2π-periodic self-similar solutions for the anisotropic affine curve shortening problem II

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Abstract

The existence of 2π-periodic positive solutions of the equation

$$u'' + u = \frac{a(x)}{u^3}$$

is studied, where a is a positive smooth 2π-periodic function. Under some non-degenerate conditions on a, the existence of solution to the equation is established.

1 Introduction and statement of the results

This paper is a continuation of [21] and studies the existence of 2π-periodic positive solutions of the equation

$$u'' + u = \frac{a(x)}{u^3} \tag{1.1}$$

for positive, smooth and 2π-periodic a. Equation (1.1) arises from self-similar solutions of the following generalized curve shortening problem

$$\frac{\partial \gamma}{\partial t} = \Phi(\theta)|k|^{\sigma-1}kN, \qquad \sigma > 0, \quad x \in \mathbb{S}^1 = \mathbb{R}/2\pi\mathbb{Z}, \tag{1.2}$$

where $\gamma(\cdot, t)$ is a planar curve, $k(\cdot, t)$ is its curvature with respect to the unit normal N, and Φ is a positive function depending on the normal angle x of the curve. This problem has been extensively studied in the last three decades, see [1-7, 9, 10, 12,

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13, 15-19, 23]. Assuming that $\gamma(\cdot, t)$ is convex, and $w(x,t)$ is its support function, then using the normal angle x to parameterize γ, equation (1.2) is equivalent to

$$\frac{\partial w}{\partial t} = \frac{-\Phi(x)}{(w'' + w)^{\sigma}}, \qquad x \in \mathbb{S}^1. \tag{1.3}$$

Self-similar solutions of (1.3) are solutions having the form $w(x,t) = \xi(t)u(x)$, which are important in understanding the long time behaviors and the structure of singularities of (1.2). It is rather easy to see that $\xi(t)u(x)$ is a self-similar solution if and only if u satisfies

$$u'' + u = \frac{a(x)}{u^{p+1}}, \quad x \in \mathbb{S}^1 \tag{1.4}$$

with $a(x) = \Phi^{\frac{1}{\sigma}}(x)$, $p+1 = \frac{1}{\sigma}$ and $|\xi(t)|^{\sigma-1}\xi(t)\xi'(t) = -C$, where C is a positive constant. Equation (1.4) also appears in image processing [23], 2-dimensional L^p-Minkowski problem [8, 20] and other problems [19].

Equation (1.4) with $p \neq 2$ has been studied by many authors. When $a \equiv 1$, all solutions of (1.4) can be classified, see [1, 5]; and see [12, 13] for some results when a is 2π-periodic. In particular, Matano-Wei [22] proved that (1.4) is solvable if $0 \le p < 7$ and a is 2π-periodic and positive. An equation closely related to (1.4) is

$$u'' + \lambda u = \frac{a(x)}{u^{\nu}}. \tag{1.5}$$

Using the Poincaré-Birkhoff fixed point theorem, del Pino, Manásevich and Montero in [11] proved that that (1.5) possesses a $2\pi-$periodic solution if $\lambda \neq \frac{(n+1)^2}{4}$, $n = 0, 1, \cdots$. An important step in establishing the existence of solutions of (1.4) and (1.5) is to get an a priori estimate for all solutions.

The case $\sigma = \frac{1}{3}$ in (1.2) is called the affine curve shortening problem. Thus $p = 2$ and equation (1.4) becomes

$$u'' + u = \frac{a(x)}{u^3} \quad x \in \mathbb{S}^1, \tag{1.6}$$

and a solution of equation (1.6) is a self-similar solution of the anisotropic affine curve shortening problem. All results mentioned above do not cover the affine case. Indeed, the situation for the affine case is quite different. It is known that there are some obstructions for the existence and one can't get a priori estimates for the solutions of (1.6) without additional assumptions on a due to the invariance of the problem. To see this, let us consider its simplest form

$$u'' + u = \frac{1}{u^3}, \quad x \in \mathbb{S}^1. \tag{1.7}$$

Equation (1.7) is invariant under an action of the special linear group $SL(2, \mathbb{R})$, and all solutions of (1.7) are given by a 2-parameter family of functions

$$u_{\varepsilon,\theta}(x) = \left(\varepsilon^2 \cos^2(x-\theta) + \varepsilon^{-2} \sin^2(x-\theta)\right)^{\frac{1}{2}}, \tag{1.8}$$

for $(\varepsilon, \theta) \in (0,1] \times [0,\pi)$. Thus the set of 2π-periodic solutions of (1.7) is not bounded. For more details on the group invariance of (1.7), see [2, 21].

To state the results for equation (1.6), we need two functions. Let a be a positive 2π-periodic and C^2-function, and let

$$A_2(\theta) = \frac{a'(\theta)}{\sqrt{a(\theta)}} + \frac{a'(\theta+\pi)}{\sqrt{a(\theta+\pi)}} \tag{1.9}$$

and

$$B_2(\theta) = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{\big(a'(\theta+t) + a'(\theta+t+\pi) - a'(\theta) - a'(\theta+\pi)\big)\sin 2t}{\sin^2 t} dt. \tag{1.10}$$

Note that by definition, $A_2(\theta)$ and $B_2(\theta)$ are $\pi-$periodic, so they are defined on $\mathbb{S}^1_1 = \mathbb{R}/\pi\mathbb{Z}$. A function a is called B-nondegenerate if $(A_2(\theta), B_2(\theta)) \neq (0,0)$ for $\theta \in \mathbb{S}^1_1$.

In [2], Ai, Chou and Wei proved that if a is a positive, B-nondegenerate and C^2-function of period π, then one can get a priori estimates for all π-periodic solutions of (1.6). That is, there exists a constant C depending on a only such that

$$\frac{1}{C} \leq u \leq C \tag{1.11}$$

for any π-periodic solution u of (1.6). This result was generalized in [21] to the case that a is 2π-periodic. In establishing a priori estimates, a major problem is to study possible blow-ups. The difference between π-periodic and 2π-periodic cases is, for a blow-up sequence $u_{\varepsilon,t}$, only single blow-up can occur when is a is $\pi-$periodic, while a is 2π-periodic, there are 2 possible blow-ups. We have to analyze the interaction between different blow-ups.

Let a is a positive and B-nondegenerate. Then the map $G(\theta) = (A_2(\theta), B_2(\theta))$ satisfies $G(\theta) \neq 0$ for $\theta \in \mathbb{S}^1_1$, and the degree $deg(G, \mathbb{S}^1_1)$ is well-defined, where $deg(G, \mathbb{S}^1_1) = deg(\overline{G}, D; 0)$, $D = \{(x,y)|x^2+y^2 \leq 1\} = \{(r\cos 2\theta, r\sin 2\theta)|0 \leq r \leq 1, \theta \in [0,\pi]\}$, $\partial D \approx \mathbb{S}^1_1 = \{(\cos 2\theta, \sin 2\theta)|\theta \in [0,\pi]\}$, $\overline{G} : D \to \mathbb{R}^2$ is a continuous extension of $G : \mathbb{S}^1_1 \to \mathbb{R}^2$, and $deg(\overline{G}, D; 0)$ is the Brouwer degree. It is well known that $deg(\overline{G}, D; 0)$ is determined by the map G.

The existence results in [2] and [21] now can be stated as follows.

Theorem 1.1 *Let a be a positive, C^2 and 2π-periodic and B-nondegenerate function. Then*

(1) equation (1.6) has a $\pi-$periodic solution if a is π-periodic and $deg(G, \mathbb{S}^1_1) \neq -1$;

(2) equation (1.6) has a $2\pi-$periodic solution if a is 2π-periodic, $\|1-a\|_{C^2} << 1$ and $deg(G, \mathbb{S}^1_1) \neq -1$.

The part (1) in the above theorem was proved in [2] and part (2) was proved in [21]. The statement in [21] is slightly different, where the degree of G is computed as a 2π-periodic map that equals to $2deg(G, \mathbb{S}^1_1)$.

The aim of this paper is to remove the assumption $\|1-a\|_{C^2} << 1$ in the case that a is $2\pi-$periodic. Namely, we will prove

Theorem 1.2 *Let a be a positive, C^2 and 2π-periodic and B-nondegenerate function. Then equation (1.6) has a $2\pi-$periodic solution provided that $deg(G, \mathbb{S}^1_1) \neq -1$.*

Now we explain briefly the reason that we made the assumption $\|1-a\|_{C^2} << 1$ in [21]. For $\pi-$peridic case, we can fix $\varepsilon << 1$ and consider the homotopy of $a_s(x) = (1-s)(1+\varepsilon a(x)) + sa(x)$. Then for $s \in [0,1]$, the function a_s is $B-$nondegenerate if a is, and thus one can get uniform a priori estimate for all π-periodic solutions of

$$u'' + u = \frac{a_s(x)}{u^3}. \tag{1.12}$$

By Lyapunov-Schmit reduction, one can solve (1.12) for small s if $deg(G, \mathbb{S}^1_1) \neq -1$, and the solution can be continued to $s = 1$ thanks to the priori estimate, see [2]. Lyapunov-Schmit reduction can be applied to 2π-periodic case as well. However, for 2π-periodic a, we do not know how to construct a homotopy like a_s as in the π-periodic case. The $B-$nondegenerate condition on a is a nonlinear restriction. Thus we imposed the assumption $\|1-a\|_{C^2} << 1$.

