FINITE-ENERGY SIGN-CHANGING SOLUTIONS WITH DIHEDRAL SYMMETRY FOR THE STATIONARY NON LINEAR SCHRÖDINGER EQUATION

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Abstract. We address the problem of the existence of finite energy solitary waves for nonlinear Klein-Gordon or Schrodinger type equations

$$\Delta u - u + f(u) = 0 \quad \text{in} \quad \mathbb{R}^N, \, u \in H^1(\mathbb{R}^N),$$

where $N \geq 2$. Under natural conditions on the nonlinearity f, we prove the existence of *infinitely many nonradial solutions* in any dimension $N \geq 2$. Our result complements earlier works of Bartsch and Willem [1] ($N = 4$ or $N \geq 6$) and Lorca-Ubilla [13] ($N = 5$) where solutions invariant under the action of $O(2) \times O(N-2)$ are constructed. In contrast, the solutions we construct are invariant under the action of $D_k \times O(N-2)$ where $D_k \subset O(2)$ denotes the subgroup generated by the rotation of angle $2\pi/k$, for some integer $k \geq 7$, but they are not invariant under the action of $O(2) \times O(N-2)$.

1. Introduction and statement of main results

Nonlinear semilinear elliptic equations of the form

$$\Delta u - u + f(u) = 0 \quad \text{in} \quad \mathbb{R}^N, \, u \in H^1(\mathbb{R}^N), \tag{1.1}$$

arise in various models in physics, mathematical physics and biology. In particular, the study of standing waves (or solitary waves) for the nonlinear Klein-Gordon or Schrödinger equations reduces to (1.1). We refer to Berestycki and Lions [3], [4], Bartsch and Willem [1] for further references and motivations.

Obviously (1.1) is equivariant with respect to the action of the group of isometries of $\mathbb{R}^N$, it is henceforth natural to ask whether all solutions of (1.1) are radially symmetric. In that regard, the classical result of Gidas, Ni and Nirenberg [7] asserts that all *positive* solutions of (1.1) are indeed radially symmetric. Therefore, nonradial solutions, if they exist, are necessarily sign-changing solutions. When the nonlinearity f is odd, Berestycki and Lions [3], [4] and Struwe [16] have obtained the existence of infinitely many *radially symmetric* sign-changing solutions under some (almost necessary) growth condition on f (we also refer to the work of Bartsch and Willem [2], Conti, Merizzi and Terracini [5] for different approaches and weaker assumptions on the nonlinearity f).

The existence of *nonradial* sign-changing solutions was first proved by Bartsch and Willem [1] in dimension $N = 4$ and $N \geq 6$. The key idea is to look for solutions invariant under the action of $O(2) \times O(N-2) \subset O(N)$ to recover some compactness property. Later on, this result was generalized by Lorca and Ubilla [13] to handle the $N = 5$ dimensional case. The proofs of both results rely on variational methods and the oddness of the nonlinearity f is needed. The question of the existence of nonradial solutions remained open in dimensions $N = 2, 3$.

In this paper, we construct unbounded sequences of solutions of (1.1) in any dimensions $N \geq 2$. The solutions we obtain are nonradial, have finite energy and are invariant under the action of $D_k \times O(N-2)$ where $D_k \subset O(2)$ is the dihedral group generated by the rotation of angle $2\pi/k$, for $k \geq 7$. Moreover, these solutions are not invariant under the action of $O(2) \times O(N-2)$ and hence they are different from the solutions constructed in [1] and [13].

We set $u_+ := \max(u, 0)$ and $u_- := \max(-u, 0)$. We will assume that the nonlinearity f can be decomposed as $f(u) = f_1(u_+) - f_2(u_-)$ where the functions $f_i : \mathbb{R} \to \mathbb{R}$ are at least $\mathcal{C}^{1,\mu}$ for some $\mu \in (0,1)$ and satisfy the following conditions :

(H.1) $f_i(0) = f_i'(0) = 0$ for $i = 1, 2$.

(H.2) For $i = 1, 2$, the equation

$$\Delta\, w_i - w_i + f_i(w_i) = 0, \tag{1.2}$$

has a unique *positive* solution w_i which tends to 0 at infinity and achieves its maximum at the origin.

(H.3) The solution w_i is *nondegenerate*, in the sense that

$$\mathrm{Ker}\,\left(\Delta - 1 + f_i'(w_i)\right) \cap L^\infty(\mathbb{R}^N) = \mathrm{Span}\,\{\partial_{x_1} w_i, \ldots, \partial_{x_N} w_i\}\;. \tag{1.3}$$

A typical example of a nonlinearity f satisfying the above assumptions is given by the function

$$f(u) = (u_+^{p_1} - c_1\, u_+^{q_1}) - (u_-^{p_2} - c_2\, u_-^{q_2})\,,$$

where $c_i \geq 0$ and $1 < q_i < p_i < \frac{N+2}{N-2}$ (we agree that $\frac{N+2}{N-2} = +\infty$ when $N = 2$). In this case, the existence of w_i is standard and follows from well known arguments in the calculus of variation while the uniqueness follows from results of Kwong [10] and Kwong and Zhang [11]. Concerning the nondegeneracy condition (which follows from uniqueness), we refer to Appendix C of [15].

The energy functional associated to (1.1) is given by

$$\mathcal{E}(u) := \frac{1}{2}\int_{\mathbb{R}^N} (|\nabla u|^2 + u^2)\, dx - \int_{\mathbb{R}^N} F(u)\, dx \tag{1.4}$$

where

$$F(u) := \int_0^u f(s)\, ds = F_1(u_+) + F_2(u_-), \qquad \text{and} \qquad F_i(u) := \int_0^u f_i(s)\, ds\,.$$

We will denote by $\mathcal{E}_i$ the energy of the function w_i. Namely

$$\mathcal{E}_i := \frac{1}{2}\int_{\mathbb{R}^N} (|\nabla w_i|^2 + w_i^2)\, dx - \int_{\mathbb{R}^N} F_i(w_i)\, dx. \tag{1.5}$$

Granted the above, we can now state the main result of this paper.

Theorem 1.1. *Assume that the nonlinearity f satisfies the assumptions (H.1)-(H.3) and that $k \geq 7$ is a fixed integer. Then, there exist $(m_i)_i$ and $(n_i)_i$, sequences of integers tending to $+\infty$, and $(u_i)_i$, a sequence of nonradial, sign-changing solutions of (1.1), whose energy $\mathcal{E}(u_i)$ is equal to*

$$\mathcal{E}(u_i) = k\,((m_i + n_i)\,\mathcal{E}_1 + n_i\,\mathcal{E}_2) + o(1)\,.$$

Moreover, the solutions u_i are invariant under the action of $D_k \times O(N-2)$ but are not invariant under the action of $O(2) \times O(N-2)$.

Some remarks are due. The sequences of integers $(m_i)_i$ and $(n_i)_i$ are not arbitrary and in fact they are related by some highly nonlinear equations. To explain these, we introduce the interaction function $\Psi_{i\to j}$ is defined by

$$\Psi_{i\to j}(s) := -\int_{\mathbb{R}^N} w_i(\cdot - s\,\mathsf{e})\,\mathrm{div}\,(f_j(w_j)\,\mathsf{e})\,dx\,.$$

where $\mathsf{e} \in \mathbb{R}^N$ is *any* unit vector. It is easy to check that this definition is independent of the choice of e. Then, the integers m_i and n_i are

$$\left(2\,\sin\frac{\pi}{k}\right)\,m_i\,\ell_i = (2n_i - 1)\,\bar{\ell}_i\,,$$

where $\ell_i, \bar{\ell}_i$ are large enough and are related by

$$\Psi_{1\to 1}(\ell_i) = \left(2\,\sin\frac{\pi}{k}\right)\,\Psi_{2\to 1}(\bar{\ell}_i)\,.$$

Observe that we do not assume that the function f is odd, and hence, oddness of the nonlinearity is not necessary for the existence of nonradial solutions of (1.1). The assumption (H.3) on the nonlinearity f reflects the techniques we use : Instead of variational methods, we are going to use *singular perturbation techniques* to prove Theorem 1.1. This might look counterintuitive since in most of singularly perturbed problems, a small parameter is needed (generally as a coefficient in front of the Laplacian or in the nonlinearity) in order to ensure that an appropriate sequence of function constitute good enough approximate solutions as the parameter ϵ tends to its limit value (generally $\epsilon = 0$).

There is no such a small parameter in (1.1). Instead, we use the *non compactness of the space of finite energy solutions* of (1.1) to build a discrete sequence of functions which are as close as wanted from being solutions. The idea is to arrange $k\times(m+n)$ copies of the entire positive solution w_1 and $k \times n$ copies of the entire negative solution $-w_2$ at carefully chosen points in $\mathbb{R}^N$ so that, as m and n tend to ∞ the corresponding function is close (in a sense to be made precise) to a solution of (1.1). We will adjust the discrete parameters m, n, the location of the points where the solutions w_1 and $-w_2$ are centered so that some global equilibrium is achieved. A similar idea has already been used by Wei and Yan [17] where infinitely many positive *bound states* for a class of nonlinear Schrodinger equations are constructed. But the motivation for this construction certainly comes from a similar construction which has been obtained by Kapouleas in the context of compact constant mean curvature surfaces of Euclidean 3-space [9]. We shall return to this later on.

Incidentally, let us mention that Malchiodi [14] has recently constructed positive (infinite energy) solutions of (1.1) by perturbing a configuration of infinitely many copies of the positive solution w_1 arranged along three rays meeting at a common point. The solutions he has constructed are bounded but they have infinite energy. Our key observation is that solutions with finite energy can be obtained using similar ideas provided one considers sign-changing solutions and this is precisely the contribution of our paper. Let us also mention that positive solutions of (1.1) with unbounded energy have also been constructed by del Pino, Kowalczyk, Pacard and Wei in [6] again using ideas which steam from similar construction in the theory of non compact constant mean curvature surfaces of Euclidean 3-space.

The proof of the main result is rather technical and, in order to help allay the complexity of the notations and present the ideas of the proof as clearly as possible, we will prove Theorem 1.1 in the case where the nonlinearity is given by

$$f(u) = |u|^{p-1}\, u \, .$$

Mutatis mutandis, the proof goes through for any nonlinearity satisfying (H.1)-(H.3). Therefore, from now on, we will be interested in solutions of

$$\Delta u - u + |u|^{p-1}\, u = 0 \quad \text{in} \quad \mathbb{R}^N , \tag{1.6}$$

which tend to 0 as $|x|$ tends to ∞. We will assume that the exponent p satisfies $1 < p < \frac{N+2}{N-2}$ when $N \geq 3$ and $1 < p$ when $N = 2$. Observe that equation (1.6) is the Euler-Lagrange equation of the functional defined by

$$\mathcal{E}\,(u) := \frac{1}{2} \int_{\mathbb{R}^N} \left(|\nabla u|^2 + u^2 \right)\, dx - \frac{1}{p+1} \int_{\mathbb{R}^N} |u|^{p+1}\, dx \, , \tag{1.7}$$

and let us recall that there exists a unique radially symmetric (in fact radially decreasing) positive solution of

$$\Delta w - w + w^p \ = 0 \quad \text{in} \quad \mathbb{R}^N ,$$

which tends to 0 as $|x|$ tends to ∞. Moreover, all positive solutions to the above problem which tend to 0 at ∞ are translations of w. The function w together with its translations will constitute the building blocks of our construction.

