**Sample Solutions of Assignment 5 for MAT3270B: 3.4-3.9**

**Section 3.4**

In each of problems find the general solution of the given differential equation

7. $y'' - 2y' + 2y = 0$
12. $4y'' + 9y = 0$
14. $9y'' + 9y' - 4y = 0$
15. $y'' + y' + 1.25y = 0$

**Answer:** 7. The characteristic equation is

$$r^2 - 2 + 2 = 0$$

Thus the possible values of $r$ are $r_1 = 1 + i$ and $r_2 = 1 - i$, and the general solution of the equation is

$$y(t) = e^t(c_1 \cos t + c_2 \sin t).$$

12. The characteristic equation is

$$4r^2 + 9 = 0$$

Thus the possible values of $r$ are $r_1 = \frac{3i}{2}$ and $r_2 = \frac{-3i}{2}$, and the general solution of the equation is

$$y(t) = c_1 \cos \frac{3t}{2} + c_2 \sin \frac{3t}{2}.$$

14. The characteristic equation is

$$9r^2 + 9r - 4 = 0$$

Thus the possible values of $r$ are $r_1 = \frac{1}{3}$ and $r_2 = \frac{-4}{3}$, and the general solution of the equation is

$$y(t) = c_1 e^{\frac{t}{3}} + c_2 e^{\frac{-4t}{3}}.$$

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15. The characteristic equation is

$$r^2 + r + 1.25 = 0$$

Thus the possible values of $r$ are $r_1 = \frac{-1}{2} + i$and $r_2 = \frac{-1}{2} - i$, and the general solution of the equation is

$$y(t) = e^{\frac{-t}{2}}(c_1 \cos t + c_2 \sin t).$$

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23. Consider the initial value problem

$$3u^{''} - u^{'} + 2u = 0,\ u(0) = 2,\ u^{'}(0) = 0.$$

a. Find the solution $u(t)$ of this problem.

b. Find the first time at which $|u(t)| = 10$.

**Answer:** (a.) The characteristic equation is

$$3r^2 - r + 2 = 0$$

Thus the possible values of $r$ are $r_1 = \frac{1+\sqrt{23}i}{6}$ and $r_2 = \frac{1-\sqrt{23}i}{6}$, and the general solution of the equation is

$$u(t) = e^{\frac{t}{6}}(c_1 \cos\frac{\sqrt{23}t}{6} + c_2 \sin\frac{\sqrt{23}t}{6}).$$

(b.) From $u(0) = 0$, we get $c_1 = 2$

From $u^{'}(0) = 0$, we get $c_2 = \frac{-2}{\sqrt{23}}$

Therefore,

$$u(t) = e^{\frac{t}{6}}(2\cos\frac{\sqrt{23}t}{6} - \frac{2}{\sqrt{23}}\sin\frac{\sqrt{23}t}{6}).$$

The first time is $t_0 = 10.7598$ such that $|u(t)| = 10$.

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27. Show that $W(e^{\lambda t}\cos\mu t, e^{\lambda t}\sin\mu t) = \mu e^{2\lambda t}$

**Answer:** Let

$$y_1 = e^{\lambda t}\cos\mu t$$

$$y_2 = e^{\lambda t}\sin\mu t$$

$$W(e^{\lambda t}\cos\mu t, e^{\lambda t}\sin\mu t) = y_2^{'}y_1 - y_1^{'}y_2$$

$$= e^{2\lambda t}\cos\mu t(\lambda\sin\mu t + \mu\sin\mu t) - e^{2\lambda t}\sin\mu t(\lambda\cos\mu t - \mu\sin\mu t)$$

$$= \mu e^{2\lambda t}$$

❑

33. If the functions $y_1$ and $y_2$ are linearly independent solutions of $y^{''} + p(t)y^{'} + q(t)y = 0$, show that between consecutive zeros of $y_1$ there is one and only one zero of $y_2$. Note that this result is illustrated by the solutions $y_1 = \cos t$ and $y_2 = \sin t$ of the equation $y^{''} + y = 0$.

**Answer:** Assume the two consecutive zeros of $y_1$ are $t_1$ and $t_2$, and $t_1 < t_2$. Then $y_1(t) < 0$ or $y_1(t) > 0$ for all $t_1 < t < t_2$.

Case A: We consider the case $y_1(t) > 0$ for all $t_1 < t < t_2$.
In this case, obviously, $y_1^{'}(t_1) > 0$ and $y_1^{'}(t_2) < 0$.

$W(y_1, y_2) = y_2^{'} y_1 - y_1^{'} y_2 \neq 0$ for all $t$, because $y_1$ and $y_2$ are linearly independent solutions of $y^{''} + p(t)y^{'} + q(t)y = 0$ , then $W(y_1, y_2)(t) > 0$ or $W(y_1, y_2)(t) < 0$ for all $t$.

If $W(y_1, y_2)(t) > 0$ for all $t$.
We get

$$W(y_1, y_2)(t_1) = -y_1^{'}(t_1)y_2(t_1) > 0$$

$$W(y_1, y_2)(t_2) = -y_1^{'}(t_2)y_2(t_2) > 0$$

Hence $y_2(t_1) > 0$ and $y_2(t_2) < 0$, so there is one zero of $y_2$.

If $W(y_1, y_2)(t) < 0$ for all $t$.
We get

$$W(y_1, y_2)(t_1) = -y_1^{'}(t_1)y_2(t_1) < 0$$

$$W(y_1, y_2)(t_2) = -y_1^{'}(t_2)y_2(t_2) < 0$$

Hence $y_2(t_1) < 0$ and $y_2(t_2) > 0$, so there is one zero of $y_2$.

Case B: We consider the case $y_1(t) < 0$ for all $t_1 < t < t_2$.
In this case, obviously, $y_1^{'}(t_1) < 0$ and $y_1^{'}(t_2) > 0$.

$W(y_1, y_2) = y_2^{'} y_1 - y_1^{'} y_2 \neq 0$ for all $t$, because $y_1$ and $y_2$ are linearly independent solutions of $y^{''} + p(t)y^{'} + q(t)y = 0$ , then $W(y_1, y_2)(t) > 0$ or $W(y_1, y_2)(t) < 0$ for all $t$.

If $W(y_1, y_2)(t) > 0$ for all $t$.
We get

$$W(y_1, y_2)(t_1) = -y_1^{'}(t_1)y_2(t_1) > 0$$

$$W(y_1, y_2)(t_2) = -y_1^{'}(t_2)y_2(t_2) > 0$$

Hence $y_2(t_1) < 0$ and $y_2(t_2) > 0$, so there is one zero of $y_2$.

If $W(y_1, y_2)(t) < 0$ for all $t$.
We get

$$W(y_1, y_2)(t_1) = -y_1^{'}(t_1)y_2(t_1) < 0$$
$$W(y_1, y_2)(t_2) = -y_1^{'}(t_2)y_2(t_2) < 0$$

Hence $y_2(t_1) > 0$ and $y_2(t_2) < 0$, so there is one zero of $y_2$.

