Sample Solutions of Assignment 4 for MAT3270B: 3.1,3.2,3.3

Section 3.1

Find the general solution of the given. differential equation

1. $y^{''} + 2y^{'} - 3y = 0$

4. $2y^{''} - 3y^{'} + y = 0$

7. $y^{''} - 9y^{'} + 9y = 0$

Answer: 1. The characteristic equation is

$$r^2 + 2r - 3 = (r + 3)(r - 1) = 0$$

Thus the possible values of r are $r_1 = -3$ and $r_2 = 1$, and the general solution of the equation is

$$y(t) = c_1 e^t + c_2 e^{3t}.$$

4. The characteristic equation is

$$2r^2 - 3r + 1 = (2r - 1)(r - 1) = 0$$

Thus the possible values of r are $r_1 = \frac{1}{2}$ and $r_2 = 1$, and the general solution of the equation is

$$y(t) = c_1 e^t + c_2 e^{\frac{t}{2}}.$$

7. The characteristic equation is

$$r^2 - 9r + 9 = (r - 4)(r - 5) = 0$$

Thus the possible values of r are $r_1 = 5$ and $r_2 = 4$, and the general solution of the equation is

$$y(t) = c_1 e^{4t} + c_2 e^{5t}.$$

❏

17. Find a differential equation whose general solution is $y = c_1e^{2t} + c_2e^{-3t}$

Answer: The the characteristic equation is

$$(r-2)(r+3) = r^2 + r - 6 = 0$$

So the equation is

$$y'' + y' - 6y = 0.$$

❏

21. Solve the initial value problem $y'' - y' - 2y = 0,\ y(0) = \alpha,\ y'(0) = 2$. Then find α so that the solution approaches zero as $t \to \infty$.

Answer: The characteristic equation is

$$r^2 - r - 2 = (r+1)(r-2) = 0$$

Thus the possible values of r are $r_1 = -1$ and $r_2 = 2$, and the general solution of the equation is

$$y(t) = c_1e^{2t} + c_2e^{-t}.$$

Using the first initial condition, we obtain

$$c_1 + c_2 = \alpha.$$

Using the second initial condition, we obtain

$$2c_1 - c_2 = 2.$$

By solving above equations we find that $c_1 = \frac{\alpha+2}{3}$ and $c_2 = \frac{2(\alpha+1)}{3}$. Hence,

$$y(t) = \frac{\alpha+2}{3}e^{2t} + c_2 = \frac{2(\alpha+1)}{3}e^{-t}.$$

From $y(t) \to 0$ as $t \to \infty$, we find $\alpha = -2$.

❏

22. Solve the initial value problem $4y'' - y = 0,\ y(0) = 2,\ y'(0) = \beta$. Then find β so that the solution approaches zero as $t \to \infty$.

Answer: The the characteristic equation is

$$4r^2 - 1 = (2r + 1)(2r - 1) = 0$$

Thus the possible values of r are $r_1 = \frac{1}{2}$ and $r_2 = \frac{-1}{2}$, and the general solution of the equation is

$$y(t) = c_1 e^{\frac{1}{2}t} + c_2 e^{-\frac{1}{2}t}.$$

Using the first initial condition, we obtain

$$c_1 + c_2 = 2.$$

Using the second initial condition, we obtain

$$2c_1 - c_2 = 2\beta.$$

By solving above equations we find that $c_1 = \beta + 1$ and $c_2 = 1 - \beta$. Hence,

$$y(t) = (\beta + 1)e^{\frac{1}{2}t} + (1 - \beta)e^{-\frac{1}{2}t}.$$

From $y(t) \to 0$ as $t \to \infty$, we find $\beta = -1$. ❑

In each of the following problem determine the value of α, if any, for which all solutions tend to zero as $t \to \infty$; Also determine the value of α, if any, for which all (nonzero) solutions become unbounded as $t \to \infty$.

23.$y'' - (2\alpha - 1)y' + \alpha(\alpha - 1)y = 0$

Answer: The characteristic equation is

$$r^2 - (2\alpha - 1)r + \alpha(\alpha - 1) = (r - \alpha)(r - (\alpha - 1)) = 0$$

Thus the possible values of r are $r_1 = \alpha$ and $r_2 = \alpha - 1$, and the general solution of the equation is

$$y(t) = c_1 e^{\alpha t} + c_2 e^{(\alpha-1)t}.$$

If we want $y(t) \to 0$ as $t \to \infty$, then $\alpha < 0$ and $\alpha - 1 < 0$. Hence, in this case $\alpha < 0$;
If we want $y(t)$ become unbounded as $t \to \infty$, then $\alpha > 0$ and $\alpha - 1 > 0$. Hence, in this case $\alpha > 1$.