As far as the asymptotic behavior of w at infinity is concerned, it is known that there exists a constant $c_{N,p} > 0$, only depending on N and p, such that

$$\lim_{r \to \infty} e^r\, r^{\frac{N-1}{2}}\, w = c_{N,p} > 0, \quad \text{and} \qquad \lim_{r \to \infty} \frac{w'}{w} = -1 \, , \tag{1.8}$$

where we have set $r := |x|$. Of importance to us will be the *interaction function* Ψ defined by

$$\Psi(s) := - \int_{\mathbb{R}^N} w(\cdot - s\, \mathsf{e})\, \mathrm{div}(w^p\, \mathsf{e})\, dx \ , \tag{1.9}$$

where $\mathsf{e} \in \mathbb{R}^N$ is *any* unit vector. It is known (see Lemma 5.1) that

$$\Psi(s) = C_{N,p}\, e^{-s}\, s^{-\frac{N-1}{2}}\, (1 + O(s^{-1})) \quad \text{as} \quad s \to \infty \, ,$$

where the constant $C_{N,p} > 0$ only depends on N and p also that

$$-(\log \Psi)'(s) = 1 + \frac{N-1}{2\, s} + O(s^{-2}) \quad \text{as} \quad s \to \infty \, . \tag{1.10}$$

Finally, the solution w is *nondegenerate* in the sense defined in (1.3) (we refer the reader to [15]).

This nondegeneracy property is crucial in our construction. Recall that being *nondegenerate* is equivalent to the fact that the L^∞-kernel of the operator

$$L_0 := \Delta - 1 + p\, w^{p-1} \, , \tag{1.11}$$

which is nothing but the linearized operator about w, is spanned by the functions

$$\partial_{x_1} w, \ldots , \partial_{x_N} w \, , \tag{1.12}$$

which naturally belong to this space.

As already mentioned, the solutions we construct are invariant under a large group of symmetries. More precisely, they will enjoy the following invariance :

$$u(x) = u(R\,x), \quad \text{for all} \quad R \in \{I_2\} \times O(N-2)\,, \tag{1.13}$$

also

$$u(R_k\,x) = u(x) \quad \text{and} \quad u(\Gamma\,x) = u(x)\,, \tag{1.14}$$

where $R_k \in O(2) \times \{I_{N-2}\}$ is the rotation of angle $2\pi/k$ in the (x_1, x_2)-plane and $\Gamma \in O(2) \times \{I_{N-2}\}$ is the symmetry with respect to the hyperplane $x_2 = 0$. Here I_n denotes the identity in $\mathbb{R}^n$.

The solutions of (1.6) we construct are small perturbations of the sum of copies of $\pm w$, centered at carefully chosen points in $\mathbb{R}^2 \times \{0\} \subset \mathbb{R}^N$. Let us now give a precise description of these points. We fix an integer $k \geq 7$ which will define the dihedral group we are working with and we assume that we are given m, n two positive integers and $\ell, \bar{\ell}$ two positive real numbers related by

$$\left(2 \sin \frac{\pi}{k}\right) m\,\ell = (2n-1)\,\bar{\ell}\,. \tag{1.15}$$

The canonical basis of $\mathbb{R}^N$ will be denoted by

$$\mathtt{e}_1 := (1, 0, \ldots, 0),\ \mathtt{e}_2 := (0, 1, 0, \ldots, 0) \quad \ldots \quad \mathtt{e}_N := (0, \ldots, 0, 1)\,.$$

We consider the regular polygon in $\mathbb{R}^2 \times \{0\} \subset \mathbb{R}^N$ whose vertices are given by the orbit of the point

$$\mathring{y}_1 := \frac{\bar{\ell}}{2 \sin \frac{\pi}{k}}\,\mathtt{e}_1 \in \mathbb{R}^N\,, \tag{1.16}$$

under the action of the group generated by R_k. Observe that, by construction, the edges of this polygon have length $\bar{\ell}$.

We now define the polygon whose vertices are the orbit of the point

$$\mathring{y}_{m+1} := \mathring{y}_1 + m\,\ell\,\mathtt{e}_1\,, \tag{1.17}$$

under the group generated by R_k. Observe that, thanks to (1.15), the edges of this polygon have length $2\,n\,\bar{\ell}$.

By construction, the distance between the points $\mathring{y}_1$ and $\mathring{y}_{m+1}$ is equal to $m\,\ell$ and we will denote by $\mathring{y}_j$, for $j = 2, \ldots, m$ the points evenly distributed on the segment between these two points. Namely

$$\mathring{y}_j := \mathring{y}_1 + (j-1)\,\ell\,\mathtt{e}_1 \quad \text{for} \quad j = 1, \ldots, m. \tag{1.18}$$

As already mentioned, the distance between the points $\mathring{y}_{m+1}$ and $R_k\,\mathring{y}_{m+1}$ is equal to $2\,n\,\bar{\ell}$, and again we distribute evenly points $\mathring{z}_h$, $h = 1, \ldots, 2n-1$, along this segment. More precisely, if we denote by

$$\mathtt{t} := -\sin \frac{\pi}{k}\,\mathtt{e}_1 + \cos \frac{\pi}{k}\,\mathtt{e}_2 \in \mathbb{R}^N\,, \tag{1.19}$$

then the points $\mathring{z}_h$ are given by

$$\mathring{z}_h := \mathring{y}_{m+1} + h\,\bar{\ell}\,\mathtt{t} \quad \text{for} \quad h = 1, \ldots, 2n-1. \tag{1.20}$$

Observe that, by construction

$$R_k\,\mathring{y}_{m+1} = \mathring{y}_{m+1} + 2\,n\,\bar{\ell}\,\mathtt{t}\,.$$

The solutions we construct will be perturbations of the function $\mathring{U}$ which is the sum of positive copies of w centered at the points $\mathring{y}_j$, for $j = 1, \ldots, m+1$, together

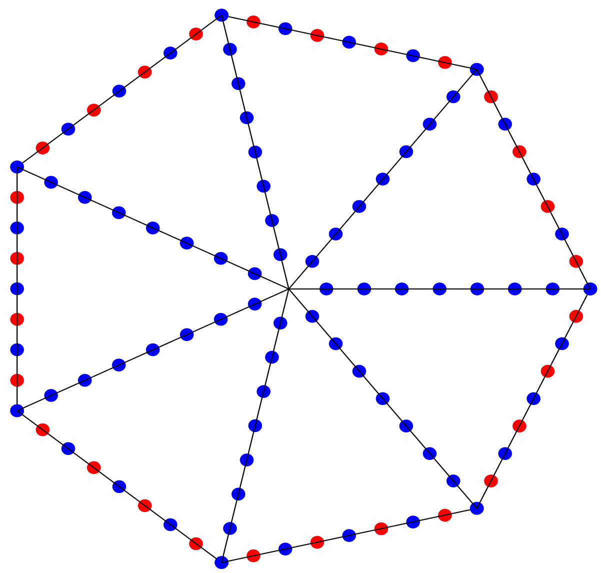

FIGURE 1. The location of the bumps. Here $k = 7, m = 8, n = 4$. The blue color indicates a positive bump, and the red color indicates a negative bump.

with their images by the rotations $R_k^i = R_k \circ \ldots \circ R_k$ (composition of R_k, h times), for $i = 1, \ldots, k-1$, and copies of $(-1)^h\, w$ (hence with alternating sign) centered at the points $\mathring{z}_h$, for $h = 1, \ldots, 2n-1$, together with their images by the rotations R_k^i, for $i = 1, \ldots, k-1$. To be more specific, we define

$$\mathring{U} := \sum_{i=0}^{k-1} \left(\sum_{j=1}^{m+1} w(\cdot - R_k^i\, \mathring{y}_j) + \sum_{h=1}^{2n-1} (-1)^h w(\cdot - R_k^i\, \mathring{z}_h) \right) , \tag{1.21}$$

A picture of the locations of the bumps is given in Figure 1.

So far, the approximate solution $\mathring{U}$ depends on two discrete parameters (m and n) and two continuous parameters (ℓ and $\bar{\ell}$) which are related by (1.15). It should be clear from the construction that the function $\mathring{U}$ we have constructed has some dihedral symmetry and (1.15) is just a translation of the fact that the length of the rays and the length of the edges of a regular polygon are related. The construction of the approximate solution $\mathring{U}$ also depends on the parameter k which defines the dihedral group under the action of which our solution will be invariant. The constraint $k \geq 7$ has a purely geometric origin, roughly speaking, we need $\frac{\pi}{2} - \frac{\pi}{k}$, which is the angle at $\mathring{y}_1$ between the edge of the exterior regular polygon and the ray joining this vertex to the origin, to be larger than $\pi/3$. Hence

$$\frac{\pi}{2} - \frac{\pi}{k} > \frac{\pi}{3} .$$

In turn, this last condition steams from the maximal number of number of tangent non overlapping disks of radius 1 which are tangent to a given disc of radius 1. When $k \leq 6$, the angle $\frac{\pi}{2} - \frac{\pi}{k}$ is too small and the copies of w centered at the points $\mathring{y}_m$ and $\mathring{z}_1$ interact too much to consider that $\mathring{U}$ is a good approximate solution. This will become clear in the proof of Lemma 5.2 where the interaction between the different copies of w is studied and where the fact that the distance from any point where the copies of $\pm w$ are centered to their closest neighbors can be estimated by $\ell + O(1)$ as ℓ tends to ∞ is used in a crucial way.

We still assume that the integer $k \geq 7$ is fixed, that m, n are two positive integers and $\ell, \bar{\ell}$ are two positive real numbers satisfying (1.15). We now further assume that ℓ and $\bar{\ell}$ are related by

$$\Psi(\ell) = \left(2 \sin \frac{\pi}{k}\right) \Psi(\bar{\ell}), \tag{1.22}$$

where Ψ is the function defined in (1.9). The origin of this second constraint on the choice of the parameters is not obvious at all. It can either be understood as a *balancing condition* which is a consequence of a conservation law for solutions of (1.6) (corresponding to the balancing formula in the framework of constant mean curvature surfaces) or it can be understood as a condition which will ensure that the approximate solution we consider is, in a sense to be made precise, very close to a genuine solution of (1.6) (we refer to §5 where this second equation will arise and where some formal justification of this constraint will be given).

