AN OPTIMAL BOUND ON THE NUMBER OF INTERIOR SPIKE SOLUTIONS FOR THE LIN-NI-TAKAGI PROBLEM

WEIWEI AO, JUNCHENG WEI, AND JING ZENG

Abstract. We consider the following singularly perturbed Neumann problem

$$\varepsilon^2\Delta u - u + u^p = 0 \quad \text{in} \quad \Omega, \quad u > 0 \quad \text{in} \quad \Omega, \quad \frac{\partial u}{\partial \nu} = 0 \quad \text{on} \quad \partial\Omega,$$

where p is subcritical and Ω is a smooth and bounded domain in $\mathbb{R}^n$ with its unit outward normal ν. Lin-Ni-Wei [18] proved that there exists ε_0 such that for $0 < \varepsilon < \varepsilon_0$ and for each integer k bounded by

$$1 \le k \le \frac{\delta(\Omega, n, p)}{(\varepsilon|\log\varepsilon|)^n} \tag{0.1}$$

where $\delta(\Omega, n, p)$ is a constant depending only on Ω, p and n, there exists a solution with k interior spikes. We show that the bound on k can be improved to

$$1 \le k \le \frac{\delta(\Omega, n, p)}{\varepsilon^n}, \tag{0.2}$$

which is optimal.

1. Introduction and statement of main results

Of concern is the following Lin-Ni-Takagi problem ([17])

$$\begin{cases} \varepsilon^2\Delta u - u + u^p = 0 \text{ in } \Omega \\ u > 0 \text{ in } \Omega \\ \frac{\partial u}{\partial \nu} = 0 \text{ on } \partial\Omega, \end{cases} \tag{1.1}$$

where p satisfies $1 < p < +\infty$ for $n = 2$ and $1 < p < \frac{n+2}{n-2}$ for $n \ge 3$ and Ω is bounded, smooth domain in $\mathbb{R}^n$ with its unit outward normal ν.

Problem (1.1) arises in many applied models concerning biological pattern formations. For instance, it gives rise to steady states in the Keller-Segel model of the chemotactic aggregation of the cellular slime molds and it also plays an important role in the Gierer-Meinhardt

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1991 *Mathematics Subject Classification.* 35J25, 35J20, 35B33, 35B40.

Key words and phrases. Singular Perturbation, localized energy method, optimal bound.

model describing the regeneration phenomena of hydra. See [9], [15] and [17] for more details.

Problem (1.1) has been studied extensively for the last twenty years. In the pioneering paper [17], Lin, Ni and Takagi proved the a priori estimates and existence of least energy solutions to (1.1), that is, a solution u_ϵ with minimal energy. Furthermore, Ni and Takagi showed in [24, 25] that, *for each $\epsilon > 0$ sufficiently small, u_ϵ has a spike at the most curved part of the boundary, i.e., the region where the mean curvature attains maximum value.*

Since the publication of [25], problem (1.1) has received a great deal of attention and significant progress has been made. More specifically, solutions with multiple boundary peaks as well as multiple interior peaks have been established. (See [4]-[5], [11]-[14], [16]-[18], [26]-[29] and the references therein.) In particular, it was established in Gui and Wei [13] that *for any two given integers $k \geq 0, l \geq 0$ and $k+l > 0$, problem (1.1) has a solution with exactly k interior spikes and l boundary spikes* for every ϵ sufficiently small. Furthermore, Lin, Ni and Wei [18] showed that there are at least $\frac{\delta(n,p,\Omega)}{(\epsilon|\log\epsilon|)^n}$ number of interior spikes. On the other hand, problem (1.1) also admits higher dimensional concentrations. (See [23].) For results in this direction, we refer to [1], [19]-[22]. In particular, we mention the results of Malchiodi and Montenegro [21, 22] on the existence of solutions concentrating on the *whole boundary* provided that the sequence ε satisfies some gap condition.

In this paper, we shall address the question of the maximal possible number of spikes, in terms of small parameter $\varepsilon > 0$, that a solution of (1.1) could have. Note that since p is subcritical, the solutions to (1.1) is uniformly bounded (Lin-Ni-Takagi [17]). Thus the energy bound for solutions of (1.1) is $O(1)$. On the other hand, each spike contributes to at least $O(\varepsilon^n)$ energy. This implies that the number of interior spikes can not exceed $O(\varepsilon^{-n})$. Our main result, Theorem 1.1 below, asserts that for every positive integer $k \leq \frac{\delta_{\Omega,n,p}}{\varepsilon^n}$, where $\delta(\Omega, n, p)$ is a constant depending only on n, p and Ω, problem (1.1) has a solution with exactly k peaks. This gives an optimal bound on the number of interior spikes.

Our proof uses a *"localized energy method"* as in [12] and [18]. There are two main difficulties. First, the distance between spikes is assumed only to be $O(\varepsilon)$. In the Liapunov-Schmidt reduction process, we have to prove that all the estimates are uniform with respect to the integer k. Second, we have to detect the difference in the energy when spikes move to the boundary of the configuration space. A crucial estimate is Lemma 5.1, in which we prove that the accumulated error can be controlled from step k to step $k+1$. To prove Lemma 5.1, we have to perform a secondary Liapunov-Schmidt reduction. This seems to be new.

We now state the main result in this paper.

Theorem 1.1. *There exists an* $\varepsilon_0 > 0$ *such that for* $0 < \varepsilon < \varepsilon_0$*, and any positive integer* k *satisfying*

$$1 \leq k \leq \frac{\delta(\Omega, n, p)}{\varepsilon^n}, \tag{1.2}$$

where $\delta(\Omega, n, p)$ *is a constant depending on* n, Ω *and* p *only, problem (1.1) has a solution* u_ε *that possesses exactly* k *local maximum points.*

Remark 1.1. As mentioned earlier, the upper bound for k is the best possible. As far as we know, the only result on the optimal upper bound for the number of spikes is the one-dimensional situation. In a series of papers [7]-[8], Felmer-Martinez-Tanaka studied the following singularly perturbed nonlinear Schrödinger equation

$$\varepsilon^2 \Delta u - V(x) u + u^p = 0, u > 0, u \in H^1(\mathbb{R}). \tag{1.3}$$

They constructed solutions to (1.3) with $\frac{C}{\varepsilon}$ number of spikes. Extension to Gierer-Meinhardt system can be found in [6]. Related construction can also be found in del Pino-Felmer-Tanaka [3].

Remark 1.2. An interesting problem is to study the *homogenization* of the measure $\varepsilon^{-n} |\nabla u|^2 dx$. We expect that it will approach some kind of Lebesgue measure. As $\varepsilon \to 0$, the locations of the maximum points should approach to some sphere-packing positions.

Remark 1.3. It is clear that the proofs of Theorem 1.1 can be applied to a large class of singularly perturbed problems

$$\begin{cases} \epsilon^2 \Delta u - u + f(u) = 0 \text{ in } \Omega \\ u > 0 \text{ in } \Omega, \ \frac{\partial u}{\partial \nu} = 0 \text{ on } \partial\Omega, \end{cases} \tag{1.4}$$

where $f(u)$ satisfies the conditions (f1)-(f3) stated in [18].

The paper is organized as follows. Notations, preliminaries and some useful estimates are explained in Section 2. Section 3 contains the study of a linear problem that is the first step in the Lyapunov-Schmidt reduction process. In Section 4, we solve a nonlinear projected problem. Section 5 contains a key estimate which majors the differences between k-th step and $(k+1)$-th step. We then set up a maximization problem in Section 6. Finally in Section 7, we show that the solution to the maximization problem is indeed a solution of (1.1) and prove Theorem 1.1.

Throughout this paper, unless otherwise stated, the letters c, C will always denote various generic constants that are independent of ε and k for ε small enough.

Acknowledgment. Juncheng Wei was supported by a GRF grant from RGC of Hong Kong.

2. Notation and Some Preliminary Analysis

In this section we introduce some notations and some preliminary analysis on approximate solutions. Our main concern is that all the estimates should be independent of k-the number of spikes.

Without loss of generality, we may assume that $0 \in \Omega$. By the following rescaling:

$$z = \varepsilon x, \ \ x \in \Omega_\varepsilon := \{\varepsilon z \in \Omega\},$$

equation (1.1) becomes

$$\begin{cases} \Delta u - u + u^p = 0 \ \text{ in } \Omega_\varepsilon \\ u > 0 \text{ in } \Omega_\varepsilon, \ \ \frac{\partial u}{\partial \nu} = 0 \ \text{ on } \partial\Omega_\varepsilon. \end{cases} \tag{2.5}$$

For $u \in H^2(\Omega_\varepsilon)$, we also put

$$S_\varepsilon(u) = \Delta u - u + u^p. \tag{2.6}$$

Associated with problem (2.5) is the energy functional

$$J_\varepsilon(u) = \frac{1}{2}\int_{\Omega_\varepsilon} (|\nabla u|^2 + u^2) - \frac{1}{p+1}\int_{\Omega_\varepsilon} u_+^{p+1}, \ \ u \in H^1(\Omega_\varepsilon), \tag{2.7}$$

where we denote $u_+ = \max(u, 0)$.

Now we define the configuration space,

$$\Lambda_k := \Big\{(Q_1, \cdots, Q_k) \in \Omega^k \ \Big| \ \min_{i\neq j} |Q_i - Q_j| \geq \rho\varepsilon, \min_{i,j,d(Q_j,\partial\Omega)\leq 10\epsilon|\ln\epsilon|} |Q_i - Q_j^*| \geq \rho\varepsilon\Big\}, \tag{2.8}$$

where $Q_j^* = Q_j + 2d(Q_j, \partial\Omega)\nu_{\bar{Q}_j}$, $\nu_{\bar{Q}_j}$ denotes the unit outer normal at $\bar{Q}_j \in \partial\Omega$, and $\bar{Q}_j$ is the unique point on $\partial\Omega$ such that $d(Q_j, \partial\Omega) = d(Q_j, \bar{Q}_j)$, and ρ is a constant which is large enough (but independent of ε). (This is possible since $d(Q_j, \partial\Omega) \leq 10\epsilon|\ln\epsilon|$.)

By the definition above, we may assume that

$$1 \leq k \leq \frac{\delta}{\varepsilon^n \rho^n} \tag{2.9}$$

for some $\delta > 0$ sufficiently small only depend on Ω ,n and p. We can get a lower bound of ρ, so we have a upper bound of k which is of $O(\frac{1}{\varepsilon^n})$. See Remark 6.1 below.

Let w be the unique solution of

$$\begin{cases} \Delta w - w + w^p = 0, \ w > 0 \text{ in } \mathbb{R}^n, \\ w(0) = \max_{y \in \mathbb{R}^n} w(y), \ w \to 0 \text{ as } |y| \to \infty. \end{cases} \tag{2.10}$$

By the well-known result of Gidas, Ni and Nirenberg [10], w is radially symmetric and is strictly decreasing, and $w'(r) < 0$ for $r > 0$. Moreover, we have the following asymptotic behavior of w:

$$\begin{cases} w(r) = A_n r^{-\frac{n-1}{2}} e^{-r}(1 + O(\frac{1}{r})) \\ w'(r) = -A_n r^{-\frac{n-1}{2}} e^{-r}(1 + O(\frac{1}{r})) \end{cases} \tag{2.11}$$

for $r > 0$ large, where A_n is a positive constant.

Let $K(r)$ be the fundamental solution of $-\Delta + 1$ centered at 0. Then we have

$$\begin{cases} w(r) = (A_0 + O(\frac{1}{r}))K(r) \\ w'(r) = -(A_0 + O(\frac{1}{r}))K(r) \end{cases} \tag{2.12}$$

for $r > 0$ large, where A_0 is a positive constant.

For $Q \in \Omega$, we define $w_{\varepsilon,Q}$ to be the unique solution of

$$\Delta v - v + w(\cdot - \frac{Q}{\varepsilon})^p = 0 \ \text{ in } \Omega_\varepsilon, \ \frac{\partial v}{\partial \nu} = 0 \ \text{ on } \partial\Omega_\varepsilon. \tag{2.13}$$

We first analyze $w_{\varepsilon,Q}$. To this end, set

$$\varphi_{\varepsilon,Q} = w(\frac{z - Q}{\varepsilon}) - w_{\varepsilon,Q}(\frac{z}{\varepsilon}). \tag{2.14}$$

We state the following lemma on the properties of $\varphi_{\varepsilon,Q}$:

Lemma 2.1. *Assume that $c\varepsilon \leq d(Q, \partial\Omega) \leq 10\varepsilon|\ln\varepsilon|$, where $c \geq \frac{\rho}{2}$. We have*

$$\varphi_{\varepsilon,Q} = -(A_0 + O(\frac{1}{\rho^{\frac{1}{2}}}))K(\frac{z - Q^*}{\varepsilon}) + O(e^{-2\rho}). \tag{2.15}$$

Proof. In Lemma 2.1 of [18], a similar estimate was proved under the condition that $C_1\varepsilon|\ln\varepsilon| \leq d(Q, \partial\Omega) \leq \delta$. Here we will relax this condition to $c\varepsilon \leq d(Q, \partial\Omega) \leq 10\varepsilon|\ln\varepsilon|$. The proof is similar. For the sake of completeness, we repeat a modification of the proof here.