❏

27. Find an equation of the form $ay'' - by' + cy = 0$ for which all solutions approach a multiple of e^{-t} as $t \to \infty$.

Answer: We select $y_1 = e^{-t}$ and $y_2 = e^{-2t}$. Let $y(t) = c_1 y_1(t) + c_2 y_2(t)$ satisfy $ay'' - by' + cy = 0$, then the characteristic equation is

$$(r+1)(r+2) = (r^2 + 3r + 2) = 0.$$

Hence the equation is $y'' + 3y' + 2y = 0$.

❏

Section 3.2

Find the Wronskian of the given pair of functions.

1. $e^{2t},\ e^{\frac{-3t}{2}}$

3. $e^{-2t},\ te^{-2t}$

6. $\cos^2\theta,\ 1 + \cos 2\theta$

Answer: The computation is easy, so we just give the final result.

1. $W = \frac{-7}{2} e^{\frac{t}{2}}$

3. $W = e^{-4t}$

6. $W = 0$

❏

In the following problems determine the longest interval in which the given initial value problem is certain to have a unique twice differentiable solution. Do not attempt to find the solution.

7. $ty'' + 3y = t, y(1) = 1, y'(1) = 2$

11. $(x-3)y'' + xy' + (\ln|x|)y = 0, y(1) = 0, y'(1) = 1$

Answer: 7. The original solution can written as

$$y'' + \frac{3}{t}y = 1.$$

and $p(t) = 0,\ q(t) = \frac{3}{t},\ g(t) = 1$. Then then only point of discontinuity of the coefficients is $t = 0$. Therefore, the longest open interval, containing the initial point $t = 1$, in which all the coefficients are continuous, is $0 < t < \infty$.

Answer: 11. The original solution can written as

$$y'' + \frac{x}{x-3}y' + \frac{\ln|x|}{x-3} = 0.$$

and $p(t) = \frac{x}{x-3},\ q(t) = \frac{\ln|x|}{x-3},\ g(t) = 0$. Then the only points of discontinuity of the coefficients is $t = 0$, and $t = 3$. Therefore, the longest open interval, containing the initial point $t = 1$, in which all the coefficients are continuous, is $0 < t < 3$.

❏

14.Verify that $y_1(t) = 1$ and $y_2(t) = t^{\frac{1}{2}}$ are solutions of the differential equation $yy'' + (y')^2 = 0$ for $t > 0$. Then show that $c_1 + c_2t^{\frac{1}{2}}$ is not, in general, a solution of this equation. Explain why this result does not contradict Theorem 3.2.2.

Answer: It is easy to verify y_1 and y_2 are solutions of the differential equation $yy'' + (y')^2 = 0$ for $t > 0$, and $y = c_1 + c_2 t^{\frac{1}{2}}$ is not a solution(in general) of this equation.
This result does not contradict Theorem 3.2.2 because this equation is nonlinear.

❏

15. Show that if $y = \phi(t)$ is a solution of the differential equation $y'' + p(t)y' + q(t)y = g(t)$, where $g(t)$ is not always zero, the $y = c\phi(t)$, where c is any constant other than 1, is not a solution. Explain why this result does not contradict the remark following Theorem 3.2.2.

Answer:

$$[c\phi(t)]'' + p(t)[c\phi(t)]' + q(t)[c\phi(t)]$$

$$= c[\phi(t)'' + p(t)\phi(t)' + q(t)\phi(t)]$$

$$= cg(t) \neq g(t)$$

if c is a constant other than 1, and $g(t)$ is not always zero.
This result does not contradict Theorem 3.2.2 because this equation is not homogeneous.

❏

17. If the Wronskian W of f and g is $3e^{4t}$, and if $f(t) = e^{2t}$, find $g(t)$.

Answer:

$$W = f(t)g'(t) - f'(t)g(t) = e^{2t}g'(t) - 2e^{2t}g(t)$$

Let $W = 3e^{4t}$, we get the following equation

$$g^{'} - 2g(t) = 3e^{2t}.$$

From the above equation, $g(t) = te^{2t} + ce^{2t}$.