In this paper, we take a different and more direct approach to prove Theorem 1.2. The key point and the main difference between this paper and [21] is that we do not consider periodic solutions of (1.6), but instead the initial value problem of (1.6). Let $u(u_0, u_0'; x)$ be the solution of (1.6) with the initial boundary condition

$$u(u_0, u_0'; 0) = u_0, \quad u'(u_0, u_0'; 0) = u_0'.$$

An important observation is that the analysis of periodic solutions in [21] is valid for solutions of the initial value problem as well, which enables us to use the degree argument to find fixed point (u_0, u_0') of the Poincaré map:

$$u(u_0, u_0'; 2\pi) = u_0, \quad u'(u_0, u_0'; 2\pi) = u_0'. \tag{1.13}$$

By the periodicity of a, (1.13) implies that $u(u_0, u_0'; x)$ is a 2π-periodic solution of (1.6). But we do not use degree theory to find zeros of the map

$$(u_0, u_0') \to (u(u_0, u_0'; 2\pi) - u_0), u'(u_0, u_0'; 2\pi) - u_0'))$$

directly, instead we introduce another map $(u_0, u_0') \to \mathcal{F}(u_0, u_0')$ defined by

$$\mathcal{F}(u_0, u_0') = (\int_0^{2\pi} \frac{a'(x)}{u^2(u_0, u_0'; x)} \cos 2x dx, \int_0^{2\pi} \frac{a'(x)}{u^2(u_0, u_0'; x)} \cos 2x dx).$$

A key fact that will be used is that $\mathcal{F}(u_0, u_0') = 0$ implies (1.13). The advantage to consider the map $\mathcal{F}(u_0, u_0')$ is that one can get the asymptotic expansion as $\lambda \to 0$ via the blow-up analysis. Having the asymptotic expansion, the existence of (u_0, u_0') satisfying $\mathcal{F}(u_0, u_0') = 0$ can be accomplished by degree theory.

The paper is organized as follows. In Section 2, it is shown that the existence of 2π-periodic solutions of (1.6) is equivalent to that $\mathcal{F}$ has zeros. We analyze the critical points of the solution $u(\lambda, \theta; x)$ in detail in Section 3. An asymptotical expansion of $\mathcal{F}$ is given in Section 4 following the blow-up analysis of the solutions $u(\lambda, \theta; x)$, and the main theorem is proved in Section 5 by making use of asymptotic behavior of the map $\mathcal{F}$ and the degree theory.

2 A Map

In this section we define a map $\mathcal{F} : (0, 1] \times \mathbb{R}/\pi\mathbb{Z} \to \mathbb{R}^2$ such that each zero of the map corresponds to a 2π-periodic solution of

$$u'' + u = \frac{a(x)}{u^3}. \tag{2.1}$$

Let $u(\lambda, \theta; x)$ be the solution of (2.1) satisfying the initial value conditions

$$u(\lambda, \theta; 0) = \sqrt{\lambda^2 \cos^2\theta + \lambda^{-2} \sin^2\theta}; \quad u'(\lambda, \theta; 0) = \frac{(\lambda^2 - \lambda^{-2}) \sin\theta \cos\theta}{\sqrt{\lambda^2 \cos^2\theta + \lambda^{-2} \sin^2\theta}}.$$

It is easy to see that any initial value $u_0 > 0, u_0' \in \mathbb{R}$ can be written as the above form for $(\lambda, \theta) \in (0, 1] \times \mathbb{R}/\pi\mathbb{Z}$. Hence all solutions of (2.1) are given by $u(\lambda, \theta; x)$. By the periodicity of a, we know that $u(\lambda, \theta; x)$ is 2π-periodic solution of (2.1) if and only if

$$u(\lambda, \theta; 2\pi) = u(\lambda, \theta; 0), \quad u'(\lambda, \theta; 2\pi) = u'(\lambda, \theta; 0). \tag{2.2}$$

We start with the following simple lemma.

Lemma 2.1 *For any (λ, θ), the solution $u(\lambda, \theta; x)$ exists for $x \in \mathbb{R}$ and there is a constant C independent of λ such that*

$$\frac{1}{C}(\lambda^2 + \lambda^{-2}) \le u^2(\lambda, \theta; x) + u'^2(\lambda, \theta; x) + \frac{a(x)}{u^2(\lambda, \theta; x)} \le C(\lambda^2 + \lambda^{-2}), \quad x \in [0, 2\pi]. \tag{2.3}$$

Proof. Let u be a solution of equation (2.1). We define $F(x) = u'^2 + u^2 + \dfrac{a(x)}{u^2(x)}$. Then by (2.1), one can easily get that

$$F'(x) = \frac{a'(x)}{u^2},$$

which implies that

$$|F'(x)| \le C|F(x)| \tag{2.4}$$

since $a(x)$ is smooth and positive. Thus

$$|(\log F)'| \le C,$$

which leads to, for all $R > 0$, there exists a constant $C(R) > 0$ such that

$$F(x) \leq C(R), \quad x \in [-R, R].$$

Now the global existence of u follows immediately.

From (2.4) and Gronwall inequality we have

$$\frac{1}{C} F(0) \leq F(x) \leq C F(0), \quad x \in [0, 2\pi]. \tag{2.5}$$

The inequalities in (2.3) follows from (2.5),

$$\frac{1}{C}(u'^2(x) + u^2(x) + \frac{1}{u^2(x)}) \leq F(x) \leq C(u'^2(x) + u^2(x) + \frac{1}{u^2(x)})$$

and

$$u'^2(0) + u^2(0) + \frac{1}{u^2(0)} = \lambda^2 + \lambda^{-2}.$$

This completes the proof. □

Set

$$A(\lambda, \theta) = \int_0^{2\pi} \frac{a'(x)}{u^2(\lambda, \theta; x)} \cos 2x dx$$
$$B(\lambda, \theta) = \int_0^{2\pi} \frac{a'(x)}{u^2(\lambda, \theta; x)} \sin 2x dx$$

and

$$\mathcal{F} : (0, 1] \times \mathbb{R}/\pi\mathbb{Z} \to \mathbb{R}^2 : \mathcal{F}(\lambda, \theta) = (A(\lambda, \theta), B(\lambda, \theta)). \tag{2.6}$$

We know that the map $\mathcal{F}$ is well defined for all (λ, θ) by Lemma 2.1. An important feature of the map $\mathcal{F}$ is that when $\lambda = 1$, the solution $u(\lambda, \theta; x)$ is independent of θ, hence so is the map $\mathcal{F}(\lambda, \theta)$. This is important for later purpose and the reason that we write the initial value in terms of λ and θ. But we will see this also causes some difficulties in choosing the blow-up point when we perform blow-up analysis at the same time.

The following result is the starting point of our approach later, which relates the 2π-periodic solutions of (2.1) to the zeros of the map $\mathcal{F}(\lambda, \theta)$.

Proposition 2.2 *Let a be a positive, C^2 and 2π-periodic function and $\mathcal{F}$ be given by (2.6). Then $u(\lambda, \theta; x)$ is a 2π-periodic solution of (2.1) if and only if $\mathcal{F}(\lambda, \theta) = 0$.*

Proof. For any solution u of (2.1), we have

$$(\frac{u^2}{2})''' + 4(\frac{u^2}{2})' = \frac{a'(x)}{u^2}. \tag{2.7}$$

Multiplying by $\cos 2x$, $\sin 2x$ and integration over $[0, 2\pi]$, we get $\mathcal{F}(\lambda, \theta) = 0$ if $u(\lambda, \theta; x)$ is a 2π-periodic solution of (2.1).

Conversely, let $\mathcal{F}(\lambda,\theta)=0$. It follows from (2.7) and integration by parts that

$$\begin{aligned} A(\lambda,\theta) &= \int_0^{2\pi} [(\frac{u^2}{2})''' + 4(\frac{u^2}{2})'] \cos 2x dx \\ &= (\frac{u^2}{2})'' \cos 2x|_0^{2\pi} \\ &= (u'^2 + u''u)|_0^{2\pi} \\ &= (u'^2 + \frac{a(x)}{u^2} - u^2)|_0^{2\pi}. \end{aligned} \tag{2.8}$$

Similarly, we have

$$\begin{aligned} B(\lambda,\theta) &= \int_0^{2\pi} [(\frac{u^2}{2})''' + 4(\frac{u^2}{2})'] \sin 2x dx \\ &= -2(\frac{u^2}{2})' \cos 2x|_0^{2\pi} \\ &= -2u'u|_0^{2\pi}. \end{aligned} \tag{2.9}$$

Hence $\mathcal{F}(\lambda,\theta)=0$ and (2.8), (2.9) imply

$$\begin{aligned} u'^2(2\pi) - u'^2(0) = u^2(2\pi) - u^2(0) + \frac{a(0)}{u^2(0)u^2(2\pi)}(u^2(2\pi) - u^2(0)), \\ u'(2\pi)u(2\pi) = u'(0)u(0). \end{aligned} \tag{2.10}$$

Let us assume $u'(0) \neq 0$ and $c = \frac{u'(2\pi)}{u'(0)}$. Then the second equality in (2.10) gives

$$\frac{u'(2\pi)}{u'(0)} = \frac{u(0)}{u(2\pi)} = c > 0. \tag{2.11}$$

Substituting (2.11) into the first equality of (2.10) we get

$$(c^2-1)u'^2(0) = (1-c^2)(1+\frac{a(0)}{c^2u^4(2\pi)})u^2(2\pi). \tag{2.12}$$

Therefore, $c=1$ and

$$u'(2\pi) = u'(0), \quad u(2\pi) = u(0). \tag{2.13}$$

The case $u'(0)=0$ can be treated similarly. Thus (2.13) always holds true, that is, $u(\lambda,\theta;x)$ is a 2π-periodic solution of (2.1). □

Remark 2.3 *Let*

$$C(\lambda,\theta) = \int_0^{2\pi} \frac{a'(x)}{u^2(\lambda,\theta;x)} dx.$$

Then it follows from (2.7) that $u(\lambda,\theta;x)$ is a 2π-periodic solution of (2.1) also implies $C(\lambda,\theta)=0$. But this equation is not independent of $\mathcal{F}(\lambda,\theta)=0$.