We need to show there is only one zero of $y_2$ between $t_1$ and $t_2$.
If otherwise, then there is at least two zero of $y_2$ between $t_1$ and $t_2$, we can select two consecutive zeros of $y_2$, say $t_3$ and $t_4$, such that $t_1 < t_3 < t_1 < t_4 < t_2$. Then from above proof we can show there is another zero of $y_1$, say $t_5$ such that $t_1 < t_3 < t_1 < t_5 < t_4 < t_2$. This contradict that $t_1$ and $t_2$ are two consecutive zeros of $y_1$.

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38. Euler Equation. An equation of the form

$$t^2y^{''} + \alpha ty^{'} + \beta y = 0, \ t > 0,$$

where $\alpha$ and $\beta$ are real constants, is called an Euler equation. Show that the substitution $x = \ln t$ transform an Euler equation into an equation with constant coefficients.

**Answer:**

$$\frac{dx}{dt} = \frac{1}{t}$$
$$\frac{d^2y}{dt^2} = \frac{1}{t^2}\frac{d^2y}{dx^2} - \frac{1}{t^2}\frac{dy}{dx}$$

We get

$$t^2y^{''} + \alpha ty^{'} + \beta y = \frac{d^2y}{dx^2} + (\alpha + 1)\frac{dy}{dx} + \beta y = 0.$$

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## Section 3.5

Find the general solutions of the given differential equations.

1. $y'' - 2y' + y = 0$

8. $16y'' + 24y' + 9y = 0$

**Answer:** 1. The characteristic equation is

$$r^2 - 2r + 1 = (r-1)(r-1) = 0$$

Thus the possible value of $r$ is $r = 1$, and the general solution of the equation is

$$y(t) = (c_1 + c_2 t)e^t.$$

8. The characteristic equation is

$$16r^2 + 24r + 90$$

Thus the possible value of $r$ is $r_1 = -\frac{3}{4}$, and the general solution of the equation is

$$y(t) = (c_1 + c_2 t)e^{\frac{-3}{4}t}.$$

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In each of the problems use the method of reduction of order to find second solution of the given equation.

23. $t^2y'' - 4ty' + 6y = 0;\ t > 0;\ y_1(t) = t^2$

25. $t^2y'' + 3ty' + y = 0;\ t > 0;\ y_1(t) = \frac{1}{t}$

26. $t^2y'' - t(t+2)y' + (t+2)y = 0;\ t > 0;\ y_1(t) = t$

**Answer:** 23. Let $y(t) = t^2 v(t)$, then

$$y^{'} = v^{'}t^2 + 2tv$$

$$y^{''} = v^{''}t^2 + 4v^{'}t + 2v$$

So we get

$$t^4 v^{''} = 0$$

. Then

$$v(t) = c_1 t + c_2$$

and so

$$y(t) = c_1 t^3 + c_2 t^2.$$

From $y_1(t) = t^2$, we find the second solution is $y_2(t) = t^3$.

25. Let $y(t) = t^{-1}v(t)$, then

$$y^{'} = v^{'}t^{-1} - t^{-2}v$$

$$y^{''} = t^{-1}v^{''}t^2 - 2v^{'}t^{-2} + 2t^{-3}v$$

So we get

$$tv^{''} + v^{'} = 0$$

. Then

$$v(t) = c_1 \ln t + c_2$$

and so

$$y(t) = c_1 t^{-1} \ln t + c_2 t^{-1}.$$

From $y_1(t) = t^{-1}$, we find the second solution is $y_2(t) = t^{-1} \ln t$.

26. Let $y(t) = tv(t)$, then

$$y^{'} = v^{'}t + v$$

$$y^{''} = v^{''}t + 2v^{'}$$

So we get

$$v^{''} - v^{'} = 0$$

. Then

$$v(t) = c_1 e^t + c_2$$

and so

$$y(t) = c_1 te^t + c_2 t.$$

From $y_1(t) = t$, we find the second solution is $y_2(t) = te^t$.

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32. The differential equation

$$y^{''} + \delta(xy^{'} + y) = 0$$

arises in the study of the turbulent flow of a uniform stream past a circular cylinder. Verify that $y_1 = exp(\frac{-\delta x^2}{2})$ is one solution and then find the general solution in the form of an integral.

**Answer:**

$$y_1 = e^{\frac{-\delta x^2}{2}}$$

$$y_1^{'} = -\delta x e^{\frac{-\delta x^2}{2}}$$

$$y_1^{''} = (\delta^2 x^2 - \delta) e^{\frac{-\delta x^2}{2}}$$

Then it is easy to see

$$y_1^{''} + \delta(xy_1^{'} + y_1) = 0.$$

Let $y = ve^{\frac{-\delta x^2}{2}}$, then

$$y^{'} = v^{'} e^{\frac{-\delta x^2}{2}} - \delta x v e^{\frac{-\delta x^2}{2}}$$

$$y^{''} = v^{''} e^{\frac{-\delta x^2}{2}} - 2\delta x v^{'} e^{\frac{-\delta x^2}{2}} + \delta^2 x^2 v e^{\frac{-\delta x^2}{2}} - \delta v e^{\frac{-\delta x^2}{2}}.$$

We get $v^{''} - \delta x v^{'} = 0$, and then $v(x) = \int_0^x c_1 e^{\frac{\delta x^2}{2}} + c_2$.

Hence,

$$y(x) = (c_1 \int_0^x e^{\frac{\delta x^2}{2}} + c_2) e^{\frac{-\delta x^2}{2}}.$$

From $y_1 = exp(\frac{-\delta x^2}{2})$, we find the second solution is

$$y(x) = e^{\frac{-\delta x^2}{2}} \int_0^x e^{\frac{\delta x^2}{2}}.$$

$\square$

In the following problem use the method of of problem 33 to find second independent solution of the given equation.

34. $t^2y^{''} + 3ty^{'} + y = 0;\ t > 0;\ y_1(t) = \frac{1}{t}$

**Answer:** The original equation can be written as

$$y^{''} + \frac{3}{t}y^{'} + \frac{1}{t^2}y = 0.$$

From the Abel's theorem

$$W(y_1, y_2) = c_1e^{-\int \frac{3dt}{t}} = \frac{c_1}{t^3} = y_1y_2^{'} - y_1^{'}y_2 = \frac{y_2^{'}}{t} + \frac{y_2}{t^2}$$

We get

$$y_2^{'} + \frac{y_2}{t} = \frac{c_1}{t^2},$$

and

$$y_2(t) = \frac{1}{t}[c_1 \ln t + c_2].$$

$\square$

38. If $a,\ b,\ c$ are positive constants, show that all solutions of $ay^{''} + by^{'} + cy = 0$ approach zero as $t \to \infty$.

**Answer:** The characteristic equation is

$$ar^2 - br + c = 0$$

Case A: $b^2 - 4ac < 0$

Thus the possible values of $r$ are $r_1 = \frac{-b+i\sqrt{4ac-b^2}}{2a}$ and $r_2 = \frac{-b-i\sqrt{4ac-b^2}}{2a}$, and the general solution of the equation is

$$y(t) = e^{\frac{-bt}{2a}}(c_1 \cos \frac{\sqrt{4ac - b^2}}{2a}t + c_2 \sin \frac{\sqrt{4ac - b^2}}{2a}t).$$

Hence, $y(t) \to 0,$ as $t \to \infty$, since $a,$ $b$ are positive constants.

