2π-periodic self-similar solutions for the anisotropic affine curve shortening problem

Meiyue Jiang * Liping Wang † and Juncheng Wei ‡

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Abstract

We study the existence of 2π-periodic positive solutions of the equation

$$u_{\theta\theta} + u = \frac{a(\theta)}{u^3},$$

where $a(\theta)$ is a positive smooth 2π-periodic function. A priori estimates and sufficient conditions for the existence of solutions of the equation are established.

1 Introduction and statement of the results

We are concerned in this paper with the equation

$$u_{\theta\theta} + u = \frac{a(\theta)}{u^3}, \quad \theta \in S^1, \tag{1.1}$$

where $a(\theta)$ is a positive smooth function on $S^1 = \mathbb{R}/2\pi\mathbb{Z}$. Equation (1.1) arises from the study of the generalized curve shortening problem, which can be derived as follows. Consider the following generalized curve shortening problem

$$\frac{\partial \gamma}{\partial t} = \Phi(\theta)|k|^{\sigma-1}kN, \qquad \sigma > 0, \quad \theta \in S^1, \tag{1.2}$$

where $\gamma(\cdot, t)$ is a planar curve, $k(\cdot, t)$ is its curvature with respect to the unit normal N, and Φ is a positive function depending on the normal angle θ of the curve. This

*LMAM, School of Mathematical Sciences, Peking University, Beijing, 100871, P. R. China, E-mail: mjiang@math.pku.edu.cn.

†Department of Mathematics, East China Normal University, Shanghai, 200241, P. R. China, E-mail: lpwang@math.ecnu.edu.cn.

‡Department of Mathematics, The Chinese University of Hong Kong, Shatin, Hong Kong. E-mail: wei@math.cuhk.edu.hk.

problem has been extensively studied in the last two decades. See [1-7,15,16,18-23,31]. Assuming that $\gamma(\cdot,t)$ is convex, then we can use the normal angle θ to parameterize γ, and equation (1.2) is equivalent to

$$\frac{\partial w}{\partial t} = \frac{-\Phi(\theta)}{(w_{\theta\theta}+w)^{\sigma}}, \qquad \theta \in S^1, \tag{1.3}$$

where $w(\theta,t)$ is the support function of $\gamma(\cdot,t)$. A self-similar solution is of the form $w(\theta,t)=\xi(t)u(\theta)$, which means that the shape of the curves does not change during the evolution governed by (1.2). Such solutions are important in understanding the long time behaviors and the structure of singularities of (1.2). It is rather easy to see that $\xi(t)u(\theta)$ is a self-similar solution if and only if u satisfies

$$u_{\theta\theta}+u = \frac{a(\theta)}{u^{p+1}}, \quad \theta \in S^1 \tag{1.4}$$

with $a(\theta)=\Phi^{\frac{1}{\sigma}}(\theta)$, $p+1=\frac{1}{\sigma}$ and

$$|\xi(t)|^{\sigma-1}\xi(t)\xi'(t) = -C,$$

where C is a positive constant. The case $\sigma^{-1}=3$ or equivalently, $p=2$, (1.2) is called the affine curve shortening problem, and equation (1.4) becomes (1.1). Thus a solution of equation (1.1) is a self-similar solution of the anisotropic affine curve shortening problem. Note that in general $a(\theta)$ can only be assumed to be 2π-periodic. Equation (1.4) also appears in image processing [31], 2-dimensional L^p-Minkowski problem [8],[26] and other problems [24].

Equation (1.1) is a special case of

$$u''+f(\theta,u)=0 \tag{1.5}$$

where $f:\mathbb{R}\times\mathbb{R}^+ \to \mathbb{R}$ is continuous, $T-$periodic in the first variable, has a singularity of repulsive type near the origin. The existence of periodic solutions of (1.5) had been studied by many people. Using the Poincaré-Birkhoff fixed point theorem, the following result was proved by del Pino, Manásevich and Montero in [17]. Let $\{\mu_n\}_{n=0}^{\infty}$ be the eigenvalues of

$$u''+\mu u=0$$

with $2T$-periodic boundary conditions:

$$u(0)=u(2T), \quad u'(0)=u'(2T),$$

that is, $\mu_n=\left(\frac{\pi n}{T}\right)^2$, $n=0,1,\cdots$. If f satisfies

$$\frac{c'}{s^{\nu}} \le -f(\theta,s) \le \frac{c}{s^{\nu}}, \qquad \forall s\in(0,\delta), \tag{$f1$}$$

for some positive constants c, c', δ and $\nu \geq 1$, and there exists a nonnegative integer n such that

$$\frac{\mu_n}{4} < \liminf_{s\to+\infty} \frac{f(\theta,s)}{s} \leq \limsup_{s\to+\infty} \frac{f(\theta,s)}{s} < \frac{\mu_{n+1}}{4} \tag{$f2$}$$

uniformly in $\theta \in [0,T]$, then problem (1.5) possesses at least one T-periodic positive solution. This result gives a Fredholm alternative-like result for the problem

$$u'' + \lambda u = \frac{a(x)}{u^\nu}, \tag{1.6}$$

which means that (1.6) possesses a $T-$periodic solution if $\lambda \neq \frac{\mu_n}{4}$ for all $n = 0, 1, \cdots$. Thus for $T = 2\pi$, problem (1.6) has at least one 2π-periodic solution for λ satisfying $\frac{n^2}{4} < \lambda < \frac{(n+1)^2}{4}$, $n = 0, 1, \cdots$.

Equation (1.4) with $p \neq 2$ has been studied by many authors. When $a \equiv 1$, all solutions of (1.4) can be classified. See Abresh and Langer [1] in the case of $p = 0$ and B. Andrews [5] in the class of general p. When a is 2π-periodic, Dohmen and Giga [18], Dohmen, Giga and Mizoguchi [19] studied the $p \leq 1$ case. Matano and Wei [28] proved that (1.4) is solvable if $0 \leq p < 7, p \neq 2$ and a is 2π-periodic.

These results do not cover the affine case, that is, equation (1.1). Indeed, the situation for the affine case is quite different. It is known that there are some obstructions for the existence and one can not get a priori estimates of the solutions of (1.1) without additional assumptions on a due to the invariance of the problem. To see this point, let us consider its simplest form

$$u_{\theta\theta} + u = \frac{1}{u^3}, \quad \theta \in S^1. \tag{1.7}$$

Equation (1.7) is invariant under the action of the special linear group $SL(2,\mathbb{R})$. For any matrix

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

in $SL(2,\mathbb{R})$, we have an "affine diffeomorphism" on S^1 given by

$$(\cos\theta, \sin\theta) \longmapsto \frac{(\cos\theta,\sin\theta)A^T}{\|(\cos\theta,\sin\theta)A^T\|} = (\cos\tilde{\theta}, \sin\tilde{\theta}),$$

that is,

$$\tan\tilde{\theta} = \frac{c + d\tan\theta}{a + b\tan\theta}, \qquad ad - bc = 1.$$

For any function u in S^1, define

$$u_A(\tilde{\theta}) = \cos\tilde{\theta}\sqrt{(a\tan\tilde{\theta} - c)^2 + (b\tan\tilde{\theta} - d)^2}u(\theta).$$

Then $u_A(\tilde{\theta})$ is a solution of (1.7) if and only if $u(\theta)$ is a solution. Starting with the trivial solution $u \equiv 1$, one can show that all solutions of (1.7) are given by a 2-parameter family of functions

$$u_{\varepsilon,t}(\theta) = \left(\varepsilon^2 \cos^2(\theta - t) + \varepsilon^{-2} \sin^2(\theta - t)\right)^{\frac{1}{2}}, \tag{1.8}$$

for $(\varepsilon, t) \in (0,1] \times [0, \pi)$. Thus the set of solutions of (1.7) is not bounded. The group $SL(2,\mathbb{R})$ plays an important role as that of the conformal group $Conf(S^2)$ in the Nirenberg's problem in geometry, which has been extensively studied and many significant results have been obtained. See [9-14, 25,27,32,34] and the references therein.

In [2], Ai, Chou and Wei considered the solvability of problem (1.1) under the assumption that *a is π-periodic.* After scaling, in this case the problem is equivalent to the equation (1.6) with $T = 2\pi$, $\lambda = \mu_1 = \frac{1}{4}$ and $\nu = 3$. Let

$$\begin{aligned} B(\theta) &= \int_0^\pi \frac{a(\theta + t) - a(\theta) - 2^{-1}a'(\theta)\sin 2t}{\sin^2 t} dt \\ &= \int_0^\pi \frac{(a'(\theta + t) - a'(\theta))\sin 2t}{\sin^2 t} dt. \end{aligned}$$

A function a is called B-nondegenerate if $B(\theta)$ never vanish at any critical point θ of the function a. They proved that if a is a positive, B-nondegenerate, C^2-function of period π, then one can get an a priori estimates of π-periodic solutions of (1.1). Moreover, if the Brouwer degree $deg(G, \mathbb{R}/\pi\mathbb{Z}, 0) \neq 0$, where

$$G(\theta) = (-B(\theta), a'(\theta)), \qquad \mathbb{R}/\pi\mathbb{Z} \to \mathbb{R}^2,$$

then problem (1.1) has a π-periodic solution.

In this paper we study equation (1.1) for 2π-periodic function a, which is more interesting and natural from geometric point of view. We will consider a slightly general form, that is, for a fixed $n \geq 2$, the existence of $n\pi$-periodic solutions of

$$u_{\theta\theta} + u = \frac{a(\theta)}{u^3} \tag{1.9}$$

with a satisfying $a(\theta + n\pi) = a(\theta)$. It is the same as the equation (1.6) with $T = \pi$, $\nu = 3$ and $\lambda = n^2$ by scaling.

Before we state our results, we comment on the difficulties in studying (1.9). A major problem is to study possible blow-ups. In the case of $\pi-$periodic $a(\theta)$, since the blow-up sequence $u_{\varepsilon,t}$ (defined at (1.8)) is $\pi-$periodic, only single blow-up can occur. However, when a is $n\pi$-periodic, there are n possible blow-ups. We have to analyze the interaction between different blow-ups.

To state our main results, for any positive C^2-function $a(\theta)$, we define

$$A_n(\theta) = \sum_{j=1}^{n} \frac{a'(\theta + (j-1)\pi)}{\sqrt{a(\theta + (j-1)\pi)}} \tag{1.10}$$

and

$$B_n(\theta)=\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\frac{\left(\sum_{j=1}^n a^{'}(\theta+t+(j-1)\pi)-\sum_{j=1}^n a^{'}(\theta+(j-1)\pi)\right)\sin 2t}{\sin^2 t}dt. \quad (1.11)$$

Following [2], a function a is called B_n-nondegenerate if $B_n(\theta)$ does not vanish whenever $A_n(\theta)=0$.

Our first result is the a priori estimates.

Theorem 1.1 *Let a be a positive, C^2 and $n\pi$-periodic function. Suppose that $a(\theta)$ is B_n-nondegenerate. Then there exists a constant C depending on a only such that*

$$\frac{1}{C}\le u\le C \quad (1.12)$$

for any $n\pi$-periodic solution u of (1.9).