In view of Proposition 2.2, in order to find 2π-periodic solutions of (2.1), it suffices to find a solution of

$$\mathcal{F}(\lambda,\theta)=0. \tag{2.14}$$

This will be accomplished by a degree argument. To this end we need an asymptotic expansion of $\mathcal{F}(\lambda,\theta)$ as $\lambda\to 0$, which follows from blow-up analysis of $u(\lambda,\theta;x)$ as $\lambda\to 0$.

3 Analysis of Critical Points of $u(\lambda,\theta;x)$

The computation of the degree of $\mathcal{F}(\lambda,\theta)$ relies on the asymptotic expansion, which is based on the blow-up analysis of $u(\lambda,\theta;y)$ as $\lambda\to 0$. Since the blow-up is made at a critical point of $u(\lambda,\theta;y)$, we first give some analysis of the critical points of $u(\lambda,\theta;y)$ in this section. This part is necessary and important in our later arguments, which differs from [21], where we only consider periodic solutions, and the existence of critical points is trivial.

From now on we always assume that a is a positive, C^2 and 2π-periodic function. And unless otherwise stated, the letter C will always denote various generic constants which are independent of k. We denote $A\sim B$ or $A=O(B)$ if there exist two positive uniform constants C_1 and C_2 such that $C_1A\le B\le C_2A$, and $o(1)$ means a quantity that goes to zero as $\lambda\to 0$.

Lemma 3.1 *Let u be a monotonic solution of*

$$u''+u=\frac{a(x+\theta)}{u^3},\quad x\in[0,T]. \tag{3.1}$$

Then for $\delta>0$, there is an $\varepsilon_0(\delta)$ independent of θ such that if $u(0)\le\varepsilon_0$ or $u(T)\le\varepsilon_0$, we have

$$\gamma=|\{x\in[0,T]|u(x)\le\varepsilon_0\}|\le\frac{\delta}{4}.$$

Proof: Let u be monotonic increasing. We fix a constant $\varepsilon_0\le\frac{1}{8}$ such that

$$\frac{a(x)}{u^3}-u\ge\frac{8}{\delta^2},\quad 0<u\le\varepsilon_0, x\in[0,2\pi]. \tag{3.2}$$

Then $I=\{x\in[0,T]|u(x)\le\varepsilon_0\}=[0,\eta]$ for some $\eta\ge 0$. By the mean value theorem, we can find $\xi\in(\frac{\eta}{2},\eta)$ such that

$$u(\eta)-u(\frac{\eta}{2})=u'(\xi)\frac{\eta}{2}.$$

Therefore,

$$0\le u'(\xi)\le\frac{2}{\eta}\varepsilon_0\le\frac{1}{4\eta}. \tag{3.3}$$

Applying the mean value theorem to u', using (3.1)-(3.3) we get

$$u'(\xi) - u'(0) \geq \frac{8}{\delta^2}\xi \geq \frac{4}{\delta^2}\eta. \tag{3.4}$$

Hence it follows from (3.3) and (3.4) that

$$\frac{4}{\delta^2}\eta \leq \frac{1}{4\eta},$$

that is, $\eta \leq \frac{\delta}{4}$, thanks to the fact $u'(0) \geq 0$. The proof is complete. □

With the aid of Lemma 3.1 now we have

Proposition 3.2 *Let u be a monotonic solution of (3.1) such that*

$$u^2(x) + u'^2(x) + \frac{a(x)}{u^2(x)} \sim \lambda^2 + \lambda^{-2}, \quad x \in [0, T]. \tag{3.5}$$

Then for $\delta > 0$, there is a λ_0 independent of θ such that for all $\lambda \leq \lambda_0$, we have $T \leq \frac{\pi}{2} + \delta$. In particular, in any interval having length more that $\frac{\pi}{2} + \delta$, the function $u(x)$ has at least one critical point.

Proof: As in Lemma 3.1, let us assume that u is monotone increasing. Let δ, ε_0 be the constants given by in Lemma 3.1 and $T_1 \in [0, T]$ such that $u(T_1) = \varepsilon_0$. Then by Lemma 3.1 we have $T_1 \leq \frac{\delta}{4}$. Now we show $T - T_1 \leq \frac{\pi}{2} + \frac{\delta}{2}$ if λ is small.

We argue by contradiction. Suppose there is a sequence $\lambda_n \to 0$, θ_n, monotone increasing functions u_n defined on a subinterval I_n of $[0, T]$ having length more than $\frac{\pi}{2} + \frac{\delta}{2}$ satisfies (3.5) and

$$u_n'' + u_n = \frac{a(x + \theta_n)}{u_n^3}. \tag{3.6}$$

After translation we may assume $I_n = [0, \frac{\pi}{2} + \frac{\delta}{2}]$. First we note

$$u_n(x) \geq \varepsilon_0, \quad x \in [0, \frac{\pi}{2} + \frac{\delta}{2}]$$

due to the fact that u_n is monotonic increasing. Then

$$|\frac{a(x)}{u_n^3(x)}| \leq C, \quad x \in [0, \frac{\pi}{2} + \frac{\delta}{2}]. \tag{3.7}$$

By the standard estimate to equation (3.6) we find

$$\|u_n\|_{C^1} \leq C(M_n + 1), \tag{3.8}$$

where $M_n = \max_{x \in [0, \frac{\pi}{2} + \delta]} u_n(x)$. It follows from (3.5) and (3.8) that $M_n \to \infty$ as $n \to \infty$. Divided (3.6) by M_n and letting $n \to \infty$, we get $\frac{u_n}{M_n} \to v$, $\max_{x \in [0, \frac{\pi}{2}]} v(x) = 1$ and

$$v'' + v = 0, \quad x \in [0, \frac{\pi}{2} + \frac{\delta}{2}]. \tag{3.9}$$

Since u_n is monotone increasing on $[0, \frac{\pi}{2}+\frac{\delta}{2}]$, so is v. This is impossible as v satisfies (3.9) and $v \gneqq 0$. The proof is finished. □

Recall that $u(\lambda, \theta; x)$ is the solution of

$$u'' + u = \frac{a(x)}{u^3} \tag{3.10}$$

satisfying the initial value conditions

$$u(\lambda, \theta; 0) = \sqrt{\lambda^2 \cos^2\theta + \lambda^{-2}\sin^2\theta}; \quad u'(\lambda, \theta; 0) = \frac{(\lambda^2 - \lambda^{-2})\sin\theta\cos\theta}{\sqrt{\lambda^2\cos^2\theta + \lambda^{-2}\sin^2\theta}}.$$

According to Lemma 2.1, we know that

$$u^2(x) + u'^2(x) + \frac{a(x)}{u^2(x)} \sim \lambda^2 + \lambda^{-2}, \quad x \in [-2\pi, 2\pi]. \tag{3.11}$$

Therefore, it follows from Proposition 3.2 that the function $u(\lambda, \theta; x)$ has critical points in $[0, \frac{\pi}{2} + \delta]$ and $[-\frac{\pi}{2} - \delta, 0]$, respectively. It is easy to see from (3.10) and (3.11) that all critical points of $u(\lambda, \theta; x)$ are nondegenerate for small λ, hence they are isolated. Using these facts now we define two functions $\Theta(\lambda, \theta)$ and $\Theta_1(\lambda, \theta)$. We restrict $\theta \in [0, \frac{\pi}{2})$ first. Let $\Theta(\lambda, \theta)$ and $\Theta_1(\lambda, \theta)$ be the first critical point of $u(\lambda, \theta; x)$ in $x \geq 0$ and $x \leq 0$, respectively. By definition we have that

$$\Theta(\lambda, 0) = \Theta_1(\lambda, 0) = 0,$$

$\Theta(\lambda, \theta)$ is a local minimizer and $\Theta_1(\lambda, \theta)$ is a local maximizer if $\theta \in [0, \frac{\pi}{2})$, since λ is small and $u'(\lambda, \theta; 0) \leq 0$. Moreover, as a consequence of (3.11) we see that

$$m(\lambda, \theta) = u(\lambda, \theta; \Theta(\lambda, \theta)) \sim \lambda, \quad \theta \in [0, \frac{\pi}{2})$$

and

$$M(\lambda, \theta) = u(\lambda, \theta; \Theta_1(\lambda, \theta)) \sim \lambda^{-1}, \quad \theta \in [0, \frac{\pi}{2}).$$

The following lemma is proved in [21].

