MUTUALLY EXCLUSIVE SPIKY PATTERN AND SEGMENTATION MODELED BY THE FIVE-COMPONENT MEINHARDT-GIERER SYSTEM*

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Abstract. We consider the five-component Meinhardt-Gierer model for mutually exclusive patterns and segmentation which was proposed in [11]. We prove rigorous results on the existence and stability of mutually exclusive spikes which are located in different positions for the two activators. Sufficient conditions for existence and stability are derived, which depend in particular on the relative size of the various diffusion constants. Our main analytical methods are the Liapunov-Schmidt reduction and nonlocal eigenvalue problems. The analytical results are confirmed by numerical simulations.

Key words. Pattern Formation, Mutual Exclusion, Stability, Steady states

AMS subject classifications. Primary 35B35, 92C15; Secondary 35B40, 92D25

1. Introduction. We analyze the five-component Meinhardt-Gierer system whose components are two activators and one inhibitor as well as two lateral activators. It has been introduced and very successfully used in various modeling aspects by Meinhardt and Gierer [11]. In particular, it can explain the phenomenon of mutual exclusion and handle segmentation in the simplest case of two different segments. This model has been reviewed and its many implications have been discussed in detail by Meinhardt in Chapter 12 of [10].

The most important features of this system can be highlighted as **lateral activation of mutually exclusive states**. To each of the local activators a lateral activator is associated in a spatially nonlocal and time-delayed way. The consequence of the presence of the two lateral activators in the system is the possibility to have stable patterns which for the two activators are mutually exclusive, or in other words, the patterns for the two activators are located in different positions. It is clear that mutually exclusive patterns are not possible for a three-component system with only two activators and one inhibitor since mutually exclusive patterns for the two activators could destabilize each other in various ways. Therefore the lateral activators are needed.

Numerical simulations of mutually exclusive patterns have been performed in [11], [10]. Many interesting features have been discovered and explained but those works do not give analytical solutions and they are not mathematically rigorous. To obtain mathematically rigorous results, in this study we show the existence and stability of mutually exclusive spikes in such a system.

The overall feedback mechanism of the system can be summarized as follows: **Lateral activation is coupled with self-activation and overall inhibition**. We will explain this in more detail after the system has been formulated quantitatively.

A widespread pattern in biology is **segmentation**. The mutual exclusion effect described in this paper is a special case of segmentation where only two different segments are present. Examples for biological segmentation are the body segments of insects or the segments of insect legs. The segments usually resemble each other strongly, but on the other hand they are different from each other. Segments may for example have an internal polarity which is often visible by bristles or hairs. This internal pattern within a segment depends on the position of the segment within the sequence in its natural state. In some biological cases a good understanding of how segment position and internal structure are related has been obtained. One famous example are surgical experiments on insects, e.g. for cockroach legs. Creating a discontinuity in the normal neighborhood of structures by cutting a leg and pasting one piece to the end of another partial leg creates a discontinuity in the segment structure as some segments are missing their natural neighbors. This forces the emergence of new stable patterns in the cockroach leg such that all segments get back their natural neighbors. However, the resulting pattern can be very different from any naturally occurring pattern.

For example for cockroach legs, if the normal sequence of structures within a segment is $123\dots9$, a combination of a partial leg 12345678 to which the piece 456789 is added first leads to the structure 12345678456789. Note the presence of the jump discontinuity in this sequence between the numbers 8 and 4. Now segment regulation adds the piece 765 which removes the discontinuity and leads to the final structure 12345678**765**456789. This is different from the original natural structure but nevertheless each segment has the same neighbors as in the natural situation.

In this example which was experimentally verified by Bohn [1], it is not the natural sequence but the normal neighborhood which is regulated. It is exactly this neighboring structure which can be modeled mathematically using the system from [11] which is considered here and this paper can be the starting point to a rigorous understanding of more complex segmentation phenomena.

Now we give a sociological application of mutual exclusion (see[11]): Consider two families. They can hardly live in exactly the same house as this would lead to overcrowding and is therefore less preferable. But if they live in the same street or neighborhood they can support, nurture and benefit each other. Thus this collaborative behavior can lead to a rather stable situation. Indeed, stable coexisting states with concentration peaks remaining close but keeping

*The work of JW is supported by an Earmarked Grant of RGC of Hong Kong. The work of MW is supported by a BRIEF Award of Brunel University. MW thanks the Department of Mathematics at CUHK for their kind hospitality.

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a certain characteristic distance from each other are typical phenomena which are observed in quantitative models of systems modelling mutual exclusion and they obviously resemble real-world behavior in this example very well.

This feedback mechanism of lateral activation coupled with overall inhibition can be quantified by formulating the effects of "activation", "lateral activation" and "inhibition" using the language of molecular reactions and invoking the law of mass action. Now we are going to discuss this in a quantitative manner. We will introduce the resulting model system first and then explain how these feedback mechanisms are represented by the terms in the model.

The original system from [11] (after re-scaling and some simplifications) can be stated as follows:

$$\begin{cases} g_{1,t} = \epsilon^2 g_{1,xx} - g_1 + \dfrac{cs_2 g_1^2}{r}, \quad g_{2,t} = \epsilon^2 g_{2,xx} - g_2 + \dfrac{cs_1 g_2^2}{r}, \\ \tau r_t = D_r r_{xx} - r + cs_2 g_1^2 + cs_1 g_2^2, \\ \tau s_{1,t} = D_s s_{1,xx} - s_1 + g_1, \quad \tau s_{2,t} = D_s s_{2,xx} - s_2 + g_2. \end{cases} \tag{1.1}$$

Here $0 < \epsilon << 1$, $D_r > 0$ and $D_s > 0$ are diffusion constants, c is a positive reaction constant and τ is nonnegative time-relaxation constant (in [11] the choice $\tau = 1$ was made).