Case B: $b^2 - 4ac = 0$

Thus the possible values of $r$ are $r_1 = \frac{-b}{2a}$, and the general solution of the equation is

$$y(t) = e^{\frac{-bt}{2a}}(c_1 + c_2 t).$$

Hence, $y(t) \to 0,$ as $t \to \infty$, since $a,$ $b$ are positive constants.

Case C: $b^2 - 4ac > 0$

Thus the possible values of $r$ are $r_1 = \frac{-b+\sqrt{b^2-4ac}}{2a}$ and $r_2 = \frac{-b-\sqrt{b^2-4ac}}{2a}$, and the general solution of the equation is

$$y(t) = c_1 e^{\frac{-b+\sqrt{b^2-4ac}}{2a}} + c_2 e^{\frac{-b-\sqrt{b^2-4ac}}{2a}}.$$

Hence, $y(t) \to 0,$ as $t \to \infty$, since $a$ is positive constant and $-b + \sqrt{b^2 - 4ac} < 0$.

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39. (a) If $a > 0$ and $c > 0$, but $b = 0$, show that the result of Problem 38 is no longer true, but that all solutions are bounded as $t \to \infty$.
(b) If $a > 0$ and $b > 0$, but $c = 0$, show that the result of Problem 38 is no longer true, but that all solutions approach a constant that depends on the initial conditions as $t \to \infty$. Determine this constants for the initial conditions $y(0) = y_0,\ y^{'}(0) = y^{'}_(0)$.

**Answer:** (a). $a > 0$ and $b > 0$, but $c = 0$

The characteristic equation is

$$ar^2 + cr = 0$$

Thus the possible values of $r$ are $r_1 = \sqrt{\frac{c}{a}}i$ and $r_2 = -\sqrt{\frac{c}{a}}i$, and the general solution of the equation is

$$y(t) = c_1 \cos\sqrt{\frac{c}{a}}t + c_2 \sin\sqrt{\frac{c}{a}}t.$$

Hence, all solutions are bounded as $t \to \infty$.

(b): $a > 0$ and $b > 0$, but $c = 0$

The characteristic equation is

$$ar^2 - br = 0$$

Thus the possible values of $r$ are $r_1 = 0$ and $r_2 = -\frac{b}{a}$, and the general solution of the equation is

$$y(t) = c_1 + c_2 e^{\frac{-bt}{a}}.$$

Hence, $y(t) \to c_1,$ as $t \to \infty$, since $a,\ b$ are positive constants. Obviously, $c_1$ depends on the initial conditions.

From $y^{'}(0) = y^{'}(0)$, we can get $c_2 = -\frac{ay_0^{'}}{b}$.

From $y(0) = y_0$, we get $c_1 = -\frac{ay_0^{'}}{b} + y_0$.

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## Section 3.6

**Answer:** In each of the following problems, find the general solutions of the given differential equations.

1. $y^{''} - 2y^{'} + 3y = 3e^{2t}$

4. $y^{''} + 2y^{'} = 3 + 4\sin 2t$

7. $2y^{''} + 3y^{'} + y = t + 3\sin t$

10. $u^{''} + \omega_0^2 u = \cos\omega_0 t$

11. $y^{''} + y^{'} + 4y = 2\sinh t$

**Answer:** 1. The characteristic equation is

$$r^2 - 2r - 3 = (r-3)(r+1) = 0$$

Thus the possible values of $r$ are $r_1 = 3$ and $r_2 = -1$, and the general solution of the homogeneous equation is

$$y(t) = c_1 e^{3t} + c_2 e^{-t}.$$

Let $Y(t) = Ae^{2t}$ where $A$ is a constant to be determined. On substituting to the original equation, we get

$$-15Ae^{2t} = 3e^{2t}.$$

So $A = -\frac{1}{5}$ and

$$y(t) = c_1 e^{3t} + c_2 e^{-t} - \frac{1}{5}e^{2t}$$

is the general solution of the original equation.

4. The characteristic equation is

$$r^2 + 2r = r(r+2) = 0$$

Thus the possible values of $r$ are $r_1 = 0$ and $r_2 = -2$, and the general solution of the homogeneous equation is

$$y(t) = c_1 + c_2 e^{-2t}.$$

Let $Y(t) = At + B\sin 2t + C\cos 2t$ where $A, B\text{and}, C$ are constants to be determined. On substituting to the original equation, we get

$$-4(B+C)\sin 2t + 4(B-C)\cos 2t + 2A = 3 + 4\sin 2t.$$

So $A = \frac{3}{2}$, $B = \frac{-1}{2}$ and $C = \frac{-1}{2}$ and

$$y(t) = c_1 + c_2 e^{-2t} + \frac{3t}{2} - \frac{1}{2}(\sin 2t + \cos 2t)$$

is the general solution of the original equation.

7. The characteristic equation is

$$2r^2 + 3r + 1 = (2r+1)(r+1) = 0$$

Thus the possible values of $r$ are $r_1 = \frac{-1}{2}$ and $r_2 = -1$, and the general solution of the homogeneous equation is

$$y(t) = c_1 e^{-t} + c_2 e^{\frac{-t}{2}}.$$

Let $Y(t) = A + Bt + Ct^2 + D\sin t + E\cos t$ where $A, B, C, D,$ and$E$ are constants to be determined. On substituting to the original equation, we get

$$4C(A+3B)+(B+6C)t+Ct^2+(-2D-3E+D)\sin t+(-2E+3D+E)\cos 2t = t^2+3\sin t.$$

So $A = 14,\ B = -6,\ \ C = 1,\ D = \frac{-3}{10}$ and $E = \frac{-9}{13}$ and

$$y(t) = c_1 e^{-t} + c_2 e^{\frac{-t}{2}} + 14t - 16Bt + t^2 - \frac{3}{10}\sin t - \frac{9}{13}\cos t$$

is the general solution of the original equation.

10. The characteristic equation is

$$r^2 + \omega_0^2 = 0$$

Thus the possible values of $r$ are $r_1 = i\omega_0$ and $r_2 = -i\omega_0$, and the general solution of the homogeneous equation is

$$u(t) = c_1 \cos\omega_0 t + c_2 \sin\omega_0 t.$$

Let $Y(t) = At\cos\omega_0 t + Bt\sin\omega_0 t$ where $A$ and $B$ are constants to be determined. On substituting to the original equation, we get

$$-2A\omega_0 \sin\omega_0 t + 2B\omega_0 \cos\omega_0 t = \cos\omega_0 t.$$

So $A = 0,\ B = \frac{1}{2\omega_0}$ and

$$u(t) = c_1 \cos\omega_0 t + c_2 \sin\omega_0 t + \frac{t}{2\omega_0}\sin\omega_0 t$$

is the general solution of the original equation.