Lemma 3.3 *As $\lambda \to 0$, the following estimate holds uniformly in θ:*

$$\Theta(\lambda, \theta) - \Theta_1(\lambda, \theta) = \frac{\pi}{2} + o(1). \tag{3.12}$$

Using the lemma above we can prove

Proposition 3.4 *Let $u(\lambda, \theta; x)$ and $\Theta(\lambda, \theta)$ be as above. Then as $\lambda \to 0$, the following estimate holds uniformly in $\theta \in [0, \frac{\pi}{2})$:*

$$\Theta(\lambda, \theta) = \theta + o(1), \quad \theta \in [0, \frac{\pi}{2}). \tag{3.13}$$

Proof: Since $\theta \in [0, \frac{\pi}{2})$, $u(\lambda, \theta; x)$ is monotone decreasing on $[\Theta_1(\lambda, \theta), \Theta(\lambda, \theta)]$. We are going to prove: $\forall \delta > 0$, $\exists \lambda_0(\delta) > 0$ such that for all $\lambda \leq \lambda_0$,

$$|\Theta(\lambda, \theta) - \theta| \leq \delta, \quad \theta \in [0, \frac{\pi}{2}). \tag{3.14}$$

To this end, first applying Lemma 3.1 to $u(\lambda, \theta; x)$ in the interval $[\Theta_1, \Theta]$, we conclude that there is an $\varepsilon_0(\delta)$ such that

$$|\{x \in [\Theta_1, \Theta] | u(\lambda, \theta; x) \leq \varepsilon_0\}| \leq \frac{\delta}{4}. \tag{3.15}$$

Let $T(\lambda, \theta)$ be given by $u(\lambda, \theta; T(\lambda, \theta)) = \varepsilon_0$. Then (3.15) implies

$$|\Theta(\lambda, \theta) - T(\lambda, \theta)| \leq \frac{\delta}{4}. \tag{3.16}$$

In order to get (3.14), it suffice to show it holds for $\theta \leq T(\lambda, \theta)$, since if $\theta \geq T(\lambda, \theta)$, (3.16) implies

$$|\Theta(\lambda, \theta) - \theta)| \leq |\Theta(\lambda, \theta) - T(\lambda, \theta)| \leq \frac{\delta}{4}.$$

In the remaining we show $\exists \lambda_0(\delta) > 0$ such that for all $\lambda \leq \lambda_0$,

$$|\frac{\pi}{2} - \theta + \Theta_1(\lambda, \theta)| \leq \frac{\delta}{2}, \quad \theta \in [0, \frac{\pi}{2}). \tag{3.17}$$

Having this estimate, (3.14) follows from Lemma 3.3 immediately.

To prove (3.17) we consider the function $v_\lambda(x) = \lambda u(\lambda, \theta; x + \Theta_1)$, which satisfies

$$v_\lambda'' + v_\lambda = \lambda \frac{a(x + \Theta_1)}{u^3(\lambda, \theta; x + \Theta_1)} := \lambda f_\lambda(x), \quad x \in [0, T(\lambda, \theta) - \Theta_1] \tag{3.18}$$

and

$$v_\lambda'(0) = \lambda u'(\lambda, \theta; \Theta_1) = 0, \tag{3.19}$$

$$v_\lambda(\theta - \Theta_1) = \lambda u(\lambda, \theta; 0) = \lambda \sqrt{\lambda^2 \cos^2 \theta + \lambda^{-2} \sin^2 \theta}. \tag{3.20}$$

Thus

$$\begin{aligned} v_\lambda(x) = & - \cos x \int_0^x \lambda f_\lambda(s) \sin s ds + \sin x \int_0^x \lambda f_\lambda(s) \cos s ds \\ & + v_\lambda(0) \cos x. \end{aligned} \tag{3.21}$$

By definition we have

$$u(\lambda, \theta; x) \geq \varepsilon_0, \quad x \in [\Theta_1, T(\lambda, \theta)],$$

$$|\frac{a(x)}{u(\lambda, \theta; x)}| \leq C(\delta), \quad x \in [\Theta_1, T(\lambda, \theta)].$$

It follows that

$$|f_\lambda(x)| \le C(\delta), \quad x \in [0, T(\lambda,\theta) - \Theta_1].$$

Hence from (3.21) we deduce that

$$v_\lambda(\theta - \Theta_1) = v_\lambda(0)\cos(\theta - \Theta_1) + o(1), \quad \lambda \to 0 \tag{3.22}$$

and

$$v'_\lambda(\theta - \Theta_1) = -v_\lambda(0)\sin(\theta - \Theta_1) + o(1), \quad \lambda \to 0. \tag{3.23}$$

On the other hand, from (3.20) we infer that

$$v_\lambda(\theta - \Theta_1) = \sin\theta + o(1) \quad \lambda \to 0 \tag{3.24}$$

holds uniformly in θ. Therefore, inserting (3.24) into (3.22) we have

$$v_\lambda(0)\cos(\theta - \Theta_1) = \sin\theta + o(1), \quad \lambda \to 0 \tag{3.25}$$

uniformly in θ, too.

Now we separate two cases:
(1) $\theta \le \frac{\delta}{8}$. In this case, using $v_\lambda(0) \sim 1$ and (3.25), one can find $\lambda_1(\delta)$ such that

$$|\cos(\theta - \Theta_1)| = \sin\frac{\delta}{4}, \quad \lambda \le \lambda_1(\delta).$$

This shows that

$$|\frac{\pi}{2} - (\theta - \Theta_1)| \le \frac{\delta}{4}, \quad \lambda \le \lambda_1(\delta). \tag{3.26}$$

(2) $\theta \ge \frac{\delta}{8}$. In this case, we see that, as $\lambda \to 0$,

$$\begin{aligned} v'_\lambda(\theta - \Theta_1) &= \lambda u'(\lambda, \theta; 0) \\ &= \lambda \frac{(\lambda^{-2} - \lambda^2)\sin\theta\cos\theta}{\sqrt{\lambda^2\cos^2\theta + \lambda^{-2}\sin^2\theta}} = -\cos\theta + o(1) \end{aligned} \tag{3.27}$$

uniformly in θ. Combining (3.23) with (3.27) we arrive at

$$v_\lambda(0)\sin(\theta - \Theta_1) = -\cos\theta + o(1). \tag{3.28}$$

Thus as a consequence of (3.25) and (3.28), we can find $\lambda_2(\delta)$ such that

$$|\frac{\pi}{2} - (\theta - \Theta_1)| \le \frac{\delta}{4}, \quad \lambda \le \lambda_2(\delta) \tag{3.29}$$

and $|v_\lambda(0) - 1| \le \frac{\delta}{2}$. Combining (3.26) with (3.29) we get (3.17). This completes the proof of (3.17) and (3.14). □

In the next section we will perform the blow-up analysis at a minimum point of $u(\lambda, \theta : x)$ as $\lambda \to 0$ for all $\theta \in [0, \pi)$. (The reason for performing blow-up analysis at a minimum point is only for the simplicity of notation since one can do the same process at a maximum point as well.) As shown before, the first critical point of $u(\lambda, \theta; x)$ in $x \geq 0$ is a minimum point if $\theta \in [0, \frac{\pi}{2})$. But this is not the case if $\theta \in [\frac{\pi}{2}, \pi)$, so we need to define the function $\Theta(\lambda, \theta)$ slightly different. Let $\Theta_2(\lambda, \theta)$ be the first critical point of $u(\lambda, \theta; x)$ in $x \geq 0$ which is a maximum point as $\theta \in [\frac{\pi}{2}, \pi)$, and let $\Theta(\lambda, \theta)$ be the critical point of $u(\lambda, \theta; x)$ next to $\Theta_2(\lambda, \theta)$. Clearly, $\Theta(\lambda, \theta)$ is a minimum point and we have

$$m(\lambda, \theta) = u(\lambda, \theta; \Theta(\lambda, \theta)) \sim \lambda,$$
$$M(\lambda, \theta) = u(\lambda, \theta; \Theta_2(\lambda, \theta)) \sim \lambda^{-1}.$$

By the same argument, one can show that the estimate (3.13) holds for $\theta \in [\frac{\pi}{2}, \pi)$. Thus we have

Proposition 3.5 *Let $u(\lambda, \theta; x)$ be as above. Then for $\lambda << 1$, there exists a function $\Theta(\lambda, \theta)$ which is a local minimum of $u(\lambda, \theta; x)$ such that $m(\lambda, \theta) = u(\lambda, \theta; \Theta(\lambda, \theta)) \sim \lambda$ and*

$$\Theta(\lambda, \theta) = \theta + o(1) \tag{3.30}$$

holds uniformly in $\theta \in [0, \pi)$.

Remark 3.6 *One can show that $\Theta(\lambda, \theta)$ is the first local minimum of $u(\lambda, \theta; x)$ in $x \geq 0$, which is C^1 in (λ, θ) via the implicit function theorem. But one should note that $\Theta(\lambda, \theta + \pi) \neq \Theta(\lambda, \theta) + \pi$ in general. However, we will see that (3.30) is sufficient for the degree argument in Section 5.*

4 Asymptotic Expansion of $\mathcal{F}(\lambda, \theta)$

In this section, following the argument in [21], we give an asymptotic expansion of the solution $\mathcal{F}(\lambda, \theta)$ given in the last section. This is crucial in the computation of the degree of $\mathcal{F}(\lambda, \theta)$ in the next section. We note that in [21], the blow-up argument was used to obtain a priori estimates of 2π-periodic solutions of (2.1). An important observation here is that the argument not only works for the periodic solutions of (2.1), but also for all solutions of (2.1).

Let $m(\lambda, \theta) = u(\lambda, \theta; \Theta(\lambda, \theta))$ and

$$U_\varepsilon(x) = \left(\varepsilon^2 \cos^2 x + \varepsilon^{-2} \sin^2 x\right)^{\frac{1}{2}} .$$

We define a transformation

$$x = \Theta(\lambda, \theta) + \psi_{\lambda, \theta}(y) = \Theta(\lambda, \theta) + \int_0^y \frac{1}{U^2{}_{\varepsilon^{-1}}(\tau)} d\tau,$$

where $\varepsilon(\lambda, \theta) = (a(\Theta(\lambda, \theta))^{-\frac{1}{4}} m(\lambda, \theta) \sim \lambda$. It induces a rule of transformation of the equation

$$u'' + u = \frac{a(x)}{u^3}, \quad x \in [0, T] \tag{4.1}$$

as follows: let

$$v(\lambda,\theta;y) = U_{\varepsilon^{-1}}(y)u(\Theta(\lambda,\theta)+\psi_{\lambda,\theta}(y)) = U_{\varepsilon^{-1}}(x-\Theta(\lambda,\theta))u(\lambda,\theta;x),$$

using

$$\frac{dy}{dx} = U_{\varepsilon}^{-2}(x-\Theta(\lambda,\theta)), \qquad \text{and}$$

$$\frac{d^2y}{dx^2} = \left(\varepsilon^2-\varepsilon^{-2}\right)\sin 2(x-\Theta(\lambda,\theta))U_{\varepsilon}^{-4}(x-\Theta(\lambda,\theta)),$$

one can verify that

$$\frac{d^2v(\lambda,\theta;y)}{dy^2} + v(\lambda,\theta;y) = \frac{a(\Theta(\lambda,\theta)+\psi_{\lambda,\theta}(y))}{v^3(\lambda,\theta;y)}. \tag{4.2}$$

Using equation (4.2), the following result on the asymptotical behavior of $u(\lambda,\theta;x)$ as $\lambda\to 0$ was proved in [21], where the periodic solutions was dealt with. The argument is valid for the solution $u(\lambda,\theta;x)$. Hence the detail of the proof is omitted.