In any case, it follows from (1.22) that $\bar{\ell}$ is implicitly given as a function of ℓ (provided this later is large enough) and that it can be expanded, in powers of ℓ, as

$$\bar{\ell} = \ell + \ln\left(2 \sin \frac{\pi}{k}\right) + O(\ell^{-1}), \tag{1.23}$$

as ℓ tends to ∞. Inserting this information back into (1.15), we find using Lemma 5.1, that

$$\frac{2n-1}{m} = 2 \sin \frac{\pi}{k} \left(1 - \ln\left(2 \sin \frac{\pi}{k}\right) \ell^{-1} + O(\ell^{-2})\right). \tag{1.24}$$

Theorem 1.1 is a direct consequence of the following result :

Theorem 1.2. *Assume that the integer $k \geq 7$ and the real number $\tau > 0$ are fixed. There exists a positive constant $\ell_0 > 0$ such that, for all $\ell \geq \ell_0$, if $\bar{\ell}$ is* the *solution of (1.22) and if n, m are positive integers satisfying (1.15) and*

$$m \leq \ell^{\tau},$$

then (1.6) has a sign changing solution u which satisfies the symmetry conditions given in (1.13) and (1.14). Moreover

$$u = \mathring{U} + o(1), \tag{1.25}$$

where $o(1) \to 0$ uniformly in $\mathbb{R}^N$ as $\ell \to \infty$, and the energy (1.7) of u is finite, given by

$$\mathcal{E}(u) = k\,(m + 2n)\,\mathcal{E}(w) + o(1), \tag{1.26}$$

where $o(1) \to 0$ as $\ell \to \infty$.

The condition $m \leq \ell^{\tau}$ is purely technical and is a drawback of our proof. Going carefully through the last arguments of the proof, it will be clear that this condition can be weakened to handle the cases where

$$m \leq e^{\nu \ell},$$

for some $\nu > 0$, small enough. This condition can be completely removed but since this would enlarge considerably the size of the paper, therefore, we have chosen not to follow this route.

Observe that, once ℓ is fixed large enough, the constant $\bar{\ell}$ is given uniquely by (1.22). Therefore, the existence of solutions of (1.6) depends on our ability to solve (1.15) for some integers m, n. We now give examples of such solutions. As already mentioned, for ℓ large enough, one can solve implicitly (1.22) as a function of ℓ and one obtains the expansion given in (1.24). Certainly, for any integer $m \geq 1$, one can choose $n \in \mathbb{N}$ such that

$$1 \leq 2n - 1 - \left(2 \, \sin \frac{\pi}{k}\right) \, m \leq 3 \, . \tag{1.27}$$

Then, provided m is chosen large enough, there will exist a unique $\ell > \ell_0$ satisfying (1.24) and (1.27) implies that there exist positive constants $C_1 < C_2$ such that

$$C_1 \, m \leq \ell \leq C_2 \, m \, . \tag{1.28}$$

Theorem 1.2 then ensures the existence of solutions of (1.6) for each such a choice.

To complete the description of our construction, let us briefly comment on the relations between this result and the corresponding construction for constant mean curvature surfaces in Euclidean 3-space. As already mentioned, the construction in the present paper follows very closely a similar construction of compact constant mean curvature surfaces given in [9]. In this framework one tries to construct compact constant mean curvature surfaces in Euclidean 3-space by connecting together spheres of radius 1 which are tangent. In the initial configuration, the center of the spheres can be arranged along the edges of a very large regular polygon and also along the rays joining the center to the vertices of the polygon. It is proven in [9] that a perturbation argument can be applied and, as a result, a compact constant mean curvature surface is obtained (provided the size of the polygon is large enough). This surface can be constructed in such a way that the pieces which are close to the rays joining the origin to the vertices are embedded and close to nodoids while the pieces which are close to the edges of the regular polygon are immersed and close to nodoids (in our framework, this corresponds to the fact that we arrange solutions with the same sign along the rays joining the origin to the vertices of the polygon and solutions with alternative sign along the edges of the polygon). A similar construction with dihedral symmetry has also been obtained by Jleli and Pacard in [8].

Remark 1.1. *For the sake of simplicity, we have chosen to present the proof of the existence of solutions which are invariant under the action of a rather large group of symmetry. However, it is clear that a more general construction (i.e. with less symmetry) for solutions of (1.6) can be obtained as is already the case for constant mean curvature surfaces* [9].

In the next section, we describe more carefully the solution predicted in the above Theorem and we give an overview of the proof and of the plan of the paper.

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2. Ansatz and sketch of the proof

We construct a finite dimensional family of approximate solutions of (1.6) which are close to $\mathring{U}$ and depend on $2n+m$ parameters which we now define. These approximate solutions are in fact equal to $\mathring{U}$ when all the parameters are set to 0. This time, instead of centering the copies of $\pm w$ at the points $\mathring{y}_j$, $\mathring{z}_h$ as well as at their images by the rotations R_k^i, for $i=1,\ldots,k-1$, we center the copies of $\pm w$ at points which are small perturbations of the points $\mathring{y}_j$, $\mathring{z}_h$. To make this precise, we set

$$y_j := \mathring{y}_j + \alpha_j\, \mathtt{e}_1, \quad \text{for} \quad j=1,\ldots,m+1 \tag{2.1}$$

and

$$z_h := \mathring{z}_h + \beta_h\, \mathtt{t} + \bar{\ell}\,\gamma_h\, \mathtt{n}, \quad \text{for} \quad h=1,\ldots,2n-1\,, \tag{2.2}$$

where $\bar{\ell}$ has been defined in (1.22), $\mathtt{t}$ has been defined in (1.19) and

$$\mathtt{n} := \cos\frac{\pi}{k}\,\mathtt{e}_1 + \sin\frac{\pi}{k}\,\mathtt{e}_2\,.$$

Observe that, by symmetry

$$\beta_h = -\beta_{2n-h} \qquad \text{and} \qquad \gamma_h = \gamma_{2n-h} \quad \text{for} \quad h=1,\ldots,n\,, \tag{2.3}$$

and in particular $\beta_n = 0$. This is due to the fact that if we rotate by an angle of $\frac{2\pi}{k}$ the point obtained reflecting z_h with respect to the plane $x_2=0$ we reach z_{2n-h}. We thus conclude that there are only $2n+m$ free parameters.

We will assume that the parameters which appear in the definition of both y_j and z_h satisfy

$$|\alpha_j| \le 1,\ j=1,\ldots,m+1, \quad |\beta_h|\le 1,\ h=1,\ldots,n-1 \quad |\gamma_h| \le 1,\ h=1,\ldots,n\,. \tag{2.4}$$

In these inequalities, the constant 1 is arbitrary and could have been replaced by any positive constant.

The set of points where the copies of w will be centered is now given by

$$\Pi := \bigcup_{i=0}^{k-1} \left(\{R_k^i\, y_j \,:\, j=1,\ldots,m+1\} \cup \{R_k^i\, z_h \,:\, h=1,\ldots,2n-1\} \right)\,. \tag{2.5}$$

We look for a solution of (1.6) of the form $u = U + \phi$, where

$$U(x) := \sum_{i=0}^{k-1} \left(\sum_{j=1}^{m+1} w(x - R_k^i\, y_j) + \sum_{h=1}^{2n-1} (-1)^h w(x - R_k^i\, z_h) \right)\,. \tag{2.6}$$

Observe that, by construction, the function U satisfies the symmetry assumption (1.13) and (1.14). We set

$$L := \Delta - 1 + p\,U^{p-1}, \tag{2.7}$$

$$E := |U|^{p-1}\,U - \sum_{i=0}^{k-1} \left(\sum_{j=1}^{m+1} w^p(\cdot - R_k^i\, y_j) + \sum_{h=1}^{2n-1} (-1)^h w^p(\cdot - R_k^i\, z_h) \right)\,, \tag{2.8}$$

and

$$N(\phi) := |U+\phi|^{p-1}(U+\phi) - |U|^{p-1}U - p\,|U|^{p-1}\,\phi. \tag{2.9}$$

With these notations, the solvability of (1.6) reduces to find parameters α_j, β_h and γ_h and a function ϕ solution of the nonlinear problem

$$L\,\phi + E + N(\phi) = 0 \quad \text{in} \quad \mathbb{R}^N\,, \tag{2.10}$$

and which tends to 0 as $|x|$ tends to ∞.

Remark 2.1. *Observe that we want to solve (2.10) in the class of functions ϕ satisfying (1.13) and (1.14). Therefore, from now on, we always assume that all the functions we consider satisfy (1.13) and (1.14) without further mentioning it.*

In order to solve this highly nonlinear problem, we apply a Liapunov-Schmidt type reduction argument : first, we solve a projected problem which allows one to reduce the solvability of (2.10) to the solvability of some finite dimensional nonlinear system (the reduced problem), then, in a second step, we will explain how to solve this reduced problem.

To proceed with the first step, we consider a cut off function $\chi \in C^\infty(0,\infty)$ such that

$$\chi(s) \equiv 1 \quad \text{for} \quad s \leq -1, \quad \text{and} \quad \chi(s) \equiv 0 \quad \text{for} \quad s \geq 0\,. \tag{2.11}$$

We fix a constant $\zeta > 0$ (independently of ℓ) so that the balls of radius $\frac{\ell-\zeta}{2}$, centered at different points of Π are mutually disjoint, for all ℓ large enough. This is possible thanks to our geometric assumption $k \geq 7$ and also thanks to (1.23) which implies that the minimum distance between two different points of Π is bounded from below by $\ell - \zeta_0$ for some constant $\zeta_0 > 0$ independent of ℓ large enough (say $\ell \geq \ell_0$). Observe that, when $k \leq 6$ the distance between y_m and z_1 can be evaluated by $2\,\sin(\frac{\pi}{2} - \frac{\pi}{k})\,\ell + O(1)$, as ℓ tends to ∞ and therefore a proper choice of ζ would not have been possible in this case since $2\,\sin(\frac{\pi}{2} - \frac{\pi}{k}) < 1$. We define the compactly supported vector field

$$\Xi := \chi\,(2\,\cdot - \ell + \zeta)\;\nabla w\,. \tag{2.12}$$

Observe that, by construction (in fact given the choice of ζ), we have

$$\int_{\mathbb{R}^N} (\mathsf{e}_i \cdot \Xi(\cdot - y))\,(\mathsf{e}_j \cdot \Xi(\cdot - z))\,dx = 0\,, \tag{2.13}$$

if $i \neq j$ or if $y \neq z \in \Pi$.

It will be convenient to define the function

$$M(c,d) := \sum_{i=0}^{k-1} \left(\sum_{j=1}^{m+1} c_j \cdot \Xi(\cdot - R_h^i\, y_j) + \sum_{h=1}^{2n-1} d_h \cdot \Xi(\cdot - R_h^i\, z_h) \right)\,, \tag{2.14}$$

as well as the operator

$$\mathcal{L}(\phi, c, d) := L\,\phi + M(c,d)\,, \tag{2.15}$$

where ϕ is a function defined on $\mathbb{R}^N$, the $(m+1)$-tuple $c = (c_1, \ldots, c_{m+1}) \in (\mathbb{R}\,\mathsf{e}_1)^{m+1}$ and the $(2n-1)$-tuple $d := (d_1, \ldots, d_{2n-1}) \in (\mathbb{R}\,\mathsf{e}_1 \oplus \mathbb{R}\,\mathsf{e}_2)^{2n-1}$.