As for the existence we have

Theorem 1.2 *Let a be a positive, C^2 and $n\pi$-periodic function. Suppose that*

$$\min_{A_n(\theta)=0} B_n(\theta)>0, \quad \textit{or} \quad \max_{A_n(\theta)=0} B_n(\theta)<0, \quad (1.13)$$

then equation (1.9) has an $n\pi$-periodic solution.

An example of a satisfying (1.13) is

$$a(\theta)=(1+b_1\cos\theta+b_2\cos 2\theta)^2, \quad (1.14)$$

where b_1 and b_2 are some constants in $(-\frac{1}{2},\frac{1}{2})$ to be determined later. It is easy to see that $A_2(\theta)=-8b_2\sin 2\theta$. Hence $A_2(\theta)=0$ if and only if $\theta=0,\frac{\pi}{2},\pi,\frac{3\pi}{2}$ and

$$a(\theta+t+\pi)+a(\theta+t)=\begin{cases} 2(1+b_2\cos 2t)^2+2(b_1\cos t)^2, & \theta=0,\pi;\\ 2(1-b_2\cos 2t)^2+2(b_1\sin t)^2, & \theta=\frac{\pi}{2},\frac{3\pi}{2}.\end{cases}$$

Then

$$\big(a(\theta+t+\pi)+a(\theta+t)\big)'=\begin{cases} 4(1+b_2\cos 2t)(-2b_2\sin 2t)-2b_1^2\sin 2t, & \theta=0,\pi;\\ 4(1-b_2\cos 2t)(2b_2\sin 2t)+2b_1^2\sin 2t, & \theta=\frac{\pi}{2},\frac{3\pi}{2}\end{cases}$$

and

$$B_2(\theta)=\begin{cases} -4(b_1^2+4b_2)\pi-8b_2^2\pi, & \theta=0,\pi;\\ 4(b_1^2+4b_2)\pi-8b_2^2\pi, & \theta=\frac{\pi}{2},\frac{3\pi}{2}.\end{cases}$$

Thus (1.13) is satisfied if $b_1^2+4b_2=0$ and $b_2\neq 0$.

Let

$$G_n(\theta) = \Big(- B_n(\theta), A_n(\theta)\Big) : \mathbb{R}/n\mathbb{Z}\pi \to \mathbb{R}^2,$$

whose Brouwer degree $deg(G_n, \mathbb{R}/n\mathbb{Z}\pi, 0)$ is well defined if a is B_n-nondegenerate. With the condition (1.13), we see that $deg(G_n, \mathbb{R}/n\mathbb{Z}\pi, 0) = 0$.

The following result concerns with the case $deg(G_n, \mathbb{R}/n\mathbb{Z}\pi, 0) \neq 0$. Let

$$\overline{A}_n(\theta) = \sum_{j=1}^{n} a'(\theta + (j-1)\pi)$$

and

$$\overline{B}_n(\theta) = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{\left(\sum_{j=1}^n a'(\theta + t + (j-1)\pi) - \sum_{j=1}^n a'(\theta + (j-1)\pi)\right)\sin 2t}{\sin^2 t} dt.$$

For $1 + \varepsilon a$ we have $A_n(\varepsilon)(\theta)$ and $B_n(\varepsilon)(\theta)$ given by (1.10) and (1.11). Then

$$(-B_n(\varepsilon)(\theta), A_n(\varepsilon)(\theta)) = (-\overline{B}_n(\theta), \overline{A}_n(\theta)) + O(\varepsilon).$$

Thus for small ε, $1 + \varepsilon a$ is B_n-nondegenerate if $\overline{G}_n(\theta) := (-\overline{B}_n(\theta), \overline{A}_n(\theta)) \neq (0,0)$ and $deg(G_n, \mathbb{R}/n\mathbb{Z}\pi, 0) = deg(\overline{G}_n, \mathbb{R}/n\mathbb{Z}\pi, 0)$ by homotopy invariance of degree.

Theorem 1.3 *Let a be a positive, C^2 and $n\pi$-periodic function such that for all θ, $\overline{G}_n(\theta) = (-\overline{B}_n(\theta), \overline{A}_n(\theta)) \neq (0,0)$ and $deg(\overline{G}_n, \mathbb{R}/n\mathbb{Z}\pi, 0) \neq 0$. Then the equation*

$$u_{\theta\theta} + u = \frac{1 + \varepsilon a(\theta)}{u^3} \tag{1.15}$$

has an $n\pi$-periodic solution if ε is small.

It is not difficult to see that if $n = 1$, the map $\overline{G}_n(\theta)$ is the same as the map G in [2]. In this case, we fix $\varepsilon << 1$ and consider the homotopy of $a_s(\theta) = (1-s)(1 + \varepsilon a(\theta)) + sa(\theta)$. For $s \in [0,1]$, the function a_s is $B-$nondegenerate, and one can get a uniform a priori estimate of the π-periodic solutions of

$$u_{\theta\theta} + u = \frac{a_s(\theta)}{u^3} \tag{1.16}$$

for $s \in [0,1]$. Using the degree argument, one can solve the problem up to $s = 1$ if $deg(G, \mathbb{R}/\mathbb{Z}\pi, 0) \neq 0$. However, for $n \geq 2$, we do not know how to construct a such homotopy. The degree argument can only be used for small ε.

We briefly sketch the idea of the proofs of our results. We only consider the case $n = 2$ since that of $n \geq 3$ is the same. To prove Theorems 1.1 and 1.2, we take a sequence $\lambda_k \to 1$ and consider the equation

$$u_{\theta\theta} + \lambda_k u = \frac{a(\theta)}{u^3}, \qquad \theta \in S^1. \tag{1.17}$$

According to [17], there is a 2π-periodic solution u_k if $\lambda_k \neq 1$. By careful analysis of blow-up, we can get asymptotic estimates of u_k and $\lambda_k - 1$. Under the condition (1.13), these estimates ensure that u_k converges to a solution of (1.1) if $\lambda_k \nearrow 1$ or $\lambda_k \searrow 1$ as $k \to \infty$. The proof of Theorem 1.3 follows from the Liapunov-Schmidt reduction and degree argument.

The paper is organized as follows. In Section 2, some asymptotical estimates are given based on blow-up analysis, and in Section 3 we give the necessary condition for the existence and sharp estimates. Theorem 1.1 and Theorem 1.2 are proved in Section 4, and in Section 5 we provide a proof of Theorem 1.3.

2 Preliminary Blow-up Analysis

Let a be a positive 2π-periodic function, λ_k be a sequence such that $\lambda_k \to 1$ as $k \to \infty$, and let u_k be a 2π-periodic solution of

$$u_{\theta\theta} + \lambda_k u = \frac{a(\theta)}{u^3}, \qquad \theta \in [0, 2\pi]. \tag{2.1}$$

The aim of this section is to derive some asymptotical estimates of u_k and $\lambda_k - 1$, which will lead to sharp estimates in the next section.

Here and after, unless otherwise stated, the letter C will always denote various generic constants which are independent of ε_k. We denote $A \sim B$ if there exist two positive uniform constants C_1 and C_2 such that $C_1 A \leq B \leq C_2 A$. $C_k = o(1)$ means that $\lim_{k\to+\infty} C_k = 0$.

We start with the following simple but useful lemma.

Lemma 2.1 *For any solution u of problem (2.1), defining $F_k(\theta) = u_\theta^2 + \lambda_k u^2 + \dfrac{a(\theta)}{u^2}$, then there is a constant C independent of k such that*

$$C^{-1} F_k(\theta_2) \leq F_k(\theta_1) \leq C F_k(\theta_2), \qquad \forall \quad \theta_1, \theta_2 \in [0, 2\pi].$$

Proof: By equation (2.1), we can easily get that

$$F_k'(\theta) = \frac{a'(\theta)}{u^2},$$

which implies that

$$|F_k'(\theta)| \leq C|F_k(\theta)|$$

since $a(\theta)$ is smooth and positive. Thus

$$|(\log F_k)'| \leq C,$$

which leads to

$$C^{-1} \leq \frac{F_k(\theta_1)}{F_k(\theta_2)} \leq C, \qquad \forall \quad \theta_1, \theta_2 \in [0, 2\pi].$$

Thus we finish the proof. □

From the above lemma, we know that u_k is bounded from above if and only if it is bounded from below. Without loss of generality, we assume that $\min_{\theta\in[0,2\pi]} u_k(\theta) \to 0$. Then $F_k(\theta) \to +\infty$ uniformly in θ as $k \to \infty$. This in particular implies that, as $k \to \infty$,

$$\text{either} \quad u_k(\theta) \quad \text{or} \quad \frac{1}{u_k(\theta)} \to +\infty \quad \text{whenever} \quad u_k'(\theta) = 0.$$

Then by equation (2.1) we see that $u_k''(\theta) \neq 0$. Hence for $k >> 1$, the local minimum and maximum points of u_k are isolated. Therefore they appear alternatively and satisfy

$$u_k(\tau_k) \sim \frac{1}{u_k(\theta_k)},$$

where τ_k and θ_k are local minimum and maximum points of u_k, respectively.

Let $\tau_k^1 < \tau_k^2$ be two consecutive local minimum points of u_k and let $\theta_k \in [\tau_k^1, \tau_k^2]$ satisfy $u_k(\theta_k) = M_k = \max_{\theta\in[\tau_k^1,\tau_k^2]} u_k(\theta)$. Then $u_k(\tau_k^1), u_k(\tau_k^2) \to 0$ and $M_k \to \infty$ as $k \to \infty$. We have the following convergence result.

Lemma 2.2 *For $k \to \infty$ we have*

$$\theta_k - \tau_k^1 \to \frac{\pi}{2}, \tag{2.2}$$

$$\tau_k^2 - \tau_k^1 \to \pi, \tag{2.3}$$

$$\frac{u_k(\frac{\tau_k^2-\tau_k^1}{\pi}\theta + \tau_k^1)}{M_k} \to \sin\theta \quad \textit{uniformly in} \quad [0,\pi]. \tag{2.4}$$

In particular, u_k has two minimum and two maximum points on $[0, 2\pi)$.