The x-indices indicate spatial derivatives. We will derive results for the system (1.1) on a bounded interval $\Omega = (-L, L)$ for $L > 0$ with Neumann boundary conditions. Some results for the system on the real line ($L = \infty$) will also be established and they will be compared with the bounded interval case.

The first two components, the **activators** g_1 and g_2 activate themselves locally which is due to the terms g_1^2 and g_2^2, respectively, in the first two equations.

The lateral activators are introduced in (1.1) by the fourth and fifth components s_1 and s_2 as follows: To both the activators, g_i, $i = 1, 2$, there are nonlocal and delayed versions s_i. Now s_1 acts as an activator to g_2 and s_2 acts as in activator to g_1 due to the terms s_2 in the first and s_1 in the second equation which have a positive feedback. The expression lateral activation is used since g_i activate g_{3-i} laterally through its nonlocal counterpart s_i rather than locally through g_i itself.

Lateral activation is finally coupled with overall inhibition as follows: The third component r acts as an **inhibitor** to both g_1 and g_2 due to the term r in the first and second equations which has a negative feedback. Note also that both the local and the nonlocal activators have a positive feedback on r due to the terms $s_2 g_1^2$ and $s_1 g_2^2$ in the third equation.

This feedback mechanism is a generalization of the well-known Gierer-Meinhardt system [6] which has one local activator coupled to an inhibitor. We recall that the classical Gierer-Meinhardt system as well as the five-component system considered here are both Turing systems [13] as they allow spatial patterns to arise out of a homogeneous steady state by the so-called Turing instability. (Some analytical results for the existence and stability of spiky Turing pattern for the Gierer-Meinhardt system have been obtained for example in [3], [4], [5], [9], [12], [14], [17], [18], [19].)

Now we state our rigorous results on the existence and stability of stationary, mutually exclusive, spiky patterns for the system (1.1).

We prove the **existence** of a spiky pattern with one spike for g_1 and one spike for g_2 which are located in different positions under the following conditions:

(i) the diffusivities of the two lateral activators are large compared to the inhibitor diffusivity and

(ii) the inhibitor diffusivity is large compared to the diffusivities of the two (local) activators.

We summarize the two main conditions (i), (ii) which guarantee the existence of mutually-exclusive spike patterns for (1.1) in the following:

$$\text{We assume that} \quad \epsilon^2 << C_1 D_r \le D_s \quad \text{for some constant } C_1 > 0. \tag{1.2}$$

We also prove the **stability** of these mutually exclusive spiky patterns provided certain conditions are met which are of the type (1.2) with C_1 replaced by some new constant C_2.

In this paper we consider a pattern displaying one spike for g_1 and one for g_2 which are located in different positions.

In particular, we prove the existence of a mutually exclusive two-spike solution to the system (1.1) if $D_s/D_r > 4$. We show that this solution is stable if (i) $D_s/D_r > 43.33$ for $L = \infty$, or in general if (5.3) holds (condition for $O(1)$ eigenvalues) and if (ii) $D_s/D_r > 4$ (condition for $o(1)$ eigenvalues).

The main results will be stated in Theorem 3.2 (Section 3) on the existence of solutions and in Theorem 2 (Section 5) as well as Theorem 3 (Section 6) on the large and small eigenvalues of the linearized problem at the solutions, respectively.

What do these results tell us about segmentation? As a first step, we have proved that in the case of two segments which we call 1 and 2 the sequence 12 can exist and be stable, and we have found sufficient conditions for this effect to happen.

The case of $n > 2$ components will lead to a system with $2n + 1$ components which is very large and not easy to handle. Even in the case $n = 2$ for the five-component system investigated in this paper the analysis becomes rather lengthy. We expect that, following our approach, we will be able to prove existence and stability of n spikes in n different locations. We do not see any major obstacle, only the proofs become more technical. We are currently working on this issue.

The outline of the paper is as follows: In Section 2, we compute the amplitudes. In Section 3, we locate the spikes and show the existence of solutions. In Section 4, we first derive the eigenvalue problem. Then we compute the large (i.e.

$O(1)$) eigenvalues and we derive sufficient conditions for the stability of solutions with respect to these. In Section 5, we solve a nonlocal eigenvalue problem which has been delayed from Section 4. In Section 6, we give the most important steps and state the main result on the stability of solutions with respect to small (i.e. $o(1)$) eigenvalues. Sufficient conditions for this stability are derived. The technical details of the analysis of small eigenvalues is delayed to the appendices. Finally, in Section 7, our results are confirmed by numerical simulations.

2. Computing the Amplitudes. We construct steady states of the form

$$g_1(x) = t_1 w\left(\frac{x-x_1}{\epsilon}\right)(1+O(\epsilon)), \quad g_2(x) = t_2 w\left(\frac{x-x_2}{\epsilon}\right)(1+O(\epsilon)),$$

where $w(y)$ is the unique positive and even homoclinic solution of the equation

$$w_{yy} - w + w^2 = 0 \tag{2.1}$$

on the real line decaying to zero at $\pm\infty$. Here we assume that the spikes for g_1 and g_2 have the same amplitude, i.e. $t_1 = t_2$. We often use different notations for the two amplitudes as this will be important later when we consider stability since there could be an instability which breaks the symmetry of having the same amplitudes. The analysis will show that t_1, t_2 and x_1, x_2 depend on ϵ but to leading order and after suitable scaling are independent of ϵ. To keep notation simple we will not explicitly indicate this dependence.