In the next section, we will define suitable function spaces in which the equation

$$\mathcal{L}\,(\phi, c, d) = h \quad \text{in} \quad \mathbb{R}^N\,, \tag{2.16}$$

admits a solution which tends to 0 as $|x|$ tends to ∞ and which satisfies the orthogonality condition

$$\int_{\mathbb{R}^N} \phi\,\mathsf{e}_1 \cdot \Xi(\cdot - y_j)\,dx = 0 \quad \text{for} \quad j = 1, \ldots, m+1, \tag{2.17}$$

and

$$\int_{\mathbb{R}^N} \phi\,\mathsf{e}_i \cdot \Xi(\cdot - z_h)\,dx = 0 \quad \text{for} \quad h = 1, \ldots, n \quad \text{and} \quad i = 1, 2. \tag{2.18}$$

Observe that, given the symmetries we impose to all the functions we deal with (see Remark 2.1), a function ϕ satisfies (2.17) and (2.18) if and only if it satisfies

$$\int_{\mathbb{R}^N} \phi \, \mathsf{e}_i \cdot \Xi(\cdot - y) \, dx = 0 \, ,$$

for all $i = 1, \ldots, N$ and all $y \in \Pi$.

In order to study the operator $\mathcal{L}$, the key idea is that the linear operator L is close to be the sum of many copies of

$$L_0 = \Delta - 1 + p \, w^{p-1} \, ,$$

centered at the points of Π and the invertibility of L_0 is well understood.

Once the linear theory will be understood, we will consider the following nonlinear projected problem : given the points y_j and z_h defined in (2.1), (2.2) and satisfying (2.4), find a function ϕ, satisfying the symmetry assumptions (1.13), (1.14), the orthogonality conditions (2.17) and (2.18) and tending to 0 as $|x|$ tends to ∞ and find vectors c_j, d_h such that

$$\mathcal{L} \, (\phi, c, d) + E + N(\phi) = 0 \quad \text{in} \quad \mathbb{R}^N \, . \tag{2.19}$$

In the next sections, we show unique solvability of (2.19) by means of a fixed point argument and we prove that the solution ϕ depends smoothly on the points y_j and z_h. To achieve this, we first study the solvability of a linear problem and then apply some standard fixed point theorem for contraction mapping to solve the nonlinear problem.

3. Linear theory

The main result of this section is concerned with the solvability of (2.16), uniformly in ℓ, as ℓ tends to ∞, and also in the parameters α_j, β_h and γ_h satisfying the constraints (2.4). We henceforth assume that the parameters $\ell, \bar{\ell}$ are chosen so that (1.22) holds and m, n are chosen so that (1.15) holds. We prove that, provided ℓ is large enough, the linear operator $\mathcal{L}$ defined in the previous section in (2.15) has nice mapping properties.

Given $\eta \in \mathbb{R}$, we consider the weighted norm

$$\|h\|_* := \sup_{x \in \mathbb{R}^N} \left| \sum_{y \in \Pi} e^{-\eta \, |x-y|} \, h(x) \right| \, , \tag{3.1}$$

where we recall that the set of points Π was defined in (2.5).

With this definition at hand, we prove the following *a priori* estimate :

Lemma 3.1. *Assume that $\eta \in (-1, 1)$ is fixed. There exist $\ell_0 > 0$ and $C > 0$ such that, for all $\ell > \ell_0$, the following inequality holds*

$$\|c\| + \|d\| \leq C \, \left(\|\mathcal{L}(\phi, c, d)\|_* + \ell^N \, e^{-\delta_0 \ell} \, \|\phi\|_* \right) \, ,$$

where we have defined

$$\delta_0 := \frac{1 - \eta}{2} \, .$$

Proof. The proof follows from some simple integration by parts. Thanks to (2.13), we obtain

$$c_1 \cdot \mathsf{e}_1 \int_{\mathbb{R}^N} (\mathsf{e}_1 \cdot \Xi(\cdot - y_1))^2 \, dx = \int_{\mathbb{R}^N} \mathcal{L}(\phi, c, d) \, (\mathsf{e}_1 \cdot \Xi(\cdot - y_1)) \, dx - \int_{\mathbb{R}^N} \phi \, L \, (\mathsf{e}_1 \cdot \Xi(\cdot - y_1)) \, dx \, .$$

Obviously,

$$\lim_{\ell \to \infty} \int_{\mathbb{R}^N} (\mathsf{e}_1 \cdot \Xi(\cdot - y_1))^2 \, dx = \frac{1}{N} \int_{\mathbb{R}^N} |\nabla w|^2 \, dx \, ,$$

therefore

$$\int_{\mathbb{R}^N} (\mathsf{e}_1 \cdot \Xi(\cdot - y_1))^2 \, dx \geq C_0 > 0 \, ,$$

for all ℓ large enough (say $\ell > \ell_0$). Thanks to (1.8) we also have

$$\left| \int_{\mathbb{R}^N} \mathcal{L}(\phi, c, d) \, (\mathsf{e}_1 \cdot \Xi(\cdot - y_1)) \, dx \right| \leq C \, \|\mathcal{L}(\phi, c, d)\|_* \, ,$$

for ℓ large enough. Finally, using the fact that

$$\left(\Delta - 1 + p \, w^{p-1} \right) \, (\mathsf{e}_1 \cdot \Xi) = 0$$

it is easy to check that there exists a constant $C > 0$ and $\delta_0 > 0$ such that

$$\left| \int_{\mathbb{R}^N} \phi \, L \, (\mathsf{e}_1 \cdot \Xi(\cdot - p_1)) \, dx \right| \leq C \, \ell^N \, e^{-\delta_0 \ell} \, \|\phi\|_* \, ,$$

where $\delta_0 = \frac{1-\eta}{2}$. To obtain the last estimate, two different effects have to be taken into account : the first one is the effect of the Laplace operator on the cutoff function and this accounts for the choice of $\delta_0 \leq \frac{1-\eta}{2}$, the second one is the difference between the two potentials $p \, U^{p-1}$ and $p \, w^{p-1}$ which appear respectively in the definition of L and L_0, this later accounts for the choice of $\delta_0 \leq \frac{p-\eta}{2}$. Collecting the above estimates, the proof of the estimate then follows at once for c_1, provided $\ell > \ell_0$. A similar proof holds for any c_j and also for d_h. We leave the details to the reader. □

Thanks to the previous estimate, we can prove the following :

Proposition 3.1. *Assume that $\eta \in (-1, 0)$ is fixed. There exist $\ell_0 > 0$ and $C > 0$ such that, for all $\ell > \ell_0$, the following inequality holds*

$$\|\phi\|_* \leq C \, \|\mathcal{L}(\phi, c, d)\|_* \, , \tag{3.2}$$

provided ϕ satisfies (2.17) and (2.18).

Proof. To begin with, we make use of the result of the previous Lemma together with the fact that $|\Xi|$ is bounded by a constant times $e^{-|x|}$. This later fact follows at once from (1.8). Since we assume that $\eta \in (-1, 0)$, we conclude that

$$\|M(c, d)\|_* \leq C \, \left(\|\mathcal{L}(\phi, c, d)\|_* + \ell^N \, e^{-\delta_0 \ell} \, \|\phi\|_* \right) \, , \tag{3.3}$$

for some constant $C > 0$, which does not depend on $\ell > \ell_0$.

It is easy to check that the function

$$W := \sum_{i=0}^{k-1} \left(\sum_{j=1}^{m+1} e^{\eta \, |\cdot - R_k^i y_j|} + \sum_{h=1}^{2n-1} e^{\eta \, |\cdot - R_k^i z_h|} \right) \, ,$$

satisfies

$$L\,W \leq \frac{1}{2}\,(\eta^2 - 1)\,W\,,$$

in $\mathbb{R}^N \setminus \cup_{y\in P} B(y,\rho)$ provided ρ is fixed large enough (independently of ℓ). Hence the function W can be used as a barrier to prove the pointwize estimate

$$|\phi|(x) \leq C\,\left(\|L\,\phi\|_* + \sum_{y\in P} \|\phi\|_{L^\infty(\partial B(y,\rho))}\right)\,W(x)\,, \tag{3.4}$$

for all $x \in \mathbb{R}^N \setminus \cup_{y\in P} B(y,\rho)$ (this is where we use the fact that $\eta \in (-1,0)$).

Granted these preliminary estimates, the proof of the result goes by contradiction. Let us assume there exist a sequence of ℓ tending to ∞ and a sequence of solutions of (2.16) for which the inequality is not true. The problem being linear, we can reduce to the case where we have a sequence $\ell^{(i)}$ tending to ∞ and sequences $\phi^{(i)}, c^{(i)}$, $d^{(i)}$ such that

$$\|\mathcal{L}(\phi^{(i)}, c^{(i)}, d^{(i)})\|_* \to 0, \quad \text{and} \quad \|\phi^{(i)}\|_* = 1.$$

But (3.3) implies that we also have

$$\|L\,\phi^{(i)}\|_* \to 0\,.$$

Then (3.4) implies that there exists $y^{(i)} \in P$ such that

$$\|\phi^{(i)}\|_{L^\infty(B(y^{(i)},\rho))} \geq C\,, \tag{3.5}$$

for some fixed constant $C > 0$. Using elliptic estimates together with Ascoli-Arzela's theorem, we can find a sequence $y^{(i)}$ and we can extract, from the sequence $\phi^{(i)}(\cdot - y^{(i)})$ a subsequence which will converge (on compact) to ϕ_∞ a solution of

$$\left(\Delta - 1 + p\,w^{p-1}\right)\,\phi_\infty = 0\,,$$

in $\mathbb{R}^N$, which is bounded by a constant times $e^{\eta\,|x|}$, with $\eta < 0$. Moreover, recall that $\phi^{(i)}$ satisfies the orthogonality conditions (2.17) and (2.18). Therefore, the limit function ϕ_∞ also satisfies

$$\int_{\mathbb{R}^N} \phi_\infty\,\nabla w\,dx = 0\,.$$

But the solution w being non degenerate, this implies that $\phi_\infty \equiv 0$, which is certainly in contradiction with (3.5) which implies that ϕ_∞ is not identically equal to 0.

Having reached a contradiction, this completes the proof of the Proposition. □

We are now in a position to prove the main result of this section :

Proposition 3.2. *Assume that $\eta \in (-1,0)$ is fixed. There exist $\ell_0 > 0$ and $C > 0$ such that, for all $\ell > \ell_0$, and for all $h \in L^\infty(\mathbb{R}^N)$ satisfying $\|h\|_* < \infty$, there exists a unique triple (ϕ, c, d) solution of*

$$\mathcal{L}\,(\phi, c, d) = h\,,$$

in $\mathbb{R}^N$, such that ϕ satisfies (2.17) and (2.18). Moreover

$$\|\phi\|_* \leq C\,\|h\|_*\,. \tag{3.6}$$

Proof. We consider the Hilbert space

$$\mathcal{H} = \left\{ \phi \in H^1(\mathbb{R}^N) \, : \, \int_{\mathbb{R}^N} \phi \, \Xi(\cdot - y) \, dx = 0, \qquad y \in \Pi \right\} ,$$

and as usual, we also assume that the function enjoy the symmetries described in Remark 2.1.