Proof. Let $\tilde{u}_k = \frac{u_k(\frac{\tau_k^2-\tau_k^1}{\pi}\theta+\tau_k^1)}{M_k}$. Then it follows from equation (2.1) that $\tilde{u}_k$ satisfies

$$\tilde{u}_{k,\theta\theta} + (\frac{\tau_k^2 - \tau_k^1}{\pi})^2\lambda_k\tilde{u}_k = (\frac{\tau_k^2 - \tau_k^1}{\pi M_k^2})^2\frac{a(\frac{\tau_k^2-\tau_k^1}{\pi}\theta + \tau_k^1)}{\tilde{u}_k^3}. \tag{2.5}$$

Integration by parts gives that

$$(\frac{\tau_k^2 - \tau_k^1}{\pi})^2\lambda_k\int_0^\pi \tilde{u}_k^2 d\theta - \int_0^\pi \tilde{u}_{k,\theta}^2 d\theta = (\frac{\tau_k^2 - \tau_k^1}{\pi M_k^2})^2\int_0^\pi \frac{a(\frac{\tau_k^2-\tau_k^1}{\pi}\theta + \tau_k^1)}{\tilde{u}_k^2}d\theta \geq 0$$

since $\tilde{u}_{k,\theta}(0) = \tilde{u}_{k,\theta}(\pi) = 0$. Hence

$$\int_0^\pi \tilde{u}_{k,\theta}^2 d\theta \leq (\frac{\tau_k^2 - \tau_k^1}{\pi})^2\lambda_k\int_0^\pi \tilde{u}_k^2 d\theta. \tag{2.6}$$

Using the fact that $0 < \tilde{u}_k \le \max \tilde{u}_k = 1$ and (2.6), we deduce that $\tilde{u}_k$ is bounded in $H^1([0,\pi])$. Thus we can assume $\tilde{u}_k \rightharpoonup \tilde{u}$ weakly in $H^1([0,\pi])$ and $\tau = \lim_{k\to\infty}(\tau_k^2 - \tau_k^1)$. Letting $k \to \infty$, one gets

$$\int_0^\pi \tilde{u}_\theta^2 d\theta \le (\frac{\tau}{\pi})^2 \int_0^\pi \tilde{u}^2 d\theta. \tag{2.7}$$

By the embedding theorem, $\tilde{u}_k \to \tilde{u}$ in $C([0,\pi])$, and so

$$\tilde{u}(0) = \lim_{k\to\infty} \frac{u_k(\tau_k^1)}{M_k} = 0, \quad \tilde{u}(\pi) = \lim_{k\to\infty} \frac{u_k(\tau_k^2)}{M_k} = 0$$

and $\tilde{u} \in H_0^1([0,\pi])$.

On the other hand, by Wirtinger's Inequality, we know

$$\int_0^\pi \tilde{u}^2 d\theta \le \int_0^\pi \tilde{u}_\theta^2 d\theta, \tag{2.8}$$

and equality holds if and only if $\tilde{u} = C\sin\theta$. It follows from (2.7) and (2.8) that

$$\tau = \lim_{k\to\infty}(\tau_k^2 - \tau_k^1) \ge \pi.$$

Applying the same argument to the interval $[\tau_k^2, 2\pi + \tau_k^1]$, we get

$$2\pi - \tau \ge \pi.$$

Therefore, $\tau = \pi$, and the equality holds in (2.8), which implies that $\tilde{u} = C\sin\theta$. The assumption that $\max_{\theta\in[0,\pi]} \tilde{u}_k(\theta) = 1$ yields that $\max_{\theta\in[0,\pi]} \tilde{u}(\theta) = 1$. Hence $\tilde{u} = \sin\theta$ and $\tilde{u}_k \to \sin\theta$ in $C^{\frac{1}{2}}[0,\pi]$. This proves (2.3) and (2.4). The proof of (2.2) follows from (2.4). The proof of Lemma 2.2 is completed. □

The following so-called Pohozaev's Identity will be used frequently in the rest of the paper.

Lemma 2.3 *Let a be a positive and 2π-periodic function. Then we have*

$$\int_0^{2\pi} \frac{a'(\theta) + 4(1-\lambda_k)u^3u'}{u^2}(\cos 2\theta + 1)d\theta = 0 \tag{2.9}$$

and

$$\int_0^{2\pi} \frac{a'(\theta) + 4(1-\lambda_k)u^3u'}{u^2}\sin 2\theta d\theta = 0 \tag{2.10}$$

for any 2π-periodic solution u of (2.1).

Proof. For any solution u of (2.1), we have

$$(\frac{u^2}{2})''' + 4(\frac{u^2}{2})' = \frac{\big(a(\theta) + (1-\lambda_k)u^4\big)'}{u^2} \tag{2.11}$$

and the lemma follows from an integration over $[0, 2\pi]$. □

Let $\varepsilon_k = u_k(\tau_k) = \min_{\theta\in[0,2\pi]} u_k(\theta) \to 0$ and let θ_k be the next local maximum point of u_k. Then $M_k = u_k(\theta_k) \to \infty$. The above lemmas lead to the following estimate.

Proposition 2.4 *There exists a uniform positive constant C such that*

$$|\lambda_k - 1| \le C\varepsilon_k^2. \tag{2.12}$$

Proof. We first prove

$$\int_0^{2\pi} \frac{1}{u_k^2} d\theta \le C. \tag{2.13}$$

To this end, let $\overline{u}_k(\theta) = u_k\big(\frac{2(\theta_k-\tau_k)}{\pi}\theta + \tau_k\big), \theta \in [0, \frac{\pi}{2}]$ and $\overline{\lambda}_k = \big(\frac{2(\theta_k-\tau_k)}{\pi}\big)^2 \lambda_k$. Then $\overline{u}_k$ satisfies

$$\overline{u}_{k,\theta\theta} + \overline{\lambda}_k \overline{u}_k = \big(\frac{2(\theta_k-\tau_k)}{\pi}\big)^2 \frac{a\big(\frac{2(\theta_k-\tau_k)}{\pi}\theta + \tau_k\big)}{\overline{u}_k^3} \tag{2.14}$$

and

$$(\frac{\overline{u}_k^2}{2})''' + 4(\frac{\overline{u}_k^2}{2})' = \big(\frac{2(\theta_k-\tau_k)}{\pi}\big)^3 \frac{a'(\frac{2(\theta_k-\tau_k)}{\pi}\theta + \tau_k)}{\overline{u}_k^2} + 4(1-\overline{\lambda}_k)\overline{u}_k \overline{u}_k'. \tag{2.15}$$

By virtue of $\overline{u}_k'(0) = 0$ and $\overline{u}_k'(\frac{\pi}{2}) = 0$ we obtain

$$\int_0^{\frac{\pi}{2}} \big((\frac{\overline{u}_k^2}{2})''' + 4(\frac{\overline{u}_k^2}{2})'\big) \sin 2\theta d\theta = 0. \tag{2.16}$$

Consequently, from (2.15) and (2.16) we deduce that

$$\begin{aligned}(\frac{2(\theta_k-\tau_k)}{\pi})^3 \int_0^{\frac{\pi}{2}} \frac{a'(\frac{2(\theta_k-\tau_k)}{\pi}\theta + \tau_k)}{\overline{u}_k^2} \sin 2\theta d\theta &= 4(\overline{\lambda}_k - 1) \int_0^{\frac{\pi}{2}} \overline{u}_k \overline{u}_k' \sin 2\theta d\theta \\ &= -4(\overline{\lambda}_k - 1) \int_0^{\frac{\pi}{2}} \overline{u}_k{}^2 \cos 2\theta d\theta.\end{aligned} \tag{2.17}$$

It follows from Lemma 2.2 that $\exists\, C > 0$ such that

$$|\int_0^{\frac{\pi}{2}} \overline{u}_k{}^2 d\theta| \le C|\int_0^{\frac{\pi}{2}} \overline{u}_k{}^2 \cos 2\theta d\theta|.$$

Hence for small δ,

$$\begin{aligned}|\overline{\lambda}_k - 1| \int_0^{\frac{\pi}{2}} \overline{u}_k^2 d\theta &\le C\Big| \int_0^{\frac{\pi}{2}} \frac{a'(\frac{2(\theta_k-\tau_k)}{\pi}\theta + \tau_k) \sin 2\theta}{\overline{u}_k^2} d\theta\Big| \\ &\le C(|\int_0^{\delta} f_k(\theta)d\theta| + |\int_{\delta}^{\frac{\pi}{2}-\delta} f_k(\theta)d\theta| + |\int_{\frac{\pi}{2}-\delta}^{\frac{\pi}{2}} f_k(\theta)d\theta|) \\ &\le C\delta \int_0^{\pi} \frac{1}{\overline{u}_k^2} d\theta + C(\delta) M_k^{-2}\end{aligned} \tag{2.18}$$

by (2.4), where $f_k(\theta) = \frac{a'(\frac{2(\theta_k-\tau_k)}{\pi}\theta + \tau_k)\sin 2\theta}{\overline{u}_k^2}$ and $C(\delta)$ is a constant depending on δ.

On the other hand, from (2.14) and (2.18) we get

$$\begin{aligned}\int_0^{\frac{\pi}{2}}(\overline{u}_k^2-\overline{u}_{k,\theta}^2)d\theta &= \big(\frac{2(\theta_k-\tau_k)}{\pi}\big)^2\int_0^{\frac{\pi}{2}}\frac{a(\frac{2(\theta_k-\tau_k)}{\pi}\theta+\tau_k)}{\overline{u}_k^2}d\theta+(1-\overline{\lambda}_k)\int_0^{\frac{\pi}{2}}\overline{u}_k^2d\theta\\ &\geq \big(\frac{2(\theta_k-\tau_k)}{\pi}\big)^2\int_0^{\frac{\pi}{2}}\frac{a(\frac{2(\theta_k-\tau_k)}{\pi}\theta+\tau_k)}{\overline{u}_k^2}d\theta-C\delta\int_0^{\frac{\pi}{2}}\frac{1}{\overline{u}_k^2}d\theta-C(\delta)M_k^{-2}\\ &\geq \frac{\min_\theta a(\theta)}{2}\int_0^{\frac{\pi}{2}}\frac{1}{\overline{u}_k^2}d\theta-C\end{aligned} \tag{2.19}$$

since δ is small. The left hand side of (2.19) can be estimated as follows. Reflecting the function $\overline{u}_k$ with respect to $\theta=\frac{\pi}{2}$, we get a new function defined on $[0,\pi]$, still denoted by $\overline{u}_k$. Then $\overline{u}_k(0)=\overline{u}_k(\pi)$. By Sobolev Inequality (See Proposition 1.3 of [2])

$$\int_0^{\pi}\frac{1}{\overline{u}_k^2}d\theta\Big(\int_0^{\pi}(\overline{u}_k^2-\overline{u}_{k,\theta}^2)d\theta\Big)\leq\pi^2.$$

Since $\overline{u}_k$ is symmetric with respect to $\theta=\frac{\pi}{2}$, we see that

$$\int_0^{\frac{\pi}{2}}\frac{1}{\overline{u}_k^2}d\theta\Big(\int_0^{\frac{\pi}{2}}(\overline{u}_k^2-\overline{u}_{k,\theta}^2)d\theta\Big)\leq\frac{\pi^2}{4}. \tag{2.20}$$

Combining (2.19) with (2.20) we obtain

$$\int_0^{\frac{\pi}{2}}\frac{1}{\overline{u}_k^2}d\theta\left(\frac{\min_\theta a(\theta)}{2}\int_0^{\frac{\pi}{2}}\frac{1}{\overline{u}_k^2}d\theta-C\right)\leq\frac{\pi^2}{4}. \tag{2.21}$$

Consequently,

$$\int_0^{\frac{\pi}{2}}\frac{1}{\overline{u}_k^2}d\theta\leq C \tag{2.22}$$

and

$$\int_{\tau_k}^{\theta_k}\frac{1}{u_k^2}d\theta\leq\frac{2(\theta_k-\tau_k)}{\pi}\int_0^{\frac{\pi}{2}}\frac{1}{\overline{u}_k^2}d\theta\leq C. \tag{2.23}$$

Similarly, let τ_k' be the minimum point of u_k next to θ_k, we have

$$\int_{\theta_k}^{\tau_k'}\frac{1}{u_k^2}d\theta\leq C. \tag{2.24}$$

It follows from (2.23) and (2.24) that

$$\int_{\tau_k}^{\tau_k'}\frac{1}{u_k^2}d\theta\leq C. \tag{2.25}$$

The same argument yields

$$\int_{\tau_k'}^{2\pi+\tau_k}\frac{1}{u_k^2}d\theta\leq C. \tag{2.26}$$

(2.13) follows from (2.25) and (2.26) since u_k has two minimum and maximum points on $[0, 2\pi)$.

