xy} + 2u_y + u_{yy} - 3u_{yx} + 4u = 0$.

(b) $9u_{xx} + 6u_{xy} + u_{yy} + u_x = 0$.

Answer: (a) It is hyperbolic since its discriminant $\mathfrak{D} = [(-1-3)/2]^2 - 1\cdot 1 = 3 > 0$.

(b) It's parabolic since its discriminant $\mathfrak{D} = (6)^2 - 9\cdot 1 = 0$. □

2. Find the regions in the xy plane where the equation

$$(1+x)u_{xx} + 2xyu_{xy} - y^2u_{yy} = 0$$

is elliptic, hyperbolic, or parabolic. Sketch them.

Answer: Indeed, its discriminant

$$\mathfrak{D} = (xy)^2 - (1+x)(-y^2) = (x^2+x+1)y^2 = [(x+1/2)^2 + 3/4]y^2,$$

so it is hyperbolic in $\{y \neq 0\}$, parabolic on $\{y = 0\}$, and elliptic nowhere. For their picture, please see Figure 3. □

3. Among all the equations of the form (1), show that the only ones that are unchanged under all rotations (rotation invariant) have the form $a(u_{xx} + u_{yy}) + bu = 0$.

Answer: Consider a rotation of independent variables:

$$^t(x', y') = \mathbf{B}^t(x, y)$$

where $\mathbf{B}$ is an orthogonal matrix. Converting to the new variables,the new coefficient matrix is

$$\mathbf{B}\mathbf{A}^t\mathbf{B}$$

where $\mathbf{A} = (a_{ij})$ is the original coefficient matrix, and $^t(a'_1, a'_2) = \mathbf{B}^t(a_1, a_2)$. Hence the only one that are unchanged under all rotations have the form $a(u_{xx} + u_{yy}) + bu = 0$. $\square$

4. What is the type of the equation

$$u_{xx} - 4u_{xy} + 4u_{yy} = 0?$$

Show by direct substitution that $u(x, y) = f(y+2x)+xg(y+2x)$ is a solution for arbitrary functions f and g.

Answer: It is parabolic since its discriminant $\mathfrak{D} = (-4/2)^2 - 1 \cdot 4 = 0$. By the chain rule, for arbitrary functions f and g, it is easy to check that $u(x, y) = f(y + 2x) + xg(y + 2x)$ satisfies the equation $u_{xx} - 4u_{xy} + 4u_{yy} = 0$ and thus is a solution. $\square$

5. Reduce the elliptic equation

$$u_{xx} + 3u_{yy} - 2u_x + 24u_y + 5u = 0$$

to the form $v_{xx} + v_{yy} + cv = 0$ by a change of dependent variable $u = ve^{\alpha x+\beta y}$ and then a change of scale $y' = \gamma y$.

Answer: Set $u = ve^{\alpha x+\beta y}$, then

$$u_x = (v_x + \alpha v)e^{\alpha x+\beta y},\ u_y = (v_y + \beta v)e^{\alpha x+\beta y},$$

and thus

$$u_{xx} = (v_{xx} + 2\alpha v_x + \alpha^2 v)e^{\alpha x+\beta y},\ u_{yy} = (v_{yy} + 2\beta v_x + \beta^2 v)e^{\alpha x+\beta y}.$$

From the equation $u_{xx} + 3u_{yy} - 2u_x + 24u_y + 5u = 0$, we have

$$(v_{xx} + 2\alpha v_x + \alpha^2 v) + 3(v_{yy} + 2\beta v_y + \beta^2 v) - 2(v_x + \alpha v) + 24(v_y + \beta v) + 5v = 0.$$

Letting $\alpha = 1, \beta = -4$ yields $v_{xx} + 3v_{yy} - 44v = 0$ and then setting $x' = x, y' = \sqrt{3}y$ yields $v_{x'x'} + v_{y'y'} - 44v = 0$. $\square$

6. Consider the equation $3u_y + u_{xy} = 0$.

(a) What is the type?

(b) Find the general solution.

(c) With the auxiliary conditions $u(x,0) = e^{-3x}$ and $u_y(x,0) = 0$, does a solution exist? Is it unique?

Answer: (a) It is hyperbolic since its discriminant $\mathfrak{D} = (1/2)^2 > 0$;

(b) Set $v = u_y$, we have $3v + v_x = 0$ which implies $v(x,y) = f(y)e^{-3x}$ and thus $u(x,y) = F(y)e^{-3x} + g(x)$, where F, g are arbitrary (differential) functions.