All functions used throughout the paper belong to the Hilbert space $H^2(-L,L)$ and the error terms are taken in the norm $H^2(-L,L)$ unless otherwise stated. After integrating (2.1), we get the relation

$$\int_R w(y)\,dy = \int_R w^2(y)\,dy \tag{2.2}$$

which will be used frequently, often without explicitly stating it. We denote

$$w_1(x) = w\left(\frac{x-x_1}{\epsilon}\right), \quad w_2(x) = w\left(\frac{x-x_2}{\epsilon}\right). \tag{2.3}$$

Note that g_1 and g_2 are small-scale variables, as $\epsilon << 1$, and r, s_1, and s_2 are large-scale (with respect to the spatial variable). For steady states, using Green functions, these slow variables, to leading order, can be expressed by an integral representation.

To get this representation, g_1 in the last three equations of (1.1) can be expanded as

$$g_1(x) = t_1\epsilon\left(\int_R w\right)\delta_{x_1}(x) + O(\epsilon^2), \quad g_1^2(x) = t_1^2\epsilon\left(\int_R w^2\right)\delta_{x_1}(x) + O(\epsilon^2),$$

where $\delta_{x_1}(x) = \delta(x-x_1)$ is the Dirac delta distribution located at x_1. Similarly, for g_2 we have

$$g_2(x) = t_2\epsilon\left(\int_R w\right)\delta_{x_2}(x) + O(\epsilon^2), \quad g_2^2(x) = t_2^2\epsilon\left(\int_R w^2\right)\delta_{x_2}(x) + O(\epsilon^2).$$

Using the Green function $G_D(x,y)$ which is defined as the unique solution of the equation

$$D\Delta G_D(x,y) - G_D(x,y) + \delta_y(x) = 0, \quad -L < x < L, \quad G_{D,x}(-L,y) = G_{D,x}(L,y) = 0, \tag{2.4}$$

we can represent $s_1(x)$ by using the fourth equation of (1.1) as

$$s_1(x) = t_1\epsilon\left(\int_R w\right)G_{D_s}(x,x_1) + O(\epsilon^2). \tag{2.5}$$

An elementary calculation gives

$$G_D(x,y) = \begin{cases} \frac{\theta}{\sinh(2\theta L)}\cosh\theta(L+x)\cosh\theta(L-y), & -L < x < y < L, \\ \frac{\theta}{\sinh(2\theta L)}\cosh\theta(L-x)\cosh\theta(L+y), & -L < y < x < L \end{cases} \tag{2.6}$$

with $\theta = 1/\sqrt{D}$. Note that

$$G_D(x,y) = \frac{1}{2\sqrt{D}}e^{-|x-y|/\sqrt{D}} - H_D(x,y), \tag{2.7}$$

where H_D is the regular part of the Green function G_D. In particular, for $L = \infty$, we have

$$G_D(x_1,x_2) = \frac{1}{2\sqrt{D}}e^{-|x-y|/\sqrt{D}} =: K_D(x_1,x_2). \tag{2.8}$$

In the same way, we derive

$$s_2(x) = t_2\epsilon\left(\int_R w\right) G_{D_s}(x, x_2) + O(\epsilon). \tag{2.9}$$

Now we compute the last two terms on the r.h.s. of the third equation of (1.1) as follows:

$$cs_2 g_1^2(x) = cs_2(x_1) t_1^2 \epsilon \left(\int_R w\right) \delta_{x_1}(x) + O(\epsilon^2) = c t_1^2 t_2 \epsilon^2 \left(\int_R w\right)^2 \delta_{x_1}(x) G_{D_s}(x_1, x_2) + O(\epsilon^3)$$

and, similarly,

$$cs_1 g_2^2(x) = c t_1 t_2^2 \epsilon^2 \left(\int_R w\right)^2 \delta_{x_2}(x) G_{D_s}(x_1, x_2) + O(\epsilon^3).$$

Now, using the third equation of (1.1), we can represent $r(x)$ by the Green function G_{D_r}

$$r(x) = c t_1 t_2 \epsilon^2 \left(\int_R w\right)^2 G_{D_s}(x_1, x_2)(t_1 G_{D_r}(x, x_1) + t_2 G_{D_r}(x, x_2)) + O(\epsilon^3). \tag{2.10}$$

Going back to the first equation in (1.1), we get

$$\epsilon^2 \Delta g_1 - g_1 + \frac{cs_2 g_1^2}{r} = t_1(\epsilon^2 \Delta w_1 - w_1) + \frac{cs_2 t_1^2 w_1^2}{r} + O(\epsilon) = t_1 \left[\frac{cs_2 t_1}{r} - 1\right] w_1^2 + O(\epsilon). \tag{2.11}$$

To have the same amplitudes of the two contributions in (2.11), we require

$$\frac{cs_2(x_1) t_1}{r(x_1)} = 1 + O(\epsilon). \tag{2.12}$$

Now we rewrite (2.12), using (2.9) and (2.10):

$$\frac{cs_2(x_1) t_1}{r(x_1)} = \frac{1}{\epsilon(\int_R w)(t_1 G_{D_r}(x_1, x_1) + t_2 G_{D_r}(x_1, x_2))} + O(\epsilon). \tag{2.13}$$

Thus, (2.12), for $x = x_1$, gives

$$t_1 G_{D_r}(x_1, x_1) + t_2 G_{D_r}(x_1, x_2) = \frac{1}{\epsilon \int_R w} + O(1). \tag{2.14}$$

In the same way, from the second equation in (1.1), we get

$$t_1 G_{D_r}(x_1, x_2) + t_2 G_{D_r}(x_2, x_2) = \frac{1}{\epsilon \int_R w} + O(1). \tag{2.15}$$

The relations (2.14), (2.15) are a linear system for the amplitudes t_1, t_2 of the spikes if their positions state that the amplitudes x_1, x_2 are known. Note that the amplitudes depend on the positions in leading order as also the Green function G_{D_r} depends on its arguments in leading order. We say that the amplitudes are strongly coupled to the positions.