Now we can prove the estimate (2.12). Using the identity (2.10) we see that

$$4|\lambda_k - 1|\left|\int_0^{2\pi} u_k(\theta)u_k'(\theta)\sin 2\theta d\theta\right| \le \left|\int_0^{2\pi} \frac{a'(\theta)\sin 2\theta}{u_k^2}\right| \le C. \tag{2.27}$$

It is easy to see from Lemma 2.2 that

$$\left|\int_0^{2\pi} u_k(\theta)u_k'(\theta)\sin 2\theta d\theta\right| = \left|\int_0^{2\pi} u_k^2(\theta)\cos 2\theta d\theta\right| \ge CM_k^2. \tag{2.28}$$

Inserting (2.28) into (2.27), we lead to

$$|\lambda_k - 1| \le \frac{C}{M_k^2} \le C\varepsilon_k^2.$$

Thus (2.12) is proved. □

Let $\varepsilon_k = \min_\theta u_k(\theta) = u_k(\tau_k) \to 0$ and

$$U_\varepsilon(\theta) = \left(\varepsilon^2 \cos^2\theta + \varepsilon^{-2}\sin^2\theta\right)^{\frac{1}{2}}.$$

We define a transformation

$$\theta = \tau_k + \psi_k(y) = \tau_k + \int_0^y \frac{1}{U^2{}_{(\varepsilon_k')^{-1}}(\tau)} d\tau, \tag{2.29}$$

where $\varepsilon_k' = a^{-\frac{1}{4}}(0)\varepsilon_k$. It induces a rule of transformation of the equation (2.1) as follows: let

$$v_k(y) = U_{\varepsilon_k'^{-1}}(y)u_k(\theta) = U_{\varepsilon_k'}^{-1}(\theta - \tau_k)u_k(\theta),$$

using

$$\frac{dy}{d\theta} = U_{\varepsilon_k'}^{-2}(\theta), \qquad \text{and}$$

$$\frac{d^2y}{d\theta^2} = \left({\varepsilon_k'}^2 - {\varepsilon_k'}^{-2}\right)\sin 2\theta U_{\varepsilon_k'}^{-4}(\theta),$$

one can verify that

$$\frac{d^2 v_k}{dy^2} + v_k = \frac{a(\tau_k + \psi_k)}{v_k^3} + (1 - \lambda_k)\frac{v_k}{U^4_{\varepsilon_k'^{-1}}(y)}. \tag{2.30}$$

Using the estimate (2.12), we can prove the following result on the asymptotical behavior of u_k.

Proposition 2.5 *Let a be a positive, C^2 and 2π-periodic function and let u_k be a solution of (2.1) such that $\varepsilon_k = \min_\theta u_k(\theta) = u_k(\tau_k) \to 0$ and $\tau_k \to \theta_0$. Then the functions $v_k, v_{k,y}, \frac{1}{v_k}$ are bounded. That is, $\exists\ C > 0$ such that*

$$\frac{1}{C} \le v_k(y) \le C, \quad |v_{k,y}(y)| \le C, \quad y \in [0, 2\pi], \tag{2.31}$$

which implies

$$C_1 U_{\varepsilon'_k}(\theta - \tau_k) \le u_k(\theta) \le C_2 U_{\varepsilon'_k}(\theta - \tau_k), \quad \theta \in [0, 2\pi]. \tag{2.32}$$

Moreover, for any $1 > \delta > 0$

$$v_k \to v_\infty \quad in \quad C^{1,\delta}([0, 2\pi]), \quad k \to \infty, \tag{2.33}$$

where the function v_∞ is given by

$$v_\infty(y) = \begin{cases} a^{\frac{1}{4}}(\theta_0), & y \in [-\frac{\pi}{2}, \frac{\pi}{2}], \\ \left(a^{\frac{1}{2}}(\theta_0)\sin^2 y + a(\theta_0 + \pi)a^{-\frac{1}{2}}(\theta_0)\cos^2 y\right)^{\frac{1}{2}}, & y \in [\frac{\pi}{2}, \frac{3\pi}{2}]. \end{cases} \tag{2.34}$$

Proof. For simplicity of notations we assume $\tau_k = 0$ and $\theta_0 = 0$. Let

$$\tilde{F}_k(y) = \frac{1}{2}\left(v_{k,y}^2 + v_k^2 + \frac{a(\psi_k(y))}{v_k^2}\right).$$

Then

$$\begin{aligned} \frac{d\tilde{F}_k}{dy} &= v_{k,y}\left(v_{k,yy} + v_k - \frac{a(\psi_k)}{v_k^3}\right) + \frac{\left(a(\psi_k)\right)_y}{2v_k^2} \\ &= (1 - \lambda_k)\frac{v_k v_{k,y}}{U^4_{\varepsilon'^{-1}_k}(y)} + \frac{a_\theta}{2v_k^2 U^2_{\varepsilon'^{-1}_k}(y)}. \end{aligned} \tag{2.35}$$

Using Proposition 2.4, we have

$$\left|\frac{(1 - \lambda_k)}{U^2_{\varepsilon'^{-1}_k}(y)}\right| \le C\frac{\varepsilon_k^2}{\varepsilon'^{-2}_k \cos^2 y + \varepsilon'^{2}_k \sin^2 y} \le C. \tag{2.36}$$

Combining (2.35) and (2.36) we get that

$$\begin{aligned} \left|\frac{(1 - \lambda_k)v_k v_{k,y}}{U^4_{\varepsilon'^{-1}_k}(y)}\right| &\le \frac{C|v_k v_{k,y}|}{\varepsilon'^{-2}_k \cos^2 y + \varepsilon'^{2}_k \sin^2 y} \\ &\le \frac{C|\tilde{F}_k|}{\varepsilon'^{-2}_k \cos^2 y + \varepsilon'^{2}_k \sin^2 y} \end{aligned}$$

and

$$\left|\frac{a_\theta}{2v_k^2 U^2_{\varepsilon'^{-1}_k}}\right| \le \frac{Ca}{v_k^2(\varepsilon'^{-2}_k \cos^2 y + \varepsilon'^{2}_k \sin^2 y)} \le \frac{C|\tilde{F}_k|}{\varepsilon'^{-2}_k \cos^2 y + \varepsilon'^{2}_k \sin^2 y}.$$

Thus

$$|\frac{d\tilde{F}_k}{dy}| \le C\tilde{F}_k \cdot \frac{1}{{\varepsilon'_k}^{-2}\cos^2 y + {\varepsilon'_k}^2 \sin^2 y}. \tag{2.37}$$

Since

$$\int_0^{2\pi} \frac{dy}{{\varepsilon'_k}^{-2}\cos^2 y + {\varepsilon'_k}^2 \sin^2 y} = \int_0^{2\pi} d\theta = 2\pi,$$

from (2.37) we derive that

$$C^{-1}\tilde{F}_k(y_2) \le \tilde{F}_k(y_1) \le C\tilde{F}_k(y_2), \qquad \text{for any} \quad y_1, y_2.$$

In particular, since $\tilde{F}_k(0) \le C$, this implies that

$$C^{-1} \le \tilde{F}_k(y) \le C,$$

that is,

$$v_k(y) \le C, \qquad |v_{k,y}(y)| \le C \quad \text{and} \quad \frac{1}{v_k} \le C, \tag{2.38}$$

which concludes (2.31) and (2.32).

From (2.31) a better estimate for $\lambda_k - 1$ can be derived:

$$|\lambda_k - 1| \le C\varepsilon_k^4 \log \frac{1}{\varepsilon_k}. \tag{2.39}$$

In fact, from (2.27), we have that

$$\begin{aligned} |\lambda_k - 1| \int_0^{2\pi} u_k^2 d\theta \le & |\int_0^{2\pi} \frac{a'(\theta)\sin 2\theta}{u_k^2}|d\theta \\ \le & C\int_0^{2\pi} \frac{|\sin 2\theta|}{u_k^2} d\theta \\ \le & C\int_0^{2\pi} \frac{|\sin 2\theta|}{U^2_{\varepsilon'_k}(\theta)} d\theta \\ \le & C{\varepsilon'_k}^2 \log\frac{1}{\varepsilon'_k} \le C\varepsilon_k^2 \log\frac{1}{\varepsilon_k}, \end{aligned}$$

which proves (2.39).

Let $f_k = (1-\lambda_k)\frac{v_k}{U^4_{{\varepsilon'_k}^{-1}}(y)}$. From (2.39), we see that for $p > 1$,

$$\int_0^{2\pi} |f_k|^p dy \le C|1-\lambda_k|^p \int_0^{2\pi} \frac{1}{U^{4p}_{{\varepsilon'_k}^{-1}}} dy \le C(\varepsilon_k^4 \log\frac{1}{\varepsilon_k})^p \varepsilon_k^{2-4p} \to 0 \tag{2.40}$$

where we have used the following estimate

$$\int_0^{2\pi} \frac{1}{({\varepsilon'_k}^{-2}\cos^2 y + {\varepsilon'_k}^2 \sin^2 y)^{2p}} dy \le C\varepsilon_k^{2-4p}.$$

Standard regularity shows that $\{v_k\}$ converges to some v_∞ in $C^{1,\alpha}$-norm for any $\alpha > 0$. Away from $y = \frac{\pi}{2}, \frac{3\pi}{2}$, $U^4_{\varepsilon_k'^{-1}}(y)$ is bounded from below, so v_∞ is C^2 if $y \neq \frac{\pi}{2}, \frac{3\pi}{2}$ and satisfies

$$v_{\infty,yy} + v_\infty = \frac{a(0)}{v_\infty^3}, \quad y \in (-\frac{\pi}{2}, \frac{\pi}{2}), \tag{2.41}$$

and

$$v_{\infty,yy} + v_\infty = \frac{a(\pi)}{v_\infty^3}, \quad y \in (\frac{\pi}{2}, \frac{3\pi}{2}). \tag{2.42}$$