Proposition 4.1 *Let a be a positive, C^2 and 2π-periodic function. Then $\exists\ C>0$ such that*

$$\frac{1}{C} \le v(\lambda,\theta;y) \le C, \quad |v_y(\lambda,\theta;y)| \le C, \quad y\in[-2\pi,2\pi], \tag{4.3}$$

and

$$\|v(\lambda,\theta;y)-v(\Theta;y)\|_{C^1([-2\pi,2\pi])} \le C\lambda^2|\log\lambda|, \quad \lambda\to 0, \tag{4.4}$$

where $v(\Theta;y)$ is the 2π-periodic function given by

$$v(\Theta;y) = \begin{cases} a^{\frac{1}{4}}(\Theta), & y\in[-\frac{\pi}{2},\frac{\pi}{2}],\\ \left(a^{\frac{1}{2}}(\Theta)\sin^2 y + a(\Theta+\pi)a^{-\frac{1}{2}}(\Theta)\cos^2 y\right)^{\frac{1}{2}}, & y\in[\frac{\pi}{2},\frac{3\pi}{2}]. \end{cases} \tag{4.5}$$

Remark 4.2 *Extending $v(\Theta;y)$ to a 2π-periodic function, then one can prove that the estimate*

$$\|v(\lambda,\theta;y)-v(\Theta;y)\|_{C^1([a,b])} \le C\lambda^2|\log\lambda| \tag{4.6}$$

holds for any bounded interval $[a,b]$, but the constant C may depend on the interval. This fact will be used later.

Recall that for $(\lambda,\theta)\in(0,1]\times\mathbb{S}^1_1$,

$$\mathcal{F}(\lambda,\theta) = (A(\lambda,\theta),B(\lambda,\theta)), \tag{4.7}$$

where

$$A(\lambda,\theta) = \int_0^{2\pi}\frac{a'(x)}{u^2(\lambda,\theta;x)}\cos 2x dx,$$

$$B(\lambda,\theta)=\int_0^{2\pi}\frac{a'(x)}{u^2(\lambda,\theta;x)}\sin 2x dx.$$

To compute the degree we need the asymptotic behavior of $\mathcal{F}(\lambda,\theta)$ as $\lambda\to 0$. Let $\Theta(\lambda,\theta)$ be the function in Proposition 3.5, $m(\lambda,\theta)=u(\lambda,\theta;\Theta(\lambda,\theta))$, $\varepsilon(\lambda,\theta)=(a(\Theta(\lambda,\theta))^{-\frac{1}{4}}m(\lambda,\theta)\sim\lambda$. For simplicity of notations we omit (λ,θ) in u and Θ. Then

$$\begin{aligned}
A(\lambda,\theta)&=\int_0^{2\pi}\frac{a'(x)}{u^2(x)}\cos 2x dx\\
&=\int_{-\Theta}^{2\pi-\Theta}\frac{a'(x+\Theta)}{u^2(x+\Theta)}\cos 2(x+\Theta)dx\\
&=\cos 2\Theta\int_{-\Theta}^{2\pi-\Theta}\frac{a'(x+\Theta)}{u^2(x+\Theta)}\cos 2x dx-\sin 2\Theta\int_{-\Theta}^{2\pi-\Theta}\frac{a'(x+\Theta)}{u^2(x+\Theta)}\sin 2x dx
\end{aligned}$$

and

$$\begin{aligned}
B(\lambda,\theta)&=\int_0^{2\pi}\frac{a'(x)}{u^2(x)}\sin 2x dx\\
&=\int_{-\Theta}^{2\pi-\Theta}\frac{a'(x+\Theta)}{u^2(x+\Theta)}\sin 2(x+\Theta)dx\\
&=\sin 2\Theta\int_{-\Theta}^{2\pi-\Theta}\frac{a'(x+\Theta)}{u^2(x+\Theta)}\cos 2x dx+\cos 2\Theta\int_{-\Theta}^{2\pi-\Theta}\frac{a'(x+\Theta)}{u^2(x+\Theta)}\sin 2x dx.
\end{aligned}$$

Using the asymptotic expansion given by Proposition 4.1 now we prove

Proposition 4.3 *Let a be a positive, C^2 and 2π-periodic function. Then as $\lambda\to 0$, we have*

$$\int_{-\Theta}^{2\pi-\Theta}\frac{a'(x+\Theta)}{u^2(x+\Theta)}\cos 2x dx=\frac{a'(\Theta)}{\sqrt{a(\Theta)}}+\frac{a'(\pi+\Theta)}{\sqrt{a(\pi+\Theta)}}+O(\lambda^2|\log\lambda|),\tag{4.8}$$

and

$$\begin{aligned}
&\int_{-\Theta}^{2\pi-\Theta}\frac{a'(x+\Theta)}{u^2(x+\Theta)}\sin 2x dx\\
=&m^2(\lambda,\theta)\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\frac{\big(a'(x+\Theta)+a'(x+\pi+\Theta)-a'(\Theta)-a'(\pi+\Theta)\big)\sin 2x}{\sin^2 x}dx+o(\lambda^2).
\end{aligned}\tag{4.9}$$

The proof is similar to that of Proposition 3.1 in [21].

Proof. Let

$$x=\psi_{\lambda,\theta}(y)=\int_0^y\frac{1}{\varepsilon^{-2}\cos^2\tau+\varepsilon^2\sin^2\tau}d\tau,$$

$$v(\lambda,\theta;y) = (\varepsilon^{-2}\cos^2 y + \varepsilon^2 \sin^2 y)^{\frac{1}{2}} u(\Theta(\lambda,\theta) + \psi_{\lambda,\theta}(y)).$$

Then

$$\cos 2x = \frac{\cos^2 y - \varepsilon^4 \sin^2 y}{\cos^2 y + \varepsilon^4 \sin^2 y},$$
$$\sin 2x = \frac{2\varepsilon^2 \sin y \cos y}{\cos^2 y + \varepsilon^4 \sin^2 y}.$$

Set $\Theta = \psi_{\lambda,\theta}(\Theta_1)$. Then using the change of variable $x = \psi_{\lambda,\theta}(y)$ and Proposition 4.1 we obtain

$$\begin{aligned}
&\int_{-\Theta}^{2\pi-\Theta} \frac{a'(x+\Theta)}{u^2(x+\Theta)} \cos 2x dx \\
=&\int_{-\Theta_1}^{2\pi-\Theta_1} \frac{a'(\psi_{\lambda,\theta}(y)+\Theta)}{v^2(\lambda,\theta;y)} \frac{\cos^2 y - \varepsilon^4 \sin^2 y}{\cos^2 y + \varepsilon^4 \sin^2 y} dy \\
=&\int_{-\Theta_1}^{2\pi-\Theta_1} \frac{a'(\psi_{\lambda,\theta}(y)+\Theta)}{v^2(\Theta;y) + O(\varepsilon^2|\log\varepsilon|)} \frac{\cos^2 y - \varepsilon^4 \sin^2 y}{\cos^2 y + \varepsilon^4 \sin^2 y} dy \\
=&\int_{-\Theta_1}^{2\pi-\Theta_1} \frac{a'(\psi_{\lambda,\theta}(y)+\Theta)}{v^2(\Theta;y)} \frac{\cos^2 y - \varepsilon^4 \sin^2 y}{\cos^2 y + \varepsilon^4 \sin^2 y} dy + O(\varepsilon^2|\log\varepsilon|) \\
=&\int_{-\frac{\pi}{2}}^{\frac{3}{2}\pi} \frac{a'(\psi_{\lambda,\theta}(y)+\Theta)}{v^2(\Theta;y)} \frac{\cos^2 y - \varepsilon^4 \sin^2 y}{\cos^2 y + \varepsilon^4 \sin^2 y} dy + O(\varepsilon^2|\log\varepsilon|).
\end{aligned} \tag{4.10}$$

In the above equality we have used the fact that the function $v(\Theta;y)$ is 2π-periodic. Now we estimate the right-hand side of (4.10). First we have

$$\begin{aligned}
&|\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{a'(\psi_{\lambda,\theta}(y)+\Theta) - a'(\Theta)}{v^2(\Theta;y)} \frac{\cos^2 y - \varepsilon^4 \sin^2 y}{\cos^2 y + \varepsilon^4 \sin^2 y} dy| \\
\leq& C\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} |a'(\psi_{\lambda,\theta}(y)+\Theta) - a'(\Theta)| dy \\
=& O(\varepsilon^2|\log\varepsilon|),
\end{aligned} \tag{4.11}$$

$$\begin{aligned}
\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{a'(\Theta)}{v^2(\Theta;y)} \frac{\cos^2 y - \varepsilon^4 \sin^2 y}{\cos^2 y + \varepsilon^4 \sin^2 y} dy &= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{a'(\Theta)}{v^2(\Theta;y)} (1 - \frac{2\varepsilon^4 \sin^2 y}{\cos^2 y + \varepsilon^4 \sin^2 y}) dy \\
&= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{a'(\Theta)}{v^2(\Theta;y)} dy + O(\varepsilon^2) \\
&= \frac{a'(\Theta)}{\sqrt{a(\Theta)}} + O(\varepsilon^2).
\end{aligned} \tag{4.12}$$