Let $\psi_\varepsilon(z)$ be the unique solution of

$$\varepsilon^2\Delta\psi_\varepsilon - \psi_\varepsilon = 0 \ \text{ in } \Omega, \ \frac{\partial\psi_\varepsilon}{\partial\nu} = 0 \ \text{ on } \partial\Omega. \tag{2.16}$$

It is easy to see that

$$0 < \psi_\varepsilon(z) \leq \psi_1(z) \leq C \ \text{ for } \varepsilon < 1. \tag{2.17}$$

On the other hand, $\varphi_{\varepsilon,Q}$ satisfies

$$\varepsilon^2\Delta v - v = 0 \ \text{ in } \Omega,\ \frac{\partial v}{\partial \nu} = \frac{\partial}{\partial \nu} w(\frac{z-Q}{\varepsilon}) \ \text{ on } \partial\Omega. \tag{2.18}$$

Using (2.11), we can see that on $\partial\Omega$,

$$\begin{aligned}\frac{\partial}{\partial \nu} w(\frac{z-Q}{\varepsilon}) &= \frac{1}{\varepsilon} w'(\frac{z-Q}{\varepsilon})\frac{\langle z-Q,\nu\rangle}{|z-Q|}\\ &= -(A_n + O(\frac{1}{\rho}))\varepsilon^{\frac{n-3}{2}}|z-Q|^{-\frac{n+1}{2}}e^{-\frac{|z-Q|}{\varepsilon}}\langle z-Q,\nu\rangle.\end{aligned}$$

We use the following comparison function:

$$\varphi_1(z) = -(A_0 - \frac{1}{\rho^{\frac{1}{2}}})K(\frac{z-Q^*}{\varepsilon}) + e^{-d\rho}\psi_\varepsilon, \tag{2.19}$$

where $d \geq 2$ is a constant.

For $z \in \partial\Omega$, $|z-Q| \geq \varepsilon^{\frac{3}{4}}$, we have

$$\frac{\partial \varphi_1(z)}{\partial \nu} = -(A_0 - \frac{1}{\rho^{\frac{1}{2}}})K'(\frac{z-Q^*}{\varepsilon})\varepsilon^{-1}\frac{\langle z-Q^*,\nu\rangle}{|z-Q^*|} + e^{-d\rho} \geq \frac{1}{2}e^{-d\rho},$$

$$\frac{\partial \varphi_{\varepsilon,Q}}{\partial \nu} \leq ce^{-\varepsilon^{-\frac{1}{4}}},$$

so

$$\frac{\partial \varphi_{\varepsilon,Q}}{\partial \nu} \leq \frac{\partial \varphi_1}{\partial \nu}.$$

For $|z-Q| \leq \varepsilon^{\frac{3}{4}}$, we have

$$\frac{\partial \varphi_1}{\partial \nu} = \frac{\partial}{\partial \nu}\{-(A_0 - \frac{1}{\rho^{\frac{1}{2}}})K(\frac{z-Q^*}{\varepsilon})\} + e^{-d\rho}.$$

Since

$$\begin{aligned}\frac{\langle z-Q,\nu\rangle}{|z-Q|} &= -(1+O(\varepsilon^{\frac{1}{2}}))\frac{\langle z-Q^*,\nu\rangle}{|z-Q^*|},\\ \frac{|z-Q|}{\varepsilon} &= (1+O(\varepsilon^{\frac{1}{2}}))\frac{|z-Q^*|}{\varepsilon},\end{aligned}$$

we obtain

$$\frac{\partial \varphi_{\varepsilon,Q}}{\partial \nu} \leq \frac{\partial \varphi_1}{\partial \nu}.$$

By the comparison principle, we have

$$\varphi_{\varepsilon,Q}(z) \leq \varphi_1(z) \text{ for } z \in \Omega. \tag{2.20}$$

Similarly, we obtain

$$\varphi_{\varepsilon,Q} \geq -(A_0 + \frac{1}{\rho^{\frac{1}{2}}})K(\frac{z-Q^*}{\varepsilon}) - e^{-d\rho}\psi_\varepsilon \text{ for } z \in \Omega. \tag{2.21}$$

□

For $\mathbf{Q} = (Q_1, \cdots, Q_k) \in \Lambda_k$, we define

$$w_{Q_i}(x) = w(x - \frac{Q_i}{\varepsilon}),\ w_{\varepsilon,\mathbf{Q}} = \sum_{i=1}^{k} w_{\varepsilon,Q_i}. \tag{2.22}$$

The next lemma analyzes $w_{\varepsilon,\mathbf{Q}}$ in Ω_ε. To this end, we divide Ω_ε into $k+1$ parts:

$$\Omega_{\varepsilon,i} = \{|x - \frac{Q_i}{\varepsilon}| \leq \frac{\rho}{2}\},\ i = 1, \cdots, k, \tag{2.23}$$

$$\Omega_{\varepsilon,k+1} = \Omega_\varepsilon \backslash \cup_{i=1}^{k} \Omega_{\varepsilon,i}. \tag{2.24}$$

Then we have the following lemma

Lemma 2.2. *For $x \in \Omega_{\varepsilon,i}$, $i = 1, \cdots, k$, we have*

$$w_{\varepsilon,\mathbf{Q}} = w_{\varepsilon,Q_i} + O(e^{-\frac{\rho}{2}}). \tag{2.25}$$

For $x \in \Omega_{\varepsilon,k+1}$, we have

$$w_{\varepsilon,\mathbf{Q}} = O(e^{-\frac{\rho}{2}}). \tag{2.26}$$

Proof. For $j \neq i$, and $x \in \Omega_{\varepsilon,i}$, we have

$$\begin{aligned} w_{\varepsilon,Q_j} &= w(x - \frac{Q_j}{\varepsilon}) - \varphi_{\varepsilon,Q_j}(\varepsilon x) \\ &= O(e^{-|x-\frac{Q_j}{\varepsilon}|} + e^{-|x-\frac{Q_j^*}{\varepsilon}|}) \\ &= O(e^{-|x-\frac{Q_j}{\varepsilon}|}) \end{aligned}$$

by the definition of the configuration set. Next we observe that given a a ball of size ρ, there are at most $c_n := 6^n$ number of non-overlapping balls of size ρ surrounding this ball. Thus we have for $x \in \Omega_{\varepsilon,i}$,

$$\begin{aligned} \sum_{j\neq i} w_{\varepsilon,Q_j}(x) &= O(\sum_{j\neq i} e^{-|x-\frac{Q_j}{\varepsilon}|}) + O(e^{-2\rho}) \\ &\leq c_n e^{-\frac{\rho}{2}} + c_n^2 e^{-\rho} + \cdots + c_n^j e^{-\frac{j\rho}{2}} + \cdots \\ &\leq \sum_{j=1}^{\infty} e^{j(\log c_n - \frac{\rho}{2})} + O(e^{-2\rho}) \\ &\leq O(e^{-(\frac{\rho}{2} - \log c_n)}) \\ &\leq O(e^{-\frac{\rho}{2}}), \end{aligned}$$

if $c_n < e^{\frac{\rho}{2}}$, which is true for ρ large enough. So this proves (2.25). The proof of (2.26) is similar.

□

Remark 2.1. *In the following sections, we will use the definition of the configuration and the estimate as above frequently.*

The following lemma is proved in Lemma 2.3 of [2].

Lemma 2.3. *Let $f \in C(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$, $g \in C(\mathbb{R}^n)$ be radially symmetric and satisfy for some $\alpha \geq 0$, $\beta \geq 0$, $\gamma_0 \in \mathbb{R}$,*

$$f(x)exp(\alpha|x|)|x|^\beta \to \gamma_0 \ \ as\ |x| \to \infty,$$
$$\int_{\mathbb{R}^n} |g(x)|exp(\alpha|x|)(1+|x|^\beta)dx < \infty.$$

Then

$$exp(\alpha|y|)|y|^\beta \int_{\mathbb{R}^n} g(x+y)f(x)dx \to \gamma_0 \int_{\mathbb{R}^n} g(x)exp(-\alpha x_1)dx \ \ as\ |y| \to \infty.$$

As in [18], we now define the following quantities:

$$B_\varepsilon(Q_j) = -\int_{\Omega_\varepsilon} w^p_{Q_j}\varphi_{\varepsilon,Q_j}dx, \quad B_\varepsilon(Q_i,Q_j) = \int_{\Omega_\varepsilon} w^p_{Q_i}w_{Q_j}dx. \tag{2.27}$$

Then we have the following:

Lemma 2.4. *For $\mathbf{Q} = (Q_1, \cdots, Q_k) \in \Lambda_k$, it holds that*

$$B_\varepsilon(Q_j) = (\gamma + O(\frac{1}{\sqrt{\rho}}))w(\frac{2d(Q_j,\partial\Omega)}{\varepsilon}) + O(e^{-(1+\xi)\rho}), \tag{2.28}$$

$$B_\varepsilon(Q_i,Q_j) = (\gamma + O(\frac{1}{\sqrt{\rho}}))w(\frac{2d(Q_j,\partial\Omega)}{\varepsilon}) + O(e^{-(1+\xi)\rho}), \tag{2.29}$$

for some $\xi > 0$ independent of ε and k for ε sufficiently small, where

$$\gamma = \int_{\mathbb{R}^n} w^p(y)e^{-y_1}dy. \tag{2.30}$$

Remark 2.2. *Note that $\gamma > 0$. See Lemma 4.7 of* [26].

Proof. By Lemma 2.2 and 2.3, the proof is similar to that of Lemma 2.5 in [18]. We omit the details. □

3. Linear Theory

In this section, we study a linear theory that allow us to perform the finite dimensional reduction procedure. The proof is similar to Section 3 of [18]. However, the main concern is to show that all the constants are independent of the number k. Fixing an integer k satisfying

$$1 \le k \le \frac{\delta}{\varepsilon^n} \tag{3.31}$$

and $\mathbf{Q} \in \Lambda_k$, we define the following functions:

$$Z_{ij} = \frac{\partial w_{Q_i}}{\partial x_j}\chi_i(x), \text{ for } i = 1, \cdots, k, \ j = 1, \cdots, n, \tag{3.32}$$

where $w_{Q_i}(x) = w(x - \frac{Q_i}{\varepsilon})$, $\chi_i(x) = \chi(\frac{2|\varepsilon x - Q_i|}{(\rho-1)\varepsilon})$ and $\chi(t)$ is a cut off function such that $\chi(t) = 1$ for $|t| \le 1$ and $\chi(t) = 0$ for $|t| \ge \frac{\rho^2}{\rho^2-1}$. Note that the support of Z_{ij} belongs to $B_{\frac{\rho^2}{2(\rho+1)}}(\frac{Q_i}{\varepsilon})$.

We consider the following linear problem: Given h, find a function ϕ satisfying

$$\begin{cases} L(\phi) := \Delta\phi - \phi + p w_{\varepsilon,\mathbf{Q}}^{p-1}\phi = h + \sum_{i=1,\cdots,k, j=1,\cdots,n} c_{ij} Z_{ij} \quad \text{in} \quad \Omega_\varepsilon \\ \frac{\partial \phi}{\partial \nu} = 0 \quad \text{on} \quad \partial\Omega_\varepsilon \\ \int_{\Omega_\varepsilon} \phi Z_{ij} = 0 \quad \text{for} \quad i = 1, \cdots, k, \ j = 1, \cdots, n. \end{cases} \tag{3.33}$$

Let

$$W := \sum_{\mathbf{Q}\in\Lambda_k} e^{-\eta|\cdot - \frac{Q_i}{\varepsilon}|}. \tag{3.34}$$

Given $0 < \eta < 1$, consider the norm

$$\|h\|_* = \sup_{x\in\Omega_\varepsilon} |W(x)^{-1} h(x)| \tag{3.35}$$

where $(Q_1, \cdots, Q_k) \in \Lambda_k$.

Proposition 3.1. *There exist positive numbers $\eta \in (0,1)$, $\varepsilon_0 > 0$, $\rho_0 > 0$ and $C > 0$, such that for all $0 < \varepsilon < \varepsilon_0$, $\rho > \rho_0$, and for any given h with $\|h\|_*$ norm bounded, there is a unique solution $(\phi, \{c_{ij}\})$ to problem (3.33). Furthermore*

$$\|\phi\|_* \le C\|h\|_*. \tag{3.36}$$

The proof of the above Proposition, which we postpone to the end of this section, is based on Fredholm alternative Theorem for compact operator and an a-priori bound for solution to (3.33) that we state (and prove) next.

Proposition 3.2. *Let h with $\|h\|_*$ bounded and assume that $(\phi, \{c_{ij}\})$ is a solution to (3.33). Then there exist positive numbers ε_0, ρ_0 and C, such that for all $0 < \varepsilon < \varepsilon_0$, $\rho > \rho_0$ and $\mathbf{Q} \in \Lambda_k$, one has*

$$\|\phi\|_* \leq C\|h\|_*, \tag{3.37}$$

where C is a positive constant independent of ε, ρ, k and $\mathbf{Q} \in \Lambda_k$.

Proof. We argue by contradiction. Assume that there exist ϕ solution to (3.33) and

$$\|h\|_* \to 0, \quad \|\phi\|_* = 1.$$

Multiplying the equation in (3.33) against Z_{ij} and integrating in Ω_ε, we get

$$\int_{\Omega_\varepsilon} L\phi Z_{ij}(x) = \int_{\Omega_\varepsilon} hZ_{ij} + c_{ij} \int_{\Omega_\varepsilon} Z_{ij}^2.$$

Given the exponential decay at infinity of $\partial_{x_i} w$ and the definition of Z_{ij}, we get

$$\int_{\Omega_\varepsilon} Z_{ij}^2 = \int_{\mathbb{R}^n} w_{x_i}^2 + O(e^{-\delta_1 \rho}), \text{ as } \rho \to \infty, \tag{3.38}$$

for some $\delta_1 > 0$. On the other hand

$$|\int_{\Omega_\varepsilon} hZ_{ij}| \leq C\|h\|_* \int_{\Omega_\varepsilon} |w_{x_i}(x - \frac{Q_i}{\varepsilon})| e^{-\eta|x - \frac{Q_i}{\varepsilon}|} dx \leq C\|h\|_*.$$

Here and in what follows, C stands for a positive constant independent of ε, and ρ, as $\varepsilon \to 0$, $\rho \to \infty$. Now if we write $\tilde{Z}_{ij}(x) = w_{x_i}(x - \frac{Q_i}{\varepsilon})$, we have

$$\begin{aligned}
-\int_{\Omega_\varepsilon} L\phi Z_{ij}(x) &= -\int_{\Omega_\varepsilon} \phi(L[Z_{ij}]) \\
&= \int_{B(\frac{Q_i}{\varepsilon}, \frac{\rho}{2})} [\Delta \tilde{Z}_{ij} - \tilde{Z}_{ij} + pw^{p-1}(x - \frac{Q_i}{\varepsilon})\tilde{Z}_{ij}]\chi_i \phi \\
&- \int_{B(z, \frac{\rho}{2})} \phi(\tilde{Z}_{ij} \Delta \chi_i + 2\nabla \chi_i \nabla Z_{ij}) \qquad (3.39) \\
&+ p\int_{B(\frac{Q_i}{\varepsilon}, \frac{\rho}{2})} (w_{\varepsilon,\mathbf{Q}}^{p-1} - w^{p-1}(x - \frac{Q_i}{\varepsilon}))\phi \tilde{Z}_{ij} \chi_i.
\end{aligned}$$

Next we estimate all the terms in the above equation.