11. The characteristic equation is

$$r^2 + r + 4 = 0$$

Thus the possible values of $r$ are $r_1 = \frac{-1}{2} + \frac{\sqrt{15}i}{2}$ and $r_2 = \frac{-1}{2} - \frac{\sqrt{15}i}{2}$, and the general solution of the homogeneous equation is

$$u(t) = e^{-\frac{t}{2}}(c_1 \cos \frac{\sqrt{15}t}{2} + c_2 \sin \frac{\sqrt{15}t}{2}).$$

Let $Y(t) = Ae^t + Be^{-t}$ where $A$ and $B$ are constants to be determined. On substituting to the original equation, we get

$$6Ae^t + 4Be^{-t} = e^t + e^{-t}.$$

So $A = \frac{1}{6},\ B = \frac{-1}{4}$ and

$$u(t) = e^{-\frac{t}{2}}(c_1 \cos \frac{\sqrt{15}t}{2} + c_2 \sin \frac{\sqrt{15}t}{2}) + \frac{1}{6}e^t - \frac{1}{4}e^{-t}$$

is the general solution of the original equation.

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In each of the following problems, find the solutions of the given initial problem.

13. $y'' + y' - 2y = 2t;\ y(0) = 0;\ y'(0) = 1$

16. $y'' - 2y' - 3y = 3te^t;\ y(0) = 1;\ y'(0) = 0$

**Answer:** 13. The characteristic equation is

$$r^2 + r - 2 = 0$$

Thus the possible values of $r$ are $r_1 = -2$ and $r_2 = 1$, and the general solution of the homogeneous equation is

$$u(t) = c_1 e^t + c_2 e^{-2t}.$$

Let $Y(t) = A + Bt$ where $A$ and $B$ are constants to be determined. On substituting to the original equation, we get

$$B - 2(A + Bt) = 2t.$$

So $A = -1,\ B = \frac{-1}{2}$ and

$$y(t) = c_1e^t + c_2e^{-2t} - t - \frac{1}{2}$$

is the general solutions of the original equations.

From $y(0) = 0$, we get $c_1 + c_2 = 1$.

From $y^{'}(0) = 1$, we get $c_1 - 2c_2 = 1$.

Hence, $c_1 = 1$ and $c_2 = \frac{-1}{2}$ and the solution of the initial problem is

$$y(t) = e^t - \frac{1}{2}e^{-2t} - t - \frac{1}{2}.$$

16. The characteristic equation is

$$r^2 - 2r - 3 = 0$$

Thus the possible values of $r$ are $r_1 = 3$ and $r_2 = -1$, and the general solution of the homogeneous equation is

$$u(t) = c_1e^{3t} + c_2e^{-t}.$$

Let $Y(t) = (A+Bt)e^{2t}$ where $A$ and $B$ are constants to be determined. On substituting to the original equation, we get

$$2B - 3A - 3Bt = 3t.$$

So $A = \frac{-2}{3},\ B = -1$ and

$$y(t) = c_1e^{t3} + c_2e^{-t} - (\frac{2}{3}t)e^{2t}$$

is the general solution of the original equation.

From $y(0) = 0$, we get $c_1 + c_2 = \frac{5}{3}$.

From $y^{'}(0) = 1$, we get $3c_1 - c_2 = \frac{7}{3}$.

Hence, $c_1 = 1$ and $c_2 = \frac{2}{3}$ and the solution of the initial problem is

$$y(t) = e^{3t} - \frac{2}{3}e^{-t} - (t + \frac{2}{3})e^{2t}.$$

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## Section 3.7

In each of Problems use the method of variation of parameters to find a particular solution of the differential equation. Then check your answer by using the method of undetermined coefficients.

2. $y'' - y' - 2y = 2e^{-t}$

4. $4y'' - 4y' + y = 16e^{\frac{t}{2}}$

**Answer:** 2. The characteristic equation is

$$r^2 - r - 2 = 0$$

Thus the possible values of $r$ are $r_1 = 2$ and $r_2 = -1$, and the general solutions of the homogeneous equation are

$$u(t) = c_1 y_1 + c_2 y_2 = c_1 e^{2t} + c_2 e^{-t}.$$

Method of variation of Parameters:

$W(y_1, y_2) = -3e^t$ then a particular solution of the original equation is

$$\begin{aligned} Y(t) &= -e^{2t} \int \frac{e^{-t}(2e^{-t})}{-3e^t} dt + e^{-t} \int \frac{e^{2t}(2e^{-t})}{-3e^t} dt \\ &= \frac{-1}{9} e^{-t} - \frac{2}{3} t e^{-t} \end{aligned}$$

Hence,

$$y(t) = c_1 e^{2t} + c_2 e^{-t} - \frac{2}{3} t e^{-t}$$

are the general solutions of the original equation.

Method of undetermined coefficients:

Let $Y(t) = Ate^{-t}$ where $A$ is constant to be determined. On substituting to the original equation, we get

$$-3e^{-t} = 2e^{-t}.$$

So $A = \frac{-2}{3}$ and

$$y(t) = c_1 e^{2t} + c_2 e^{-t} - \frac{1}{9} e^{-t} - \frac{2}{3} t e^{-t}$$

is the general solution of the original equation.

4. The original equation can written as:

$$y^{''} - y^{'} + \frac{1}{4} y = 4e^{\frac{t}{2}}$$

The characteristic equation is

$$r^2 - r + \frac{1}{4} = 0$$

Thus the possible values of $r$ are $r = \frac{1}{2}$, and the general solutions of the homogeneous equation are

$$u(t) = c_1 y_1 + c_2 y_2 = c_1 e^{\frac{t}{2}} + c_2 t e^{\frac{t}{2}}.$$

<u>Method of variation of Parameters:</u>

$W(y_1, y_2) = e^t$ then a particular solution of the original equation is

$$\begin{aligned} Y(t) &= -e^{\frac{t}{2}} \int \frac{t e^{\frac{t}{2}} (4e^{\frac{t}{2}})}{e^t} dt + t e^{\frac{t}{2}} \int \frac{e^{\frac{t}{2}} (4e^{\frac{t}{2}})}{e^t} dt \\ &= 2t^2 e^{\frac{t}{2}} \end{aligned}$$

Hence,

$$y(t) = c_1 e^{\frac{t}{2}} + c_2 t e^{\frac{t}{2}} + 2t^2 e^{\frac{t}{2}}$$

are the general solutions of the original equation.

<u>Method of undetermined coefficients:</u>

Let $Y(t) = At^2 e^{\frac{t}{2}}$ where $A$ is constant to be determined. On substituting to the original equation, we get $A = 2$ and

$$y(t) = c_1 e^{\frac{t}{2}} + c_2 t e^{\frac{t}{2}} + 2t^2 e^{\frac{t}{2}}$$

are the general solutions of the original equation.

❏

In each of Problems find the general solution of the given differential equation.