Note that the system (2.14), (2.15) has a unique solution t_1, t_2 since by (2.6)

$$G_{D_r}(x_1, x_1) G_{D_r}(x_2, x_2) - (G_{D_r}(x_1, x_2))^2 = \frac{\theta_r^2}{\sinh^2(2\theta_r L)} \cosh\theta_r(L - x_1)\cosh\theta_r(L + x_2)$$

$$\times[\cosh\theta_r(L + x_1)\cosh\theta_r(L - x_2) - \cosh\theta_r(L - x_1)\cosh\theta_r(L + x_2)] > 0$$

for $-L < x_2 < x_1 < L$, where $\theta_r = 1/\sqrt{D_r}$.

By symmetry, for $x_1 = -x_2$, we have $t_1 = t_2$. This is the case we are interested in. But we have not shown that there are such positions x_1, x_2, yet. This will be done in the next section.

For the special case $L = \infty$, we have $G_{D_r}(x_1, x_2) = \frac{1}{2\sqrt{D_r}} e^{-|x-y|/\sqrt{D_r}}$ and (2.14), (2.15) in this case are given by

$$t_1 + t_2 e^{-|x_1 - x_2|/\sqrt{D_r}} = \frac{2\sqrt{D_r}}{\epsilon \int_R w}, \quad t_2 + t_1 e^{-|x_1 - x_2|/\sqrt{D_r}} = \frac{2\sqrt{D_r}}{\epsilon \int_R w}.$$

Finally, we summarize the main result of this section

LEMMA 2.1. *Assume that $\epsilon > 0$ is small enough. Then for spike-solutions of (1.1) of the type*

$$g_1(x) = t_1 w\left(\frac{x-x_1}{\epsilon}\right)(1+O(\epsilon)), \quad g_2(x) = t_2 w\left(\frac{x-x_2}{\epsilon}\right)(1+O(\epsilon)),$$

where $w(y)$ is the unique positive and even solution of the equation

$$w_{yy} - w + w^2 = 0$$

on the real line decaying to zero at $\pm\infty$, the amplitudes t_1 and t_2 are given as the unique solution of the system

$$t_1 G_{D_r}(x_1,x_1) + t_2 G_{D_r}(x_1,x_2) = \frac{1}{\epsilon \int_R w} + O(1), \quad t_1 G_{D_r}(x_1,x_2) + t_2 G_{D_r}(x_2,x_2) = \frac{1}{\epsilon \int_R w} + O(1),$$

where G_D is the Green function defined in (2.4).

3. Existence of Mutually Exclusive Spikes. In this section, we use the Liapunov-Schmidt reduction method to rigorously prove the existence of mutually exclusive spikes. We will get a sufficient condition on the locations of the spikes.

The problem here is that the linearization of the r.h.s. of the first equation in (1.1) around w_1 has an approximate nontrivial kernel. This comes from the fact that a derivative of the equation (2.1) with respect to y gives

$$(w_y)_{yy} - w_y + 2ww_y = 0.$$

Thus, w_y belongs to the kernel of the linearization of (2.1) around w. Note that the function w_y represents the translation mode. Therefore a direct application of the implicit function theorem is not possible, but one has to deal with this kernel first. This is the goal in this section.

Recall that for given $g_1, g_2 \in H^2_N(\Omega_\epsilon)$, where $\Omega_\epsilon = (-L/\epsilon, L/\epsilon)$ and $H^2_N(\Omega_\epsilon)$ denotes the space of all functions in $H^2(\Omega_\epsilon)$ satisfying the Neumann boundary condition, by the fourth equation of (1.1) s_1 is uniquely determined, by the fifth equation s_2 is uniquely determined, and finally by the third equation r is uniquely determined. Therefore, the steady state problem is reduced to solving the first two equations.

We are looking for solutions which satisfy

$$g_1(x) = t_1 w\left(\frac{x-x_1}{\epsilon}\right)(1+O(\epsilon)), \quad g_2(x) = t_1 w\left(\frac{x+x_1}{\epsilon}\right)(1+O(\epsilon))$$

with $g_1(x) = g_2(-x)\,(x_1 > 0)$. By this reflection symmetry the problem is reduced to determining just one function: $g_1(x) = t_1 w_1(x) + v$.

We are now going to determine this function in **two steps**. Denoting the r.h.s. of the first equation of (1.1) by $S_\epsilon[t_1 w_1 + v]$, which is well-defined for steady states, our problem can be written as follows: $S_\epsilon[t_1 w_1 + v] = 0$, where $S_\epsilon : H^2_N(\Omega_\epsilon) \to L^2(\Omega_\epsilon)$.