The first term is 0 since

$$\Delta \tilde{Z}_{ij} - \tilde{Z}_{ij} + pw^{p-1}(x - \frac{Q_i}{\varepsilon})\tilde{Z}_{ij} = 0.$$

The second integral can be estimated as follows

$$\begin{aligned}\left|\int_{\Omega_\varepsilon} \phi(\tilde{Z}_{ij}\Delta\chi_i + 2\nabla\chi_i\nabla\tilde{Z}_{ij})\right| &\le C\|\phi\|_* \int_{\frac{\rho-1}{2}}^{\frac{\rho^2}{2(\rho+1)}} e^{-(1+\eta)s} s^{\frac{-(n-1)}{2}} ds \\ &\le Ce^{-(1+\xi)\frac{\rho}{2}}\|\phi\|_*,\end{aligned}$$

for some $\xi > 0$. Finally, we observe that in $B(\frac{Q_i}{\varepsilon}, \frac{\rho}{2})$ the following holds

$$|w_{\varepsilon,\mathbf{Q}}^{p-1} - w_{Q_i}^{p-1}(x)| \le Cw_{Q_i}^{p-2}(x)\left[\sum_{j\ne i} w(x - \frac{Q_j}{\varepsilon})\right].$$

Thus we obtain

$$\left|\int_{B(\frac{Q_i}{\varepsilon},\frac{\rho}{2})} (w_{\varepsilon,\mathbf{Q}}^{p-1} - w_{Q_i}^{p-1}(x))\phi\tilde{Z}_{ij}\chi_i\right| \le Ce^{-\xi\frac{\rho}{2}}\|\phi\|_*$$

for some $\xi > 0$, depending on n and p. We then conclude that

$$|c_{ij}| \le C\left[e^{-\xi\frac{\rho}{2}}\|\phi\|_* + \|h\|_*\right]. \tag{3.40}$$

Let now $\eta \in (0,1)$. It is easy to check that the function W (defined at (3.34)) satisfies

$$L\,W \le \frac{1}{2}\,(\eta^2 - 1)\,W\,,$$

in $\Omega_\varepsilon \setminus \cup_{i=1}^k B(\frac{Q_i}{\varepsilon}, \rho_1)$ provided ρ_1 is large enough but independent of ρ. Hence the function W can be used as a barrier to prove the pointwise estimate

$$|\phi|(x) \le C\left(\|L\,\phi\|_* + \sup_i \|\phi\|_{L^\infty(\partial B(\frac{Q_i}{\varepsilon},\rho_1))}\right) W(x)\,, \tag{3.41}$$

for all $x \in \Omega_\varepsilon \setminus \cup_{i=1}^k B(\frac{Q_i}{\varepsilon}, \rho_1)$.

Granted these preliminary estimates, the proof of the result goes by contradiction. Let us assume there exist a sequence of ε tending to 0, ρ tending to ∞ and a sequence of solutions of (3.33) for which the inequality is not true. The problem being linear, we can reduce to the case where we have a sequence $\varepsilon^{(n)}$ tending to 0, $\rho^{(n)}$ tending to ∞ and sequences $h^{(n)}$, $\phi^{(n)}$, $\{c_{ij}^{(n)}\}$ such that

$$\|h^{(n)}\|_* \to 0, \quad \text{and} \quad \|\phi^{(n)}\|_* = 1.$$

By (3.40), we can get that

$$\|\sum_{ij} c_{ij}^{(n)} Z_{ij}\|_* \to 0\,.$$

Then (3.41) implies that there exists $Q_i^{(n)} \in \Lambda_k$ such that

$$\|\phi^{(n)}\|_{L^\infty(B(Q_i^{(n)},\frac{\rho}{2}))} \geq C\,, \tag{3.42}$$

for some fixed constant $C > 0$. Using elliptic estimates together with Ascoli-Arzela's theorem, we can find a sequence $Q_i^{(n)}$ and we can extract, from the sequence $\phi^{(n)}(\cdot - \frac{Q_i^{(n)}}{\varepsilon})$ a subsequence which will converge (on compact sets) to ϕ_∞ a solution of

$$\left(\Delta - 1 + pw^{p-1}\right)\,\phi_\infty = 0\,,$$

in $\mathbb{R}^n$, which is bounded by a constant times $e^{-\eta\,|x|}$, with $\eta > 0$. Moreover, recall that $\phi^{(n)}$ satisfies the orthogonality conditions in (3.33). Therefore, the limit function ϕ_∞ also satisfies

$$\int_{\mathbb{R}^n} \phi_\infty\,\nabla w\,dx = 0\,.$$

By the nondegeneracy of solution w, we have that $\phi_\infty \equiv 0$, which is certainly in contradiction with (3.42) which implies that ϕ_∞ is not identically equal to 0.

Having reached a contradiction, this completes the proof of the Proposition. □

We can now prove Proposition 3.1.

Proof of Proposition 3.1. Consider the space

$$\mathcal{H} = \{u \in H^1(\Omega_\varepsilon)\;:\;\int_{\Omega_\varepsilon} uZ_{ij} = 0, \quad (Q_1,\cdots,Q_k) \in \Lambda_k\}.$$

Notice that the problem (3.33) in ϕ gets re-written as

$$\phi + K(\phi) = \bar{h} \quad \text{in} \quad \mathcal{H} \tag{3.43}$$

where $\bar{h}$ is defined by duality and $K : \mathcal{H} \to \mathcal{H}$ is a linear compact operator. Using Fredholm's alternative, showing that equation (3.43) has a unique solution for each $\bar{h}$ is equivalent to showing that the equation has a unique solution for $\bar{h} = 0$, which in turn follows from Proposition 3.2. The estimate (3.36) follows directly from Proposition 3.2. This concludes the proof of Proposition 3.1.

In the following, if ϕ is the unique solution given by Proposition 3.1, we set

$$\phi = \mathcal{A}(h). \tag{3.44}$$

Estimate (3.36) implies

$$\|\mathcal{A}(h)\|_* \leq C\|h\|_*. \tag{3.45}$$

4. The non linear projected problem

For small ε, large ρ, and fixed points $\mathbf{Q} \in \Lambda_k$, we show solvability in ϕ, $\{c_{ij}\}$ of the non linear projected problem

$$\begin{cases} \Delta(w_{\varepsilon,\mathbf{Q}}+\phi)-(w_{\varepsilon,\mathbf{Q}}+\phi)+(w_{\varepsilon,\mathbf{Q}}+\phi)^p=\sum_{i=1,\cdots,k,j=1,\cdots,n} c_{ij}Z_{ij} \quad \text{in} \quad \Omega_\varepsilon \\ \frac{\partial\phi}{\partial\nu}=0, \quad \text{on} \quad \partial\Omega_\varepsilon \\ \int_{\Omega_\varepsilon}\phi Z_{ij}=0 \quad \text{for } i=1,\cdots,k, \ j=1,\cdots,n. \end{cases} \tag{4.1}$$

The first equation in (4.1) can be rewritten as

$$L(\phi):=\Delta\phi-\phi+pw_{\varepsilon,\mathbf{Q}}^{p-1}\phi=S_\varepsilon(w_{\varepsilon,\mathbf{Q}})+N(\phi)+\sum_{i=1,\cdots,k,j=1,\cdots,n} c_{ij}Z_{ij}, \tag{4.2}$$

where

$$S_\varepsilon(w_{\varepsilon,\mathbf{Q}})=\Delta w_{\varepsilon,\mathbf{Q}}-w_{\varepsilon,\mathbf{Q}}+w_{\varepsilon,\mathbf{Q}}^p, \tag{4.3}$$

$$N(\phi)=(w_{\varepsilon,\mathbf{Q}}+\phi)^p-w_{\varepsilon,\mathbf{Q}}^p-pw_{\varepsilon,\mathbf{Q}}^{p-1}\phi. \tag{4.4}$$

We have the validity of the following result:

Proposition 4.1. *There exist positive numbers ε_0, ρ_0, C and $\xi>0$ such that for all $\varepsilon\le\varepsilon_0$, $\rho\ge\rho_0$, and for any $\mathbf{Q}\in\Lambda_k$, there is a unique solution $(\phi_{\varepsilon,\mathbf{Q}},\{c_{ij}\})$ to problem (4.1). Furthermore $\phi_{\varepsilon,\mathbf{Q}}$ is C^1 in $\mathbf{Q}$ and we have*

$$\|\phi_{\varepsilon,\mathbf{Q}}\|_*\le Ce^{-\frac{(1+\xi)}{2}\rho}. \tag{4.5}$$

Proof. The proof relies on the contraction mapping theorem in the $\|\cdot\|_*$-norm introduced above. Observe that ϕ solves (4.1) if and only if

$$\phi=\mathcal{A}\left(S_\varepsilon(w_{\varepsilon,\mathbf{Q}})+N(\phi)\right) \tag{4.6}$$

where $\mathcal{A}$ is the operator introduced in (3.44). In other words, ϕ solves (4.1) if and only if ϕ is a fixed point for the operator

$$T(\phi):=\mathcal{A}\left(S_\varepsilon(w_{\varepsilon,\mathbf{Q}})+N(\phi)\right).$$

Given $r>0$, define

$$\mathcal{B}=\{\phi\in C^2(\Omega_\varepsilon)\ :\ \|\phi\|_*\le re^{-\frac{(1+\xi)}{2}\rho},\ \int_{\Omega_\varepsilon}\phi Z_{ij}=0\}.$$

We will prove that T is a contraction mapping from $\mathcal{B}$ in itself.

To do so, we claim that

$$\|S_\varepsilon(w_{\varepsilon,\mathbf{Q}})\|_*\le Ce^{-\frac{(1+\xi)}{2}\rho} \tag{4.7}$$

and

$$\|N(\phi)\|_* \leq C\left[\|\phi\|_*^2 + \|\phi\|_*^p\right], \tag{4.8}$$

for some fixed function C independent of ρ and ε. We postpone the proof of the estimates above to the end of the proof of this Proposition. Assuming the validity of (4.7) and (4.8) and taking into account (3.45), we have for any $\phi \in \mathcal{B}$

$$\begin{aligned}\|T(\phi)\|_* \leq \ & C\left[\|S_\varepsilon(w_{\varepsilon,\mathbf{Q}}) + N(\phi)\|_*\right] \leq C\left[e^{-\frac{(1+\xi)}{2}\rho} + r^2 e^{-(1+\xi)\rho} + r^p e^{-\frac{p(1+\xi)}{2}\rho}\right] \\ & \leq r e^{-\frac{(1+\xi)}{2}\rho}\end{aligned}$$

for a proper choice of r in the definition of $\mathcal{B}$, since $p > 1$.

Take now ϕ_1 and ϕ_2 in $\mathcal{B}$. Then it is straightforward to show that

$$\begin{aligned}\|T(\phi_1) - T(\phi_2)\|_* & \leq C\|N(\phi_1) - N(\phi_2)\|_* \\ & \leq C\left[\|\phi_1\|_*^{\min(1,p-1)} + \|\phi_2\|_*^{\min(1,p-1)}\right]\|\phi_1 - \phi_2\|_* \\ & \leq \tfrac{1}{2}\|\phi_1 - \phi_2\|_*.\end{aligned}$$

This means that T is a contraction mapping from $\mathcal{B}$ into itself.

To conclude the proof of this Proposition we are left to show the validity of (4.7) and (4.8). We start with (4.7).

Fix $Q_i \in \Lambda_k$ and consider the region $|x - \frac{Q_i}{\varepsilon}| \leq \frac{\rho}{2+\sigma}$, where σ is a small positive number to be chosen later. In this region the error $S_\varepsilon(w_{\varepsilon,\mathbf{Q}})$ can be estimated in the following way

$$\begin{aligned}|S_\varepsilon(w_{\varepsilon,\mathbf{Q}})| & \leq C\left[w^{p-1}(x - \frac{Q_i}{\varepsilon})\sum_{j\neq i} w(x - \frac{Q_i}{\varepsilon}) + \sum_{j\neq i} w^p(x - \frac{Q_j}{\varepsilon})\right] \\ & \leq Cw^{p-1}(x - \frac{Q_i}{\varepsilon})e^{-(\frac{1}{2}+\frac{\sigma}{2(2+\sigma)})\rho} \\ & \leq Cw^{p-1}(x - \frac{Q_i}{\varepsilon})e^{-(\frac{1}{2}+\frac{\sigma}{4(2+\sigma)})\rho}\, e^{-\frac{\sigma}{4(2+\sigma)}\rho} \\ & \leq Cw^{p-1}(x - \frac{Q_i}{\varepsilon})e^{-\frac{1+\xi}{2}\rho}\end{aligned} \tag{4.9}$$

for a proper choice of $\xi > 0$.