Standard arguments imply that

$$\phi \in \mathcal{H} \longmapsto \frac{1}{2} \int_{\mathbb{R}^N} (|\nabla \Phi|^2 + \phi^2) \, dx + \int_{\mathbb{R}^N} \phi \, h \, dx \, ,$$

has a unique minimizer $\phi \in \mathcal{H}$ (here we implicitly use the fact that $\eta < 0$ so that the last term is a continuous linear functional on $\mathcal{H}$). Then ϕ is a weak solution of

$$\Delta \phi - \phi - h \in \bigoplus_{i=0}^{k-1} \big(\mathrm{Span}\, \{ \mathsf{e}_1 \cdot \Xi(\cdot - R_k^i y_j) \, : \, j = 1, \ldots, m+1 \} \oplus \mathrm{Span}\, \{ \mathsf{e}_j \cdot \Xi(\cdot - R_k^i z_h) \, : \, j = 1, 2 \, , \, h = 1, \ldots, 2n-1 \} \big) \, .$$

Setting $\mathcal{L}_0(\phi, c, d) := \Delta \phi - \phi + M(c, d)$, we have solved

$$\mathcal{L}_0(\phi, c, d) = h \, ,$$

for some $c_j \in \mathbb{R} \mathsf{e}_1$ and $d_h \in \mathbb{R}\mathsf{e}_1 \oplus \mathbb{R}\mathsf{e}_2$. The solvability of

$$\mathcal{L}(\phi, c, d) = h \, ,$$

in $\mathcal{H} \times (\mathbb{R} \mathsf{e}_1)^{m+1} \times (\mathbb{R} \mathsf{e}_1 \oplus \mathbb{R} \mathsf{e}_2)^{2n-1}$ can then by rephrased in the invertibility of the operator $I + K$ in $\mathcal{H} \times (\mathbb{R} \mathsf{e}_1)^{m+1} \times (\mathbb{R} \mathsf{e}_1 \oplus \mathbb{R} \mathsf{e}_2)^{2n-1}$, where by definition

$$K(\phi, c, d) := (\mathcal{L}_0)^{-1} (p \, U^{p-1} \, \phi) \, . \tag{3.7}$$

Using the fact that U decays exponentially at ∞, it is easy to check that the operator K is compact, hence the invertibility of (3.7) follows from the application of Fredholm theory. Injectivity follows from the results of Proposition 3.1 and Lemma 3.1. Fredholm alternative implies that $I + K$ is therefore an isomorphism provided ℓ is chosen large enough.

So far, we have obtained a solution which belongs to $H^1(\mathbb{R}^N)$ but elliptic regularity implies that $\phi \in L^\infty(\mathbb{R}^N)$. One can then use the result of Lemma 3.1 to conclude that

$$\|c\| + \|d\| \leq C \, \left(\|h\|_* + \ell^N \, e^{-\delta_0 \ell} \, \|\phi\|_* \right) \, .$$

Writing

$$L \, \phi = h - M(c, d) \, ,$$

and using the function W defined in the proof of the previous Proposition as a barrier, we also get

$$\|\phi\|_* \leq C \, (\|h\|_* + \|c\| + \|d\|) \, .$$

Collecting these two estimates, we conclude that

$$\|\phi\|_* \leq C \, \left(\|h\|_* + \ell^N \, e^{-\delta_0 \ell} \, \|\phi\|_* \right) \, ,$$

and hence

$$\|\phi\|_* \leq C \, \|h\|_* \, ,$$

if ℓ is chosen large enough. This completes the proof of the existence of the solution. The uniqueness follows at once from the result of Proposition 3.1 and Lemma 3.1. □

4. The non linear projected problem

We keep the notations and assumptions of the previous sections. In this section, we prove that we can apply some fixed point theorem for contraction mapping to solve the nonlinear problem

$$\mathcal{L}(\phi, c, d) + E + N(\phi) = 0 \quad \text{in} \quad \mathbb{R}^N , \tag{4.1}$$

provided the parameter ℓ is chosen large enough. This is the contain of the following :

Proposition 4.1. *Assume that $\eta \in (-1, 0)$ is fixed. Then, there exist $\ell_0 > 0$ and $C > 0$ such that, for all $\ell \geq \ell_0$, there exists a (unique) solution (ϕ, c, d) to problem (4.1) such that ϕ satisfies (2.17) and (2.18). This solution depends continuously on the parameters of the construction (namely α_j, β_h and γ_h) and satisfies*

$$\|\phi\|_* \leq C\, e^{-\delta_1 \ell} , \tag{4.2}$$

where $\delta_1 = \min(1, \frac{p+\eta}{2})$.

Before giving the proof of this result, let us briefly comment on the value of the constant δ_1. Observe that, given $p > 1$ it is always possible to choose $\eta \in (-1, 0)$ such that

$$p + \eta > 1 ,$$

and hence, $\delta_1 > \frac{1}{2}$ for this choice. We shall assume from now on that η is chosen so that $\delta_1 > \frac{1}{2}$.

Proof. The proof relies on a classical fixed point argument for contraction mapping together with the following identities. First of all, it is easy to check that there exists $C_0 > 0$ such that

$$\|E\|_* \leq C_0 \left(e^{-\ell} + e^{-\frac{p+\eta}{2}\ell} \right) .$$

In view of this estimate, we see that we can chose $\delta_1 = \min(1, \frac{p+\eta}{2})$ in the statement of the result. We set

$$C_\ell := (4 C_0 e^{-\delta_1 \ell}) / \|\mathcal{L}^{-1}\| .$$

It is easy to check that there exist $\delta_2 > 0$ and $C_2 > 0$ such that

$$\|N(\phi_1) - N(\phi_2)\|_* \leq C_2\, e^{-\delta_2 \ell}\, \|\phi_1 - \phi_2\|_* ,$$

for all ϕ_1, ϕ_2 satisfiying $\|\phi_j\|_* \leq C_\ell$.

There is no difficulty in proving the first estimate. The second estimate is slightly more delicate and follows from the observation that either one tries to get the estimate where $|\phi_1| + |\phi_2| \leq |U|/2$ in which case one can use the inequality

$$|N(\phi_2) - N(\phi_1)| \leq C\, |U|^{p-2}\, (|\phi_1| + |\phi_2|)\, |\phi_2 - \phi_1| ,$$

or one tries to get the estimate where $|\phi_1| + |\phi_1| \geq |U|/2$ in which case one can use the inequality

$$|N(\phi_2) - N(\phi_1)| \leq C\, (|\phi_1|^{p-1} + |\phi_2|^{p-1}) |\phi_2 - \phi_1| .$$

We leave the details to the reader.

The result of Proposition 3.2 allows one to rewrite (4.1) as a fixed point problem

$$(\phi, c, d) = -\mathcal{L}^{-1}\, (E + N(\phi))$$

Provided ℓ is chosen large enough, the above estimates readily yield the existence of a unique fixed point in the ball of radius C_ℓ in the space $\mathcal{H}\times(\mathbb{R}\,\mathtt{e}_1)^{m+1}\times(\mathbb{R}\,\mathtt{e}_1\oplus\mathbb{R}\,\mathtt{e}_2)^{2n-1}$. This completes the proof of the existence of a solution.

Observe that elliptic estimates imply that the solution we get satisfies

$$\|\phi\|_* + \|\nabla\phi\|_* + \|\nabla^2\phi\|_* \leq C\, e^{-\delta_1\ell}. \tag{4.3}$$

It remains to check that this solution depends continuously on the parameters. Assume that we have two solutions corresponding to two sets of parameters. Say

$$L\,\phi + M(c,d) = E + N(\phi)$$

corresponding to the points y_j and z_h and

$$\dot L\,\dot\phi + \dot M(\dot c,\dot d) + \dot E + \dot N(\dot\phi) = 0$$

corresponding to the points $\dot y_j$ and $\dot z_h$ (we will adorn all functions and operators with a $\dot{}$ when they correspond to the points $\dot y_j$ and $\dot z_h$). Observe that, by construction, $\dot\phi$ is L^2-orthogonal to $\mathtt{e}_j\cdot\dot\Xi$ while ϕ is L^2-orthogonal to $\mathtt{e}_j\cdot\Xi$. We choose γ and δ so that $\dot{\bar\phi} := \dot\phi - M(\gamma,\delta)$ satisfies the same orthogonality condition as ϕ (namely is a function L^2-orthogonal to Ξ) and we can write the second equation as

$$L\,\dot{\bar\phi} + M(\dot c,\dot d) + (\dot L - L)\,\dot\phi + L(M(\gamma,\delta)) + (\dot M(\dot c,\dot d) - M(\dot c,\dot d)) + \dot E + \dot N(\dot\phi) = 0\,.$$

Taking the difference with the first equation, we get

$$\begin{aligned}\mathcal{L}\,(\phi-\dot{\bar\phi}, c-\dot c, d-\dot d) \;&=\; (\dot L - L)\,\dot\phi + L(M(\gamma,\delta)) + (\dot M(\dot c,\dot d) - M(\dot c,\dot d))\\ &+\; (\dot E - E) + (\dot N(\dot\phi) - N(\phi))\,.\end{aligned}$$

Using the arguments we have already used to prove the existence of a solution together with (4.3), it is easy to check that

$$\begin{aligned}\|\phi-\dot{\bar\phi}\|_* + \|c-\dot c\| + \|d-\dot d\| \;&\leq\; C\,e^{-\delta_1\,\ell}\,(\sup_j|\dot y_j - y_j| + \sup_h|\dot z_h - z_h|)\\ &+\; C\,(|\gamma|+|\delta|) + C\,e^{-\delta_2\ell}\,\|\dot{\bar\phi}-\phi\|_*\,.\end{aligned}$$

Observe that we have the estimate

$$\|\gamma\| + \|\delta\| \leq C\,\|\phi\|_*\,(\sup_j|\dot y_j - y_j| + \sup_h|\dot z_h - z_h|)\,.$$

and hence, we conclude that

$$\|\phi-\dot{\bar\phi}\|_* + \|c-\dot c\| + \|d-\dot d\| \leq C\,e^{-\delta_1\,\ell}\,(\sup_j|\dot y_j - y_j| + \sup_h|\dot z_h - z_h|)\,.$$

This shows that the solution depends continuously on the parameters defining the points where the copies of $\pm w$ are centered. Indeed, this even proves that the solution is Lipschitz with respect to these parameters. □

Let us summarize what we have obtained so far. Given points y_j and z_h defined in (2.1) and (2.2), satisfying constraint (2.4), the previous Proposition 4.1 guarantees the existence of a unique solution (ϕ,c,d) of (4.1). Moreover, we have some estimate on the function ϕ the L^∞-weighted norm and classical elliptic regularity theory implies that these estimates extend to higher derivatives of ϕ. The function $u = U+\phi$ will then be the solution of (1.6) we were looking for if we can show that there exists a configuration for the points y_j and z_h for which the parameters c_j and d_h are all equal to zero.