First Step. Determine a small $v \in H^2(\Omega_\epsilon)$ with $\int_\Omega v \frac{dw_1}{dx}\, dx = 0$ such that

$$S_\epsilon[t_1 w_1 + v] = \beta\epsilon \frac{dw_1}{dx}. \tag{3.1}$$

Second Step. Choose x_1 such that

$$\beta = 0. \tag{3.2}$$

We begin with the **first** step. To this end, we need to study the linearized operator

$$\tilde{L}_{\epsilon,x_1} : H^2(\Omega_\epsilon) \to L^2(\Omega_\epsilon) \quad \text{defined by} \quad \tilde{L}_{\epsilon,x_1} := S_\epsilon'[t_1 w_1]\phi,$$

where $S_\epsilon'[t_1 w_1]$ denotes the Frechet derivative of the operator S_ϵ at $t_1 w_1$.

We define the approximate kernel and co-kernel, respectively, as follows:

$$\mathcal{K}_{\epsilon,x_1} := \text{span}\left\{\epsilon\frac{dw_1}{dx}\right\} \subset H^2(\Omega_\epsilon), \quad \mathcal{C}_{\epsilon,x_1} := \text{span}\left\{\epsilon\frac{dw_1}{dx}\right\} \subset L^2(\Omega_\epsilon).$$

By projection, we define the operator

$$L_{\epsilon,x_1} = \pi^\perp_{\epsilon,x_1} \circ \tilde{L}_{\epsilon,x_1} : \mathcal{K}^\perp_{\epsilon,x_1} \to \mathcal{C}^\perp_{\epsilon,x_1},$$

where $\pi^\perp_{\epsilon,x_1}$ is the orthogonal projection in $L^2(\Omega_\epsilon)$ onto $\mathcal{C}^\perp_{\epsilon,x_1}$.

Then we have the following key result for the Liapunov-Schmidt reduction.

PROPOSITION 3.1. *There exist positive constants* $\bar{\epsilon}$, $\bar{\delta}, \lambda$ *such that we have for all* $\epsilon \in (0, \bar{\epsilon})$, $x_1 \in \Omega$ *with* $\min(|L + x_1|, |L - x_1|) > \bar{\delta}$,

$$\|L_{\epsilon,x_1}\phi\|_{L^2(\Omega_\epsilon)} \geq \lambda \|\phi\|_{H^2(\Omega_\epsilon)} \quad \text{for all} \quad \phi \in \mathcal{K}^\perp_{\epsilon,x_1}. \tag{3.3}$$

Further, the map L_{ϵ,x_1} *is surjective.*

Proof of Proposition 3.1: We proceed by deriving a contradiction.

Suppose that (3.3) is false. Then there exist sequences $\{\epsilon_k\}$, $\{{x_1}^k\}$, $\{\phi^k\}$ with $\epsilon_k \to 0$, ${x_1}^k \in \Omega$, $\min(|L + x_1^k|, |L - x_1^k|) > \bar{\delta}$, $\phi^k = \phi_{\epsilon_k} \in K^\perp_{\epsilon_k, x_1^k}$, $k = 1, 2, \ldots$ such that

$$\|L_{\epsilon_k, {x_1}^k}\phi^k\|_{L^2(\Omega_{\epsilon_k})} \to 0, \quad \text{as } k \to \infty, \quad \|\phi^k\|_{H^2(\Omega_{\epsilon_k})} = 1, \quad k = 1,\, 2,\, \ldots\,. \tag{3.4}$$

At first (after rescaling) ϕ_ϵ is only defined on Ω_ϵ. However, by a standard result (compare [7]) it can be extended to R such that its norm in $H^2(R)$ is still bounded by a constant independent of ϵ and x_1 for ϵ small enough. It is then a standard procedure to show that this extension converges strongly in $H^2(\Omega_\epsilon)$ to some limit ϕ_1 with $\|\phi_1\|_{L^2(R)} = 1$. For the details of the argument, we refer to [8].

The same analysis is performed for w_2 and its perturbation $\phi_{\epsilon,2}$. Then $\Phi = (\phi_1, \phi_2)^T$ solves the system

$$L_0\phi_1 - \frac{1}{\int_R w\, dy}\left[2\hat{t}_1 G_{D_r}(x_1, x_1)\left(\int_R w\phi_1\, dy\right) + 2\hat{t}_1 G_{D_r}(x_1, x_2)\left(\int_R w\phi_2\, dy\right)\right.$$

$$\left. + \hat{t}_2 G_{D_r}(x_1, x_2)\left(\int \phi_1\, dy\right) - \hat{t}_1 G_{D_r}(x_1, x_2)\left(\int \phi_2\, dy\right)\right] = 0, \tag{3.5}$$

$$L_0\phi_2 - \frac{1}{\int_R w\, dy}\left[2\hat{t}_2 G_{D_r}(x_2, x_2)\left(\int w\phi_2\, dy\right) + 2\hat{t}_2 G_{D_r}(x_1, x_2)\left(\int w\phi_1\, dy\right)\right.$$

$$\left. + \hat{t}_1 G_{D_r}(x_1, x_2)\left(\int_R \phi_2\, dy\right) - \hat{t}_2 G_{D_r}(x_1, x_2)\left(\int_R \phi_1\, dy\right)\right] = 0, \tag{3.6}$$

where $L_0\phi = \epsilon^2\phi_{yy} - \phi + 2w\phi$ and

$$\alpha_\epsilon = \left(\frac{1}{\epsilon \int_R w\, dy}\right) \quad \text{and} \quad \hat{t}_i = (\alpha_\epsilon)^{-1} t_i. \tag{3.7}$$

This system is the special case with $\lambda = 0$ of (4.7), (4.8) derived in Section 4. To avoid doing this computation twice we have delayed it to Section 4, where the more general case is considered.