Consider now the region $|x - \frac{Q_i}{\varepsilon}| > \frac{\rho}{2+\sigma}$, for all i. Since $0 < \mu < p-1$, we write $\mu = p - 1 - M$. From the definition of $S_\varepsilon(w_{\varepsilon,\mathbf{Q}})$, we get in the region under consideration

$$\begin{aligned}|S_\varepsilon(w_{\varepsilon,\mathbf{Q}})| &\le C\left[\sum_j w^p(x-\frac{Q_j}{\varepsilon})\right] \le C\left[\sum_j e^{-\mu|x-\frac{Q_j}{\varepsilon}|}\right] e^{-(p-\mu)\frac{\rho}{2+\sigma}} \\ &\le \left[\sum_j e^{-\mu|x-\frac{Q_j}{\varepsilon}|}\right] e^{-\frac{1+M}{2+\sigma}\rho} \\ &\le \left[\sum_j e^{-\mu|x-\frac{Q_j}{\varepsilon}|}\right] e^{-\frac{1+\xi}{2}\rho} \qquad (4.10)\end{aligned}$$

for some $\xi > 0$, if we chose M and σ small enough. From (4.9) and (4.10) we get (4.7).

We now prove (4.8). Let $\phi \in \mathcal{B}$. Then

$$|N(\phi)| \le |(w_{\varepsilon,\mathbf{Q}} + \phi)^p - w_{\varepsilon,\mathbf{Q}}^p - pw_{\varepsilon,\mathbf{Q}}^{p-1}\phi| \le C(\phi^2 + |\phi|^p). \qquad (4.11)$$

Thus we have

$$\begin{aligned}|(\textstyle\sum_j e^{-\eta|x-\frac{Q_j}{\varepsilon}|})^{-1} N(\phi)| &\le C\|\phi\|_*\,(|\phi| + |\phi|^{p-1}) \\ &\le C(\|\phi\|_*^2 + \|\phi\|_*^p).\end{aligned}$$

This gives (4.8).

For the C^1 regularity of $\phi_{\varepsilon,\mathbf{Q}}$, see Lemma 4.1 in [18]. This concludes the proof of the Proposition. □

5. An improved estimate

In this section, we present a key estimate on the difference between the solutions in the $k-$th step and $(k+1)-$th step.

For $(Q_1, \cdots, Q_k) \in \Lambda_k$, we denote $u_{\varepsilon,Q_1,\cdots,Q_k}$ as $w_{\varepsilon,Q_1,\dots,Q_k} + \phi_{\varepsilon,Q_1,\dots,Q_k}$, where $\phi_{\varepsilon,Q_1,\cdots,Q_k}$ is the unique solution given by Proposition 4.1. The estimate below says that the difference between $u_{\varepsilon,Q_1,\cdots,Q_{k+1}}$ and $u_{\varepsilon,Q_1,\cdots,Q_k} + u_{\varepsilon,Q_{k+1}}$ is small globally in $H^1(\Omega_\varepsilon)$ norm.

We now write

$$\begin{aligned}u_{\varepsilon,Q_1,\cdots,Q_{k+1}} &= u_{\varepsilon,Q_1,\cdots,Q_k} + u_{\varepsilon,Q_{k+1}} + \varphi_{k+1} \qquad (5.12)\\ &= \bar{W} + \varphi_{k+1},\end{aligned}$$

where

$$\bar{W} = u_{\varepsilon,Q_1,\cdots,Q_k} + u_{\varepsilon,Q_{k+1}}.$$

By Proposition 4.1, we can easily derive that

$$\|\varphi_{k+1}\|_* \le Ce^{-\frac{(1+\xi)}{2}\rho}. \qquad (5.13)$$

However the estimate (5.13) is not good enough. We need the following key estimate for φ_{k+1}:

Lemma 5.1. *Let ρ, ε be as in Proposition 4.1. Then it holds*

$$\int_{\Omega_\varepsilon} (|\nabla \varphi_{k+1}|^2 + \varphi_{k+1}^2) \le Ce^{-(1+\xi)\rho}, \tag{5.14}$$

for some constant $C > 0, \xi > 0$ independent of ε, ρ, k and $\mathbf{Q} \in \Lambda_{k+1}$.

Proof. To prove (5.14), we need to perform a secondary decomposition.

We first recall the following fact: it is well-known that the principal eigenfunction ϕ_0 of the following linearized operator:

$$\Delta\phi - \phi + pw^{p-1}\phi = \lambda_1\phi \tag{5.15}$$

is even and exponentially decaying, where λ_1 is the first eigenvalue. We fix ϕ_0 such that $\max_{y\in\mathbb{R}^n}\phi_0 = 1$. Denote by $\phi_i = \chi_i\phi_0(x - \frac{Q_i}{\varepsilon})$, where χ_i is the cut-off function introduced in Section 3.

By the equations satisfied by φ_{k+1}, we have

$$\bar{L}\varphi_{k+1} = \bar{S} + \sum_{i=1,\cdots,k+1,j=1,\cdots,n} c_{ij}Z_{ij} \tag{5.16}$$

for some constants $\{c_{ij}\}$, where

$$\bar{L} = \Delta - 1 + p\tilde{W}^{p-1},$$

$$\tilde{W}^{p-1} = \begin{cases} \frac{(\bar{W}+\varphi_{k+1})^p - \bar{W}^p}{p\varphi_{k+1}}, \text{ if } \varphi_{k+1} \neq 0 \\ \bar{W}^{p-1}, \text{ if } \varphi_{k+1} = 0, \end{cases}$$

and

$$\bar{S} = (u_{\varepsilon,Q_1,\cdots,Q_k} + u_{\varepsilon,Q_{k+1}})^p - u^p_{\varepsilon,Q_1,\cdots,Q_k} - u^p_{\varepsilon,Q_{k+1}}.$$

The L^2-norm of $\bar{S}$ is estimated first: Observe that

$$\begin{aligned} |\bar{S}| &= |(u_{\varepsilon,Q_1,\cdots,Q_k} + u_{\varepsilon,Q_{k+1}})^p - u^p_{\varepsilon,Q_1,\cdots,Q_k} - u^p_{\varepsilon,Q_{k+1}}| \\ &\le C(p|u_{\varepsilon,Q_1,\cdots,Q_k}|^{p-1}u_{\varepsilon,Q_{k+1}} + p|u_{\varepsilon,Q_{k+1}}|^{p-1}u_{\varepsilon,Q_1,\cdots,Q_k}). \end{aligned}$$

By the estimate in Proposition 4.1, we have the following estimate of the first term above

$$\begin{aligned} &\int_{\Omega_\varepsilon} |u_{\varepsilon,Q_1,\cdots,Q_k}|^{2(p-1)} u^2_{\varepsilon,Q_{k+1}} dx \\ &\le C\int_{\Omega_\varepsilon} w^{2(p-1)}_{\varepsilon,Q_1,\cdots,Q_k} w^2_{\varepsilon,Q_{k+1}} dx + O(e^{-(1+\xi)\rho}) \\ &\le Ce^{-(1+\xi)\rho}. \end{aligned}$$

The second term can be estimated similarly. So we have

$$\|\bar{S}\|_{L^2(\Omega_\varepsilon)} \le ce^{-(1+\xi)\frac{\rho}{2}}. \tag{5.17}$$

By the estimate (5.13), we have the following estimate

$$\tilde{W} = \sum_{i=1}^{k+1} w(x - \frac{Q_i}{\epsilon}) + O(e^{-(1+\xi)\frac{\rho}{2}}). \tag{5.18}$$

Decompose φ_{k+1} as

$$\varphi_{k+1} = \psi + \sum_{i=1}^{k+1} c_i \phi_i + \sum_{i=1,\cdots,k+1, j=1,\cdots,n} d_{ij} Z_{ij} \tag{5.19}$$

for some c_i, d_{ij} such that

$$\int_{\Omega_\varepsilon} \psi \bar{L} \phi_i dx = \int_{\Omega_\varepsilon} \psi Z_{ij} dx = 0,\ i = 1, ..., k,\ j = 1, ..., n. \tag{5.20}$$

Since

$$\varphi_{k+1} = \phi_{\varepsilon, Q_1, \cdots, Q_{k+1}} - \phi_{\varepsilon, Q_1, \cdots, Q_k} - \phi_{\varepsilon, Q_{k+1}}, \tag{5.21}$$

we have for $i = 1, \cdots, k$,

$$\begin{aligned} d_{ij} &= \int_{\Omega_\varepsilon} \varphi_{k+1} Z_{ij} \\ &= \int_{\Omega_\varepsilon} (\phi_{\varepsilon, Q_1, \cdots, Q_{k+1}} - \phi_{\varepsilon, Q_1, \cdots, Q_k} - \phi_{\varepsilon, Q_{k+1}}) Z_{ij} \\ &= -\int_{\Omega_\varepsilon} \phi_{\varepsilon, Q_{k+1}} Z_{ij} \end{aligned}$$

and

$$\begin{aligned} d_{k+1,j} &= \int_{\Omega_\varepsilon} \varphi_{k+1} Z_{k+1,j} \\ &= \int_{\Omega_\varepsilon} (\phi_{\varepsilon, Q_1, \cdots, Q_{k+1}} - \phi_{\varepsilon, Q_1, \cdots, Q_k} - \phi_{\varepsilon, Q_{k+1}}) Z_{k+1,j} \\ &= -\int_{\Omega_\varepsilon} \phi_{\varepsilon, Q_1, \cdots, Q_k} Z_{k+1,j}, \end{aligned}$$

where we use the orthogonality conditions satisfied by $\phi_{\varepsilon, Q_1, \cdots, Q_k}$ and $\phi_{\varepsilon, Q_{k+1}}$. So by Proposition 4.1, we have

$$\begin{cases} |d_{ij}| \leq c e^{-(1+\xi)\frac{\rho}{2}} e^{-\eta \frac{|Q_i - Q_{k+1}|}{\varepsilon}} \text{ for } i = 1, \cdots, k \\ |d_{k+1,j}| \leq c e^{-(1+\xi)\frac{\rho}{2}} \sum_{i=1}^{k} e^{-\eta \frac{|Q_i - Q_{k+1}|}{\varepsilon}} \end{cases} \tag{5.22}$$

for some $\eta > 0$.

By (5.19), we can rewrite (5.16) as

$$\bar{L}\psi + \sum_{i=1}^{k+1} c_i \bar{L}\phi_i + \sum_{i=1,\cdots,k+1,j=1,\cdots,n} d_{ij}\bar{L}Z_{ij} = \bar{S} + \sum_{i=1,\cdots,k+1,j=1,\cdots,n} c_{ij}Z_{ij}. \tag{5.23}$$

To obtain the estimates for the coefficients c_i , we use the equation (5.23).

First, multiplying (5.23) by ϕ_i and integrating over Ω_ε, we have

$$c_i \int_{\Omega_\varepsilon} \bar{L}(\phi_i)\phi_i = -\sum_{j=1}^{n} d_{ij} \int_{\Omega_\varepsilon} \bar{L}(Z_{ij})\phi_i + \int_{\Omega_\varepsilon} \bar{S}\phi_i \tag{5.24}$$

where

$$\begin{cases} |\int_{\Omega_\varepsilon} \bar{S}\phi_i| \leq ce^{-(1+\xi)\frac{\rho}{2}} e^{-\eta\frac{|Q_i-Q_{k+1}|}{\varepsilon}} \text{ for } i=1,\cdots,k \\ |\int_{\Omega_\varepsilon} \bar{S}\phi_{k+1}| \leq ce^{-(1+\xi)\frac{\rho}{2}} \sum_{i=1}^{k} e^{-\eta\frac{|Q_i-Q_{k+1}|}{\varepsilon}}. \end{cases} \tag{5.25}$$

From (5.18) we see that

$$\int_{\Omega_\varepsilon} \bar{L}(\phi_i)\phi_i = -\lambda_1 \int_{\mathbb{R}^n} \phi_0^2 + O(e^{-(1+\xi)\frac{\rho}{2}}). \tag{5.26}$$

Combining (5.22) and (5.24)-(5.26), we have

$$\begin{cases} |c_i| \leq ce^{-(1+\xi)\frac{\rho}{2}} e^{-\eta\frac{|Q_i-Q_{k+1}|}{\varepsilon}},\ i=1,...,k \\ |c_{k+1}| \leq ce^{-(1+\xi)\frac{\rho}{2}} \sum_{i=1}^{k} e^{-\eta\frac{|Q_i-Q_{k+1}|}{\varepsilon}}. \end{cases} \tag{5.27}$$

Next let us estimate ψ. Multiplying (5.23) by ψ and integrating over Ω_ε, we find

$$\int_{\Omega_\varepsilon} \bar{L}(\psi)\psi = \int_{\Omega_\varepsilon} \bar{S}\psi - \sum_{i=1,\cdots,k+1,j=1,\cdots,n} d_{ij} \int_{\Omega_\varepsilon} \bar{L}(Z_{ij})\psi. \tag{5.28}$$

We claim that

$$\int_{\Omega_\varepsilon} [-\bar{L}(\psi)\psi] \geq c_0 \|\psi\|^2_{H^1(\Omega_\varepsilon)} \tag{5.29}$$

for some constant $c_0 > 0$.