In the next section, we find a precise expansion of the parameters c_j and d_h in terms of the free parameters in the construction (namely the parameters α_j, β_h and γ_h which have been used to define the points y_j and z_h). This expansion is obtained by projecting, in $L^2(\mathbb{R}^N)$, the equation (4.1) into the space spanned by $\mathsf{e}_j \cdot \Xi(\cdot - y)$, for $y \in \Pi$, as was already done in the proof of Lemma 3.1. We also explain how to solve the projected problem and this will complete the prof of Theorem 1.2. Observe that we have $2n+m$ free parameters $\alpha_1, \dots, \alpha_{m+1}$, $\beta_1, \dots, \beta_{n-1}$ and $\gamma_1, \dots, \gamma_n$ and and, given the symmetries of the solution, there are obvious relations between d_{2n-h} and d_h, for $h = 1, \dots, n$. In particular d_n is collinear to $\mathtt{n}$ and hence the number of equations to solve is also equal to $2n+m$.

5. Projections of the error and the proof of the Theorem

Again, we keep the notations of the previous section.

We define the following $\bar{n} \times \bar{n}$ matrix

$$T := \begin{pmatrix} 2 & -1 & 0 & \dots & 0 \\ -1 & 2 & \ddots & \ddots & \vdots \\ 0 & \ddots & \ddots & \ddots & 0 \\ \vdots & \ddots & \ddots & 2 & -1 \\ 0 & \dots & 0 & -1 & 2 \end{pmatrix} \in \mathcal{M}_{\bar{n}\times\bar{n}}\,, \tag{5.1}$$

It is easy to check that the inverse of T is the matrix whose entries are given by

$$\left(T^{-1}\right)_{ij} = \min(i,j) - \frac{ij}{\bar{n}+1}\,.$$

We define the vectors $S^{\downarrow}$ and $S^{\uparrow}$ by

$$T\,S^{\downarrow} := (0,\dots,0,1) \in \mathbb{R}^{\bar{n}} \qquad T\,S^{\uparrow} := (1,0,\dots,0) \in \mathbb{R}^{\bar{n}}\,. \tag{5.2}$$

We will remove the subscripts corresponding to the dimension of these matrices and vectors since there will be no confusion.

It will be convenient to adopt the following notations $\alpha := (\alpha_1, \dots, \alpha_{m+1})$, $\beta := (\beta_1, \dots, \beta_{n-1})$ and $\gamma := (\gamma_1, \dots, \gamma_n)$, where the parameters α_j, β_h and γ_h are the one involved in the construction of the points y_j and z_h which were given in (2.1) and (2.2). As usual, we assume that these parameters satisfy (2.4).

As in the introduction, we define

$$\Psi(s) := -\int_{\mathbb{R}^N} w(\cdot - s\,\mathsf{e})\,\mathrm{div}(w^p\,\mathsf{e})\,dx \tag{5.3}$$

where $\mathsf{e} \in \mathbb{R}^N$ is a unit vector. The proof of our result is based on the following key Lemma :

Lemma 5.1. *The following equality holds for any unit vector* e *and any* $s > 0$

$$\Psi(s)\,\mathsf{e} = -p\int_{\mathbb{R}^N} w(\cdot - s\,\mathsf{e})\,w^{p-1}\,\nabla w\,dx\,.$$

Moreover, there exists a constant $C_{N,p} > 0$ *only depending on* N *and* p *such that, the following expansion holds*

$$\Psi(s) := C_{N,p}\,e^{-s}\,s^{-\frac{N-1}{2}}\left(1 + O(s^{-1})\right) \quad \text{as} \quad s \to \infty.$$

The proof of the above result is by now standard, we refer to [14] and [12] for details.

Finally, we define

$$\kappa := -(\log \Psi)'(\ell) , \qquad \bar\kappa := -(\log \Psi)'(\bar\ell) ,$$

and

$$\lambda_1 := \kappa - 2\,\bar\kappa\,\sin\frac{\pi}{k} \qquad \text{and} \qquad \lambda_2 := \kappa + \bar\kappa\,\sin\frac{\pi}{k} - \frac{1}{\bar\ell}\cot\frac{\pi}{k}\cos\frac{\pi}{k} .$$

The main result of this section is the proof of the following :

Proposition 5.1. *Let ϕ be the solution of (4.1) which has been obtained in Proposition 4.1. The coefficients c_j and d_h are all equal to 0 if and only if the parameters α, β, γ are solutions of the nonlinear system*

$$\begin{aligned} \alpha &= \frac{\lambda_1}{\kappa}\,\alpha_1\,S^{\uparrow} + \frac{1}{\kappa}\left(\bar\kappa\,\beta_1 + \cot\frac{\pi}{k}\,\gamma_1 + \lambda_2\,\alpha_{m+1}\right) S^{\downarrow} + e^{-\delta_2 \ell} A + Q \in \mathbb{R}^{m+1} , \\ \beta &= -\sin\frac{\pi}{k}\,\alpha_{m+1}\,S^{\uparrow} + e^{-\delta_2 \ell} A + Q \in \mathbb{R}^{n-1} , \\ \gamma &= \frac{1}{\bar\ell}\cos\frac{\pi}{k}\,\alpha_{m+1}\,S^{\uparrow} + \gamma_n\,S^{\downarrow} + e^{-\delta_2 \ell} A + Q \in \mathbb{R}^{n} , \end{aligned} \tag{5.4}$$

where $\delta_2 > 0$, $A = A(\alpha, \beta, \gamma)$ and $Q = Q(\alpha, \beta, \gamma)$ denote smooth vector valued functions (which vary from line to line), uniformly bounded as $\ell \to \infty$ and the Taylor expansion of Q with respect to α, β, γ does not involve any constant nor any linear term.

Proof. Observe that

$$\int_{\mathbb{R}^N} (\mathsf{e}_j \cdot \Xi(\cdot - y))(\mathsf{e}_i \cdot \Xi(\cdot - z))\,dx = \frac{1}{N}\int_{\mathbb{R}^N} |\nabla w|^2\,dx\,(1 + o(1)) ,$$

if $i = j$ and $y = z \in \Pi$ and is equal to 0 otherwise, as was already mentioned in the proof of Lemma 3.1.

Now, we use

$$\int_{\mathbb{R}^N} (\mathcal{L}(\phi, c, d) + E + N(\phi))\;(\mathsf{e}_i \cdot \Xi(\cdot - y))\,dx = 0$$

so that all c_i and d_h are zero, if and only if

$$\int_{\mathbb{R}^N} (L\,\phi + E + N(\phi))\;(\mathsf{e}_i \cdot \Xi(\cdot - y))\,dx = 0$$

for all $y \in P$. Using the Lemmas below, it is easy to check that this reduces to the solvability a nonlinear system in α, β and γ. With little work, we find that this system can be written in the desired form using the inverse of the matrices T.

Observe that we implicitly use the fact that the integers n and m are bounded by ℓ^τ, for some fixed $\tau > 0$ so that the norms of the inverses of the matrices T under consideration blow up at most polynomialy in terms of ℓ and this can easily be absorbed since the error tends to 0 exponentially fast in terms of ℓ. □

The proof of this result is based on the following technical Lemmas. First, using elementary geometry, we find the :

Lemma 5.2. *The following expansions hold*

$$\frac{1}{\Psi(\ell)}\int_{\mathbb{R}^N}\Xi(\cdot-y_1)\,E\,dx \;=\; \Big(\kappa\,(\alpha_1-\alpha_2)+2\,\sin\frac{\pi}{k}\,\bar\kappa\,\alpha_1\Big)\;\mathtt{e}_1+e^{-\delta_2\ell}\,A+Q\,,$$

and, for $j=2,\ldots,m$,

$$\frac{1}{\Psi(\ell)}\int_{\mathbb{R}^N}\Xi(\cdot-y_j)\,E\,dx \;=\; \kappa\;(-\alpha_{j-1}+2\alpha_j-\alpha_{j+1})\;\mathtt{e}_1+e^{-\delta_2\ell}\,A+Q\,,$$

$$\begin{aligned}\frac{1}{\Psi(\ell)}\int_{\mathbb{R}^N}\Xi(\cdot-y_{m+1})\,E\,dx \;&=\; \Big(\kappa\,(\alpha_{m+1}-\alpha_m)-\bar\kappa\,(\beta_1+\sin\frac{\pi}{k}\,\alpha_{m+1})\\ &-\;\cot\frac{\pi}{k}(\gamma_1-\frac{1}{\bar\ell}\cos\frac{\pi}{k}\alpha_{m+1})\Big)\;\mathtt{e}_1+e^{-\delta_2\ell}A+Q.\end{aligned}$$

We also have

$$\begin{aligned}\frac{1}{\Psi(\bar\ell)}\int_{\mathbb{R}^N}\Xi(\cdot-z_1)\,E\,dx \;&=\; \bar\kappa\,(2\,\beta_1-\;\beta_2+\sin\frac{\pi}{k}\,\alpha_{m+1})\,\mathtt{t}\\ &+\;(\gamma_2-2\,\gamma_1+\frac{1}{\bar\ell}\,\cos\frac{\pi}{k}\,\alpha_{m+1})\,\mathtt{n}+e^{-\delta_2\ell}\,A+Q\,,\end{aligned}$$

and

$$\begin{aligned}\frac{(-1)^n}{\Psi(\bar\ell)}\int_{\mathbb{R}^N}\Xi(\cdot-z_h)\,E\,dx \;&=\; \bar\kappa\;(\beta_{h-1}-2\beta_h+\beta_{h+1})\,\mathtt{t}\\ &-\;(\gamma_{h-1}-2\gamma_h+\gamma_{h+1})\,\mathtt{n}+e^{-\delta_2\ell}\,A+Q\,,\end{aligned}$$

for all $h=2,\ldots,n-1$, *and finally*

$$\frac{(-1)^n}{\Psi(\bar\ell)}\int_{\mathbb{R}^N}\Xi(\cdot-z_n)\,E\,dx=2\,(\gamma_n-\gamma_{n-1})\,\mathtt{n}+e^{-\delta_2\ell}\,A+Q\,,$$

where $\delta_2>0$, $A=A(\alpha,\beta,\gamma)$ *and* $Q=Q(\alpha,\beta,\gamma)$ *denote smooth vector valued functions (which vary from line to line), uniformly bounded as* $\ell\to\infty$ *and the Taylor expansion of* Q *with respect to* α,β,γ *does not involve any constant nor any linear term.*

Proof. Given $y\in\Pi$, we would like to estimate

$$\int_{\mathbb{R}^N}\Xi(\cdot-y)\,E\,dx\,.$$

Observe that, given the structure of U and the fact that the function w decays exponentially, we can write using Taylor's expansion

$$\int_{\mathbb{R}^N}\Xi(\cdot-y)\,E\,dx=\sum_{z\in\Pi_y}\epsilon_z\int_{\mathbb{R}^N}w(\cdot-z)\,p\,w^{p-1}(\cdot-y)\,\Xi(\cdot-y)\,dx+e^{-\delta_3\ell}\,A$$

where Π_y is the set of *closest neighbors* of y in Π, namely the set of points in Π whose distance from y is equal to $\ell+O(1)$. Here $\epsilon_z=\pm1$ according to the sign which is used in front of $w(\cdot-z)$ in the definition of U and $\delta_3>1$.