Hence

$$v_\infty(y) = a^{\frac{1}{4}}(0) \quad if \quad y \in (-\frac{\pi}{2}, \frac{\pi}{2})$$

since $v_\infty(0) = \lim v_k(0) = a^{\frac{1}{4}}(0)$ and $v'_\infty(0) = \lim v'_k(0) = 0$. It follows that

$$v_\infty(\frac{\pi}{2}) = v_\infty(0) = a^{\frac{1}{4}}(0), \quad v'_\infty(\frac{\pi}{2}) = v'_\infty(0) = 0.$$

Therefore, from (2.42) we get

$$v_\infty(y) = \left(a^{\frac{1}{2}}(0)\sin^2 y + a(\pi)a^{-\frac{1}{2}}(0)\cos^2 y\right)^{\frac{1}{2}}, \quad y \in [\frac{\pi}{2}, \frac{3\pi}{2}].$$

□

Remark: From equation (2.30) the following estimate can be obtained:

$$\|v_k - v_\infty\|_{C^1([0,2\pi])} \leq C\varepsilon_k^{\frac{4}{3}} |\log \varepsilon_k|^{\frac{4}{3}}. \tag{2.43}$$

Indeed, for $y \in [-\frac{\pi}{2}, \frac{\pi}{2}]$ we have

$$\begin{aligned} \frac{d^2(v_k - v_\infty)}{dy^2} + (v_k - v_\infty) &= \frac{a(\psi_k) - a(0)}{v_k^3} + \frac{a(0)}{v_k^3} - \frac{a(0)}{v_\infty^3} + \frac{(1-\lambda_k)v_k}{U^4_{\varepsilon_k'^{-1}}(y)} \\ &= \frac{a(\psi_k) - a(0)}{v_k^3} + c_k(y)(v_k - v_\infty) + \frac{(1-\lambda_k)v_k}{U^4_{\varepsilon_k'^{-1}}(y)}, \end{aligned} \tag{2.44}$$

where c_k is a bounded function due to (2.31). It follows from

$$\begin{aligned} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} |a(\psi_k(y)) - a(0)|^q dy &= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} |a(\theta) - a(0)|^q \frac{a^{\frac{1}{2}}(0)\varepsilon_k^2}{a(0)\sin^2\theta + \varepsilon_k^4 \cos^2\theta} d\theta \\ &= a^{\frac{1}{2}}(0)\varepsilon_k^2 \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{|a(\theta) - a(0)|^q}{|\sin\theta|^q} \frac{|\sin\theta|^q}{a(0)\sin^2\theta + \varepsilon_k^4\cos^2\theta} d\theta \\ &\leq a^{\frac{1}{2}}(0)\varepsilon_k^2 \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{|a(\theta) - a(0)|^q}{|\sin\theta|^q} \frac{|\sin\theta|}{a(0)\sin^2\theta + \varepsilon_k^4\cos^2\theta} d\theta \\ &\leq C\varepsilon_k^2 |\log \varepsilon_k| \end{aligned} \tag{2.45}$$

for $q>1$. Let $q=\frac{4}{3}$ and applying the L^p-estimate to (2.44), taking $v_k(0)=v_\infty(0)$ into account, we have

$$\|v_k-v_\infty\|_{W^{2,\frac{4}{3}}([-\frac{\pi}{2},\frac{\pi}{2}])}\le\Big(\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}|a(\psi_k(y))-a(0)|^{\frac{4}{3}}dy\Big)^{\frac{3}{4}}+\Big(\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\Big|\frac{(1-\lambda_k)v_k}{U^4_{\varepsilon_k'^{-1}}(y)}\Big|^{\frac{4}{3}}dy\Big)^{\frac{3}{4}}$$
$$\le C\varepsilon_k^{\frac{4}{3}}|\log\varepsilon_k|^{\frac{4}{3}} \tag{2.46}$$

according to (2.40) and (2.45). Using the same argument we can obtain

$$\|v_k-v_\infty\|_{W^{2,\frac{4}{3}}([\frac{\pi}{2},\frac{3\pi}{2}])}\le C\varepsilon_k^{\frac{4}{3}}|\log\varepsilon_k|^{\frac{4}{3}}. \tag{2.47}$$

Now the estimate (2.43) is a consequence of (2.46), (2.47) and the embedding theorem.

3 Sharp blow-up estimates

In this section we will use Pohozaev Identities to get a sharp estimate of λ_k-1. Let $a(\theta)$ be a 2π-period positive C^2 function and $\lambda_k\to1$, and let u_k be a 2π-periodic solution of

$$u_{k,\theta\theta}+\lambda_k u_k=\frac{a(\theta)}{u_k^3},\quad \theta\in[0,2\pi]. \tag{3.1}$$

The main result of this section is

Proposition 3.1 *Assume that* $\min_{\theta\in[0,2\pi]}u_k(\theta)=u_k(\tau_k)=\varepsilon_k\to0$ *and* $\tau_k\to\theta_0$. *Then*

$$A_2(\theta_0)=\frac{a'(\theta_0)}{\sqrt{a(\theta_0)}}+\frac{a'(\pi+\theta_0)}{\sqrt{a(\pi+\theta_0)}}=0, \tag{3.2}$$

and

$$\lambda_k-1$$
$$=\frac{\varepsilon_k^4}{2\pi a^2(0)}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\frac{\big(a'(\theta+\theta_0)+a'(\theta+\pi+\theta_0)-a'(\theta_0)-a'(\pi+\theta_0)\big)\sin2\theta}{\sin^2\theta}d\theta+o(\varepsilon_k^4)$$
$$=\frac{\varepsilon_k^4}{2\pi a^2(0)}B_2(\theta_0)+o(\varepsilon_k^4). \tag{3.3}$$

Proof. For simplicity of notations, we assume $\theta_k = 0$. This can be achieved by translation. Then $\theta_0 = 0$. By (2.9) and Proposition 2.4 we get that

$$\begin{aligned}\int_0^{2\pi} \frac{a'(\theta)}{u_k^2}(\cos 2\theta + 1)d\theta &= 4\int_0^{2\pi}(\lambda_k - 1)u_k u_k'(\cos 2\theta + 1)d\theta \\ &= 4(\lambda_k - 1)\int_0^{2\pi} u_k^2 \sin 2\theta d\theta \\ &= 4(\lambda_k - 1)\int_0^{2\pi} (\frac{u_k}{M_k})^2 \sin 2\theta d\theta \cdot M_k^2 \\ &= 4(\lambda_k - 1)\Big(\int_0^{2\pi} \sin^2\theta \sin 2\theta d\theta + o(1)\Big)M_k^2 \\ &= o\big(|1-\lambda_k|M_k^2\big) = o(1). \end{aligned} \tag{3.4}$$

On the other hand, as $k \to +\infty$, using the change of variable $\theta = \psi_k(y)$ we have

$$\begin{aligned}\int_0^{2\pi} \frac{a'(\theta)}{u_k^2(\theta)}(\cos 2\theta + 1)d\theta &= \int_{-\frac{\pi}{2}}^{\frac{3\pi}{2}} \frac{a'(\theta)}{u_k^2(\theta)}(\cos 2\theta + 1)d\theta \\ &= \int_{-\frac{\pi}{2}}^{\frac{3\pi}{2}} \frac{a'(\psi_k(y))}{v_k^2(y)} \frac{2\cos^2 y}{\cos^2 y + a^{-1}(0)\varepsilon_k^4 \sin^2 y}dy \\ &= 2\Big(\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{a'(0)}{v_\infty^2(y)}dy + \int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} \frac{a'(\pi)}{v_\infty^2(y)}dy\Big) + o(1) \\ &= 2\pi\Big(\frac{a'(0)}{\sqrt{a(0)}} + \frac{a'(\pi)}{\sqrt{a(\pi)}}\Big) + o(1) \end{aligned} \tag{3.5}$$

by Lebesgue Dominated Convergence Theorem and Proposition 2.5. This proves (3.2).

The proof of (3.3) is more involved.

By (2.10) we see that

$$\int_0^{2\pi} \frac{a'(\theta)}{u_k^2(\theta)} \sin 2\theta d\theta = -4(\lambda_k - 1)\int_0^{2\pi} u_k^2(\theta)\cos 2\theta d\theta. \tag{3.6}$$

From $u_k(\psi_k(y))({\varepsilon'_k}^{-2}\cos^2 y + {\varepsilon'_k}^2 \sin^2 y)^{\frac{1}{2}} \to v_\infty(y)$ in C^1 we have

$$\begin{aligned}&|u_k(\psi_k(y))({\varepsilon'_k}^{-2}\cos^2 y + {\varepsilon'_k}^2 \sin^2 y)^{\frac{1}{2}} - v_\infty(y)| \\ =&|u_k(\theta)({\varepsilon'_k}^2\cos^2\theta + {\varepsilon'_k}^{-2}\sin^2\theta)^{-\frac{1}{2}} - v_\infty(\psi_k^{-1}(\theta))| = o(1). \end{aligned} \tag{3.7}$$

Hence

$$\begin{aligned}\int_0^{2\pi} u_k^2(\theta)\cos 2\theta d\theta =& \int_0^{2\pi} (v_\infty^2(\psi_k^{-1}(\theta)) + o(1))({\varepsilon'_k}^2\cos^2\theta + {\varepsilon'_k}^{-2}\sin^2\theta)\cos 2\theta d\theta \\ =&\varepsilon_k^{-2} a^{\frac{1}{2}}(0)\int_{-\frac{\pi}{2}}^{\frac{3\pi}{2}} v_\infty^2(\psi_k^{-1}(\theta))\sin^2\theta\cos 2\theta d\theta + o(\varepsilon_k^{-2}). \end{aligned} \tag{3.8}$$

For $\theta \in [-\frac{\pi}{2}, \frac{\pi}{2}]$, we have $\psi_k^{-1}(\theta) \in [-\frac{\pi}{2}, \frac{\pi}{2}]$, so $v_\infty^2(\psi_k^{-1}(\theta)) = a^{\frac{1}{2}}(0)$ and

$$a^{\frac{1}{2}}(0) \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} v_\infty^2(\psi_k^{-1}(\theta)) \sin^2\theta \cos 2\theta d\theta = -\frac{\pi}{4} a(0). \tag{3.9}$$

Similarly, if $\theta \in [\frac{\pi}{2}, \frac{3\pi}{2}]$, we have $\psi_k^{-1}(\theta) \in [\frac{\pi}{2}, \frac{3\pi}{2}]$ and

$$\begin{aligned} v_\infty^2(\psi_k^{-1}(\theta)) &= a^{\frac{1}{2}}(0) \sin^2 \psi_k^{-1}(\theta) + a(\pi) a^{-\frac{1}{2}}(0) \cos^2 \psi_k^{-1}(\theta) \\ &= \frac{\varepsilon_k'^4 a(\pi) a^{-\frac{1}{2}}(0) \cos^2\theta + a^{\frac{1}{2}}(0) \sin^2\theta}{\varepsilon_k'^4 \cos^2\theta + \sin^2\theta}. \end{aligned}$$