Now, adding (3.5) and (3.6), we obtain

$$L_0(\phi_1 + \phi_2) - w^2\left(\frac{2\int_R w(\phi_1 + \phi_2)\, dy}{\int_R w^2\, dy}\right) = 0.$$

This implies by Theorem 1.4 of [15] that $\phi_1 = -\phi_2$, and, setting $\phi := \phi_1$, for ϕ we must have

$$L_0\phi - \frac{4}{4 - c_0}\frac{w^2}{\int w^2\, dy}\int w\phi\, dy = \lambda\phi, \tag{3.8}$$

where $0 < c_0 < 2$ (compare (5.1) for $\lambda = 0$). Now by Theorem 1.4 of [15] we must have $\phi = 0$. This contradicts $\|\phi\|_{L^2(R)} = 1$. Therefore, (3.3) must be true.

By the Closed Range Theorem it follows that the map L_{ϵ,x_1} is surjective. (The details are given for example in [8].) .3cm

Based on this key result for the Liapunov-Schmidt reduction it is now fairly standard (see for example the works [8] and [16]) to derive that there exists a small $v \in H^2(\Omega_\epsilon)$ with $\int_\Omega v \frac{dw_1}{dx}\, dx = 0$ such that

$$S[t_1 w_1 + v] = \beta\epsilon\frac{dw_1}{dx}.$$

This completes the **first** step.

We now turn to the **second** step. We have to show that $\beta = 0$ for a certain x_1. This amounts to showing that

$$\int_\Omega S[t_1 w_1 + v](x)\epsilon\frac{dw_1}{dx}\, dx = 0$$

for a certain x_1. Note that computing x_1 in fact means determining the locations of the spikes.

To this end, we have to expand $S[t_1 w_1 + v](x_1 + \epsilon y)$.

We compute

$$S[t_1 w_1 + v](x_1 + \epsilon y) = t_1 \left[\frac{c s_2(x_1 + \epsilon y) t_1}{r(x_1 + \epsilon y)} - 1 \right] w_1^2(x_1 + \epsilon y) + O(\epsilon^2).$$

Using (2.9), (2.10) and the expansions

$$G_D(x_1 + \epsilon y, x_2) = G_D(x_1, x_2) + G_{D,x_1}(x_1, x_2)\epsilon y + O(\epsilon^2 |y|^2)$$

and

$$G_D(x_1 + \epsilon y, x_1) = G_D(x_1, x_1) - \frac{1}{2D}\epsilon|y| - \frac{1}{2} H_{D,x_1}(x_1, x_1)\epsilon y + O(\epsilon^2 |y|^2),$$

where we have used (2.7), we get

$$\frac{c s_2(x_1 + \epsilon y) t_1}{r(x_1 + \epsilon y)} = \frac{G_{D_r}(x_1, x_1) + G_{D_r}(x_1, -x_1)}{G_{D_s}(x_1, -x_1)}$$

$$\times \frac{G_{D_s}(x_1, -x_1) + \frac{1}{2} G_{D_s,x_1}(x_1, -x_1)\epsilon y + O(\epsilon^2 |y|^2)}{G_{D_r}(x_1, x_1) + G_{D_r}(x_1, -x_1) - \epsilon|y|/(2D) + \frac{1}{2}(-H_{D_r,x_1}(x_1, x_1) + G_{D_r,x_1}(x_1, -x_1))\epsilon y}$$

$$= 1 + \frac{G_{D_s,x_1}(x_1, -x_1)}{2 G_{D_s}(x_1, -x_1)}\epsilon y - \frac{G_{D_r,x_1}(x_1, -x_1) - H_{D_r,x_1}(x_1, x_1)}{2[G_{D_r}(x_1, x_1) + G_{D_r}(x_1, -x_1)]}\epsilon y + O(\epsilon^2 y^2) + \text{even term in } y. \tag{3.9}$$

This implies

$$\int_\Omega S[w_1 + v](x)\epsilon \frac{dw_1}{dx}\, dx =$$

$$= \frac{1}{2}\left[\frac{G_{D_s,x_1}(x_1, -x_1)}{G_{D_s}(x_1, -x_1)} - \frac{G_{D_r,x_1}(x_1, -x_1) - H_{D_r,x_1}(x_1, x_1)}{G_{D_r}(x_1, x_1) + G_{D_r}(x_1, -x_1)} \right] \epsilon y \int_R y w^2 \frac{dw}{dy}\, dy + \epsilon^2 W_\epsilon(x_1),$$

where $W_\epsilon(x_1) = O(\epsilon)$, uniformly for $0 \le x_1 \le L$.

Using (2.6), we further compute

$$F(x_1) := \frac{G_{D_s,x_1}(x_1, -x_1)}{G_{D_s}(x_1, -x_1)} - \frac{G_{D_r,x_1}(x_1, -x_1) - H_{D_r,x_1}(x_1, x_1)}{G_{D_r}(x_1, x_1) + G_{D_r}(x_1, -x_1)}$$

$$= -\theta_s \frac{\sinh 2\theta_s (L - x_1)}{\cosh^2 \theta_s (L - x_1)} - \theta_r \frac{\sinh 2\theta_r x_1 - \sinh 2\theta_r (L - x_1)}{\cosh \theta_r (L - x_1)[\cosh \theta_r (L - x_1) + \cosh \theta_r (L + x_1)]},$$

where $\theta = 1/\sqrt{D}$. We have to determine x_1 such that $F(x_1) = 0$. Note that

$$F(0) = -\theta_s \frac{\sinh 2\theta_s L}{\cosh^2 \theta_s L} + \theta_r \frac{\sinh 2\theta_r L}{2\cosh^2 \theta_r L} > 0$$

if

$$\frac{\theta_s}{\theta_r} < \frac{1}{2} \frac{\tanh \theta_r L}{\tanh \theta_s L}. \tag{3.10}$$

The inequality (3.10) is satisfied if, for fixed L, θ_r is large compared to θ_s.