Since the approximate solution is exponentially decaying away from the points $\frac{Q_i}{\varepsilon}$, we have

$$\int_{\Omega_\varepsilon \setminus \cup_i B_{\frac{\rho-1}{2}}(\frac{Q_i}{\varepsilon})} \bar{L}(\psi)\psi \geq \frac{1}{2} \int_{\Omega_\varepsilon \setminus \cup_i B_{\frac{\rho-1}{2}}(\frac{Q_i}{\varepsilon})} |\nabla\psi|^2 + |\psi|^2. \tag{5.30}$$

Now we only need to prove the above estimates in the domain $\cup_i B_{\frac{\rho-1}{2}}(\frac{Q_i}{\varepsilon})$. We prove it by contradiction. Otherwise, there exists a sequence $\rho_n \to$

$+\infty$, and $Q_i^{(n)}$ such that

$$\int_{B_{\frac{\rho n-1}{2}}(\frac{Q_i^{(n)}}{\varepsilon})} |\nabla\psi_n|^2 + |\psi_n|^2 = 1, \int_{B_{\frac{\rho n-1}{2}}(\frac{Q_i^{(n)}}{\varepsilon})} \bar{L}(\psi_n)\psi_n \to 0, \text{ as } n\to\infty.$$

Then we can extract from the sequence $\psi_n(\cdot - \frac{Q_i^{(n)}}{\varepsilon})$ a subsequence which will converge weakly in $H^1(\mathbb{R}^n)$ to ψ_∞, such that

$$\int_{\mathbb{R}^n} |\nabla\psi_\infty|^2 + |\psi_\infty|^2 - pw^{p-1}\psi_\infty^2 = 0, \tag{5.31}$$

and

$$\int_{\mathbb{R}^n} \psi_\infty\phi_0 = \int_{\mathbb{R}^n} \psi_\infty \frac{\partial w}{\partial x_i} = 0, \text{ for } i = 1, \cdots, n. \tag{5.32}$$

From (5.31) and (5.32), we deduce that $\psi_\infty = 0$.

Hence

$$\psi_n \rightharpoonup 0 \text{ weakly in } H^1(\mathbb{R}^n). \tag{5.33}$$

So

$$\int_{B_{\frac{\rho n-1}{2}}(\frac{Q_i^{(n)}}{\varepsilon})} p\tilde{W}^{p-1}\psi_n^2 \to 0 \text{ as } n\to\infty. \tag{5.34}$$

We have

$$\|\psi_n\|_{H^1(B_{\frac{\rho n-1}{2}})} \to 0 \text{ as } n\to\infty. \tag{5.35}$$

This contradicts the assumption

$$\|\psi_n\|_{H^1} = 1. \tag{5.36}$$

So we get that

$$\int_{\Omega_\varepsilon} [-\bar{L}(\psi)\psi] \geq c_0\|\psi\|^2_{H^1(\Omega_\varepsilon)}. \tag{5.37}$$

From (5.28) and (5.37), we get

$$\begin{aligned} \|\psi\|^2_{H^1(\Omega_\epsilon)} &\leq c(\sum_{ij} |d_{ij}||\int_{\Omega_\epsilon} \bar{L}(Z_{ij})\psi| + |\int_{\Omega_\epsilon} \bar{S}\psi|) && (5.38)\\ &\leq c(\sum_{ij} |d_{ij}|\|\psi\|_{H^1(\Omega_\epsilon)} + \|\bar{S}\|_{L^2(\Omega_\epsilon)}\|\psi\|_{H^1(\Omega_\epsilon)}). && (5.39) \end{aligned}$$

So

$$\|\psi\|_{H^1(\Omega_\epsilon)} \leq c(\sum_{ij} |d_{ij}| + \|\bar{S}\|_{L^2(\Omega_\epsilon)}). \tag{5.40}$$

From (5.27) (5.22) (5.17) and (5.40), we get that

$$\begin{aligned} \|\varphi_{k+1}\|_{H^1(\Omega_\epsilon)} &\leq c(e^{-\frac{\rho}{2}(1+\xi)} + \|\bar{S}\|_{L^2}) && (5.41)\\ &\leq ce^{-\frac{\rho}{2}(1+\xi)}. && (5.42) \end{aligned}$$

□

6. The Reduced Problem: A Maximization Procedure

In this section, we study a maximization problem. Fix $\mathbf{Q} \in \Lambda_k$, we define a new functional

$$\mathcal{M}_\varepsilon(\mathbf{Q}) = J_\varepsilon(u_{\varepsilon,\mathbf{Q}}) = J_\varepsilon[w_{\varepsilon,\mathbf{Q}} + \phi_{\varepsilon,\mathbf{Q}}] : \Lambda_k \to \mathbb{R}. \tag{6.43}$$

Define

$$C_k^\varepsilon = \max_{\mathbf{Q}\in\Lambda_k}\{\mathcal{M}_\varepsilon(\mathbf{Q})\}. \tag{6.44}$$

Since $\mathcal{M}_\varepsilon(\mathbf{Q})$ is continuous in $\mathbf{Q}$, the maximization problem has a solution. Let $\mathcal{M}_\varepsilon(\bar{\mathbf{Q}})$ be the maximum where $\bar{\mathbf{Q}} = (\bar{Q}_1, \cdots, \bar{Q}_k) \in \bar{\Lambda}_k$, that is

$$\mathcal{M}_\varepsilon(\bar{Q}_1, \cdots, \bar{Q}_k) = \max_{\mathbf{Q}\in\Lambda_k} \mathcal{M}_\varepsilon(\mathbf{Q}), \tag{6.45}$$

and we denote the solution by $u_{\varepsilon,\bar{Q}_1,\cdots,\bar{Q}_k}$.

A consequence of Lemma 5.1 is the following:

Proposition 6.1. *Suppose that $k < \frac{\delta}{\epsilon^n}$ where δ is sufficiently small (but independent of ϵ). Then it holds*

$$C_{k+1}^\varepsilon > C_k^\varepsilon + I(w) - \frac{\gamma}{4}e^{-\rho}, \tag{6.46}$$

where $I(w)$ is the energy of w,

$$I(w) = \frac{1}{2}\int_{\mathbb{R}^n}(|\nabla w|^2 + w^2) - \frac{1}{p+1}\int_{\mathbb{R}^n} w^{p+1}. \tag{6.47}$$

and $\gamma > 0$ is defined at (2.30).

Proof. We prove it by contradiction. Assume that on the contrary we have

$$C_{k+1}^\varepsilon \le C_k^\varepsilon + I(w) - \frac{\gamma}{4}e^{-\rho}. \tag{6.48}$$

First we claim:

Given $(Q_1, \cdots, Q_k) \in \bar{\Lambda}_k$, there exists $Q_{k+1} \in \Omega$, such that

$$B_{3\rho\varepsilon}(Q_{k+1}) \cap \{Q_1, \cdots, Q_k, \partial\Omega\} = \emptyset. \tag{6.49}$$

In fact, if not, we have $k\cdot|B_1|\cdot(3\rho)^n \ge \frac{|\Omega|}{2\varepsilon^n}$. So $k \ge \frac{|\Omega|}{2\times 3^n\rho^n\varepsilon^n|B_1|} = \frac{C_{\Omega,n}}{\rho^n\varepsilon^n}$. By the assumption, we have $k \le \frac{\delta}{\rho^n\varepsilon^n}$ where δ is sufficiently small. This is a contradiction if we choose δ so small such that $\delta < C_{\Omega,n}$. So the claimed is proved.

Assume that $(\bar{Q}_1, \cdots, \bar{Q}_k) \in \bar{\Lambda}_k$ is such that $\mathcal{M}_\varepsilon(\bar{Q}_1, \cdots, \bar{Q}_k) = \max_{\mathbf{Q}\in\Lambda_k} \mathcal{M}_\varepsilon(\mathbf{Q}) = C_k^\varepsilon$, and we denote the solution by $u_{\varepsilon,\bar{Q}_1,\cdots,\bar{Q}_k}$. Let

Q_{k+1} be a point satisfying (6.49). (The existence of C_k^ε follows from continuity of $\mathcal{M}_\varepsilon$.)

Next we consider the solution concentrates at $(\bar{Q}_1, \cdots, \bar{Q}_k, Q_{k+1})$. As in Section 5, we decompose the solution as

$$u_{\varepsilon,\bar{Q}_1,\cdots,\bar{Q}_k,Q_{k+1}} = u_{\varepsilon,\bar{Q}_1,\cdots,\bar{Q}_k} + u_{\varepsilon,Q_{k+1}} + \varphi_{k+1}. \tag{6.50}$$

By the definition of C_k^ε, it is easy to see that

$$C_{k+1}^\varepsilon \geq J_\varepsilon(u_{\varepsilon,\bar{Q}_1,\cdots,Q_{k+1}}). \tag{6.51}$$

Define a cut-off function $\tilde{\chi}$ such that $\tilde{\chi}(x) = \tau(dist(x, \partial B_{\frac{3\rho}{2}}(\frac{Q_{k+1}}{\varepsilon})))$, where τ is a cutoff function, $\tau(t) = 0$ if $t \leq \frac{1}{2}$, $\tau(t) = 1$ if $t \geq 1$.

Let us define, now, $\mu = \tilde{\chi} u_{\varepsilon,\bar{Q}_1,\cdots,Q_{k+1}}$. Then we evaluate $J_\varepsilon(\mu)$:

$$\begin{aligned}
&J_\varepsilon(\mu) = J_\varepsilon(\tilde{\chi} u_{\varepsilon,\bar{Q}_1,\cdots,Q_{k+1}}) \\
&= \frac{1}{2}\int_{\Omega_\varepsilon} |\tilde{\chi}\nabla u_{\varepsilon,\bar{Q}_1,\cdots,Q_{k+1}} + u_{\varepsilon,\bar{Q}_1,\cdots,Q_{k+1}}\nabla\tilde{\chi}|^2 + \tilde{\chi}^2 u^2_{\varepsilon,\bar{Q}_1,\cdots,Q_{k+1}} dx \\
&-\frac{1}{p+1}\int_{\Omega_\varepsilon} \tilde{\chi}^{p+1} u^{p+1}_{\varepsilon,\bar{Q}_1,\cdots,Q_{k+1}} dx \\
&= \frac{1}{2}\int_{\Omega_\varepsilon} (|\nabla u_{\varepsilon,\bar{Q}_1,\cdots,Q_{k+1}}|^2 + u^2_{\varepsilon,\bar{Q}_1,\cdots,Q_{k+1}}) dx - \frac{1}{p+1}\int_{\Omega_\varepsilon} u^{p+1}_{\varepsilon,\bar{Q}_1,\cdots,Q_{k+1}} dx \\
&+\frac{1}{2}\int_{\Omega_\varepsilon} |\nabla\tilde{\chi}|^2 u^2_{\varepsilon,\bar{Q}_1,\cdots,Q_{k+1}} dx + \frac{1}{4}\int_{\Omega_\varepsilon} \nabla\tilde{\chi}^2 \nabla u^2_{\varepsilon,\bar{Q}_1,\cdots,Q_{k+1}} dx \\
&+\frac{1}{2}\int_{\Omega_\varepsilon} (\tilde{\chi}^2 - 1)(|\nabla u_{\varepsilon,\bar{Q}_1,\cdots,Q_{k+1}}|^2 + u^2_{\varepsilon,\bar{Q}_1,\cdots,Q_{k+1}}) dx \\
&+\frac{1}{p+1}\int_{\Omega_\varepsilon} (1 - \tilde{\chi}^{p+1}) u^{p+1}_{\varepsilon,\bar{Q}_1,\cdots,Q_{k+1}} dx \\
&= J_\varepsilon(u_{\varepsilon,\bar{Q}_1,\cdots,Q_{k+1}}) + \frac{1}{2}\int_{\Omega_\varepsilon} (|\nabla\tilde{\chi}|^2 - \frac{1}{2}\Delta\tilde{\chi}^2) u^2_{\varepsilon,\bar{Q}_1,\cdots,Q_{k+1}} dx \\
&+\frac{1}{2}\int_{\Omega_\varepsilon} (\tilde{\chi}^2 - 1)(|\nabla u_{\varepsilon,\bar{Q}_1,\cdots,Q_{k+1}}|^2 + u^2_{\varepsilon,\bar{Q}_1,\cdots,Q_{k+1}}) dx \\
&+\frac{1}{p+1}\int_{\Omega_\varepsilon} (1 - \tilde{\chi}^{p+1}) u^{p+1}_{\varepsilon,\bar{Q}_1,\cdots,Q_{k+1}} dx.
\end{aligned}$$

By the definition of the cut-off function $\tilde{\chi}$ and taking into account the exponentially decaying away from the spikes of the function $u_{\varepsilon,\bar{Q}_1,\cdots,Q_{k+1}}$, we have

$$
\begin{aligned}
&|\frac{1}{2}\int_{\Omega_\varepsilon}(|\nabla\tilde{\chi}|^2-\frac{1}{2}\Delta\tilde{\chi}^2)u^2_{\varepsilon,\bar{Q}_1,\cdots,Q_{k+1}}dx+\frac{1}{p+1}\int_{\Omega_\varepsilon}(1-\tilde{\chi}^{p+1})u^{p+1}_{\varepsilon,\bar{Q}_1,\cdots,Q_{k+1}}dx\\
&+\frac{1}{2}\int_{\Omega_\varepsilon}(\tilde{\chi}^2-1)(|\nabla u_{\varepsilon,\bar{Q}_1,\cdots,Q_{k+1}}|^2+u^2_{\varepsilon,\bar{Q}_1,\cdots,Q_{k+1}})dx|\le Ce^{-(1+\xi)\rho}
\end{aligned}
$$

for some $\xi>0$. So we get

$$
J_\varepsilon(\mu)=J_\varepsilon(u_{\varepsilon,\bar{Q}_1,\cdots,Q_{k+1}})+O(e^{-(1+\xi)\rho}) \tag{6.52}
$$

for some $\xi>0$.