❏

19. If $W(f, g)$ is the Wronskian of f and g, and if $u = 2f - g,\ v = f + 2g$, find the Wronskian $W(u, v)$ of u and v in term of $W(f, g)$.

Answer:

$$W(u, v) = uv^{'} - u^{'}v$$

$$= (2f - g)(f^{'} - 2g^{'}) - (2f^{'} - g^{'})(f + 2g)$$

$$= 5fg^{'} - 5f^{'}g$$

$$= 5W(f, g).$$

❏

20. If the Wronskian of f and g is $t\cos t - \sin t$ and if $u = f + 3g,\ v = f - g$, find the Wronskian of u and v.

Answer:

$$W(u, v) = uv^{'} - u^{'}v$$

$$= (f + 3g)(f^{'} - g^{'}) - (f^{'} + 3g^{'})(f - g)$$

$$= -4fg^{'} + 4f^{'}g$$

$$= -4W(f, g) = -4(t\cos t - \sin t).$$

❏

In the following problems verify that the function y_1 and y_2 are solutions of the given differential equation. Do they constitute a fundamental set of solutions?

23. $y^{''} + 4y = 0,\ y_1(t) = \cos 2t,\ y_2(t) = \sin 2t$
25. $x^2y^{''} - x(x+2)y^{'} + (x+2)y = 0,\ x > 0,\ y_1(x) = x,\ y_2(x) = xe^x$

Answer: 23.

$$y_1(t) = \cos 2t,\ y_1^{'}(t) = -2\sin 2t,\ y_1^{''}(t) = -4\cos 2t$$

$$y_2(t) = \sin 2t,\ y_2^{'}(t) = 2\cos 2t,\ y_2^{''}(t) = -4\sin 2t$$

From above equation, we can verify that the function y_1 and y_2 are solutions of the given differential equation $y^{''} + 4y = 0$.
They constitute a fundamental set solutions because $W(y_1, y_2) = 2$.
25.

$$y_1(x) = x,\ y_1^{'}(x) = 1,\ y_1(x) = 0$$

$$y_2(x) = xe^x,\ y_2^{'}(x) = (1+x)e^x,\ y_2^{''}(x) = (1+x)e^x$$

From above equation, we can verify that the function y_1 and y_2 are solutions of the given differential equation $x^2y^{''} - x(x+2)y^{'} + (x+2)y = 0$.
They constitute a fundamental set solutions because $W(y_1, y_2) = x^2e^x$.

❏

Section 3.3

In the following problems determine whether the given pair of functions is linearly independent or linearly dependent.

3. $f(t) = e^{\lambda t}\cos\mu t,\ g(t) = e^{\lambda t}\sin\mu t,\ \mu \neq 0$
4. $f(x) = e^{3x},\ g(x) = e^{3(x-1)}$

Answer:

3.

$$W(f,g) = fg^{'} - f^{'}g$$

$$= e^{2\lambda t}\cos\mu t(\lambda\sin\mu t - \mu\sin\mu t) - e^{2\lambda t}\sin\mu t(\lambda\cos\mu t - \mu\sin\mu t)$$

$$= -\mu e^{2\lambda t} \neq 0 \quad \text{for } \mu \neq 0$$

Hence, the given pair of functions is linearly independent.

4.

$$W(f,g) = fg^{'} - f^{'}g$$

$$= e^{3x}3e^{3(x-1)} - 3e^{3x}e^{3(x-1)} = 0$$

Hence, the given pair of functions is linearly dependent. ❏

9. The Wronskian of two functions is $W(t) = t\sin^2 t$. Are the functions linearly independent or linearly dependent? Why?

Answer: The functions is linearly independent because W is not always zero. ❏

11. If the functions y_1 and y_2 are linearly independent solutions of $y^{''} + p(t)y^{'} + q(t)y = 0$, prove that c_1y_1 and c_2y_2 are also linearly independent solutions, provided that neither c_1 nor c_2 is zero.

Answer: Obviously, $W(c_1y_1, c_2y_2) = c_1c_2W(y_1, y_2)$. so c_1y_1 and c_2y_2 are also linearly independent solutions, provided that neither c_1 nor c_2 is zero.

❏

13. If the functions y_1 and y_2 are linearly independent solutions of $y'' + p(t)y' + q(t)y = 0$, determine under what conditions the function $y_3 = a_1y_1 + a_2y_2$ and $y_4 = b_1y_1 + b_2y_2$ also form a linearly independent set of solutions.