Observe that, given $\mathtt{e}\in\mathbb{R}^N$ with $|\mathtt{e}|=1$ and $\mathtt{a}\in\mathbb{R}^N$. The following expansion holds

$$\Psi(|\tilde\ell\mathtt{e}+\mathtt{a}|)\frac{(\tilde\ell\mathtt{e}+\mathtt{a})}{|\tilde\ell\mathtt{e}+\mathtt{a}|}\;=\;\Psi(\tilde\ell)\,\Big(\mathtt{e}-\tilde\kappa\,\mathtt{a}^{||}+\frac{1}{\tilde\ell}\,\mathtt{a}^{\perp}+\,O(|\mathtt{a}|^2)\Big)$$

as $\tilde{\ell} \to \infty$. Here, we have decomposed $\mathtt{a} = \mathtt{a}^{||} + \mathtt{a}^{\perp}$ where $\mathtt{a}^{||}$ is collinear to $\mathtt{e}$ and $\mathtt{a}^{\perp}$ is orthogonal to $\mathtt{e}$. Here

$$\tilde{\kappa} := -(\log \Psi)'(\tilde{\ell}) .$$

Using Lemma 5.1, together with the previous expansion, we find

$$\begin{aligned} p \int_{\mathbb{R}^N} w(\cdot - (\tilde{\ell}\, \mathtt{e} + \mathtt{a})) w^{p-1}\, \Xi \, dx &= -\Psi(\tilde{\ell}) \left(\mathtt{e} - \tilde{\kappa}\, \mathtt{a}^{||} + \frac{1}{\tilde{\ell}}\, \mathtt{a}^{\perp} \right) \\ &+ e^{-\delta_3 \tilde{\ell}} A(\mathtt{a}) + \Psi(\tilde{\ell})\, Q(\mathtt{a}) . \end{aligned} \tag{5.5}$$

This fact, together with (1.16), gives that in the expansion of $\int_{\mathbb{R}^N} \Xi(\cdot - z)\, E\, dx$ the main terms are the interaction of z with closest neighbors.

We recall that Γ denotes the symmetry with respect to the $x_2 = 0$ hyperplane.

Proof of the first estimate. In Π, the closest neighbors of y_1 are y_2, $R_k\, y_1$ and $R_k^{-1}\, y_1$. It follows from the definition of the points in Π as well as the definition of $\bar{\ell}$ given in (1.22) that

$$y_2 - y_1 = \ell\, \mathbf{e}_1 + (\alpha_2 - \alpha_1)\, \mathbf{e}_1 , \qquad R_k\, y_1 - y_1 = \bar{\ell}\, \mathtt{t} + 2 \sin \frac{\pi}{k}\, \alpha_1\, \mathtt{t} ,$$

and

$$R_k^{-1}\, y_1 - y_1 = \bar{\ell}\, \mathtt{t}^* + 2 \sin \frac{\pi}{k}\, \alpha_1\, \mathtt{t}^* ,$$

where $\mathtt{t}^* := \Gamma\, \mathtt{t}$ and where we have used the identity

$$R_k\, \mathtt{e}_1 - \mathtt{e}_1 = 2 \sin \frac{\pi}{k}\, \mathtt{t} .$$

Using the expansion (5.5), we get

$$\begin{aligned} \int_{\mathbb{R}^N} \Xi(\cdot - y_1)\, E\, dx &= \big(\Psi(\ell)\, \mathtt{e}_1 + \Psi(\bar{\ell})\, (\mathtt{t} + \mathtt{t}^*)\big) - \Psi(\ell)\, \kappa\, (\alpha_2 - \alpha_1)\, \mathtt{e}_1 \\ &- \Psi(\bar{\ell})\, 2 \sin \tfrac{\pi}{k}\, \bar{\kappa}\, \alpha_1\, (\mathtt{t} + \mathtt{t}^*) + e^{-\delta_3 \ell} A + \Psi(\ell)\, Q . \end{aligned}$$

The first estimate in Lemma 5.2 follows from the fact that ℓ and $\bar{\ell}$ are related by (1.22) and $\mathtt{t} + \mathtt{t}^* = -2 \sin \frac{\pi}{k}\, \mathtt{e}_1$.

Proof of the second estimate. In Π, the closest neighbors of y_j are y_{j-1} and y_{j+1} (observe that, thanks to the fact that $k \geq 7$, the distance between y_m and z_1 can be estimated by $2 \sin\theta\, \ell + O(1)$ where $\theta = \frac{\pi}{4} - \frac{\pi}{2k} > \frac{\pi}{6}$ and hence is much larger than $\ell + O(1)$ and hence, the closest neighbors of y_m are again y_{m-1} and y_{m+1}). Since

$$y_{j+1} - y_j = \ell\, \mathbf{e}_1 + (\alpha_{j+1} - \alpha_j)\, \mathbf{e}_1 , \qquad \text{and} \qquad y_{j-1} - y_j = -\ell\, \mathbf{e}_1 + (\alpha_{j-1} - \alpha_j)\, \mathbf{e}_1 ,$$

we can make use of (5.5) and conclude that

$$\int_{\mathbb{R}^N} \Xi(\cdot - y_j)\, E\, dx = -\Psi(\ell)\, \kappa\, (\alpha_{j-1} - 2\alpha_j + \alpha_{j+1})\, \mathtt{e}_1 + e^{-\delta_3 \ell} A + \Psi(\ell)\, Q ,$$

and this completes the proof of the second estimate.

Proof of the third estimate. The closest neighbors of y_{m+1} in Π are y_m, z_1 and $R_k^{-1} z_{2n-1} = \Gamma\, z_1$. We have

$$y_m - y_{m+1} = -\ell\, \mathtt{e}_1 + (\alpha_m - \alpha_{m+1})\, \mathtt{e}_1,$$

$$z_1 - y_{m+1} = \bar\ell\, \mathtt{t} + \left(\beta_1 - \sin\frac{\pi}{k}\,\alpha_{m+1}\right)\, \mathtt{t} + \left(\bar\ell\,\gamma_1 - \cos\frac{\pi}{k}\,\alpha_{m+1}\right)\, \mathtt{n}\,,$$

and

$$\Gamma\, z_1 - y_{m+1} = \bar\ell\, \mathtt{t}^* + \left(\beta_1 - \sin\frac{\pi}{k}\,\alpha_{m+1}\right)\, \mathtt{t}^* + \left(\bar\ell\,\gamma_1 - \cos\frac{\pi}{k}\,\alpha_{m+1}\right)\, \mathtt{n}^*\,,$$

where $\mathtt{n}^* = \Gamma\,\mathtt{n}$. Making use of (5.5), we get

$$\begin{aligned}\int_{\mathbb{R}^N} \Xi(\cdot - y_{m+1})\, E\, dx \;&=\; -\big(\Psi(\ell)\,\mathtt{e}_1 + \Psi(\bar\ell)\,(\,\mathtt{t}^* + \mathtt{t})\big) - \Psi(\ell)\,\kappa\,(\alpha_m - \alpha_{m+1})\,\mathtt{e}_1 \\ &+\; \Psi(\bar\ell)\,\bar\kappa\,(\beta_1 + \sin\tfrac{\pi}{k}\,\alpha_{m+1})\,(\mathtt{t}^* + \mathtt{t}) \\ &-\; \Psi(\bar\ell)\,(\,\gamma_1 - \tfrac{1}{\bar\ell}\,\cos\tfrac{\pi}{k}\,\alpha_{m+1})\,(\mathtt{n}^* + \mathtt{n}) \\ &+\; e^{-\delta_3\ell}\, A + \Psi(\ell)\, Q\,.\end{aligned}$$

Be careful that the copies of w come with positive signs at y_{m+1} and y_m while they come with negative signs at z_1 and $\Gamma\, z_1$. The formula follows at once from the fact that $\mathtt{t}\cdot\mathtt{e}_1 = -\sin\frac{\pi}{k}$, $\mathtt{t}^* + \mathtt{t} = -2\,\sin\frac{\pi}{k}\,\mathtt{e}_1$ and $\mathtt{n}^* + \mathtt{n} = 2\,\cos\frac{\pi}{k}\,\mathtt{e}_1$.

Proof of the fourth estimate. The closest neighbors of z_1 in Π are y_{m+1} and z_2. We can write

$$y_{m+1} - z_1 = -\bar\ell\,\mathtt{t} - \left(\beta_1 + \sin\frac{\pi}{k}\,\alpha_{m+1}\right)\,\mathtt{t} - \bar\ell\,\left(\gamma_1 - \frac{1}{\bar\ell}\,\sin\frac{\pi}{k}\,\alpha_{m+1}\right)\,\mathtt{n}\,,$$

and

$$z_2 - z_1 = \bar\ell\,\mathtt{t} + (\beta_2 - \beta_1)\,\mathtt{t} + \bar\ell\,(\gamma_2 - \gamma_1)\,\mathtt{n}\,.$$

Arguing as above, we get

$$\begin{aligned}\int_{\mathbb{R}^N} \Xi(\cdot - z_1)\, E\, dx \;&=\; \Psi(\bar\ell)\,\bar\kappa\,(2\beta_1 - \beta_2 + \sin\tfrac{\pi}{k}\,\alpha_{m+1})\,\mathtt{t} \\ &+\; \Psi(\bar\ell)\,(\gamma_2 - 2\gamma_1 + \tfrac{1}{\bar\ell}\,\cos\tfrac{\pi}{k}\,\alpha_{m+1})\,\mathtt{n} \\ &+\; e^{-\delta_3\ell}\, A + \Psi(\ell)\, Q\,,\end{aligned}$$

Be careful that the copies of w come with alternative signs. The proof of the fourth estimate then follows at once.

Proof of the fifth and sixth estimate. For $h = 2, \ldots, n$, we have

$$z_{h-1} - z_h = -\bar\ell\,\mathtt{t} + (\beta_{h-1} - \beta_h)\,\mathtt{t} + \bar\ell\,(\gamma_{h-1} - \gamma_h)\,\mathtt{n}\,,$$

and

$$z_{h+1} - z_h = \bar\ell\,\mathtt{t} + (\beta_{h+1} - \beta_h)\,\mathtt{t} + \bar\ell\,(\gamma_{h+1} - \gamma_h)\,\mathtt{n}\,.$$

Applying (5.5), we conclude that

$$\begin{aligned}(-1)^h \int_{\mathbb{R}^N} \Xi(\cdot - z_h)\, E\, dx \;&=\; \Psi(\bar\ell)\;\;\bar\kappa\;(\beta_{h-1} - 2\beta_h + \beta_{h+1})\;\mathtt{t} \\ &-\; \Psi(\bar\ell)\;\;(\gamma_{h-1} - 2\gamma_h + \gamma_{h+1})\;\mathtt{n} \\ &+\; e^{-\delta_3\ell}\, A + \Psi(\ell)\;\; Q\,.\end{aligned}$$

Again, one should be careful that the copies of w come with alternative signs. This completes the proof of the sixth estimate. The fifth estimate follows from similar considerations. □

The next result is easier to get. It reads :

Lemma 5.3. *The following expansions hold*

$$\int_{\mathbb{R}^N} \Xi(\cdot - y)\,(L\,\phi)\,dx = \Psi(\ell)\,e^{-\delta_2 \ell}\,A\,,$$

and

$$\int_{\mathbb{R}^N} \Xi(\cdot - y)\,N(\phi)\,dx = \Psi(\ell)\,e^{-\delta_2 \ell}\,A\,,$$

where $\delta_2 > 0$ and $A = A(\alpha, \beta, \gamma)$ denote smooth vector valued functions (which vary from line to line), uniformly bounded as $\ell \to \infty$.