5. $y'' + y = \tan t,\ 0 < t < \frac{\pi}{2}$

8. $y'' + 4y = 3\csc 2t,\ 0 < t < \frac{\pi}{2}$

**Answer:** 5.The characteristic equation is

$$r^2 + 1 = 0$$

Thus the possible values of $r$ are $r_1 = i$ and $r_2 = -i$, and the general solution of the equation is

$$y(t) = c_1 \cos t + c_2 \sin t.$$

$W(y_1, y_2) = 1$ then a particular solution of the original equation is

$$\begin{aligned} Y(t) &= -\cos t \int \sin t \tan t dt + \sin t \int \cos t \tan t dt \\ &= -\cos t \ln \tan t + \sec t,\ 0 < t < \frac{\pi}{2} \end{aligned}$$

Hence,

$$y(t) = c_1 \cos t + c_2 \sin t - \cos t \ln \tan t + \sec t,\ 0 < t < \frac{\pi}{2}$$

are the general solutions of the original equation.

8. The characteristic equation is

$$r^2 + 4 = 0$$

Thus the possible values of $r$ are $r_1 = 2i$ and $r_2 = -2i$, and the general solution of the equation is

$$y(t) = c_1 \cos 2t + c_2 \sin 2t.$$

$W(y_1, y_2) = 2$ then a particular solution of the original equation is

$$\begin{aligned} Y(t) &= -\cos 2t \int \frac{3 \sin 2t \csc 2t}{2} dt + \sin 2t \int \frac{3 \cos 2t \csc 2t}{2} dt \\ &= -\frac{3t}{2} \cos 2t + \frac{3}{4} \sin 2t \ln \sin t,\ 0 < t < \frac{\pi}{2} \end{aligned}$$

Hence,

$$y(t) = c_1 \cos 2t + c_2 \sin 2t - \frac{3t}{2} \cos 2t + \frac{3}{4} \sin 2t \ln \sin t, \ 0 < t < \frac{\pi}{2}$$

are the general solutions of the original equation.

❏

In each of the problems verify that the given functions $y_1$ and $y_2$ satisfy the corresponding homogeneous equation; then find a particular solution of the given nonhomogeneous equation.

13. $t^2 y'' - 2y = 3t^2 - 1, \ t > 0; \ y_1 = t^2, \ y_2 = t^{-1}$

15. $ty'' + (1+t)y' + 4y = t^2 e^{2t}, \ y_1 = 1 + t, \ y_2 = e^t$

**Answer:** 5. It is easy to check that $y_1$ and $y_2$ satisfy

$$t^2 y'' - 2y = 3t^2 - 1, \ t > 0.$$

$W(y_1, y_2) = -3$ and let

$$\begin{aligned} Y(t) &= -t^2 \int \frac{t^{-1}(3t^2 - 1)}{-3} + t^{-1} \int \frac{t^2(3t^2 - 1)}{-3} \\ &= \frac{3}{10} t^4 - \frac{1}{3} t^2 \ln t + \frac{1}{9} t^2, \ t > 0. \end{aligned}$$

then a particular solution of the original equation is

$$y(t) = \frac{3}{10} t^4 - \frac{1}{3} t^2 \ln t, \ t > 0.$$

8. It is easy to check that $y_1$ and $y_2$ satisfy

$$ty'' + (1+t)y' + 4y = t^2 e^{2t}$$

$W(y_1, y_2) = te^t$ and let

$$\begin{aligned} Y(t) &= -(1+t) \int \frac{e^t(t^2 e^{2t})}{te^t} + e^t \int \frac{(1+t)(t^2 e^{2t})}{te^t} \\ &= (\frac{1}{2} t^2 - \frac{5}{4} t + \frac{5}{4}) e^{2t}. \end{aligned}$$

then a particular solution of the original equation is

$$y(t) = (\frac{1}{2}t^2 - \frac{5}{4}t + \frac{5}{4})e^{2t}.$$

❏

22. By choosing the lower limit of the integration in Equ.(28) in the text as the initial point $t_0$, show that $Y(t)$ becomes

$$Y(t) = \int_{t_0}^{t} \frac{y_1(s)y_2(t) - y_1(t)y_2(s)}{y_1(s)y_2'(s) - y_1'(s)y_2(s)} g(s)ds.$$

Show that $Y(t)$ is a solution of the initial value problem

$$L[y] = g(t),\ y(t_0) = 0,\ y'(t_0) = 0.$$

Thus $Y$ can be identified with $v$ in problem 21.

**Answer:**

$$Y(t) = -y_1(t) \int \frac{y_2(s)g(s)}{W(y_1, y_2)(s)} ds + y_2(t) \int \frac{y_1(s)g(s)}{W(y_1, y_2)(s)} ds$$

$$= \int_{t_0}^{t} \frac{-y_1(t)y_2(s)g(s)}{y_1(s)y_2'(s) - y_2(s)y_1'(s)} ds + \int_{t_0}^{t} \frac{y_2(t)y_1(s)g(s)}{y_1(s)y_2'(s) - y_2(s)y_1'(s)} ds$$

$$= \int_{t_0}^{t} \frac{y_1(s)y_2(t) - y_2(s)y_1(t)}{y_1(s)y_2'(s) - y_2(s)y_1'(s)} g(s)ds$$

Hence,

$$Y(0) = \int_{t_0}^{t_0} \frac{y_1(s)y_2(t) - y_2(s)y_1(t)}{y_1(s)y_2'(s) - y_2(s)y_1'(s)} g(s)ds$$

From

$$Y(t) = -y_1(t) \int_{t_0}^{t} \frac{y_2(s)g(s)}{W(y_1, y_2)(s)} ds + y_2(t) \int_{t_0}^{t} \frac{y_1(s)g(s)}{W(y_1, y_2)(s)} ds,$$

we get

$$Y'(t) = -y_1'(t)\int_{t_0}^{t} \frac{y_2(s)g(s)}{W(y_1,y_2)(s)}ds + y_2'(t)\int_{t_0}^{t} \frac{y_1(s)g(s)}{W(y_1,y_2)(s)}ds$$

then

$$Y'(t_0) = -y_1'(t_0)\int_{t_0}^{t_0} \frac{y_2(s)g(s)}{W(y_1,y_2)(s)}ds + y_2'(t_0)\int_{t_0}^{t_0} \frac{y_1(s)g(s)}{W(y_1,y_2)(s)}ds = 0$$

$$Y''(t) = -y_1''(t)\int_{t_0}^{t} \frac{y_2(s)g(s)}{W(y_1,y_2)(s)}ds + y_2''(t)\int_{t_0}^{t} \frac{y_1(s)g(s)}{W(y_1,y_2)(s)}ds + g(t)$$

Hence, $Y''(t) + p(t)Y'(t) + q(t)Y(t)$

$$= [y_2''(t) + p(t)y_2'(t) + q(t)y_2(t)]\int_{t_0}^{t} \frac{y_1(s)g(s)}{W(y_1,y_2)(s)}ds$$

$$-[y_1''(t) + p(t)y_1'(t) + q(t)y_1(t)]\int_{t_0}^{t} \frac{y_2(s)g(s)}{W(y_1,y_2)(s)}ds + g(t) \equiv g(t).$$