Then

$$\begin{aligned} & a^{\frac{1}{2}}(0) \int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} v_\infty^2(\psi_k^{-1}(\theta)) \sin^2\theta \cos 2\theta d\theta \\ =& a^{\frac{1}{2}}(0) \int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} \frac{\varepsilon_k'^4 a(\pi) a^{-\frac{1}{2}}(0) \cos^2\theta + a^{\frac{1}{2}}(0) \sin^2\theta}{\varepsilon_k'^4 \cos^2\theta + \sin^2\theta} \sin^2\theta \cos 2\theta d\theta \\ =& a(0) \int_{\pi}^{2\pi} \sin^2\theta \cos 2\theta d\theta + o(1) \\ =& -\frac{\pi}{4} a(0) + o(1). \end{aligned} \tag{3.10}$$

Substituting (3.9) and (3.10) into (3.8), we get

$$\int_0^{2\pi} u_k^2(\theta) \cos 2\theta d\theta = -\frac{\pi}{2} a(0) \varepsilon_k^{-2} + o(\varepsilon_k^{-2}). \tag{3.11}$$

Now we estimate the left hand side of (3.6). It follows from (3.7) that

$$\begin{aligned} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{a'(\theta)\sin 2\theta}{u_k^2(\theta)} d\theta &= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{a'(\theta)}{u_k^2(\theta)(\varepsilon_k'^2 \cos^2\theta + \varepsilon_k'^{-2} \sin^2\theta)^{-1}} \frac{\sin 2\theta}{(\varepsilon_k'^2 \cos^2\theta + \varepsilon_k'^{-2}\sin^2\theta)} d\theta \\ &= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} (\frac{1}{v_\infty^2(\psi_k^{-1}(\theta))} + O(\varepsilon_k^{\frac{4}{3}} |\log \varepsilon_k|^{\frac{4}{3}})) \frac{\varepsilon_k'^2 a'(\theta) \sin 2\theta}{\varepsilon_k'^4 \cos^2\theta + \sin^2\theta} d\theta \\ &= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{1}{v_\infty^2(\psi_k^{-1}(\theta))} \frac{\varepsilon_k'^2 a'(\theta)\sin 2\theta}{\varepsilon_k'^4 \cos^2\theta + \sin^2\theta} d\theta + o(\varepsilon_k^2) \\ &= \frac{\varepsilon_k'^2}{a^{\frac{1}{2}}(0)} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{a'(\theta)\sin 2\theta}{\varepsilon_k'^4 \cos^2\theta + \sin^2\theta} d\theta + o(\varepsilon_k^2) \\ &= \frac{\varepsilon_k'^2}{a^{\frac{1}{2}}(0)} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{a'(\theta) - a'(0)}{\sin\theta} \frac{\sin\theta \sin 2\theta}{\varepsilon_k'^4 \cos^2\theta + \sin^2\theta} d\theta + o(\varepsilon_k^2) \\ &= \frac{\varepsilon_k^2}{a(0)} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{a'(\theta) - a'(0)}{\sin^2\theta} \sin 2\theta d\theta + o(\varepsilon_k^2), \end{aligned} \tag{3.12}$$

where we have used (2.43).

Similarly, we have

$$\begin{aligned}\int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} \frac{a'(\theta)\sin 2\theta}{u_k^2}d\theta &= \int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} \frac{1}{v_\infty^2(\psi_k^{-1}(\theta))}\frac{{\varepsilon'_k}^2 a'(\theta)\sin 2\theta}{{\varepsilon'_k}^4\cos^2\theta+\sin^2\theta}d\theta + o(\varepsilon_k^2)\\ &= {\varepsilon'}_k^2\int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} \frac{a'(\theta)\sin 2\theta}{{\varepsilon'}_k^4 a(\pi)a^{-\frac{1}{2}}(0)\cos^2\theta + a^{\frac{1}{2}}(0)\sin^2\theta}d\theta + o(\varepsilon_k^2)\\ &= {\varepsilon'}_k^2\int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} \frac{a'(\theta)-a'(\pi)}{\sin\theta}\frac{\sin\theta\sin 2\theta}{{\varepsilon'}_k^4 a(\pi)a^{-\frac{1}{2}}(0)\cos^2\theta + a^{\frac{1}{2}}(0)\sin^2\theta}d\theta + o(\varepsilon_k^2)\\ &= \frac{\varepsilon_k^2}{a(0)}\int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} \frac{a'(\theta)-a'(\pi)}{\sin^2\theta}\sin 2\theta d\theta + o(\varepsilon_k^2)\\ &= \frac{\varepsilon_k^2}{a(0)}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{a'(\theta+\pi)-a'(\pi)}{\sin^2\theta}\sin 2\theta d\theta + o(\varepsilon_k^2). \end{aligned} \tag{3.13}$$

Combining (3.6) and (3.13) we can obtain that

$$\int_{\pi}^{2\pi} \frac{a'(\theta)\sin 2\theta}{u_k^2}d\theta = \frac{\varepsilon_k^2}{a(0)}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{a'(\theta)-a'(0)+a(\theta+\pi)-a'(\pi)}{\sin^2\theta}\sin 2\theta d\theta + o(\varepsilon_k^2).$$

This completes the proof of Proposition 3.1. □

4 Proof of Theorem 1.1 and Theorem 1.2

In this section, we prove Theorem 1.1 and Theorem 1.2. We only discuss the case $n = 2$ since for $n > 2$, the proof is the same.

Proof of Theorem 1.1: This is an immediate consequence of Proposition 3.1. Indeed, if there is a sequence u_k of

$$u_{k,\theta\theta} + u_k = \frac{a(\theta)}{u_k^3}, \quad \theta \in S^1 \tag{4.1}$$

such that $\varepsilon_k = \min_{\theta\in S^1} u_k(\theta) = u_k(\tau_k) \to 0$. Let $\tau_k \to \theta_0$, then by Proposition 3.1, we have

$$A_2(\theta_0) = \frac{a'(\theta_0)}{\sqrt{a(\theta_0)}} + \frac{a'(\pi+\theta_0)}{\sqrt{a(\pi+\theta_0)}} = 0 \tag{4.2}$$

and

$$\begin{aligned}&\frac{\varepsilon_k^4}{2\pi a^2(\theta_0)}\bigg(\int_0^{\pi} \frac{\big(a'(\theta)+a'(\theta+\theta_0)-a'(0)-a'(\pi)\big)\sin 2\theta}{\sin^2\theta}\bigg)d\theta + o(\varepsilon_k^4)\\ =&\frac{\varepsilon_k^4}{2\pi a^2(\theta_0)}B_2(\theta_0) + o(\varepsilon_k^4) = 0.\end{aligned}$$

Hence

$$B_2(\theta_0) = 0. \tag{4.3}$$

This contradicts the assumption that a is B_2-nondegenerate and proves Theorem 1.1. $\square$

Proof of Theorem 1.2: Without loss of generality we suppose $\min_{A_2(\theta)=0} B_2(\theta) > 0$. We take a sequence $\lambda_k < 1$ and $\lambda_k \to 1$. According to [17], there is a 2π-periodic solution u_k of

$$u_{\theta\theta} + \lambda_k u = \frac{a(\theta)}{u^3}. \tag{4.4}$$

If $\varepsilon_k = \min_{\theta \in S^1} u_k(\theta) = u_k(\tau_k) \to 0$, then by Proposition 3.1 we have $A_2(\theta_0) = 0$ and

$$\lambda_k - 1 = \frac{\varepsilon_k^4}{2\pi a^2(\theta_0)} B_2(\theta_0) + o(\varepsilon_k^4). \tag{4.5}$$

So $B_2(\theta_0) < 0$, which is impossible. Thus the sequence u_k are uniformly bounded from below and above. By taking a limit, we obtain a solution u of (1.1). $\square$

5 Proof of Theorem 1.3

In this section, we will use Liapunov-Schmidt reduction and degree theory to prove Theorem 1.3. Similar approach has been used by Rey-Wei ([29], [30]) and Wei-Xu ([33]). For simplicity we only give the proof of 2π-periodic case and that of $n > 2$ is similar and hence it is omitted.

Let ε be a small positive number and

$$S[u] := u_{\theta\theta} + u - \frac{1 + \varepsilon a(\theta)}{u^3}. \tag{5.1}$$

In order to prove Theorem 1.3 we need to find a solution of $S[u] = 0$. In following we consider $S[u] = 0$ as a perturbation of

$$u_{\theta\theta} + u = \frac{1}{u^3}. \tag{5.2}$$

It is known that all solutions of (5.2) are given by

$$U_{\Lambda,t}(\theta) = \left(\Lambda^2 \cos^2(\theta - t) + \Lambda^{-2} \sin^2(\theta - t)\right)^{\frac{1}{2}},$$

where $(\Lambda, t) \in (0, 1] \times S^1$. We are going to find a 2π-periodic solution u of $S[u] = 0$ having the form

$$u(\theta) = U_{\Lambda,t}(\theta) + \phi(\theta), \tag{5.3}$$

where $(\Lambda, t) \in (0, 1] \times S^1$ and $\phi(\theta)$ is relatively small.

Substituting (5.3) into the equation (5.1) we obtain

$$S[U_{\Lambda,t} + \phi] = S[U_{\Lambda,t}] + L[\phi] + N[\phi], \tag{5.4}$$

where

$$S[U_{\Lambda,t}] = \frac{d^2 U_{\Lambda,t}}{d\theta^2} + U_{\Lambda,t} - \frac{1+\varepsilon a(\theta)}{U_{\Lambda,t}^3} = -\frac{\varepsilon a(\theta)}{U_{\Lambda,t}^3},$$

$$L[\phi] = \phi_{\theta\theta} + \phi + \frac{3\phi}{U_{\Lambda,t}^4},$$

and

$$N[\phi] = -\left(\frac{1+\varepsilon a(\theta)}{(U_{\Lambda,t}+\phi)^3} - \frac{1+\varepsilon a(\theta)}{U_{\Lambda,t}^3} + \frac{3\phi}{U_{\Lambda,t}^4}\right).$$

It is easy to see that

$$\{\phi \in C^2(S^1) | L[\phi] = 0\} = span\{\frac{\partial U_{\Lambda,t}}{\partial \Lambda}, \frac{\partial U_{\Lambda,t}}{\partial t}\}.$$

One should note that the linear operator L depends on (Λ, t).