In the limit $L \to 0$ the condition (3.10) converges to $\frac{\theta_s}{\theta_r} < 1/\sqrt{2}$. In the limit $L \to \infty$, (3.10) gives $\frac{\theta_s}{\theta_r} < 1/2$. For general $L \in (0, \infty)$ we can write (3.10) as follows: $\frac{\theta_s}{\theta_r} < \alpha(L)$ with $\frac{1}{2} < \alpha(L) < \frac{1}{\sqrt{2}}$.

Going back to the original diffusion constants, the inequality (3.10) is equivalent to

$$\frac{D_s}{D_r} > 4 \frac{\tanh^2 \theta_s L}{\tanh^2 \theta_r L}. \tag{3.11}$$

In the limit $L \to 0$, (3.11) gives $\frac{D_s}{D_r} > 2$ and, in the limit $L \to \infty$, we can write (3.11) as follows: $\frac{D_s}{D_r} > 4$.

For all $L \in (0, \infty)$ we can write (3.11) as follows: $\frac{D_s}{D_r} > \beta(L)$ for some continuous function $\beta(L) \in (2, 4)$.

Note that (3.11) holds if

$$\frac{D_s}{D_r} > 4. \tag{3.12}$$

This is not the optimal condition, but it is rather handy and easy to check.

On the other hand,

$$F(L/2) = -\theta_s \frac{\sinh \theta_s L}{\cosh^2(\theta_s L/2)} < 0.$$

By the intermediate value theorem, under the condition (3.11), there exists an $x_1 \in (0, L/2)$ such that $F(x_1) = 0$. There exists no such $x_1 \in [L/2, L)$ since the function F is negative in that interval.

Note that $F(L/2) \to 0$ as $\theta_s \to 0$. This implies that $x_1 \to L/2$ as $\theta_s \to 0$.

We now show that the zero $x_1 \in [0, L/2]$ of F is unique by proving that $F'(x_1) < 0$ for $x_1 \in (0, L/2)$ if

$$\frac{\theta_s}{\theta_r} < \frac{\tanh(\theta_r L/2)}{\sqrt{2}\tanh(\theta_s L/2)}. \tag{3.13}$$

We compute

$$F'(x_1) = 2\theta_s^2 \frac{1}{\cosh^2 \theta_s(L - x_1)} - \theta_r^2 \frac{1}{\cosh^2 \theta_r(L - x_1)}$$

$$-\theta_r^2 \frac{[\cosh \theta_r(L - x_1) + \cosh \theta_r(L + x_1)]^2 - [\sinh \theta_r(L - x_1) + \sinh \theta_r(L + x_1)]^2}{[\cosh \theta_r(L - x_1) + \cosh \theta_r(L + x_1)]^2}.$$

Therefore, taking into consideration only the first two terms and noting that the last term is negative, we have $F'(x_1) < 0$ if (3.13) holds, and in this case, the solution for x_1 is unique.

Note that (3.13) holds if $\frac{\theta_s}{\theta_r} < \frac{1}{\sqrt{2}}$ or, equivalently, $\frac{D_s}{D_r} > 2$.

Therefore (3.10) and (3.13) are both true if $\frac{\theta_s}{\theta_r} < \frac{1}{2}$ or, equivalently, $\frac{D_s}{D_r} > 4$.

Now for (3.13), since $F'(x_1) \neq 0$, a standard degree argument shows that for $\epsilon << 1$ there exists a unique x_1^ϵ depending on ϵ such that $\int_\Omega S[w_1 + v](x)\epsilon \frac{dw_1}{dx}\, dx = 0$. Further, $x_1^\epsilon \to x_1$ as $\epsilon \to 0$, where x_1 satisfies

$$\frac{G_{D_s, x_1}(x_1, -x_1)}{G_{D_s}(x_1, -x_1)} - \frac{G_{D_r, x_1}(x_1, -x_1) - H_{D_r, x_1}(x_1, x_1)}{G_{D_r}(x_1, x_1) + G_{D_r}(x_1, -x_1)} = 0.$$

Thus we have shown existence and at the same time located the positions of the spikes. We summarize this result in the following theorem:

THEOREM 3.2. *There exist mutually exclusive, spiky steady states to (1.1) in $(-L, L)$ with Neumann boundary conditions such that*

$$g_1^\epsilon(x) = t_1^\epsilon w\left(\frac{x - x_1^\epsilon}{\epsilon}\right)(1 + O(\epsilon)), \quad g_2^\epsilon(x) = t_1^\epsilon w\left(\frac{x + x_1^\epsilon}{\epsilon}\right)(1 + O(\epsilon)) \tag{3.14}$$

with

$$t_1^\epsilon = \frac{1}{\epsilon \int_R w\, dy\, (G_{D_r}(x_1, x_1) + G_{D_r}(x_1, -x_1))} + O(1) \tag{3.15}$$

and $x_1^\epsilon \to x_1$ as $\epsilon \to 0$, where

$$\frac{G_{D_s, x_1}(x_1, -x_1)}{G_{D_s}(x_1, -x_1)} - \frac{G_{D_r, x_1}(x_1, -x_1) - H_{D_r, x_1}(x_1, x_1)}{G_{D_r}(x_1, x_1) + G_{D_r}(x_1, -x_1)} = 0. \tag{3.16}$$

If $D_s/D_r > 4$ equation (3.16) has a unique solution $x_1 \in (0, L/2]$ and no solution in $(L/2, L]$. Further, $x_1 \to L/2$ as $\theta_s \to 0$.