Therefore, $Y(t)$ is a solution of the initial value problem

$$L[y] = g(t),\ y(t_0) = 0,\ y'(t_0) = 0.$$

❏

23. (a) Use the result of Problem 22 to show that the solution of the initial value problem

$$y'' + y = g(t),\ y(t_0) = 0,\ y'(t_0) = 0$$

is

$$y = \int_{t_0}^{t} \sin(t-s)g(s)ds$$

(b) Find the solution of the initial value problem

$$y'' + y = g(t),\ y(0) = y_0,\ y'(0) = y_0'.$$

**Answer:** The characteristic equation is

$$r^2 + 1 = 0$$

Thus the possible values of $r$ are $r_1 = i$ and $r_2 = -i$, and the general solution of the homogeneous equation is

$$y(t) = c_1 \cos t + c_2 \sin t.$$

(a) From the result in Problem 22, we get the solution of the initial problem

$$Y(t) = \int_{t_0}^{t} \frac{y_1(s)y_2(t) - y_1(t)y_2(s)}{y_1(s)y_2'(s) - y_1'(s)y_2(s)} g(s)ds$$

$$= \int_{t_0}^{t} \frac{\cos s \sin t - \cos t \sin s}{\cos s \cos s + \sin s \sin s} g(s)ds$$

$$= \int_{t_0}^{t} (\cos s \sin t - \cos t \sin s) g(s)ds$$

$$= \int_{t_0}^{t} \sin(t-s) g(s)ds$$

(b) Because

$$y(t) = c_1 \cos t + c_2 \sin t$$

satisfy the corresponding homogenous equation, let $y(0) = y_0$ and $y'(0) = y_0'$, then we get $c_1 = y_0$ and $c_2 = y_0'$. Hence, the solution of the initial problem is

$$y(t) = y_0 \cos t + y_0' \sin t + \int_{t_0}^{t} \sin(t-s) g(s)ds.$$

❏

28. The method of reduction of order (Section 3.5) can also be used for the nonhomogeneous equation

$$y''(t) + p(t)y'(t) + q(t)y(t) = g(t), \qquad (i)$$

provided one solution $y_1$ of the corresponding homogeneous equation is known. Let $y = v(t)y_1(t)$ and show that $y$ satisfies Equ.(i) if and only if

$$y_1 v^{''} + [2y_1^{'}(t) + p(t)y_1(t)]v^{'} = g(t), \qquad (ii)$$

Equation (ii) is a first order linear equation for $v^{'}$. Solving this equation, integrating the result, and then multiplying by $y_1(t)$ lead to the general solution of Eq.(i).

**Answer:** Let $y = v(t)y_1(t)$, then

$$y^{'}(t) = v^{'}(t)y_1(t) + v(t)y_1^{'}(t)$$

$$y^{''}(t) = v^{''}(t)y_1(t) + 2v^{'}(t)y_1^{'}(t) + v(t)y_1^{''}(t)$$

So,

$$\begin{aligned} & y^{''}(t) + p(t)y^{'}(t) + q(t)y(t) \\ = & \, y_1 v^{''} + [2y_1^{'}(t) + p(t)y_1(t)]v^{'} + v(y_1^{''}(t) + p(t)y_1^{'}(t) + q(t)y_1(t)) \\ = & \, g(t) \end{aligned}$$

if $y_1$ is the solution of the corresponding homogeneous equation and

$$y_1 v^{''} + [2y_1^{'}(t) + p(t)y_1(t)]v^{'} = g(t).$$

❏

In each of the problems use the method outlined in Problem 28 to solve the given differential equation.

29. $t^2 y^{''} - 2ty^{'} + 2y = 4t^2, \; t > 0; \; y_1(t) = t$

30. $4y^{''} + 7ty^{'} + 5y = t, \; t > 0; \; y_1(t) = t$

**Answer:** 29. Use the method in Problem 28, the original equation can be written as

$$y^{''}(t) - \frac{2}{t}y^{'}(t) + \frac{2}{t^2}y(t) = 4.$$

Let $v$ satisfies

$$y_1 v'' + [2y_1'(t) + p(t)y_1(t)]v' = g(t)$$

then $tv'' = 4$ and $v = 4t \ln t + c_1 t$.
So $y_2 = y_1 v = 4t^2 \ln t + c_1 t^2$ is the solution of $t^2 y'' - 2ty' + 2y = 4t^2,\ t > 0$.
Hence, the general solution are

$$y_2 = c_1 t^2 + c_2 t + 4t^2 \ln t$$

29. Use the method in Problem 28, the original equation can be written as

$$y''(t) + \frac{7}{t} y'(t) + \frac{5}{t^2} y(t) = \frac{1}{t}.$$

Let $v$ satisfies

$$y_1 v'' + [2y_1'(t) + p(t)y_1(t)]v' = g(t)$$

then $v'' + \frac{5}{t} v' = 1$ and $v = \frac{1}{12} t^2 + c_1 t^{-4}$.
So $y_2 = y_1 v = \frac{1}{12} t + c_1 t^{-5}$ is the solution of $4y'' + 7ty' + 5y = t,\ t > 0$.
Hence, the general solution are

$$y_2 = c_1 t^{-5} + c_2 t^{-1} + \frac{1}{12} t.$$

❏

## Section 3.8

In the following problem, determine $\omega_0$, $R$ and $\delta$ so as to write the given expression in the form $u = R \cos \omega_0 t - \delta$.

3. $u = 4 \cos 3t - 2 \sin 3t$

**Answer:** $R = \sqrt{4^2 + (-2)^2} = 2\sqrt{5}$ and $\delta = \arctan(\frac{-2}{4}) \cong= -0.4636$
Hence $u = 2\sqrt{5}\cos 3t - \delta$ with $\delta = \arctan(\frac{-2}{4}) \cong= -0.4636$.

❏

5. A mass weighing 2 lb stretches a spring 6 in. If the mass is pulled down an additional 3 in. and then released, and if there is no damping, determine the position $u$ of the mass at time $t$. Plot $u$ versus $t$. Find the frequency, period, and amplitude of the motion.

**Answer:** Generally, the motion the mass is described by the following equation

$$mu^{''}(t) + \gamma u^{'}(t) + ku(t) = F(t).$$

Since nothing is said in the statement of the problem about an external force, we assume that $F(t) = 0$.
Also, $\gamma = 0$ because there is no damping.

To determine $m$ note that

$$m = \frac{w}{g} = \frac{2lb}{32ft/sec^2} = \frac{1}{16}\frac{lb - sec^2}{ft}.$$

The spring constant $k$ is be found from the statement that the mass stretches the spring 6 in, or by $\frac{1}{2}$ ft. Thus

$$k = \frac{2lb}{1/2ft} = 4\frac{lb}{ft}.$$

The equation of motion of the mass is

$$u^{''} + 64u(t) = 0.$$

The initial conditions are $u(0) = \frac{1}{4}$ and $u^{'}(0) = 0$.
The general solution is

$$u = A\cos(8t) + B\sin(8t).$$

The solution satisfies the initial conditions $u(0) = \frac{1}{4}$ and $u^{'} = 0$, so we can get $A = \frac{1}{4}$ and $B = 0$.
Hence, the position $u$ of the mass at time $t$ is

$$\frac{1}{4}\cos(8t)ft, t\text{in sec};\ \omega = 8rad/sec,\ T = \pi/4sec,\ R = 1/4ft.$$

❏

19.Assume that the system described by the equation $mu^{''} + \gamma u^{'} + ku = 0$ is either critically damped or overdamped. Show that the mass can pass through the equilibrium position at most once, regardless of the initial conditions.