On the other hand, one can see that

$$
\mu=\mu_1+\mu_2, \tag{6.53}
$$

with

$$
\mu_1=\begin{cases}\tilde{\chi}u_{\varepsilon,\bar{Q}_1,\cdots,Q_{k+1}} & \text{if } x\in B_{\frac{3\rho}{2}}(\frac{Q_{k+1}}{\varepsilon}) \text{ and } dist(x,\partial B_{\frac{3\rho}{2}}(\frac{Q_{k+1}}{\varepsilon}))\ge\frac{1}{2}\\ 0 & \text{otherwise,}\end{cases} \tag{6.54}
$$

and

$$
\mu_2=\begin{cases}\tilde{\chi}u_{\varepsilon,\bar{Q}_1,\cdots,Q_{k+1}} & \text{if } x\in \Omega_\varepsilon\backslash B_{\frac{3\rho}{2}}(\frac{Q_{k+1}}{\varepsilon}) \text{ and } dist(x,\partial B_{\frac{3\rho}{2}}(\frac{Q_{k+1}}{\varepsilon}))\ge\frac{1}{2}\\ 0 & \text{otherwise .}\end{cases} \tag{6.55}
$$

From the definition of μ_1 and μ_2, we have

$$
J_\varepsilon(\mu)=J_\varepsilon(\mu_1+\mu_2)=J_\varepsilon(\mu_1)+J_\varepsilon(\mu_2). \tag{6.56}
$$

So we need to evaluate $J_\varepsilon(\mu_1)$ and $J_\varepsilon(\mu_2)$ separately.

First let us consider $J_\varepsilon(\mu_1)$:

$$
\begin{aligned}
J_\varepsilon(\mu_1) &= \frac{1}{2}\int_{B_{\frac{3\rho-1}{2}}(\frac{Q_{k+1}}{\varepsilon})} |\nabla\tilde{\chi} u_{\varepsilon,\bar{Q}_1,\cdots,Q_{k+1}} + \nabla u_{\varepsilon,\bar{Q}_1,\cdots,Q_{k+1}}\tilde{\chi}|^2 + |\tilde{\chi} u_{\varepsilon,\bar{Q}_1,\cdots,Q_{k+1}}|^2 \\
&-\frac{1}{p+1}\int_{B_{\frac{3\rho-1}{2}}(\frac{Q_{k+1}}{\varepsilon})} (\tilde{\chi} u_{\varepsilon,\bar{Q}_1,\cdots,Q_{k+1}})^{p+1} dx \\
&= \frac{1}{2}\int_{B_{\frac{3\rho-1}{2}}(\frac{Q_{k+1}}{\varepsilon})} \tilde{\chi}^2(|\nabla u_{\varepsilon,\bar{Q}_1,\cdots,Q_{k+1}}|^2 + u^2_{\varepsilon,\bar{Q}_1,\cdots,Q_{k+1}}) \\
&-\frac{1}{p+1}\int_{B_{\frac{3\rho-1}{2}}(\frac{Q_{k+1}}{\varepsilon})} \tilde{\chi}^{p+1} u^{p+1}_{\varepsilon,\bar{Q}_1,\cdots,Q_{k+1}} dx \\
&+\frac{1}{2}\int_{B_{\frac{3\rho-1}{2}}(\frac{Q_{k+1}}{\varepsilon})} (|\nabla\tilde{\chi}|^2 - \frac{1}{2}\Delta\tilde{\chi}^2) u^2_{\varepsilon,\bar{Q}_1,\cdots,Q_{k+1}} dx \\
&= \frac{1}{2}\int_{B_{\frac{3\rho-1}{2}}(\frac{Q_{k+1}}{\varepsilon})} (|\nabla u_{\varepsilon,\bar{Q}_1,\cdots,Q_{k+1}}|^2 + u^2_{\varepsilon,\bar{Q}_1,\cdots,Q_{k+1}}) dx \\
&-\frac{1}{p+1}\int_{B_{\frac{3\rho-1}{2}}(\frac{Q_{k+1}}{\varepsilon})} u^{p+1}_{\varepsilon,\bar{Q}_1,\cdots,Q_{k+1}} dx + O(e^{-(1+\xi)\rho})
\end{aligned}
$$

$$
\begin{aligned}
&= \frac{1}{2}\int_{B_{\frac{3\rho-1}{2}}(\frac{Q_{k+1}}{\varepsilon})} (|\nabla u_{\varepsilon,Q_{k+1}}|^2 + u^2_{\varepsilon,Q_{k+1}}) - \frac{1}{p+1}\int_{B_{\frac{3\rho-1}{2}}(\frac{Q_{k+1}}{\varepsilon})} u^{p+1}_{\varepsilon,Q_{k+1}} \\
&+[\frac{1}{2}\int_{B_{\frac{3\rho-1}{2}}(\frac{Q_{k+1}}{\varepsilon})} (|\nabla u_{\varepsilon,\bar{Q}_1,\cdots,Q_{k+1}}|^2 + u^2_{\varepsilon,\bar{Q}_1,\cdots,Q_{k+1}}) dx \\
&-\frac{1}{2}\int_{B_{\frac{3\rho-1}{2}}(\frac{Q_{k+1}}{\varepsilon})} (|\nabla u_{\varepsilon,Q_{k+1}}|^2 + u^2_{\varepsilon,Q_{k+1}}) \\
&-\frac{1}{p+1}\int_{B_{\frac{3\rho-1}{2}}(\frac{Q_{k+1}}{\varepsilon})} u^{p+1}_{\varepsilon,\bar{Q}_1,\cdots,Q_{k+1}} dx + \frac{1}{p+1}\int_{B_{\frac{3\rho-1}{2}}(\frac{Q_{k+1}}{\varepsilon})} u^{p+1}_{\varepsilon,Q_{k+1}} dx] \\
&+O(e^{-(1+\xi)\rho}).
\end{aligned}
$$

Using (5.13) and (5.14), we obtain

$$\begin{aligned}
&|\frac{1}{2}\int_{B_{\frac{3\rho-1}{2}}(\frac{Q_{k+1}}{\varepsilon})}(|\nabla u_{\varepsilon,\bar{Q}_1,\cdots,Q_{k+1}}|^2+u^2_{\varepsilon,\bar{Q}_1,\cdots,Q_{k+1}})-\frac{1}{2}\int_{B_{\frac{3\rho-1}{2}}(\frac{Q_{k+1}}{\varepsilon})}(|\nabla u_{\varepsilon,Q_{k+1}}|^2+u^2_{\varepsilon,Q_{k+1}})\\
&-\frac{1}{p+1}\int_{B_{\frac{3\rho-1}{2}}(\frac{Q_{k+1}}{\varepsilon})}u^{p+1}_{\varepsilon,\bar{Q}_1,\cdots,Q_{k+1}}dx+\frac{1}{p+1}\int_{B_{\frac{3\rho-1}{2}}(\frac{Q_{k+1}}{\varepsilon})}u^{p+1}_{\varepsilon,Q_{k+1}}dx|\\
&=|\int_{B_{\frac{3\rho-1}{2}}(\frac{Q_{k+1}}{\varepsilon})}\nabla u_{\varepsilon,Q_{k+1}}\nabla(u_{\varepsilon,\bar{Q}_1,\cdots,\bar{Q}_k}+\varphi_{k+1})+u_{\varepsilon,Q_{k+1}}(u_{\varepsilon,\bar{Q}_1,\cdots,\bar{Q}_k}+\varphi_{k+1})\\
&-u^p_{\varepsilon,Q_{k+1}}(u_{\varepsilon,\bar{Q}_1,\cdots,\bar{Q}_k}+\varphi_{k+1})dx|+O(e^{-(1+\xi)\rho})\\
&=|\int_{\partial B_{\frac{3\rho-1}{2}}(\frac{Q_{k+1}}{\varepsilon})}\frac{\partial u_{\varepsilon,Q_{k+1}}}{\partial\nu}(u_{\varepsilon,\bar{Q}_1,\cdots,\bar{Q}_k}+\varphi_{k+1}) \qquad (6.57)\\
&-\int_{B_{\frac{3\rho-1}{2}}(\frac{Q_{k+1}}{\varepsilon})}S_\varepsilon(u_{\varepsilon,Q_{k+1}})(u_{\varepsilon,\bar{Q}_1,\cdots,\bar{Q}_k}+\varphi_{k+1})|+O(e^{-(1+\xi)\rho})\\
&\leq C\|S_\varepsilon(u_{\varepsilon,Q_{k+1}})\|_{L^2(B_{\frac{3\rho-1}{2}}(\frac{Q_{k+1}}{\varepsilon}))}(\|(u_{\varepsilon,\bar{Q}_1,\cdots,\bar{Q}_k}\|_{L^2(B_{\frac{3\rho-1}{2}}(\frac{Q_{k+1}}{\varepsilon}))}+\|\varphi_{k+1})\|_{L^2})\\
&+O(e^{-(1+\xi)\rho}).
\end{aligned} \tag{6.58}$$

By (3.40), Proposition 4.1 and Lemma 5.1, we infer that

$$\begin{aligned}
&\|S_\varepsilon(u_{\varepsilon,Q_{k+1}})\|_{L^2(B_{\frac{3\rho-1}{2}}(\frac{Q_{k+1}}{\varepsilon}))}(\|(u_{\varepsilon,\bar{Q}_1,\cdots,\bar{Q}_k}\|_{L^2(B_{\frac{3\rho-1}{2}}(\frac{Q_{k+1}}{\varepsilon}))}+\|\varphi_{k+1})\|_{L^2})\\
&\leq Ce^{-(1+\xi)\rho}.
\end{aligned} \tag{6.59}$$

Again by Lemma 2.2 and Proposition 4.1, we have

$$\begin{aligned}
&|\frac{1}{2}\int_{\Omega_\varepsilon\setminus B_{\frac{3\rho-1}{2}}(\frac{Q_{k+1}}{\varepsilon})}(|\nabla u_{\varepsilon,Q_{k+1}}|^2+u^2_{\varepsilon,Q_{k+1}})-\frac{1}{p+1}\int_{\Omega_\varepsilon\setminus B_{\frac{3\rho-1}{2}}(\frac{Q_{k+1}}{\varepsilon})}u^{p+1}_{\varepsilon,Q_{k+1}}|\\
&\leq ce^{-(1+\xi)\rho}.
\end{aligned}$$

Combining the above, we obtain

$$J_\varepsilon(\mu_1)=J_\varepsilon(u_{\varepsilon,Q_{k+1}})+O(e^{-(1+\xi)\rho}). \tag{6.60}$$

Similar to (6.57), we have

$$\begin{aligned}
J_\varepsilon(u_{\varepsilon,Q_{k+1}}) &= J_\varepsilon(w_{\varepsilon,Q_{k+1}}+\phi_{\varepsilon,Q_{k+1}}) \qquad (6.61)\\
&= J_\varepsilon(w_{\varepsilon,Q_{k+1}})+O(e^{-(1+\xi)\rho}).
\end{aligned}$$

By the definition of $w_{\varepsilon,Q_{k+1}}$, we get

$$
\begin{aligned}
&J_\varepsilon(w_{\varepsilon,Q_{k+1}})\\
&=\frac{1}{2}\int_{\Omega_\varepsilon} w^p_{Q_{k+1}} w_{\varepsilon,Q_{k+1}}dx-\frac{1}{p+1}\int_{\Omega_\varepsilon} w^{p+1}_{\varepsilon,Q_{k+1}}dx\\
&=\frac{1}{2}\int_{\Omega_\varepsilon} w^{p+1}_{Q_{k+1}}dx-\frac{1}{p+1}\int_{\Omega_\varepsilon} w^{p+1}_{Q_{k+1}}dx\\
&-\frac{1}{2}\int_{\Omega_\varepsilon} w^p_{Q_{k+1}}\varphi_{\varepsilon,Q_{k+1}}dx-\frac{1}{p+1}\int_{\Omega_\varepsilon} w^{p+1}_{\varepsilon,Q_{k+1}}-w^{p+1}_{Q_{k+1}}dx.
\end{aligned}
$$

Note that

$$
\begin{aligned}
&\int_{\Omega_\varepsilon}(\frac{1}{2}-\frac{1}{p+1})w^{p+1}_{Q_{k+1}}dx\\
&=\int_{\mathbb{R}^n}(\frac{1}{2}-\frac{1}{p+1})w^{p+1}_{Q_{k+1}}dx-\int_{\mathbb{R}^n\setminus\Omega_\varepsilon}(\frac{1}{2}-\frac{1}{p+1})w^{p+1}_{Q_{k+1}}dx\\
&=I(w)+O(e^{-(1+\xi)\rho})
\end{aligned}
$$

and

$$
\begin{aligned}
|\int_{\Omega_\varepsilon}\frac{1}{p+1}w^{p+1}_{\varepsilon,Q_{k+1}}-\frac{1}{p+1}w^{p+1}_{Q_{k+1}}+w^p_{Q_{k+1}}\varphi_{\varepsilon,Q_{k+1}}dx| \quad &\le C\int_{\Omega_\varepsilon} w^{p-1}_{Q_{k+1}}\varphi^2_{\varepsilon,Q_{k+1}}dx\\
&\le Ce^{-(1+\xi)\rho}.
\end{aligned}
$$

So by Lemma 2.4, we get

$$
\begin{aligned}
J_\varepsilon(w_{\varepsilon,Q_{k+1}}) \quad &=I(w)-\frac{1}{2}B_\varepsilon(Q_{k+1})+O(e^{-(1+\xi)\rho}) \qquad (6.62)\\
&=I(w)+O(e^{-(1+\xi)\rho}).
\end{aligned}
$$

(6.60) and (6.62) yield

$$
\begin{aligned}
J_\varepsilon(u_{\varepsilon,Q_{k+1}}) &= I(w)-\frac{1}{2}B_\varepsilon(Q_{k+1})+O(e^{-(1+\xi)\rho}) \qquad (6.63)\\
&= I(w)+O(e^{-(1+\xi)\rho}).
\end{aligned}
$$