It follows from (4.11) and (4.12) that

$$\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{a'(\psi_{\lambda,\theta}(y)+\Theta)}{v^2(\Theta;y)} \frac{\cos^2 y - \varepsilon^4 \sin^2 y}{\cos^2 y + \varepsilon^4 \sin^2 y} dy = \frac{a'(\Theta)}{\sqrt{a(\Theta)}} + O(\varepsilon^2|\log\varepsilon|). \tag{4.13}$$

Similarly we have

$$\int_{\frac{\pi}{2}}^{\frac{3}{2}\pi} \frac{a'(\psi_{\lambda,\theta}(y)+\Theta)}{v^2(\Theta;y)} \frac{\cos^2 y - \varepsilon^4 \sin^2 y}{\cos^2 y + \varepsilon^4 \sin^2 y} dy = \frac{a'(\Theta+\pi)}{\sqrt{a(\Theta+\pi)}} + O(\varepsilon^2 |\log \varepsilon|). \tag{4.14}$$

Inserting (4.13) and (4.14) into (4.10) we get (4.8) as $\lambda \sim \varepsilon$.

Now we prove (4.9). The proof is more involved. First we obtain

$$\begin{aligned}
&\int_{-\Theta}^{2\pi-\Theta} \frac{a'(x+\Theta)}{u^2(x+\Theta)} \sin 2x dx \\
&= \int_{-\Theta_1}^{2\pi-\Theta_1} \frac{a'(\psi_{\lambda,\theta}(y)+\Theta)}{v^2(\lambda,\theta;y)} \frac{2\varepsilon^2 \sin y \cos y}{\cos^2 y + \varepsilon^4 \sin^2 y} dy \\
&= \int_{-\Theta_1}^{2\pi-\Theta_1} \frac{a'(\psi_{\lambda,\theta}(y)+\Theta)}{v^2(\Theta;y) + O(\varepsilon^2|\log\varepsilon|)} \frac{2\varepsilon^2 \sin y \cos y}{\cos^2 y + \varepsilon^4 \sin^2 y} dy \\
&= \int_{-\Theta_1}^{2\pi-\Theta_1} (\frac{a'(\psi_{\lambda,\theta}(y)+\Theta)}{v^2(\Theta;y)} + O(\varepsilon^2|\log\varepsilon|)) \frac{2\varepsilon^2 \sin y \cos y}{\cos^2 y + \varepsilon^4 \sin^2 y} dy
\end{aligned} \tag{4.15}$$

By elementary computations one has

$$\int_{-\Theta_1}^{2\pi-\Theta_1} |\frac{2\varepsilon^2 \sin y \cos y}{\cos^2 y + \varepsilon^4 \sin^2 y}| dy = \int_{-\frac{\pi}{2}}^{\frac{3}{2}\pi} |\frac{2\varepsilon^2 \sin y \cos y}{\cos^2 y + \varepsilon^4 \sin^2 y}| dy = O(\varepsilon^2 |log\varepsilon|). \tag{4.16}$$

So from (4.15) and (4.16) we see that

$$\begin{aligned}
&\int_{-\Theta}^{2\pi-\Theta} \frac{a'(x+\Theta)}{u^2(x+\Theta)} \sin 2x dx \\
&= \int_{-\Theta_1}^{2\pi-\Theta_1} \frac{a'(\psi_{\lambda,\theta}(y)+\Theta)}{v^2(\Theta;y)} \frac{2\varepsilon^2 \sin y \cos y}{\cos^2 y + \varepsilon^4 \sin^2 y} dy + O(\varepsilon^4 |\log\varepsilon|^2) \\
&= \int_{-\frac{\pi}{2}}^{\frac{3}{2}\pi} \frac{a'(\psi_{\lambda,\theta}(y)+\Theta)}{v^2(\Theta;y)} \frac{2\varepsilon^2 \sin y \cos y}{\cos^2 y + \varepsilon^4 \sin^2 y} dy + O(\varepsilon^4 |\log\varepsilon|^2).
\end{aligned} \tag{4.17}$$

In order to get (4.9) we express the integral in the right-hand side of (4.17) in terms of x, that is, we use the change of variable $y = \psi^{-1}_{\lambda,\theta}(x)$. Then

$$\begin{aligned}
&\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{a'(\psi_{\lambda,\theta}(y)+\Theta)}{v^2(\Theta;y)} \frac{2\varepsilon^2 \sin y \cos y}{\cos^2 y + \varepsilon^4 \sin^2 y} dy \\
=&\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{a'(x+\Theta)}{v^2(\Theta;\psi^{-1}_{\lambda,\theta}(x))} \frac{\varepsilon^2 \sin 2x}{\varepsilon^4 \cos^2 x + \sin^2 x} dx \\
=&\frac{\varepsilon^2}{a^{\frac{1}{2}}(\Theta)} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{a'(x+\Theta)\sin 2x}{\varepsilon^4 \cos^2 x + \sin^2 x} dx \\
=&\frac{\varepsilon^2}{a^{\frac{1}{2}}(\Theta)} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{a'(x+\Theta)-a'(\Theta)}{\sin x} \frac{\sin x \sin 2x}{\varepsilon^4 \cos^2 x + \sin^2 x} dx \\
=&\frac{\varepsilon^2}{a^{\frac{1}{2}}(\Theta)} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{a'(x+\Theta)-a'(\Theta)}{\sin^2 x} \sin 2x dx + o(\varepsilon^2) \\
=&m^2(\lambda,\theta) \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{a'(x+\Theta)-a'(\Theta)}{\sin^2 x} \sin 2x dx + o(\varepsilon^2)
\end{aligned} \tag{4.18}$$

by Lebesgue's dominated convergent theorem. Similarly, we have

$$\begin{aligned}
&\int_{\frac{\pi}{2}}^{\frac{3}{2}\pi} \frac{a'(\psi_{\lambda,\theta}(y)+\Theta)}{v^2(\Theta;y)} \frac{2\varepsilon^2 \sin y \cos y}{\cos^2 y + \varepsilon^4 \sin^2 y} dy \\
=&\int_{\frac{\pi}{2}}^{\frac{3}{2}\pi} \frac{a'(x+\Theta)}{v^2(\Theta;\psi^{-1}_{\lambda,\theta}(x))} \frac{\varepsilon^2 \sin 2x}{\varepsilon^4 \cos^2 y + \sin^2 x} dx \\
=&\varepsilon^2 \int_{\frac{\pi}{2}}^{\frac{3}{2}\pi} \frac{a'(x+\Theta)\sin 2x}{\varepsilon^4 a^{-\frac{1}{2}}(\Theta) a(\Theta+\pi) \cos^2 x + a^{\frac{1}{2}}(\Theta) \sin^2 x} dx \\
=&\varepsilon^2 \int_{\frac{\pi}{2}}^{\frac{3}{2}\pi} \frac{a'(x+\Theta)-a'(\Theta)}{\sin x} \frac{\sin x \sin 2x}{\varepsilon^4 a^{-\frac{1}{2}}(\Theta) a(\Theta+\pi) \cos^2 x + a^{\frac{1}{2}}(\Theta) \sin^2 x} dx \\
=&\frac{\varepsilon^2}{a^{\frac{1}{2}}(\Theta)} \int_{\frac{\pi}{2}}^{\frac{3}{2}\pi} \frac{a'(x+\Theta)-a'(\Theta)}{\sin^2 x} \sin 2x dx + o(\varepsilon^2) \\
=&\frac{\varepsilon^2}{a^{\frac{1}{2}}(\Theta)} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{a'(x+\Theta+\pi)-a'(\Theta+\pi)}{\sin^2 x} \sin 2x dx + o(\varepsilon^2) \\
=&m^2(\lambda,\theta) \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{a'(x+\Theta+\pi)-a'(\Theta+\pi)}{\sin^2 x} \sin 2x dx + o(\varepsilon^2).
\end{aligned} \tag{4.19}$$

Finally, putting (4.17), (4.18) and (4.19) together and taking $\lambda \sim \varepsilon$ into account we get (4.9). This completes the proof of Proposition 4.3. □

5 Proof of Theorem 1.2

Now we are ready to find solutions of $\mathcal{F}(\lambda,\theta)=0$ via the degree theory and prove the main theorem of this paper, Theorem 1.2. Recall that the map $G(\theta)=(A_2(\theta),B_2(\theta))$ in Theorem 1.2 and $\mathcal{F}(\lambda,\theta$ for fixed λ are π-periodic and defined on $\mathbb{S}^1_1=\mathbb{R}/\pi\mathbb{Z}$.

We first need

Lemma 5.1 *For $0<\lambda<<1$, we have*

$$\mathcal{F}(\lambda,\theta)=(A(\lambda,\theta),B(\lambda,\theta))\neq 0, \quad \theta\in\mathbb{S}^1_1. \tag{5.1}$$

Hence $deg(\mathcal{F}(\lambda,\theta),\mathbb{S}^1_1)$ is well-defined.