(c) Setting $y = 0$ yields

$$\begin{aligned} e^{-3x} &= u(x,0) = F(0)e^{-3x} + g(x) \\ 0 &= u_y(x,0) = F'(0)e^{-3x}. \end{aligned}$$

Therefore,

$$u(x,y) = (F(y) + 1 - F(0))e^{-3x},$$

where $F(y)$ satisfy $F'(0) = 0$. Put $F(y) = ny^2, n = 1, 2, ...,$ which satisfies $F'(0) = 0$ for any n, we obtain infinite solutions $u(x,y) = (ny^2 + 1)e^{-3x}$ of the problem. $\square$

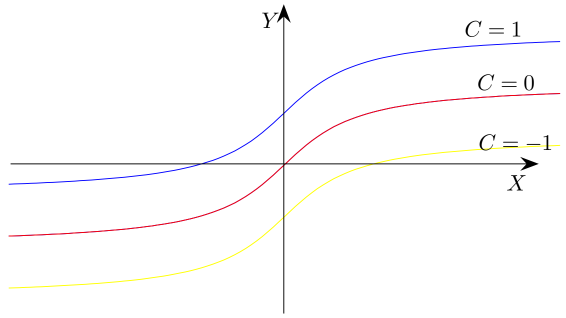

Figure 1: for Exercise 1.2.3

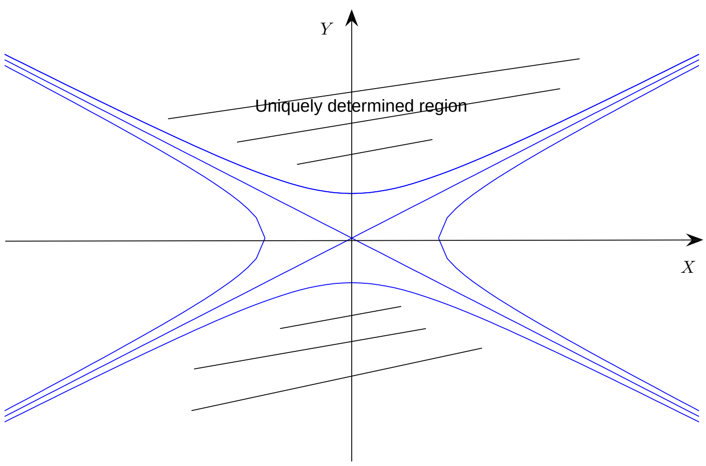

Figure 2: for Exercise 1.2.6(b)

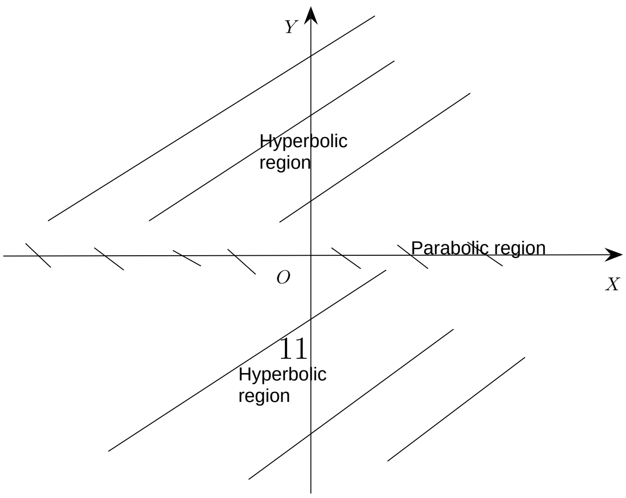

Figure 3: for Exercise 1.6.2

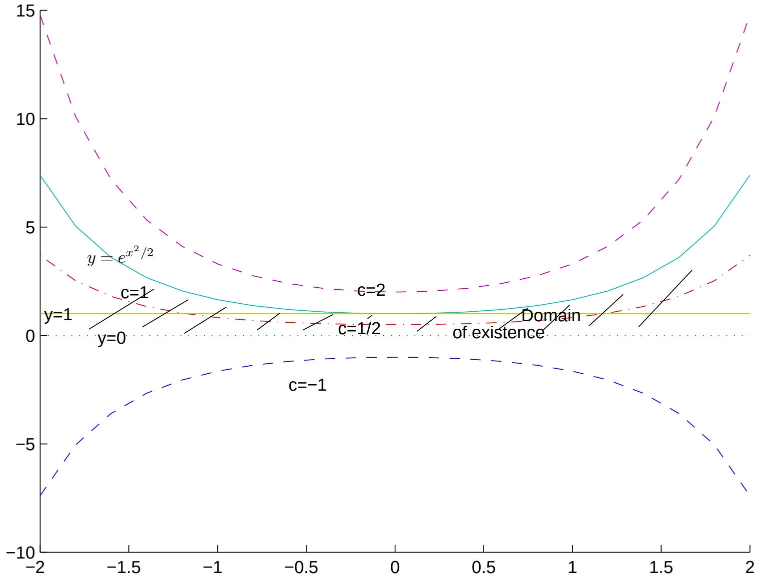

Figure 4: for extra problem 1