For $h \in C(S^1)$ we consider the following linear problem:

$$\begin{cases} L[\phi] = h + c_1 \frac{\partial U_{\Lambda,t}}{\partial \Lambda} + c_2 \frac{\partial U_{\Lambda,t}}{\partial t}, \quad \theta \in S^1, \\ \int_0^{2\pi} \phi(\theta) \frac{\partial U_{\Lambda,t}}{\partial \Lambda} d\theta = \int_0^{2\pi} \phi(\theta) \frac{\partial U_{\Lambda,t}}{\partial t} d\theta = 0, \end{cases} \tag{5.5}$$

where $(c_1, c_2) \in \mathbb{R}^2$. From linear Fredholm theorem we know that (5.5) is solvable if and only if (c_1, c_2) satisfies

$$\begin{cases} \int_0^{2\pi} -h \frac{\partial U_{\Lambda,t}}{\partial \Lambda} d\theta + c_1 \int_0^{2\pi} (\frac{\partial U_{\Lambda,t}}{\partial \Lambda})^2 d\theta + c_2 \int_0^{2\pi} \frac{\partial U_{\Lambda,t}}{\partial t} \frac{\partial U_{\Lambda,t}}{\partial \Lambda} d\theta = 0, \\ \int_0^{2\pi} -h \frac{\partial U_{\Lambda,t}}{\partial t} d\theta + c_1 \int_0^{2\pi} \frac{\partial U_{\Lambda,t}}{\partial t} \frac{\partial U_{\Lambda,t}}{\partial \Lambda} d\theta + c_2 \int_0^{2\pi} (\frac{\partial U_{\Lambda,t}}{\partial t})^2 d\theta = 0. \end{cases} \tag{5.6}$$

It is easy to see that (c_1, c_2) is uniquely determined by (5.6). Moreover, if (5.6) is satisfied, then the solution is unique and there is a positive constant C which depends on the lower bound of Λ only such that

$$\|\phi\|_{C(S^1)} \le C \|h\|_{C(S^1)} \tag{5.7}$$

and

$$|c_1| + |c_2| \le C \|h\|_{C(S^1)}. \tag{5.8}$$

The estimate (5.7) is a consequence of the fact that $h \to \phi$ is a bounded linear operator from $C(S^1) \to C(S^1)$, and (5.8) follows from (5.6).

Using the above discussion, now we can solve the nonlinear problem:

$$\begin{cases} L(\phi) = -S[U_{\Lambda,t}] - N(\phi) + c_1 \frac{\partial U_{\Lambda,t}}{\partial \Lambda} + c_2 \frac{\partial U_{\Lambda,t}}{\partial t}, \qquad \theta \in S^1, \\ \int_0^{2\pi} \phi(\theta) \frac{\partial U_{\Lambda,t}}{\partial \Lambda} d\theta = \int_0^{2\pi} \phi(\theta) \frac{\partial U_{\Lambda,t}}{\partial t} d\theta = 0, \end{cases} \tag{5.9}$$

for some coefficients c_1 and c_2. Namely we have

Lemma 5.1 *For $\Lambda_0 > 0$, there exist $\varepsilon_0 > 0$ and C which is independent of ε such that for any $\varepsilon < \varepsilon_0$, $\Lambda_0 \le \Lambda \le 1$ and $t \in S^1$, problem (5.9) has a unique solution $\phi = \phi_{\Lambda,t}$ satisfying*

$$\|\phi\|_{C(S^1)} \le C\varepsilon. \tag{5.10}$$

Moreover, the maps $(\Lambda, t) \to \phi_{\Lambda,t}$ and $(\Lambda, t) \to (c_1(\Lambda, t), c_2(\Lambda, t))$ are continuous.

Proof. We will use the contraction mapping principle to prove the lemma. To this end, we write the first equation of problem (5.9) in its equivalent form:

$$\phi = A\big(- S[U_{\Lambda,t}] - N[\phi]\big) := B(\phi). \tag{5.11}$$

For a positive constant $\varepsilon_1 \le \frac{\Lambda_0}{2}$, define a convex set in $C(S^1)$ by

$$Z := \left\{\phi \,\middle|\, \quad \phi \quad \text{is} \quad 2\pi-\text{periodic}, \|\phi\|_{C(S^1)} \le \varepsilon_1, \right.$$
$$\left. \int_0^{2\pi} \phi(\theta)\frac{\partial U_{\Lambda,t}}{\partial \Lambda} d\theta = \int_0^{2\pi} \phi(\theta)\frac{\partial U_{\Lambda,t}}{\partial t} d\theta = 0\right\}.$$

It follows from the mean value theorem that

$$\|N(\phi)\|_{C(S^1)} \le C(\Lambda_0)\|\frac{\varepsilon a\phi}{U^4_{\Lambda,t}} + \frac{\phi^2}{U^5_{\Lambda,t}}\|_{C(S^1)} \le C(\Lambda_0)(\varepsilon_1\varepsilon + \varepsilon_1^2), \quad \forall \quad \phi \in Z. \tag{5.12}$$

We know that $S[U_{\Lambda,t}] = -\frac{\varepsilon a}{U^3_{\Lambda,t}}$, thus for $\phi, \phi_1 \in Z$,

$$\begin{aligned} \|B(\phi)\|_{C(S^1)} &\le C(\Lambda_0)(\|S[U_{\Lambda,t}\|_{C(S^1)} + \|N(\phi)\|_{C(S^1)}) \\ &\le C(\Lambda_0)(\varepsilon + \varepsilon\varepsilon_1 + \varepsilon_1^2) \end{aligned} \tag{5.13}$$

and

$$\begin{aligned} \|B(\phi_1) - B(\phi)\|_{C(S^1)} &\le C(\Lambda_0)\|N[\phi_1] - N[\phi]\|_{C(S^1)} \\ &\le C(\Lambda_0)(\varepsilon + \|\phi_1\|_{C(S^1)} + \|\phi\|_{C(S^1)})\|\phi_1 - \phi\|_{C(S^1)} \\ &\le C(\Lambda_0)(\varepsilon + 2\varepsilon_1)\|\phi_1 - \phi\|_{C(S^1)}. \end{aligned} \tag{5.14}$$

Letting $\varepsilon_0 = \frac{\varepsilon_1}{4C(\Lambda_0)}$, $\varepsilon_1 < \frac{1}{4C(\Lambda_0)}$, then (5.13) and (5.14) imply that the operator B is a contraction mapping from Z to Z. Hence B has a unique fixed point $\phi_{\Lambda,t} \in Z$ and

$$\|\phi_{\Lambda,t}\|_{C(S^1)} = \|B(\phi)\|_{C(S^1)} \le C(\Lambda_0)\varepsilon.$$

The continuity of $\phi_{\Lambda,t}$, $(c_1(\Lambda, t), c_2(\Lambda, t))$ on parameters Λ, t also follows from the contraction mapping theorem. Hence Lemma 5.1 holds. □

The proof of Theorem 1.3 will be finished if for $\varepsilon \le \varepsilon_0$ we can find some $(\Lambda, t) \in (0, 1] \times S^1$ such that $(c_1(\Lambda, t, \varepsilon), c_2(\Lambda, t, \varepsilon)) = 0$ in problem (5.9). This will be accomplished by degree theory. In order to use the degree theory we need the asymptotic expansions of $c_1(\Lambda, t)$ and $c_2(\Lambda, t)$ as $\Lambda \to 0$.

First similar to (5.6), $c_1(\Lambda,t,\varepsilon)$ and $c_2(\Lambda,t,\varepsilon)$ satisfy

$$\begin{cases} \int_0^{2\pi}(S[U_{\Lambda,t}]+N(\phi))\frac{\partial U_{\Lambda,t}}{\partial\Lambda}d\theta + c_1\int_0^{2\pi}(\frac{\partial U_{\Lambda,t}}{\partial\Lambda})^2 d\theta + c_2\int_0^{2\pi}\frac{\partial U_{\Lambda,t}}{\partial t}\frac{\partial U_{\Lambda,t}}{\partial\Lambda}d\theta = 0, \\ \int_0^{2\pi}(S[U_{\Lambda,t}]+N(\phi))\frac{\partial U_{\Lambda,t}}{\partial t}d\theta + c_1\int_0^{2\pi}\frac{\partial U_{\Lambda,t}}{\partial t}\frac{\partial U_{\Lambda,t}}{\partial\Lambda}d\theta + c_2\int_0^{2\pi}(\frac{\partial U_{\Lambda,t}}{\partial t})^2 d\theta = 0. \end{cases} \tag{5.15}$$

Noting that

$$\int_0^{2\pi}\frac{\partial U_{\Lambda,t}}{\partial\Lambda}\cdot\frac{\partial U_{\Lambda,t}}{\partial t}d\theta = \int_0^{2\pi}\frac{\Lambda\cos^2\theta-\Lambda^{-3}\sin^2\theta}{\Lambda^2\cos^2\theta+\Lambda^{-2}\sin^2\theta}(\Lambda^{-2}-\Lambda^2)\sin\theta\cos\theta d\theta = 0,$$

thus (5.15) leads to

$$\begin{cases} c_1(\lambda,t,\varepsilon) = -\frac{\int_0^{2\pi}(S[U_{\Lambda,t}]+N(\phi))\frac{\partial U_{\Lambda,t}}{\partial\Lambda}d\theta}{\int_0^{2\pi}(\frac{\partial U_{\Lambda,t}}{\partial\Lambda})^2}, \\ c_2(\lambda,t,\varepsilon) = -\frac{\int_0^{2\pi}(S[U_{\Lambda,t}]+N(\phi))\frac{\partial U_{\Lambda,t}}{\partial t}d\theta}{\int_0^{2\pi}(\frac{\partial U_{\Lambda,t}}{\partial t})^2}. \end{cases} \tag{5.16}$$

Let

$$\begin{cases} \overline{c}_1(\lambda,t,\varepsilon) = -\frac{\int_0^{2\pi}S[U_{\Lambda,t}]\frac{\partial U_{\Lambda,t}}{\partial\Lambda}d\theta}{\int_0^{2\pi}(\frac{\partial U_{\Lambda,t}}{\partial\Lambda})^2}d\theta, \\ \overline{c}_2(\lambda,t,\varepsilon) = -\frac{\int_0^{2\pi}S[U_{\Lambda,t}]\frac{\partial U_{\Lambda,t}}{\partial t}d\theta}{\int_0^{2\pi}(\frac{\partial U_{\Lambda,t}}{\partial t})^2}. \end{cases} \tag{5.17}$$

Lemma 5.2 *For $\Lambda\to 0$ we have*

$$\overline{c}_1(\Lambda,t,\varepsilon) = \varepsilon(-\frac{\overline{B}_2(t)}{2\pi}\Lambda^5 + o(\Lambda^5)) \tag{5.18}$$

and

$$\overline{c}_2(\Lambda,t,\varepsilon) = \varepsilon(\frac{\overline{A}_2(t)}{2}\Lambda^2 + o(\Lambda^2)). \tag{5.19}$$

Proof: From the definition of $U_{\Lambda,t}$ we have

$$\begin{aligned} \int_0^{2\pi}(\frac{\partial U_{\Lambda,t}}{\partial\Lambda})^2 d\theta &= \int_0^{2\pi}\frac{(\Lambda\cos^2(\theta-t)-\Lambda^{-3}\sin^2(\theta-t))^2}{\Lambda^2\cos^2(\theta-t)+\Lambda^{-2}\sin^2(\theta-t)}d\theta \\ &= \Lambda^{-2}\int_0^{2\pi}(\Lambda^2\cos^2\theta+\Lambda^{-2}\sin^2\theta)d\theta \\ &+ 4\Lambda^{-2}\int_0^{2\pi}\frac{\cos^4\theta-\cos^2\theta}{\Lambda^2\cos^2\theta+\Lambda^{-2}\sin^2\theta}d\theta = \pi\Lambda^{-4}+O(1) \end{aligned} \tag{5.20}$$

and

$$\frac{-1}{\varepsilon}\int_0^{2\pi} S[U_{\Lambda,t}]\frac{\partial U_{\Lambda,t}}{\partial\Lambda}d\theta = \int_0^{2\pi} a(\theta+t)\frac{\Lambda\cos^2\theta - \Lambda^{-3}\sin^2\theta}{(\Lambda^2\cos^2\theta+\Lambda^{-2}\sin^2\theta)^2}d\theta. \tag{5.21}$$