Proof. This can be easily proved by contradiction. Suppose there is a sequence $\lambda_k\to 0$, $\theta_k\in\mathbb{S}^1_1$ such that $\mathcal{F}(\lambda_k,\theta_k)=0$. Then we have

$$\int_{-\Theta_k}^{2\pi-\Theta_k}\frac{a'(x+\Theta_k)}{u^2(x+\Theta_k)}\cos 2x dx=0 \tag{5.2}$$

and

$$\int_{-\Theta_k}^{2\pi-\Theta_k}\frac{a'(x+\Theta_k)}{u^2(x+\Theta_k)}\sin 2x dx=0, \tag{5.3}$$

where $\Theta_k=\Theta(\lambda_k,\theta_k)$. Then Proposition 4.3, (5.2) and (5.3) yield

$$A_2(\Theta_k)=\frac{a'(\Theta_k)}{\sqrt{a(\Theta_k)}}+\frac{a'(\pi+\Theta_k)}{\sqrt{a(\pi+\Theta_k)}}\to 0 \tag{5.4}$$

and

$$B_2(\Theta_k)=\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\frac{\big(a'(x+\Theta_k)+a'(x+\pi+\Theta_k)-a'(\Theta_k)-a'(\pi+\Theta_k)\big)\sin 2x}{\sin^2 x}dx\to 0. \tag{5.5}$$

But we know that $A_2^2(\theta)+B_2^2(\theta)$ is continuous and does not vanish for $\theta\in\mathbb{S}^1_1$, there is a constant c_0 such that

$$A_2^2(\theta)+B_2^2(\theta)\geq c_0>0, \quad \theta\in\mathbb{S}^1_1. \tag{5.6}$$

Clearly, (5.6) contradicts to (5.4) and (5.5). □

Proposition 5.2 *For $0<\lambda<<1$, the following equality holds:*

$$deg(\mathcal{F}(\lambda,\theta),\mathbb{S}^1_1)=deg(G(\theta),\mathbb{S}^1_1)+1. \tag{5.7}$$

Proof of Proposition 5.2: the case $A_2(\theta)\equiv 0$.

We use homotopy invariance of the degree to prove (5.7). To this end, set

$$\mathcal{F}_1(\lambda,\theta)=(A_2(\theta),\lambda^2B_2(\theta))=(0,\lambda^2B_2(\theta))$$

and

$$H_1(s,\lambda,\theta)=se^{-2i\theta}\mathcal{F}(\lambda,\theta)+(1-s)\mathcal{F}_1(\lambda,\theta).$$

Here we denote $e^{i\theta}(x,y)=(\cos\theta x-\sin\theta y,\cos\theta y+\sin\theta x)$. It is obvious that $H_1(s,\lambda,\theta)$ is continuous in (s,λ,θ) and π-periodic in θ.

Claim: For $\lambda<<1$,

$$H_1(s,\lambda,\theta)\neq 0,\quad s\in[0,1],\theta\in\mathbb{S}^1_1.$$

Indeed, if $H_1(s,\lambda,\theta)=0$, then

$$se^{-2i\Theta(\lambda,\theta)}\mathcal{F}(\lambda,\theta)+(1-s)e^{2i(\theta-\Theta(\lambda,\theta))}(0,\lambda^2B_2(\theta))=0. \tag{5.8}$$

It follows from Proposition 4.3 that the second coordinate equation in (5.8) is

$$sm^2(\lambda,\theta)B_2(\Theta(\lambda,\theta))+(1-s)\lambda^2B_2(\theta)\cos 2(\theta-\Theta(\lambda,\theta))+o(\lambda^2)=0. \tag{5.9}$$

Hence

$$B_2(\theta)=0 \tag{5.10}$$

provided by $m^2(\lambda,\theta)\sim\lambda^2$ and $\theta-\Theta(\lambda,\theta)=o(1)$. But we know that $B_2(\theta)$ does not vanish since a is B-nondegenerate and $A_2\equiv 0$, a contradiction.

By the definition of the degree, it is easy to see that

$$deg(\mathcal{F}_1(\lambda,\theta),\mathbb{S}^1_1)=deg(G(\theta),\mathbb{S}^1_1)=0.$$

Then it follows from the claim and the homotopy invariance of the degree that

$$deg(e^{-2i\theta}\mathcal{F}(\lambda,\theta),\mathbb{S}^1_1)=deg(\mathcal{F}_1(\lambda,\theta),\mathbb{S}^1_1)=0. \tag{5.11}$$

The left-hand side of (5.11) equals to $-1+deg(\mathcal{F}(\lambda,\theta),\mathbb{S}^1_1)$. So as a result of (5.11),

$$deg(\mathcal{F}(\lambda,\theta),\mathbb{S}^1_1)=1=deg(G(\theta),\mathbb{S}^1_1)+1.$$

This completes the proof. □

Next we turn to the case that $A_2(\theta)$ is not identically zero. After shifting, in this case we may assume $A_2(0)=A_2(\pi)\neq 0$. It is more involved since we are not able to prove $H_1(s,\lambda,\theta)\neq 0$ anymore as before due to the fact that in the asymptotic expansion in Proposition 4.3, the orders of λ in (4.8) and (4.9) are different. Some modifications of the homotopy are needed.

Set

$$\mathcal{F}_2(\lambda,\theta)=(A_2(\Theta(\lambda,\theta)),\lambda^2B_2(\Theta(\lambda,\theta)))$$

and

$$H_2(s,\lambda,\theta)=se^{-2i\Theta(\lambda,\theta)}\mathcal{F}(\lambda,\theta)+(1-s)\mathcal{F}_2(\lambda,\theta):=(K_1(s,\lambda,\theta),K_2(s,\lambda,\theta)).$$

It is obvious that $H_2(s,\lambda,\theta)$ is continuous in (s,λ,θ), but is not π-periodic in θ in general, thus the degree of this map is not well-defined. However, in view of the asymptotic estimate $\Theta(\lambda,\theta)$ given by Proposition 3.5, $H_2(s,\lambda,\theta)$ is very close to a π-periodic one. This enables us to complete the proof by several steps via the difference of argument of the map $\theta\to H_2(s,\lambda,\theta)$ at $\theta=\pi$ and $\theta=0$.

Lemma 5.3 *For* $0 < \lambda << 1$,

$$H_2(s, \lambda, \theta) \neq 0, \quad \theta \in [0, \pi], \quad s \in [0, 1]. \tag{5.12}$$

Proof. According to Proposition 4.3,

$$\begin{aligned} e^{-2i\Theta(\lambda,\theta)}\mathcal{F}(\lambda, \theta) &= (\int_{-\Theta}^{2\pi-\Theta} \frac{a'(x+\Theta)}{u^2(x+\Theta)} \cos 2x dx, \int_{-\Theta}^{2\pi-\Theta} \frac{a'(x+\Theta)}{u^2(x+\Theta)} \cos 2x dx) \\ &= (A_2(\Theta), m^2(\lambda, \theta)B_2(\Theta)) + (O(\lambda^2|\log \lambda|), o(\lambda^2)). \end{aligned}$$

Thus we get

$$H_2(s, \lambda, \theta) = (A_2(\Theta), (sm^2(\lambda, \theta) + (1-s)\lambda^2)B_2(\Theta)) + (O(\lambda^2|\log \lambda|), o(\lambda^2)).$$

The conclusion follows as in Lemma 5.1. □

For fixed (s, λ), we know that the degree of $H_2(s, \lambda, \theta)$ may not be well-defined, so instead of the degree we consider

$$\tilde{d}(H_2(s, \lambda, \theta)) = \frac{1}{2\pi} \int_0^\pi \frac{-(\frac{d}{d\theta}K_1)K_2 + (\frac{d}{d\theta}K_2)K_1}{K_1^2 + K_2^2} d\theta.$$

It is known that the integral on the right-hand side is the difference of argument of the map $\theta \to H_2(s, \lambda, \theta)$ at $\theta = \pi$ and $\theta = 0$, and $\tilde{d}(H_2(s, \lambda, \theta))$ is the same as the degree $deg(H_2(s, \lambda, \theta), \mathbb{S}_1^1)$ if $H_2(s, \lambda, \theta)$ is π-periodic in θ. It enjoys similar properties such as homotopy invariance as the degree.

Lemma 5.4 *For* $0 < \lambda << 1$, *we have*

$$\tilde{d}(H_2(1, \lambda, \theta)) = -1 + deg(\mathcal{F}(\lambda, \theta), \mathbb{S}_1^1) + o(1). \tag{5.13}$$

Proof. From the definition we see that

$$\begin{aligned} \tilde{d}(H_2(1, \lambda, \theta)) &= \tilde{d}(e^{-2i\Theta(\lambda,\theta)}\mathcal{F}(\lambda, \theta)) \\ &= -\frac{\Theta(\lambda, \pi) - \Theta(\lambda, 0)}{\pi} + \tilde{d}(\mathcal{F}(\lambda, \theta)) \\ &= -1 + deg(\mathcal{F}(\lambda, \theta), \mathbb{S}_1^1) + o(1) \end{aligned} \tag{5.14}$$

thanks to Proposition 3.5 and the fact that $\mathcal{F}(\lambda, \theta)$ is $\pi-$periodic in θ. □

Lemma 5.5 *Let* $A_2(0) = A_2(\pi) \neq 0$. *Then for* $0 < \lambda << 1$,

$$\tilde{d}(H_2(0, \lambda, \theta)) = \tilde{d}(H_2(1, \lambda, \theta)) + o(1). \tag{5.15}$$

Proof. Let $0 < \lambda << 1$ be fixed and $R = \{(s,\theta)|s \in [0,1], \theta \in [0,\pi]\}$. By Lemma 5.3 we have

$$H_2(s,\lambda,\theta) = (K_1(s,\lambda,\theta), K_2(s,\lambda,\theta)) \neq 0, (s,\theta) \in R.$$