**Answer:**

Case A: If the system is critically damped, then $\gamma = 2\sqrt{km}$.

The characteristic equation of the original differential equation is

$$mr^2 + \gamma r + k = 0$$

Thus the possible value of $r$ is $r = \frac{-\gamma}{2m}$, and the general solution of the homogeneous equation is

$$y(t) = (c_1 + c_2 t)e^{\frac{-\gamma}{2m}t}.$$

Obviously, $y(t)$ at most have one zero, regardless the coefficients of $c_1$ and $c_2$, because $y(t)$ always nondecreasing or nonincreasing. Hence, the mass can pass through the equilibrium position at most once, regardless of the initial conditions.

Case B: If the system is overdamped, then $\gamma > 2\sqrt{km}$.

The characteristic equation of the original differential equation is

$$mr^2 + \gamma r + k = 0$$

Thus the possible values of $r$ are $r_1 = \frac{-\gamma+\sqrt{\gamma^2-4mk}}{2m} < 0$ and $r_2 = \frac{-\gamma-\sqrt{\gamma^2-4mk}}{2m} < 0$ for $\gamma > 2\sqrt{km}$, and the general solution of the homogeneous equation is

$$\begin{aligned} y(t) &= c_1 e^{\frac{-\gamma+\sqrt{\gamma^2-4mk}}{2m}t} + c_2 e^{\frac{-\gamma-\sqrt{\gamma^2-4mk}}{2m}t} \\ &= e^{\frac{-\gamma}{2m}}(c_1 e^{\frac{\sqrt{\gamma^2-4mk}}{2m}t} + c_2 e^{\frac{-\sqrt{\gamma^2-4mk}}{2m}t}). \end{aligned}$$

Obviously, $y(t)$ at most have one zero, regardless the coefficients of $c_1$ and $c_2$, then the mass can pass through the equilibrium position at most once, regardless of the initial conditions. ❏

24. The position of a certain spring-mass system satisfied the initial value problem

$$\frac{3}{2}u^{''} + ku = 0,\ u(0) = 2,\ u^{'}(0) = v.$$

If the period and amplitude of the resulting motion are observed to be $\pi$ and 3, respectively, determine the value of $k$ and $v$.

**Answer:** The period of the motion is

$$T = 2\pi(\frac{m}{k})^{\frac{1}{2}} = 2\pi(\frac{3/2}{k})^{\frac{1}{2}} = \pi.$$

So we get $k = 6$ and the equation can written as $u^{''} + 4u = 0$.
Obviously, the general solution of this equation is

$$u(t) = A\cos 2t + B\sin 2t.$$

From $u(0) = 2$, then $A = 2$.
From the amplitude of the resulting motion is 3, $R = \sqrt{A^2 + B^2} = 3$ and then $B = \pm\sqrt{5}$.
Hence,

$$u(t) = 2\cos 2t + \pm\sqrt{5}\sin 2t$$

and

$$v = u^{'}(0) = \pm 2\sqrt{5}$$

❏

## Section 3.9

17. Consider a vibration system described by the initial value problem

$$u^{''} + \frac{1}{4}u^{'} + 2u = 2\cos\omega t,\ u(0) = 0,\ u^{'}(0) = 2.$$

(a.) Determine the steady-state part of the solution of this problem.

(b.) Find the amplitude $A$ of the steady-state solution in terms of $\omega$.

(c.) Plot $A$ versus $\omega$.

(d.) Find the maximum value of $A$ and the frequency $\omega$ for which it occurs.

**Answer:** The characteristic equation of the original differential equation is

$$r^2 + \frac{1}{4}r + 2 = 0$$

Thus the possible values of $r$ are $r_1 = -\frac{1}{8} + \frac{\sqrt{127}}{8}i$ $r_2 = -\frac{1}{8} - \frac{\sqrt{127}}{8}i$, and the general solution of the homogeneous equation is

$$u(t) = e^{-\frac{1}{8}t}(c_1\cos\frac{\sqrt{127}}{8}t + c_2\sin\frac{\sqrt{127}}{8}t).$$

The motion of this system can be described by

$$u(t) = e^{-\frac{1}{8}t}(c_1\cos\frac{\sqrt{127}}{8}t + c_2\sin\frac{\sqrt{127}}{8}t) + R\cos(\omega t - \delta).$$

where

$$R = 2/\sqrt{(2-\omega^2)^2 + \frac{\omega^2}{16}}$$

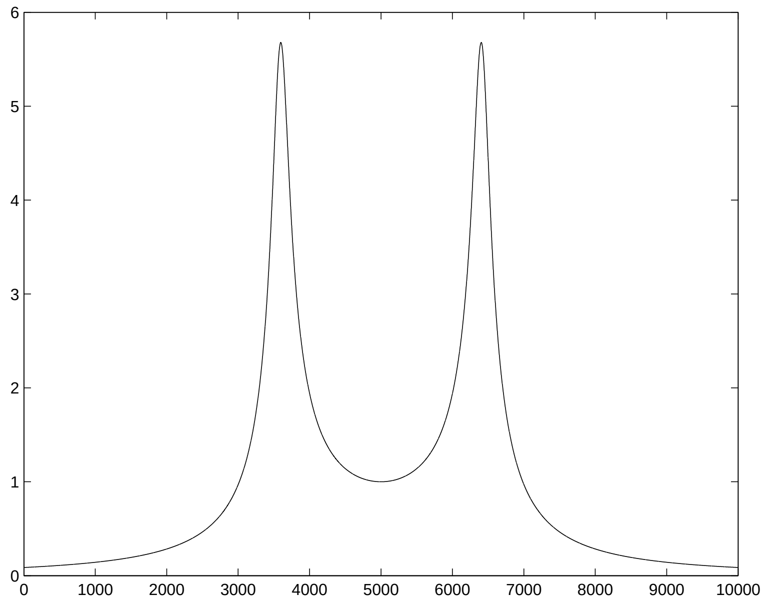


FIGURE 1. for problem 17
$\omega = 0.7$

$$\cos\delta = \frac{2-\omega^2}{\sqrt{(2-\omega^2)^2+\frac{\omega^2}{16}}}$$

and

$$\sin\delta = \frac{\frac{1}{4}\omega}{\sqrt{(2-\omega^2)^2+\frac{\omega^2}{16}}}$$

(a.) The steady-state part of the solution of this problem is

$$u = \frac{[32(2-\omega^2)\cos\omega t + 8\omega\sin\omega t]}{64-63\omega^2+16\omega^4}.$$

(b.)The amplitude $A$ of the steady-state is

$$A = \frac{8}{\sqrt{64-63\omega^2+16\omega^4}}.$$

(c.) For graph of $A$ versus $\omega$, see Figure 1.