Let

$$\theta = \psi_\Lambda(y) = \int_0^y \frac{d\tau}{\Lambda^{-2}\cos^2\tau+\Lambda^2\sin^2\tau},$$

then $\psi : S^1 \to S^1$ is a diffeomorphism and

$$\frac{\cos^2\theta}{(\Lambda^2\cos^2\theta+\Lambda^{-2}\sin^2\theta)^2}d\theta = \frac{1}{\Lambda^2}\cos^2 y dy, \tag{5.22}$$

$$\frac{\sin^2\theta}{(\Lambda^2\cos^2\theta+\Lambda^{-2}\sin^2\theta)^2}d\theta = \Lambda^2\sin^2 y dy \tag{5.23}$$

and

$$\frac{\sin 2\theta}{\Lambda^2\cos^2\theta+\Lambda^{-2}\sin^2\theta}d\theta = \frac{\sin 2y}{\Lambda^{-2}\cos^2 y+\Lambda^2\sin^2 y}dy. \tag{5.24}$$

Inserting (5.22) and (5.23) into (5.21), as $\Lambda \to 0$ we get

$$\begin{aligned}
\frac{-1}{\varepsilon}\int_0^{2\pi} S[U_{\Lambda,t}]\frac{\partial U_{\Lambda,t}}{\partial\Lambda}d\theta &= \frac{1}{\Lambda}\int_0^{2\pi} a(\psi_\Lambda(y)+t)\cos 2y dy \\
&= -\frac{1}{2\Lambda}\int_0^{2\pi} a'(\psi_\Lambda(y)+t)\psi_\Lambda'(y)\sin 2y dy \\
&= -\frac{\Lambda}{2}\int_0^{2\pi} a'(\theta+t)\frac{\sin 2\theta}{\Lambda^4\cos^2\theta+\sin^2\theta}d\theta \\
&= -\frac{\Lambda}{2}\int_0^{\pi} [a'(\theta+t)-a'(t)]\frac{\sin 2\theta}{\Lambda^4\cos^2\theta+\sin^2\theta}d\theta \\
&- \frac{\Lambda}{2}\int_\pi^{2\pi} [a'(\theta+t)-a'(t)]\frac{\sin 2\theta}{\Lambda^4\cos^2\theta+\sin^2\theta}d\theta \\
&= -\frac{\Lambda}{2}\int_0^{\pi} \frac{[a'(\theta+t)-a'(t)]}{\sin\theta}\frac{\sin\theta\sin 2\theta}{\sin^2\theta+\Lambda^4\cos^2\theta}d\theta \\
&- \frac{\Lambda}{2}\int_\pi^{2\pi} \frac{[a'(\theta+t)-a'(t+\pi)]}{\sin\theta}\frac{\sin\theta\sin 2\theta}{\sin^2\theta+\Lambda^4\cos^2\theta}d\theta \\
&= -\frac{\Lambda}{2}\overline{B}_2(t)+o(\Lambda)
\end{aligned} \tag{5.25}$$

by the dominated convergence theorem. Hence it follows from (5.17), (5.20) and (5.25) that

$$\overline{c}_1(\Lambda,t,\varepsilon) = \varepsilon(-\frac{\overline{B}_2(t)}{2\pi}\Lambda^5 + o(\Lambda^5)). \tag{5.26}$$

Similarly,

$$
\begin{aligned}
\int_0^{2\pi} (\frac{\partial U_{\Lambda,t}}{\partial t})^2 d\theta &= \int_0^{2\pi} \frac{(\Lambda^{-2}-\Lambda^2)^2\cos^2(\theta-t)\sin^2(\theta-t)}{\Lambda^2\cos^2(\theta-t)+\Lambda^{-2}\sin^2(\theta-t)}d\theta \\
&= \int_0^{2\pi} \frac{(\Lambda^{-2}-\Lambda^2)^2\cos^2\theta\sin^2\theta}{\Lambda^2\cos^2\theta+\Lambda^{-2}\sin^2\theta}d\theta \\
&= \Lambda^2(\Lambda^2-\Lambda^{-2})^2\int_0^{2\pi}(\cos^2\theta - \frac{\Lambda^2\cos^4\theta}{\Lambda^2\cos^2\theta+\Lambda^{-2}\sin^2\theta})d\theta \\
&= \Lambda^2(\Lambda^2-\Lambda^{-2})^2\int_0^{2\pi}\cos^2\theta d\theta + O(1) \\
&= \pi\Lambda^{-2} + O(1)
\end{aligned}
\tag{5.27}
$$

and

$$
\begin{aligned}
\frac{-1}{\varepsilon}\int_0^{2\pi} S[U_{\Lambda,t}]\frac{\partial U_{\Lambda,t}}{\partial t}d\theta &= \int_0^{2\pi} a(\theta+t)\frac{(\Lambda^{-2}-\Lambda^2)\sin\theta\cos\theta}{(\Lambda^2\cos^2\theta+\Lambda^{-2}\sin^2\theta)^2}d\theta \\
&= \frac{\Lambda^{-2}-\Lambda^2}{2}\int_0^{2\pi} a(\psi_\Lambda(y)+t)\sin 2y dy \\
&= \frac{\Lambda^{-2}-\Lambda^2}{4}\int_0^{2\pi} \frac{da(\psi_\Lambda(y)+t)}{dy}(\cos 2y+1)dy \\
&= \frac{\Lambda^{-2}-\Lambda^2}{4}\int_0^{2\pi} a'(\psi_\Lambda(y)+t)\frac{\cos 2y+1}{\Lambda^{-2}\cos^2 y+\Lambda^2\sin^2 y}dy \\
&= \frac{1-\Lambda^4}{2}\Big[\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} a'(\psi_\Lambda(y)+t)\frac{\cos^2 y}{\cos^2 y+\Lambda^4\sin^2 y}dy \\
&+ \int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} a'(\psi_\Lambda(y)+t)\frac{\cos^2 y}{\cos^2 y+\Lambda^4\sin^2 y}dy\Big] \\
&= \frac{a'(t)+a'(\pi+t)}{2}\pi + o(1) \\
&= \frac{\overline{A}_2(t)}{2}\pi + o(1)
\end{aligned}
\tag{5.28}
$$

provided by $\psi_\Lambda(y) \to 0$ for $y \in (-\frac{\pi}{2}, \frac{\pi}{2})$ and $\psi_\Lambda(y) \to \pi$ for $y \in (\frac{\pi}{2}, \frac{3\pi}{2})$. So from (5.17), (5.27) and (5.28) we have

$$
\overline{c}_2(\Lambda, t, \varepsilon) = \varepsilon(\frac{\overline{A}_2(t)}{2}\Lambda^2 + o(\Lambda^2)). \tag{5.29}
$$

This completes the proof of Lemma 5.2. □

Now we fix Λ_0 such that

$$
|(\overline{c}_1(\Lambda_0, t, \varepsilon), \overline{c}_2(\Lambda_0, t, \varepsilon)| \geq \frac{\varepsilon}{4\pi}[\overline{A}_2{}^2(t)\Lambda_0^4 + \overline{B}_2{}^2(t)\Lambda_0^{10}]^{\frac{1}{2}}, \quad \forall t \in S^1.
$$

Lemma 5.3 *For $\varepsilon \to 0$ we have*

$$c_1(\Lambda_0, t, \varepsilon) = \overline{c}_1(\Lambda_0, t, \varepsilon) + O(\varepsilon^2) \tag{5.30}$$

and

$$c_2(\Lambda_0, t, \varepsilon) = \overline{c}_1(\Lambda_0, t, \varepsilon) + O(\varepsilon^2). \tag{5.31}$$

Proof: From (5.12) and the definitions of $c_1(\Lambda_0, t, \varepsilon)$ we have

$$\begin{aligned} |c_1(\Lambda_0, t, \varepsilon) - \overline{c}_1(\Lambda_0, t, \varepsilon)| &= \left| \frac{\int_0^{2\pi} N(\phi) \frac{\partial U_{\Lambda,t}}{\partial \Lambda} d\theta}{\int_0^{2\pi} (\frac{\partial U_{\Lambda,t}}{\partial \Lambda})^2} \right| \\ &\le C\varepsilon^2 \end{aligned}$$

for some constant C depending on Λ_0. The proof of (5.31) is the same. $\square$

Set $\Lambda^2 = \lambda^2 + 1 - \sqrt{\lambda^2 + 2}\lambda$ with $\lambda \geq 0$ and

$$(c_1(\lambda, t), c_2(\lambda, t)) = (c_1(\Lambda, t), c_2(\Lambda, t)), \quad (\overline{c}_1(\lambda, t), \overline{c}_2(\lambda, t)) = (\overline{c}_1(\Lambda, t), \overline{c}_2(\Lambda, t)).$$

We see that $\lambda \to \infty$ as $\Lambda \to 0$ and $\lambda = 0$ if and only if $\Lambda = 1$. For $\lambda = 0$ or $\Lambda = 1$, it is easy to see that $c_i(\lambda, t)$ and $\overline{c}_i(\lambda, t)$ are independent of t, $i = 1, 2$. Therefore we have continuous maps from $D(\lambda_0) = \{(X, Y) \in \mathbb{R}^2 | X^2 + Y^2 \leq \lambda_0^2\}$ to $\mathbb{R}^2$ given by

$$F(X, Y) = (c_1(\lambda, t), c_2(\lambda, t)), \quad \overline{F}(X, Y) = (\overline{c}_1(\lambda, t), \overline{c}_2(\lambda, t)),$$

where $(X, Y) = (\lambda \cos t, \lambda \sin t)$ and $\Lambda_0^2 = \lambda_0^2 + 1 - \sqrt{\lambda_0^2 + 2}\lambda_0$.

Proof of Theorem 1.3: Let

$$G_\varepsilon(X, Y) = (-\varepsilon \frac{\overline{B}_2(t)}{2\pi} \Lambda^5, \varepsilon \frac{\overline{A}_2(t)}{2} \Lambda^2) : D(\lambda_0) \to \mathbb{R}^2.$$

It follows from the homotopy invariant of the degree that, for ε small,

$$\begin{aligned} deg(F, D(\lambda_0), 0) &= deg(\overline{F}, D(\lambda_0), 0) \\ &= deg(G_\varepsilon, D(\lambda_0), 0) \\ &= deg(G_2, \mathbb{R}/2\pi\mathbb{Z}, 0) \neq 0. \end{aligned}$$

Therefore, there exists $(\lambda, t) \in D(\lambda_0)$ such that $(c_1(\lambda, t), c_2(\lambda, t)) = (0, 0)$. This completes the proof of Theorem 1.3. $\square$

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