Now let us consider $J_\varepsilon(\mu_2)$:

$$
\begin{aligned}
&J_\varepsilon(\mu_2)\\
&=\frac{1}{2}\int_{\Omega_\varepsilon\backslash B_{\frac{3\rho+1}{2}}(\frac{Q_{k+1}}{\varepsilon})}|\nabla\tilde{\chi}u_{\varepsilon,\bar{Q}_1,\cdots,Q_{k+1}}+\nabla u_{\varepsilon,\bar{Q}_1,\cdots,Q_{k+1}}\tilde{\chi}|^2+|\tilde{\chi}u_{\varepsilon,\bar{Q}_1,\cdots,Q_{k+1}}|^2dx\\
&-\frac{1}{p+1}\int_{\Omega_\varepsilon\backslash B_{\frac{3\rho+1}{2}}(\frac{Q_{k+1}}{\varepsilon})}(\tilde{\chi}u_{\varepsilon,\bar{Q}_1,\cdots,Q_{k+1}})^{p+1}dx\\
&=\frac{1}{2}\int_{\Omega_\varepsilon\backslash B_{\frac{3\rho+1}{2}}(\frac{Q_{k+1}}{\varepsilon})}(|\nabla u_{\varepsilon,\bar{Q}_1,\cdots,Q_{k+1}}|^2+u^2_{\varepsilon,\bar{Q}_1,\cdots,Q_{k+1}})dx\\
&-\frac{1}{p+1}\int_{\Omega_\varepsilon\backslash B_{\frac{3\rho+1}{2}}(\frac{Q_{k+1}}{\varepsilon})}u^{p+1}_{\varepsilon,\bar{Q}_1,\cdots,Q_{k+1}}dx+O(e^{-(1+\xi)\rho})\\
&=\frac{1}{2}\int_{\Omega_\varepsilon\backslash B_{\frac{3\rho+1}{2}}(\frac{Q_{k+1}}{\varepsilon})}|\nabla u_{\varepsilon,\bar{Q},\cdots,\bar{Q}_k}|^2+u^2_{\varepsilon,\bar{Q}_1,\cdots,\bar{Q}_k}dx\\
&-\frac{1}{p+1}\int_{\Omega_\varepsilon\backslash B_{\frac{3\rho+1}{2}}(\frac{Q_{k+1}}{\varepsilon})}u^{p+1}_{\varepsilon,\bar{Q}_1,\cdots,\bar{Q}_k}dx\\
&+[\frac{1}{2}\int_{\Omega_\varepsilon\backslash B_{\frac{3\rho+1}{2}}(\frac{Q_{k+1}}{\varepsilon})}(|\nabla u_{\varepsilon,\bar{Q}_1,\cdots,Q_{k+1}}|^2+u^2_{\varepsilon,\bar{Q}_1,\cdots,Q_{k+1}})dx\\
&-\frac{1}{2}\int_{\Omega_\varepsilon\backslash B_{\frac{3\rho+1}{2}}(\frac{Q_{k+1}}{\varepsilon})}|\nabla u_{\varepsilon,\bar{Q},\cdots,\bar{Q}_k}|^2+u^2_{\varepsilon,\bar{Q}_1,\cdots,\bar{Q}_k}dx\\
&-\frac{1}{p+1}\int_{\Omega_\varepsilon\backslash B_{\frac{3\rho+1}{2}}(\frac{Q_{k+1}}{\varepsilon})}u^{p+1}_{\varepsilon,\bar{Q}_1,\cdots,Q_{k+1}}dx+\frac{1}{p+1}\int_{\Omega_\varepsilon\backslash B_{\frac{3\rho+1}{2}}(\frac{Q_{k+1}}{\varepsilon})}u^{p+1}_{\varepsilon,\bar{Q}_1,\cdots,\bar{Q}_k}dx]\\
&+O(e^{-(1+\xi)\rho}).
\end{aligned}
$$

Similar to (6.57), we can get

$$
\begin{aligned}
&|\frac{1}{2}\int_{\Omega_\varepsilon\setminus B_{\frac{3\rho+1}{2}}(\frac{Q_{k+1}}{\varepsilon})}(|\nabla u_{\varepsilon,\bar{Q}_1,\cdots,Q_{k+1}}|^2+u^2_{\varepsilon,\bar{Q}_1,\cdots,Q_{k+1}})dx\\
&-\frac{1}{2}\int_{\Omega_\varepsilon\setminus B_{\frac{3\rho+1}{2}}(\frac{Q_{k+1}}{\varepsilon})}|\nabla u_{\varepsilon,\bar{Q},\cdots,\bar{Q}_k}|^2+u^2_{\varepsilon,\bar{Q}_1,\cdots,\bar{Q}_k}dx\\
&-\frac{1}{p+1}\int_{\Omega_\varepsilon\setminus B_{\frac{3\rho+1}{2}}(\frac{Q_{k+1}}{\varepsilon})}u^{p+1}_{\varepsilon,\bar{Q}_1,\cdots,Q_{k+1}}dx+\frac{1}{p+1}\int_{\Omega_\varepsilon\setminus B_{\frac{3\rho+1}{2}}(\frac{Q_{k+1}}{\varepsilon})}u^{p+1}_{\varepsilon,\bar{Q}_1,\cdots,\bar{Q}_k}dx|\\
&=|\int_{\Omega_\varepsilon\setminus B_{\frac{3\rho+1}{2}}(\frac{Q_{k+1}}{\varepsilon})}S_\varepsilon(u_{\varepsilon,\bar{Q}_1\cdots,\bar{Q}_k})(u_{\varepsilon,Q_{k+1}}+\varphi_{k+1})dx|+e^{-(1+\xi)\rho}\\
&=|\sum_{i=1,\cdots,k,\ j=1,\cdots,n}c_{ij}\int_{\Omega_\varepsilon\setminus B_{\frac{3\rho+1}{2}}(\frac{Q_{k+1}}{\varepsilon})}Z_{ij}(u_{\varepsilon,Q_{k+1}}+\varphi_{k+1})dx|.
\end{aligned}
$$

By Lemma 5.1, (5.19), (5.27), (5.22) and (3.40), we have

$$
\begin{aligned}
&|\sum_{i=1,\cdots,k,\ j=1,\cdots,n}c_{ij}\int_{\Omega_\varepsilon\setminus B_{\frac{3\rho+1}{2}}(\frac{Q_{k+1}}{\varepsilon})}Z_{ij}\varphi_{k+1}dx|\\
&=|\sum_{i=1,\cdots,k,\ j=1,\cdots,n}c_{ij}\int_{\Omega_\varepsilon\setminus B_{\frac{3\rho+1}{2}}(\frac{Q_{k+1}}{\varepsilon})}Z_{ij}(\sum c_i\phi_i+\sum_{ij}d_{ij}Z_{ij})dx|\\
&\le c\sup_{ij}|c_{ij}|\sum(|c_i|+|d_{ij}|)\\
&\le ce^{-(1+\xi)\rho},
\end{aligned}
$$

$$
|\sum_{i=1,\cdots,k,\ j=1,\cdots,n}c_{ij}\int_{\Omega_\varepsilon\setminus B_{\frac{3\rho+1}{2}}(\frac{Q_{k+1}}{\varepsilon})}Z_{ij}u_{\varepsilon,Q_{k+1}}dx|\le ce^{-(1+\xi)\rho},
$$

and

$$
|\int_{B_{\frac{3\rho+1}{2}}(\frac{Q_{k+1}}{\varepsilon})}|\nabla u_{\varepsilon,\bar{Q}_1,\cdots,\bar{Q}_k}|^2+u^2_{\varepsilon,\bar{Q}_1,\cdots,\bar{Q}_k}-\frac{1}{p+1}u^{p+1}_{\varepsilon,\bar{Q}_1,\cdots,\bar{Q}_k}dx|\le Ce^{-(1+\xi)\rho}.
$$

Recalling that

$$
C^k_\varepsilon=J_\varepsilon(u_{\varepsilon,\bar{Q}_1,\cdots,\bar{Q}_k}), \tag{6.64}
$$

we get

$$
J_\varepsilon(\mu_2)=C^\varepsilon_k+O(e^{-(1+\xi)\rho}). \tag{6.65}
$$

Thus combining (6.51), (6.52), (6.56), (6.60), (6.63) and (6.65), we have

$$\begin{aligned} J_\varepsilon(\mu) &= J_\varepsilon(\mu_1+\mu_2) \\ &= J_\varepsilon(\mu_1)+J_\varepsilon(\mu_2) \\ &= C_k^\varepsilon + I(w) + O(e^{-(1+\xi)\rho}) \\ &= J_\varepsilon(u_{\varepsilon,\bar{Q}_1,\cdots,Q_{k+1}}) + O(e^{-(1+\xi)\rho}) \\ &\leq C_{k+1}^\varepsilon + O(e^{-(1+\xi)\rho}). \end{aligned}$$

Thus,

$$C_{k+1}^\varepsilon \geq C_k^\varepsilon + I(w) + O(e^{-(1+\xi)\rho}),$$

a contradiction with the assumption (6.48). □

Remark 6.1. *From the proof above, we may take* $\delta(n,p,\Omega) = \frac{\delta_0}{\rho_0^n} << \frac{|\Omega|}{2\times 3^n |B_1|\rho_0^n}$ *for some* $\delta_0 > 0$ *small, where* ρ_0 *is as in Section 4.*

Next we have the following Proposition:

Proposition 6.2. *The maximization problem*

$$\max_{\mathbf{Q}\in\bar{\Lambda}_k} \mathcal{M}_\varepsilon(\mathbf{Q}) \tag{6.66}$$

has a solution $\mathbf{Q}^\varepsilon \in \Lambda_k^\circ$*, i.e., the interior of* Λ_k*.*

Proof. We prove it by contradiction again. If $\mathbf{Q}^\varepsilon = (\bar{Q}_1,\cdots,\bar{Q}_k) \in \partial\Lambda_k$, then either there exists (i,j) such that $|Q_i - Q_j| = \varepsilon\rho$ or $|Q_i - Q_j^*| = \varepsilon\rho$. Without loss of generality, we assume $(i,j) = (i,k)$. We have

$$\begin{aligned} J_\varepsilon(u_{\varepsilon,\bar{Q}_1,\cdots,\bar{Q}_k}) &= J_\varepsilon(u_{\varepsilon,\bar{Q}_1,\cdots,\bar{Q}_{k-1}} + u_{\varepsilon,\bar{Q}_k} + \varphi_k) \\ &= J_\varepsilon(u_{\varepsilon,\bar{Q}_1,\cdots,\bar{Q}_{k-1}} + u_{\varepsilon,\bar{Q}_k}) + \int_{\Omega_\varepsilon} \nabla(u_{\varepsilon,\bar{Q}_1,\cdots,\bar{Q}_{k-1}} + u_{\varepsilon,\bar{Q}_k})\nabla\varphi_k \\ &+ (u_{\varepsilon,\bar{Q}_1,\cdots,\bar{Q}_{k-1}} + u_{\varepsilon,\bar{Q}_k})\varphi_k - (u_{\varepsilon,\bar{Q}_1,\cdots,\bar{Q}_{k-1}} + u_{\varepsilon,\bar{Q}_k})^p\varphi_k dx \\ &+ O(\|\varphi_k\|_{H^1}^2) \\ &= J_\varepsilon(u_{\varepsilon,\bar{Q}_1,\cdots,\bar{Q}_{k-1}} + u_{\varepsilon,\bar{Q}_k}) \\ &- \int_{\Omega_\varepsilon} S_\varepsilon(u_{\varepsilon,\bar{Q}_1,\cdots,\bar{Q}_{k-1}} + u_{\varepsilon,\bar{Q}_k})\varphi_k dx + O(\|\varphi_k\|_{H^1}^2). \end{aligned}$$

Observe that

$$\begin{aligned} &S_\varepsilon(u_{\varepsilon,\bar{Q}_1,\cdots,\bar{Q}_{k-1}} + u_{\varepsilon,\bar{Q}_k}) \\ &= (u_{\varepsilon,\bar{Q}_1,\cdots,\bar{Q}_{k-1}} + u_{\varepsilon,\bar{Q}_k})^p - u^p_{\varepsilon,\bar{Q}_1,\cdots,\bar{Q}_{k-1}} - u^p_{\varepsilon,\bar{Q}_k} + \sum_{i=1,\cdots,k,j=1,\cdots,n} c_{ij}Z_{ij}, \end{aligned}$$

for some $\{c_{ij}\}$ which satisfies

$$|c_{ij}| \le ce^{-(1+\xi)\frac{\rho}{2}}. \tag{6.67}$$

Using Lemma 5.1, we obtain

$$\begin{aligned}
&|\sum_{ij} c_{ij}\int_{\Omega_\varepsilon} Z_{ij}\varphi_k dx| \\
&= |\sum_{ij} c_{ij}\int_{\Omega_\varepsilon} Z_{ij}(\psi + \sum c_i\phi_i + \sum_{ij} d_{ij}Z_{ij})dx| \\
&= |\sum_{ij} c_{ij}\int_{\Omega_\varepsilon} Z_{ij}(c_i\phi_i + d_{ij}Z_{ij})dx| \\
&\le \sup_{ij}|c_{ij}|\sum_{ij}(|c_i| + |d_{ij}|) \\
&\le ce^{-(1+\xi)\rho}
\end{aligned}$$

by (5.27), (5.22) and (6.67).