(d.)The maximum value of $A$ is

$$A = \frac{64}{\sqrt{127}}$$

and the corresponding frequency

$$\omega = \frac{3\sqrt{14}}{8}.$$

❑

18. Consider the forced but undamped system described by the initial value problem

$$u'' + u = 3\cos\omega t, \ u(0) = 0, \ u'(0) = 0.$$

a. Find the solution $u(t)$ for $\omega \neq 1$.

b. Plot the solution $u(t)$ versus $t$ for $\omega = 0.7$, $\omega = 0.8$ and $\omega = 0.9$. Describe how the response $u(t)$ changes as $\omega$ varies in this interval. What happens as $\omega$ takes on values closer and closer to 1? Note that the natural frequency of the unforced system is $\omega_0 = 1$.

**Answer:** (a.) $\omega_0 = \sqrt{k/m} = 1$, if $\omega \neq \omega_0$, then the general solution is

$$u = c_1 \cos t + c_2 \sin t + \frac{3}{1-\omega^2}\cos\omega t.$$

From $u(0) = 0, \ u'(0) = 0$, we get $c_1 = -\frac{3}{1-\omega^2}, \ c_2 = 0$.

Hence the solution is

$$u(t) = \frac{3}{1-\omega^2}(\cos\omega t - \cos t).$$

(b.) For the graphs of the solution $u(t)$ versus $t$ for $\omega = 0.7$, $\omega = 0.8$ and $\omega = 0.9$, see Figure 2., Figure 3. and Figure 4.

We can write above solution as:

$$u(t) = (\frac{3}{1-\omega^2}\sin\frac{(1-\omega)t}{2})\sin\frac{(1+\omega)t}{2}.$$

If $|1-\omega|$ is small, the $|1+\omega|$ is much greater than $|1-\omega|$. Consequently, $\sin\frac{(1+\omega)t}{2}$ is rapidly oscillating function compared to $\sin\frac{(1-\omega)t}{2}$. Thus the motion is a rapid oscillation with frequency $\frac{1+\omega}{2}$ but with a

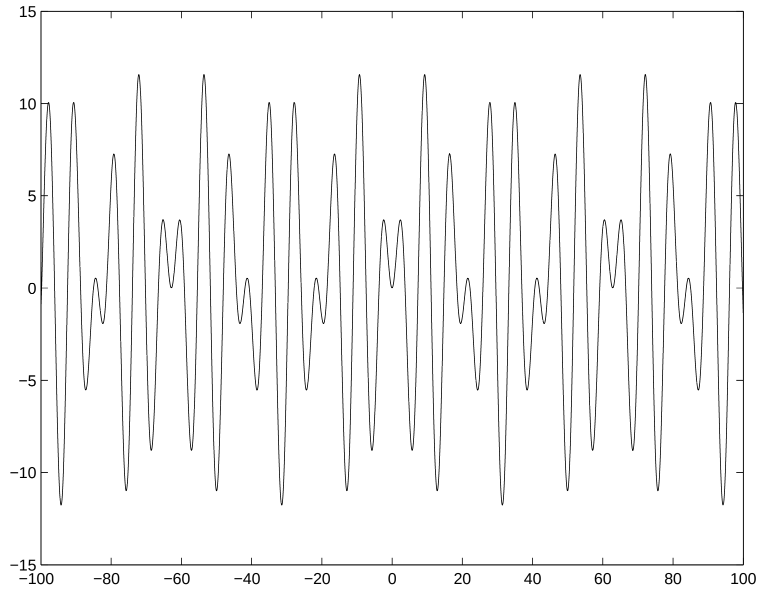


FIGURE 2. for problem 18
$\omega = 0.7$

slowly varying sinusoidal amplitude $\frac{3}{1-\omega^2} \sin \frac{(1-\omega)t}{2}$.

The amplitude of $u(t)$ gets larger and lager as $w$ varies from $\omega = 0.7$, $\omega = 0.8$ to $\omega = 0.9$, and closer and closer to 1, the natural frequency of the unforced system.

❑

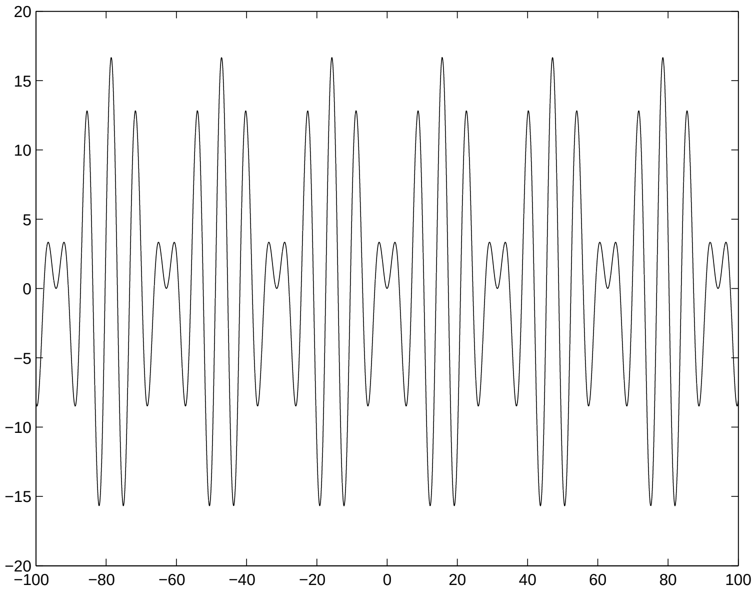


FIGURE 3. for problem 18
$\omega = 0.8;$

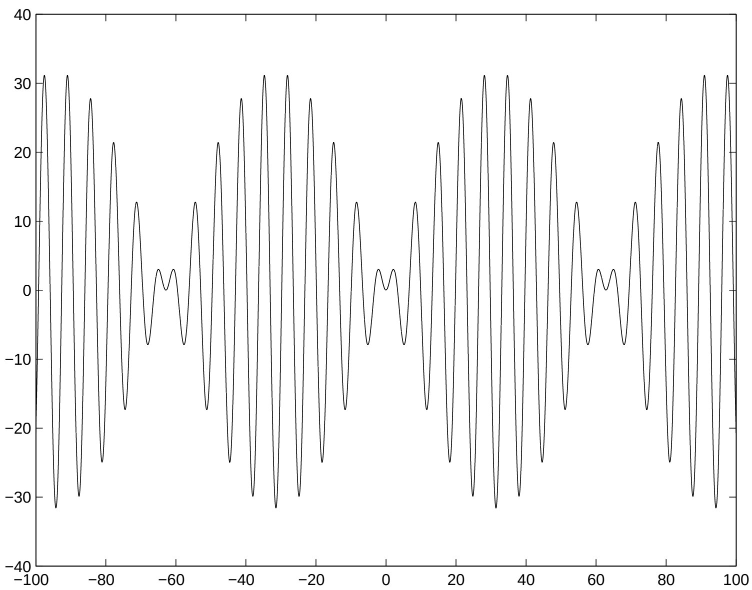


FIGURE 4. for problem 18
$\omega = 0.9$