Answer: $W(y_3, y_4) = (a_1y_1 + a_2y_2)(b_1y_1' + b_2y_2') - (a_1y_1' + a_2y_2')(b_1y_1 + b_2y_2) = (a_1b_2 - a_2b_1)W(y_1, y_2)$. So if y_3 and y_4 also form a linearly independent set of solutions, then $W(y_3, y_4)$ is not always zero. Hence $(a_1b_2 - a_2b_1) \neq 0$.

❏

19. Show that if p is differentiable and $p(t) > 0$, then the Wronskian $W(t)$ of two solutions of $[p(t)y']' + q(t)y = 0$ is $W(t) = \frac{c}{p(t)}$, where c is constant.

Answer: The original equation can be written as

$$p(t)y'' + p'(t)y' + q(t)y = 0$$

$\Rightarrow$

$$y'' + \frac{p'(t)}{p(t)}y' + \frac{q(t)}{p(t)}y = 0$$

From Abel's theorem $W(t) = c\exp\left[-\int \frac{p'(t)}{p(t)}dt\right] = ce^{-\ln p(t)} = \frac{c}{p(t)}$. ❏

20. If y_1 and y_2 are linearly independent solutions $ty^{''} + 2y^{'} + te^t y = 0$ and if $W(y_1, y_2)(1) = 2$, find the value of $W(y_1, y_2)(5)$.

Answer: The original equation can be written as

$$y^{''} + \frac{2}{t}y^{'} + e^t y = 0$$

From Abel's theorem $W(y_1, y_2)(t) = c \exp\left[-\int \frac{2}{t} dt\right] = \frac{c}{t^2}$.
We can find $c = 2$ by $W(y_1, y_2)(1) = 2$. Hence, $W(y_1, y_2)(5) = \frac{2}{25}$ ❑

In the following problems through 24 to 26 assume that p and q are continuous, and that the functions y_1 and y_2 are solutions of the differential equation $y^{''} + p(t)y^{'} + q(t)y = 0$ on an open interval I.

24. Prove that if y_1 and y_2 are zero at the same point in I, then they cannot be a fundamental set of solutions on that interval.

Answer: $W(y_1, y_2) = y_1 y_2^{'} - y_1^{'} y_2 = 0$ at some point in I because y_1 and y_2 are zero at the same point in I. Hence, from Theorem 3.3.3 they cannot be a fundamental set of solutions on I.

❑

25. Prove that if y_1 and y_2 have maxima or minima at the same point in I, then they cannot be a fundamental set of solutions on that interval.

Answer: From y_1 and y_2 have maxima or minima at the same point in I, saying t_0, we can get $y_1^{'}(t_0) = y_2^{'}(t_0) = 0$. Therefore $W(y_1, y_2) = y_1 y_2^{'} - y_1^{'} y_2 = 0$ at t_0 in I. Hence, from Theorem 3.3.3 they cannot be a fundamental set of solutions on I.

❑

26. Prove that if y_1 and y_2 have a common point of inflection t_0 in I, then they cannot be a fundamental set of solutions on that interval unless both p and q are zero at t_0.

Answer: If y_1 and y_2 have a common point of inflection t_0 in I, then $y_1'(t_0) = y_2'(t_0) = 0$. Therefore $W(y_1, y_2) = y_1 y_2' - y_1' y_2 = 0$ at t_0 in I. Hence, from Theorem 3.3.3 they cannot be a fundamental set of solutions on I.

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Supplement Problem: Consider the following two functions:

$$y_1(t) = \begin{cases} t^2, & t \leq 0 \\ 0 & t > 0 \end{cases} \tag{1}$$

$$y_2(t) = \begin{cases} 0, & t \leq 0 \\ t^2, & t > 0 \end{cases} \tag{2}$$

Show that y_1, y_2 is linearly independent but $W[y_1, y_2] \equiv 0$. What is wrong?

Answer: If there exist two constants k_1 and k_2 such that $k_1 y_1 + k_2 y_2 = 0$, then $[k_1 y_1 + k_2 y_2](1) = k_2 y_2(1) = k_2 = 0$. Similarly, $k_1 = 0$. So y_1, y_2 is linearly independent. This result does not contradict Theorem 3.3.1 because Theorem 3.3.1 does not include this case.

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