Proof. The proof of the first estimate is easy and follows the line of the proof of Lemma 3.1.

The proof of the second estimate follows from the estimate in Proposition 4.1 together with the choice of η so that $\delta_1 > \frac{1}{2}$. Details are left to the reader. □

We now explain how (5.4) can be solved. We claim that this system is equivalent to

$$\begin{aligned} \alpha &= e^{-\tilde\delta_2 \ell}\,A + Q\,, \\ \beta &= e^{-\tilde\delta_2 \ell}\,A + Q\,, \\ \gamma &= e^{-\tilde\delta_2 \ell}\,A + Q\,, \end{aligned} \tag{5.6}$$

where $\tilde\delta_2 > 0$ and $A = A(\alpha, \beta, \gamma)$ and $Q = Q(\alpha, \beta, \gamma)$ satisfy the usual assumptions.

Observe that the system (5.4) is almost of the correct form. In fact, using the second and third equation together with the expression of $S^\uparrow$ and $S^\downarrow$ one checks that

$$\beta_1 = -\frac{n-1}{n}\,\sin\frac{\pi}{k}\,\alpha_{m+1} + e^{-\tilde\delta_2 \ell}\,A + Q\,,$$

and also

$$\gamma_1 = \frac{1}{\ell}\,\cos\frac{\pi}{k}\,\alpha_{m+1} + e^{-\tilde\delta_2 \ell}\,A + Q\,,$$

and

$$\gamma_n = \frac{1}{\ell}\,\cos\frac{\pi}{k}\,\alpha_{m+1} + e^{-\tilde\delta_2 \ell}\,A + Q\,,$$

where $\tilde\delta_2 > 0$. Hence we get

$$\bar\kappa\,\beta_1 + \cot\frac{\pi}{k}\,\gamma_1 + \lambda_2\,\alpha_{m+1} = \left(\kappa + \frac{\bar\kappa}{n}\,\sin\frac{\pi}{k}\right)\,\alpha_{m+1} + e^{-\tilde\delta_2 \ell}\,A + Q\,.$$

Introducing these in the first equation, we are left to solve a coupled system in α_1 and α_{m+1}. This system reads

$$\left\{ \begin{array}{rcl} \left(1 + 2\,(m+1)\,\frac{\bar\kappa}{\kappa}\,\sin\frac{\pi}{k}\right)\,\alpha_1 - \left(1 + \frac{1}{n}\,\frac{\bar\kappa}{\kappa}\,\sin\frac{\pi}{k}\right)\,\alpha_{m+1} & = & e^{-\tilde\delta_2 \ell}\,A + Q\,, \\ -\left(1 - 2\,\frac{\bar\kappa}{\kappa}\,\sin\frac{\pi}{k}\right)\,\alpha_1 + \left(1 - \frac{1}{n}\,(m+1)\,\frac{\bar\kappa}{\kappa}\,\sin\frac{\pi}{k}\right)\,\alpha_{m+1} & = & e^{-\tilde\delta_2 \ell}\,A + Q\,, \end{array} \right.$$

where $\tilde{\delta}_2 > 0$ (possibly different from the previous one). This system can be solved to be put in diagonal form provided D, the determinant of the 2 by 2 system on the left had side, is non zero. But we have

$$D = \frac{\bar{\kappa}}{\kappa^2} \sin\frac{\pi}{k} \frac{m+2}{n} \left((2n-1)\kappa - 2m \sin\frac{\pi}{k} \bar{\kappa} \right) .$$

Using (1.15) and (1.10), we get

$$D = \frac{\bar{\kappa}}{\kappa^2} \sin\frac{\pi}{k} \frac{m+2}{n} (2n-1) \left(\frac{\ell - \bar{\ell}}{\ell} + O(\ell^{-2}) \right) ,$$

which, thanks to (1.23), is certainly bounded from below by some constant times m/ℓ for all ℓ large enough. This completes the proof of the claim.

It is now straightforward to prove, using Browder's fixed point theorem, that

Lemma 5.4. *There exist $C > 0$ and $\ell_0 > 0$ such that, for all $\ell \geq \ell_0$, there exists a solution of (5.6) such that*

$$|\alpha| + |\beta| + |\gamma| \leq C\, e^{-\tilde{\delta}_2\, \ell} .$$

The proof of this last lemma is standard and left to the reader. Observe that, with some more care, one can prove that the solution in Proposition 4.1 depends smoothly on the parameters and then one can use (increasing the value of ℓ_0 if this is necessary) one can use a fixed point theorem for contraction mapping to prove Lemma 5.4. This has the advantage to prove some local uniqueness for the solution of (5.6) and in turn, this shows the unique (local) solvability of the nonlinear equation once the parameters m, n and $\ell, \bar{\ell}$ solutions of (1.15) and (1.22) are fixed.

This last result completes the proof of Theorem 1.2.

To complete the paper, we now explain how to formally justify this constraint we impose on the choice of the parameters ℓ and $\bar{\ell}$. Let us recall that if u is a solution of (1.6) then

$$\operatorname{div} \left((a \cdot \nabla u)\, \nabla u - \frac{1}{2} (|\nabla u|^2 + u^2)\, a + \frac{1}{p+1} |u|^{p+1}\, a \right) = 0 ,$$

for any fixed vector $a \in \mathbb{R}^N$ (just multiply (1.6) by $a \cdot \nabla u$). In particular, the divergence theorem implies that, for any smooth domain $\Omega \subset \mathbb{R}^N$, the vector

$$Y(u, \Omega) := \int_{\partial\Omega} \left((\nabla u \cdot \nu)\, \nabla u - \frac{1}{2} (|\nabla u|^2 + u^2)\, \nu + \frac{1}{p+1} |u|^{p+1}\, \nu \right) d\sigma ,$$

is equal to 0. Here ν is the outside unit vector field to $\partial\Omega$. If, as we hope, a function of the form

$$U = w + \sum_i \epsilon_i\, w(\cdot - z_i) + \mathcal{O}(e^{-\frac{3\ell}{2}}) ,$$

where $\epsilon_i \in \{\pm 1\}$ and where the points z_i have the property that

$$|z_i| = \ell + O(1)$$

is in $B_{\ell/2}$, the ball of radius $\ell/2$ centered at the origin, close to a genuine solution of (1.6), then the above vector should be *reasonably close to* 0 as ℓ tends to ∞. Taking $\Omega = B_{\ell/2}$ and $u = U$ we find that

$$Y(U, B_{\ell/2}) = \sum_i \epsilon_i\, \Psi(|z_i|)\, \frac{z_i}{|z_i|} + O(e^{-\delta \ell}),$$

for some $\delta > 1$, as ℓ tends to 0. Therefore, it is reasonable to ask that

$$\sum_i \epsilon_i \, \Psi(|z_i|) \, \frac{z_i}{|z_i|} = 0 \, .$$

This is precisely the *balancing condition* we were referring to. Applying this to the approximate solution $\mathring{U}$ at the points y_1 and y_{m+1} leads to (1.22).

References

[1] T. Bartsch and M. Willem, Infinitely many nonradial solutions of a Euclidean scalar field equation, *J. Funct. Anal.* 117(1993), 447-460.

[2] T. Bartsch and M. Willem, Infinitely many radial solutions of a semilinear elliptic problem on $\mathbb{R}^N$, *Arch. Rat. Mech. Anal.* 124(1993), 261-276.

[3] H. Berestycki and P.L. Lions, Nonlinear scalar field equations, I, Existence of a ground state, *Arch. Rat. Mech. Anal.* 82(1981), 313-347.

[4] H. Berestycki and P.L. Lions, Nonlinear scalar field equations, II, *Arch. Rat. Mech. Anal.* 82(1981), 347-375.

[5] M. Conti, L. Merizzi and S. Terracini, Radial solutions of superlinear problems in $\mathbb{R}^N$, *Arch. Rat. Mech. Anal.* 153(2000), 291-316.

[6] M. del Pino, M. Kowalczyk, F. Pacard and J. Wei, The Toda system and multiple-end solutions of autonomous planar elliptic problems, preprint 2007.

[7] B. Gidas, W.M. Ni and L. Nirenberg, Symmetry of positive solutions of nonlinear elliptic equations in R^n. *Mathematical Analysis and Applications*, Part A, Adv. Math. Suppl. Studies Vol. **7A**, pp. 369-402, Academic Press, New York, 1981.

[8] M. Jleli and F. Pacard, An end-to-end construction for compact constant mean curvature surfaces. Pacific Journal of Maths, 221, no. 1, (2005) 81-108.

[9] N. Kapouleas, Compact constant mean curvature surfaces in Euclidean three-space. J. Differential Geom. 33 (1991), no. 3, 683-715.

[10] M.K. Kwong, Uniquness of positive solutions of $\Delta u - u + u^p = 0$ in $\mathbb{R}^N$. *Arch. Rational Mech. Anal.* **105** (1991), 243-266.

[11] M.K. Kwong, and L. Zhang, Uniqueness of the positive solution of $\Delta u + f(u) = 0$ in an annulus. *Diff. Int. Eqns* **4** (1991), no. 3, 583-599.

[12] F.H. Lin, W.M. Ni and J.C. Wei, On the number of interior peak solutions for a singularly perturbed Neumann problem, *Comm. Pure Appl. Math.* 60 (2007), 252-281.

[13] S. Lorca and P. Ubilla, Symmetric and nonsymmetric solutions for an elliptic equation on $\mathbb{R}^N$, *Nonlinear Anal.* 58(2004), 961-968.

[14] A. Malchiodi. *Some new entire solutions of semilinear elliptic equations in* $\mathbb{R}^N$. Preprint 2007.

[15] W.M. Ni and I. Takagi, Locating the peaks of least-energy solutions to a semilinear Neumann problem. *Duke Math. J.* 70 (1993), no. 2, 247-281.

[16] M. Struwe, Multiple solutions of differential equations without the Palais-Smale condition, *Math. Ann.* 261(1992), 399-412.

[17] J.C. Wei and S. Yan, Infinitely many positive solutions for the nonlinear Schrödinger equations in $\mathbb{R}^N$, preprint 2008.

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