Hence

$$\frac{1}{2\pi}\int_{\partial R} \frac{-dK_1K_2 + dK_2K_1}{K_1^2 + K_2^2} = 0, \tag{5.16}$$

the differentials in (5.16) are w.r.t. the variables s and θ. It follows from (5.16) that

$$\begin{aligned}
&\tilde{d}(H_2(0,\lambda,\theta)) - \tilde{d}(H_2(1,\lambda,\theta))\\
=&\frac{1}{2\pi}\int_0^1 \frac{-dK_1(s,\lambda,0)K_2(s,\lambda,0) + dK_2(s,\lambda,0)K_1(s,\lambda,0)}{K_1^2 + K_2^2}\\
-&\frac{1}{2\pi}\int_0^1 \frac{-dK_1(s,\lambda,\pi)K_2(s,\lambda,\pi) + dK_2(s,\lambda,\pi)K_1(s,\lambda,\pi)}{K_1^2 + K_2^2}\\
=&I + II.
\end{aligned} \tag{5.17}$$

By Proposition 4.3 and $\Theta(\lambda,\pi) = \pi + o(1)$ we see that for $s \in [0,1]$,

$$\begin{aligned}
&|\frac{d}{ds}K_1(s,\lambda,0)|^2 + |\frac{d}{ds}K_2(s,\lambda,0)|^2\\
=&|H_2(1,\lambda,0) - H_2(0,\lambda,0)|^2\\
=&|e^{-2i\Theta(\lambda,\pi)}\mathcal{F}(\lambda,\pi) - \mathcal{F}_2(\lambda,\pi)|^2 = o(\lambda^4\log^2\lambda)
\end{aligned}$$

and

$$K_1^2(s,\lambda,0) + K_2^2(s,\lambda,0) = A_2(0)^2 + o(1).$$

Therefore,

$$|I| \leq \int_0^1 \frac{(|\frac{d}{ds}K_1|^2 + |\frac{d}{ds}K_2|^2)^{\frac{1}{2}}}{(K_1^2 + K_2^2)^{\frac{1}{2}}}ds = o(1). \tag{5.18}$$

Similarly,

$$|II| = o(1). \tag{5.19}$$

Now plug (5.18) and (5.19) into (5.17) we obtain (5.15). □

Lemma 5.6 *Let $A_2(0) = A_2(\pi) \neq 0$. Then for $0 < \lambda << 1$, the following holds:*

$$\tilde{d}(H_2(0,\lambda,\theta)) = deg(G(\theta), \mathbb{S}_1^1) + o(1), \tag{5.20}$$

where $G(\theta) = (A_2(\theta), B_2(\theta))$.

Proof. From the definition we have

$$
\begin{aligned}
\tilde{d}(H_2(0,\lambda,\theta)) &= \tilde{d}(\mathcal{F}_2(\lambda,\theta)) \\
&= \frac{\lambda^2}{2\pi}\int_0^{\pi} \frac{\frac{d}{d\theta}A_2(\Theta)\cdot B_2(\Theta) - \frac{d}{d\theta}B_2(\Theta)\cdot A_2(\Theta)}{A_2(\Theta)^2+\lambda^4 B_2(\Theta)^2} d\theta \\
&= \frac{\lambda^2}{2\pi}\int_0^{\Theta(\lambda,\pi)} \frac{\frac{d}{d\Theta}A_2(\Theta)\cdot B_2(\Theta) - \frac{d}{d\Theta}B_2(\Theta)\cdot A_2(\Theta)}{A_2(\Theta)^2+\lambda^4 B_2(\Theta)^2} d\Theta \\
&= \frac{\lambda^2}{2\pi}[\int_0^{\pi} - \int_{\Theta(\lambda,\pi)}^{\pi}] \frac{\frac{d}{d\Theta}A_2(\Theta)\cdot B_2(\Theta) - \frac{d}{d\Theta}B_2(\Theta)\cdot A_2(\Theta)}{A_2(\Theta)^2+\lambda^4 B_2(\Theta)^2} d\Theta \\
&= deg(G(\lambda,\theta),\mathbb{S}_1^1) + o(1) \\
&= deg(G(\theta),\mathbb{S}_1^1) + o(1),
\end{aligned}
\tag{5.21}
$$

where $G(\lambda,\theta) = (A_2(\theta), \lambda^2 B_2(\theta))$, since $G(\lambda,\theta)$ is $\pi-$periodic in θ. (5.21) is a consequence of

$$deg(G(\lambda,\theta),\mathbb{S}_1^1) = deg(G(\theta),\mathbb{S}_1^1)$$

and

$$\lambda^2 |\int_{\Theta(\lambda,\pi)}^{\pi} \frac{(\frac{d}{d\Theta}A_2(\Theta))B_2(\Theta) - (\frac{d}{d\Theta}(B_2(\Theta))A_2(\Theta)}{A_2(\Theta)^2+\lambda^4 B_2(\Theta)^2} d\Theta| = o(1)$$

due to $\Theta(\lambda,\pi) = \pi + o(1)$ and $A_2(\pi) \neq 0$. □

Proof of Proposition 5.2: the case $A_2(\theta) \not\equiv 0$.

We assume that $A_2(\pi) \neq 0$. From Lemma 5.1, for $\lambda << 1$ we know $\mathcal{F}(\lambda,\theta) \neq (0,0)$ and $deg(\mathcal{F}(\lambda,\theta),\mathbb{S}_1^1)$ is well-defined. Combining Lemmas 5.4-5.6 we arrive at

$$
\begin{aligned}
deg(\mathcal{F}(\lambda,\theta),\mathbb{S}_1^1) &= \tilde{d}(H(1,\lambda,\theta)) + 1 + o(1) \\
&= \tilde{d}(H(0,\lambda,\theta)) + 1 = o(1) \\
&= deg(G(\theta),\mathbb{S}_1^1) + 1 + o(1).
\end{aligned}
\tag{5.22}
$$

But both $deg(\mathcal{F}(\lambda,\theta),\mathbb{S}_1^1)$ and $deg(G(\theta),\mathbb{S}_1^1)$ are integers, there must hold

$$deg(\mathcal{F}(\lambda,\theta),\mathbb{S}_1^1) = deg(G(\theta),\mathbb{S}_1^1) + 1. \tag{5.23}$$

This finishes the proof. □

Proof of Theorem 1.2: Set $\Lambda = 1-\lambda$. Then as $\Lambda \to 1$, the degree $deg(\mathcal{F}(1-\Lambda,\theta);\mathbb{S}_1^1,0)$ is well-defined. We define a continuous map from $D(\Lambda) = \{(X,Y) \in \mathbb{R}^2 | X^2+Y^2 \leq \Lambda^2\}$ to $\mathbb{R}^2$ as

$$\mathcal{G}(X,Y) = (A(1-\Lambda,\theta), B(1-\Lambda,\theta)) = \mathcal{F}(1-\Lambda,\theta),$$

where $(X,Y) = (\Lambda\cos\theta, \Lambda\sin\theta)$. This is a well-defined and continuous map due to the fact that $\mathcal{F}(\lambda,\theta)$ is independent of θ when $\lambda = 1$ or $\Lambda = 0$. It follows from the assumption of Theorem 1.2, $deg(\mathcal{F}(1-\Lambda,\theta);\mathbb{S}_1^1,0) \neq 0$, hence by the basic property of degree, see [14], $\mathcal{F}(1-\Lambda,\theta) = 0$ has a solution (Λ,θ). The proof is finished. □

The following result, which was proved by a different approach in [21], is an immediate consequence of Theorem 1.2.

Corollary 5.7 *Let a be a positive, C^2 and 2π-periodic and B-nondegenerate function. Suppose that*

$$\min_{A_2(\theta)=0} B_2(\theta) > 0, \quad \text{or} \ \max_{A_2(\theta)=0} B_2(\theta) < 0, \tag{5.24}$$

then equation

$$u'' + u = \frac{a(x)}{u^3} \tag{5.25}$$

has a 2π-periodic solution.

Proof: Let (5.24) be satisfied. Then it is rather straightforward to see

$$deg(G(\theta), \mathbb{S}_1^1) = 0. \tag{5.26}$$

Consequently, Theorem 1.2 can be applied, and equation (5.25) has a 2π-periodic solution. □

We conclude with a generalization of Theorem 1.2. Consider $n\pi$-periodic solution of equation (5.25), where a is a positive, C^2 and $n\pi$-periodic function. The same argument works for this case. To state the result, as in 2π-periodic case, we need two functions A_n and B_n given by

$$A_n(\theta) = \sum_{j=1}^{n} \frac{a'(\theta + (j-1)\pi)}{\sqrt{a(\theta + (j-1)\pi)}} \tag{5.27}$$

and

$$B_n(\theta) = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{\left(\sum_{j=1}^{n} a'(\theta + t + (j-1)\pi) - \sum_{j=1}^{n} a'(\theta + (j-1)\pi)\right) \sin 2t}{\sin^2 t} dt. \tag{5.28}$$

They are $\pi-$periodic. A function a is called B-nondegenerate if $B_n(\theta)$ does not vanish whenever $A_n(\theta) = 0$. Let $G : \mathbb{S}_1^1 \to \mathbb{R}^2$, $G(\theta) = (A_n(\theta), B_n(\theta)), \theta \in \mathbb{S}_1^1$. We know that $deg(G, \mathbb{S}_1^1)$ if a is B-nondegenerate. Then we have

Theorem 5.8 *Let a be a positive, C^2 and $n\pi$-periodic and B-nondegenerate function. Then equation (5.25) has an $n\pi-$periodic solution provided that $deg(G, \mathbb{S}_1^1) \neq -1$.*

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