Using the estimate (5.17) and Lemma 5.1, we find

$$\begin{aligned}
&|\int_{\Omega_\varepsilon}((u_{\varepsilon,\bar{Q}_1,\cdots,\bar{Q}_{k-1}} + u_{\varepsilon,\bar{Q}_k})^p - u^p_{\varepsilon,\bar{Q}_1,\cdots,\bar{Q}_{k-1}} - u^p_{\varepsilon,\bar{Q}_k})\varphi_k dx| \\
&\le c\|\bar{S}\|_{L^2}\|\varphi_k\|_{H^1} \le ce^{-(1+\xi)\rho}.
\end{aligned}$$

This implies that

$$\begin{aligned}
&J_\varepsilon(u_{\varepsilon,\bar{Q}_1,\cdots,\bar{Q}_k}) \\
&= J_\varepsilon(u_{\varepsilon,\bar{Q}_1,\cdots,\bar{Q}_{k-1}} + u_{\varepsilon,\bar{Q}_k}) \\
&- \int_{\Omega_\varepsilon} S_\varepsilon(u_{\varepsilon,\bar{Q}_1,\cdots,\bar{Q}_{k-1}} + u_{\varepsilon,\bar{Q}_k})\varphi_k dx + O(\|\varphi_k\|^2_{H^1}) \\
&= J_\varepsilon(u_{\varepsilon,\bar{Q}_1,\cdots,\bar{Q}_{k-1}} + u_{\varepsilon,\bar{Q}_k}) + O(e^{-(1+\xi)\rho}).
\end{aligned}$$

Next we estimate

$$\begin{aligned}
&J_\varepsilon(u_{\varepsilon,\bar{Q}_1,\cdots,\bar{Q}_{k-1}} + u_{\varepsilon,\bar{Q}_k}) \\
&= J_\varepsilon(u_{\varepsilon,\bar{Q}_1,\cdots,\bar{Q}_{k-1}}) + J_\varepsilon(u_{\varepsilon,\bar{Q}_k}) \\
&+ \int_{\Omega_\varepsilon} \nabla u_{\varepsilon,\bar{Q}_1,\cdots,\bar{Q}_{k-1}}\nabla u_{\varepsilon,\bar{Q}_k} + u_{\varepsilon,\bar{Q}_1,\cdots,\bar{Q}_{k-1}}u_{\varepsilon,\bar{Q}_k}dx \\
&-\frac{1}{p+1}\int_{\Omega_\varepsilon}(u_{\varepsilon,\bar{Q}_1,\cdots,\bar{Q}_{k-1}} + u_{\varepsilon,\bar{Q}_k})^{p+1} - u^{p+1}_{\varepsilon,\bar{Q}_1,\cdots,\bar{Q}_{k-1}} - u^{p+1}_{\varepsilon,\bar{Q}_k}dx,
\end{aligned}$$

and

$$
\begin{aligned}
&\int_{\Omega_\varepsilon} \nabla u_{\varepsilon,\bar{Q}_1,\cdots,\bar{Q}_{k-1}} \nabla u_{\varepsilon,\bar{Q}_k} + u_{\varepsilon,\bar{Q}_1,\cdots,\bar{Q}_{k-1}} u_{\varepsilon,\bar{Q}_k} dx \\
&-\frac{1}{p+1}\int_{\Omega_\varepsilon} (u_{\varepsilon,\bar{Q}_1,\cdots,\bar{Q}_{k-1}} + u_{\varepsilon,\bar{Q}_k})^{p+1} - u^{p+1}_{\varepsilon,\bar{Q}_1,\cdots,\bar{Q}_{k-1}} - u^{p+1}_{\varepsilon,\bar{Q}_k} dx \\
&= \int_{\Omega_\varepsilon} (u^p_{\varepsilon,\bar{Q}_{k+1}} - \sum_{l=1}^{n} c_{kl} Z_{kl}) u_{\varepsilon,\bar{Q}_1,\cdots,\bar{Q}_{k-1}} dx \\
&- \int_{\Omega_\varepsilon} u^p_{\varepsilon,\bar{Q}_1,\cdots,\bar{Q}_{k-1}} u_{\varepsilon,\bar{Q}_k} + u^p_{\varepsilon,\bar{Q}_k} u_{\varepsilon,\bar{Q}_1,\cdots,\bar{Q}_{k-1}} dx + O(e^{-(1+\xi)\rho}) \\
&= -\int_{\Omega_\varepsilon} u^p_{\varepsilon,\bar{Q}_1,\cdots,\bar{Q}_{k-1}} u_{\varepsilon,\bar{Q}_k} dx + O(e^{-(1+\xi)\rho})
\end{aligned}
$$

by (3.40) in Section 3.

The above three identities imply that

$$
\begin{aligned}
J_\varepsilon(u_{\varepsilon,\bar{Q}_1,\cdots,\bar{Q}_k}) &= J_\varepsilon(u_{\varepsilon,\bar{Q}_1,\cdots,\bar{Q}_{k-1}}) + J_\varepsilon(u_{\varepsilon,\bar{Q}_k}) \\
&\quad - \int_{\Omega_\varepsilon} u^p_{\varepsilon,\bar{Q}_1,\cdots,\bar{Q}_{k-1}} u_{\varepsilon,\bar{Q}_k} dx + O(e^{-(1+\xi)\rho}).
\end{aligned} \tag{6.68}
$$

Since

$$
\begin{aligned}
&\int_{\Omega_\varepsilon} u^p_{\varepsilon,\bar{Q}_1,\cdots,\bar{Q}_{k-1}} u_{\varepsilon,\bar{Q}_k} dx \\
&= \int_{\Omega_\varepsilon} (\sum_{i=1}^{k-1} w_{\varepsilon,\bar{Q}_i} + \phi_{\varepsilon,\bar{Q}_1,\cdots,\bar{Q}_{k-1}})^p (w_{\varepsilon,\bar{Q}_k} + \phi_{\varepsilon,\bar{Q}_k}) dx \\
&\geq \int_{\Omega_\varepsilon} w^p_{\varepsilon,\bar{Q}_i} w_{\varepsilon,\bar{Q}_k} dx + O(e^{-(1+\xi)\rho}),
\end{aligned} \tag{6.69}
$$

using (6.63) (6.68) and (6.69) , one can get

$$
J_\varepsilon(u_{\varepsilon,\bar{Q}_1,\cdots,\bar{Q}_k}) \quad \leq C^\varepsilon_{k-1} + I(w) - \frac{1}{2} B_\varepsilon(\bar{Q}_k) - \int_{\Omega_\varepsilon} w^p_{\varepsilon,\bar{Q}_i} w_{\varepsilon,\bar{Q}_k} dx + O(e^{-(1+\xi)\rho}).
$$

If either there exists (i,k) such that $|Q_i - Q_k| = \varepsilon\rho$ or $|Q_k - Q^*_k| = \varepsilon\rho$, by Lemma 2.4, we can get that

$$
J_\varepsilon(u_{\varepsilon,\bar{Q}_1,\cdots,\bar{Q}_k}) \leq C^\varepsilon_{k-1} + I(w) - (\frac{\gamma}{2} + O(\frac{1}{\sqrt{\rho}}))e^{-\rho} + O(e^{-(1+\xi)\rho}). \tag{6.70}
$$

Thus

$$
C^\varepsilon_k = \mathcal{M}_\varepsilon(\mathbf{Q}^\varepsilon) \leq C^\varepsilon_{k-1} + I(w) - \frac{\gamma}{4} e^{-\rho}.
$$

We reach a contradiction with Proposition 6.1.

□

7. Proof of Theorem 1.1

In this section, we apply the results in Section 4, Section 5 and Section 6 to prove Theorem 1.1. The proof is similar to [18].

Proof of Theorem 1.1: By Proposition 4.1 in Section 4, there exists ε_0, ρ_0 such that for $0 < \varepsilon < \varepsilon_0, \rho > \rho_0$, we have C^1 map which, to any $\mathbf{Q} \in \Lambda_k$, associates $\phi_{\varepsilon,\mathbf{Q}}$ such that

$$S_\varepsilon(w_{\varepsilon,\mathbf{Q}} + \phi_{\varepsilon,\mathbf{Q}}) = \sum_{i=1,\cdots,k, j=1,\cdots,n} c_{ij} Z_{ij}, \quad \int_{\Omega_\varepsilon} \phi_{\varepsilon,\mathbf{Q}} Z_{ij} dx = 0, \tag{7.71}$$

for some constants $\{c_{ij}\} \in \mathbb{R}^{kn}$.

From Proposition 6.2 in Section 6, there is a $\mathbf{Q}^\varepsilon \in \Lambda_k^\circ$ that achieves the maximum for the maximization problem in Proposition 6.2. Let $u_\varepsilon = w_{\varepsilon,\mathbf{Q}^\varepsilon} + \phi_{\varepsilon,\mathbf{Q}^\varepsilon}$. Then we have

$$D_{Q_{ij}}|_{Q_i=Q_i^\varepsilon} \mathcal{M}_\varepsilon(\mathbf{Q}^\varepsilon) = 0, \quad i = 1, \cdots, k, \quad j = 1, \cdots, n. \tag{7.72}$$

Hence we have

$$\begin{aligned} &\int_{\Omega_\varepsilon} \nabla u_\varepsilon \nabla \frac{\partial(w_{\varepsilon,\mathbf{Q}} + \phi_{\varepsilon,\mathbf{Q}})}{\partial Q_{ij}}|_{Q_i=Q_i^\varepsilon} + u_\varepsilon \frac{\partial(w_{\varepsilon,\mathbf{Q}} + \phi_{\varepsilon,\mathbf{Q}})}{\partial Q_{ij}}|_{Q_i=Q_i^\varepsilon} \\ &- u_\varepsilon^p \frac{\partial(w_{\varepsilon,\mathbf{Q}} + \phi_{\varepsilon,\mathbf{Q}})}{\partial Q_{ij}}|_{Q_i=Q_i^\varepsilon} = 0, \end{aligned}$$

which gives

$$\sum_{i=1,\cdots,k,\ j=1,\cdots,n} c_{ij} \int_{\Omega_\varepsilon} Z_{ij} \frac{\partial(w_{\varepsilon,\mathbf{Q}} + \phi_{\varepsilon,\mathbf{Q}})}{\partial Q_{sl}}|_{Q_s=Q_s^\varepsilon} = 0, \tag{7.73}$$

for $s = 1, \cdots, k, l = 1, \cdots, n$. We claim that (7.73) is a diagonally dominant system. In fact, since $\int_{\Omega_\varepsilon} \phi_{\varepsilon,\mathbf{Q}} Z_{sl} dx = 0$, we have that

$$\int_{\Omega_\varepsilon} Z_{sl} \frac{\partial \phi_{\varepsilon,\mathbf{Q}}}{\partial Q_{ij}}|_{Q_i=Q_i^\varepsilon} = -\int_{\Omega_\varepsilon} \phi_{\varepsilon,\mathbf{Q}} \frac{\partial Z_{sl}}{\partial Q_{ij}} = 0, \text{ if } s \neq i.$$

If $s = i$, we have

$$\begin{aligned} |\int_{\Omega_\varepsilon} Z_{il} \frac{\partial \phi_{\varepsilon,\mathbf{Q}}}{\partial Q_{ij}}|_{Q_i=Q_i^\varepsilon}| &= |-\int_{\Omega_\varepsilon} \phi_{\varepsilon,\mathbf{Q}} \frac{\partial Z_{il}}{\partial Q_{ij}}| \\ &\leq C\varepsilon^{-1} \|\phi_{\varepsilon,\mathbf{Q}}\|_* = O(\varepsilon^{-1} e^{-\frac{\rho}{2}(1+\xi)}). \end{aligned}$$

For $s \neq i$, we have

$$\int_{\Omega_\varepsilon} Z_{sl} \frac{\partial w_{\varepsilon,\mathbf{Q}}}{\partial Q_{ij}} = O(\varepsilon^{-1} e^{-\frac{\eta|Q_i - Q_s|}{\varepsilon}}).$$

For $s=i$, recall the definition of Z_{ij}, we have

$$\int_{\Omega_\varepsilon} Z_{sl}\frac{\partial w_{\varepsilon,\mathbf{Q}}}{\partial Q_{sj}} = -\varepsilon^{-1}\delta_{lj}\int_{\mathbb{R}^n}(\frac{\partial w}{\partial y_j})^2 + O(\varepsilon^{-1}e^{-\rho}). \tag{7.74}$$

For each (s,l), the off-diagonal term gives

$$\begin{aligned}&\sum_{s\neq i}\int_{\Omega_\varepsilon} Z_{sl}\frac{\partial(w_{\varepsilon,\mathbf{Q}}+\phi_{\varepsilon,\mathbf{Q}})}{\partial Q_{ij}}|_{Q_i=Q_i^\varepsilon} + \sum_{s=i,l\neq j}\int_{\Omega_\varepsilon} Z_{sl}\frac{\partial(w_{\varepsilon,\mathbf{Q}}+\phi_{\varepsilon,\mathbf{Q}})}{\partial Q_{sj}}|_{Q_i=Q_i^\varepsilon}\\ &= \varepsilon^{-1}(O(e^{-\eta\rho})+O(e^{-\frac{\rho}{2}})+O(e^{-\rho})) \qquad (7.75)\\ &= \varepsilon^{-1}O(e^{-\eta\rho}),\end{aligned}$$

for some $\eta>0$.

So from (7.74) and (7.75), we can see that equation (7.73) becomes a system of homogeneous equations for c_{sl}, and the matrix of the system is nonsingular. So $c_{sl}=0$ for $s=1,\cdots,k, l=1,\cdots,n$. Hence $u_\varepsilon = w_{\varepsilon,\mathbf{Q}^\varepsilon}+\phi_{\varepsilon,\mathbf{Q}^\varepsilon}$ is a solution of (2.5).

Similar to the argument in Section 6 of [18], one can get that $u_\varepsilon>0$ and it has exactly k local maximum points for ε small and ρ large enough.

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Department of Mathematics, Chinese University of Hong Kong, Shatin, Hong Kong. *wwao@math.cuhk.edu.hk*

Department of Mathematics, Chinese University of Hong Kong, Shatin, Hong Kong,and Department of Mathematics, University of British Columbia, Vancouver, B.C., Canada, V6T 1Z2. *wei@math.cuhk.edu.hk*

Department of Mathematics, Fujian Normal University, Fuzhou, Fujian, China