Finally, we compute the equation for x_1 in the limit $L \to \infty$. In this limit, x_1 satisfies

$$\frac{\theta_s}{\theta_r} = \frac{e^{-2\theta_r x_1}}{1 + e^{-2\theta_r x_1}} + O(e^{-CL})$$

for some $C > 0$ independent of x_1. This is equivalent to

$$e^{2|x_1|/\sqrt{D_r}} = \sqrt{\frac{D_s}{D_r}} - 1 + O(e^{-CL}). \tag{3.17}$$

This concludes our study of existence. In the following sections we consider the stability issue.

4. Stability I: The Eigenvalue Problem and the Large Eigenvalues. Now we study the (linearized) stability of this mutually exclusive steady state. To this end, we first derive the linearized operator around the steady state $(g_1^\epsilon, g_2^\epsilon, r^\epsilon, s_1^\epsilon, s_2^\epsilon)$ given in Theorem 1.

We perturb the steady state as follows:

$$g_1 = g_1^\epsilon + \phi_1^\epsilon e^{\lambda t}, \quad g_2 = g_2^\epsilon + \phi_2^\epsilon e^{\lambda t}, \quad r = r^\epsilon + \psi^\epsilon e^{\lambda t},$$

$$s_1 = s_1^\epsilon + \eta_1^\epsilon e^{\lambda t}, \quad s_2 = s_2^\epsilon + \eta_2^\epsilon e^{\lambda t}.$$

By linearization we obtain the following eigenvalue problem (dropping superscripts ϵ):

$$\begin{cases} \lambda_\epsilon \phi_1 = \epsilon^2 \phi_{1,xx} - \phi_1 + \dfrac{c\eta_2 g_1^2}{r} + \dfrac{2cs_2 g_1 \phi_1}{r} - \dfrac{cs_2 g_1^2 \psi}{r^2} \\ \lambda_\epsilon \phi_2 = \epsilon^2 \phi_{2,xx} - \phi_2 + \dfrac{c\eta_1 g_2^2}{r} + \dfrac{2cs_1 g_2 \phi_2}{r} - \dfrac{cs_1 g_2^2 \psi}{r^2}, \\ \tau\lambda_\epsilon \psi = D_r \psi_{xx} - \psi + c\eta_2 g_1^2 + 2cs_2 g_1 \phi_1 + c\eta_1 g_2^2 + 2cs_1 g_2 \phi_2, \\ \tau\lambda_\epsilon \eta_1 = D_s \eta_{1,xx} - \eta_1 + \phi_1, \\ \tau\lambda_\epsilon \eta_2 = D_s \eta_{2,xx} - \eta_2 + \phi_2. \end{cases} \tag{4.1}$$

where all components belong to the space $H_N^2(\Omega)$.

We now analyze the case $\lambda_\epsilon \to \lambda_0 \neq 0$ (large eigenvalues). After re-scaling and taking the limit $\epsilon \to 0$ in (4.1) and noting that ϕ_i converges locally in $H^2(-L/\epsilon, L/\epsilon)$, we get for the first two components, using the approximations of g_1 and g_2 given in Theorem 3.2:

$$\epsilon^2 \Delta\phi_1 - \phi_1 + \frac{2cs_2(x_1)t_1 w_1 \phi_1}{r(x_1)} - \frac{cs_2(x_1)t_1^2 w_1^2}{r^2(x_1)}\psi(x_1) + \frac{c\eta_2(x_1)t_1^2 w_1^2}{r(x_1)} = \lambda\phi_1, \tag{4.2}$$

$$\epsilon^2 \Delta\phi_2 - \phi_2 + \frac{2cs_1(x_2)t_2 w_2 \phi_2}{r(x_2)} - \frac{cs_1(x_2)t_2^2 w_2^2}{r^2(x_2)}\psi(x_2) + \frac{c\eta_2(x_2)t_2^2 w_2^2}{r(x_2)} = \lambda\phi_1. \tag{4.3}$$

Now, in (4.2) and (4.3) we calculate the terms $\psi(x)$ and $\eta_1(x)$ and $\eta_2(x)$, respectively. To get $\psi(x)$, using the Green function G_{D_r}, we solve the linear equation for ψ given by

$$D_r \psi_{xx} - \psi + 2cs_2 t_1 w_1 \phi_1 + 2cs_1 t_2 w_2 \phi_2 + c\eta_2 t_1^2 w_1^2 + c\eta_1 t_2^2 w_2^2 = 0,$$

where again for g_1 and g_2 we have used the asymptotic expansions of Theorem 3.2. For simplicity, we study the case $\tau = 0$. Then the stability result extends to small τ as well, since we know that $|\lambda_\epsilon| \leq C$ for all eigenvalues such that $\lambda_\epsilon > -c_0$ for some small $c_0 > 0$, which can be shown by a simple argument based on quadratic forms. This gives

$$\psi(x) \sim \left[2cs_2(x_1)t_1\epsilon(\int_R w\phi_1\,dy) + c\eta_2(x_1)t_1^2\epsilon \int_R w^2\,dy\right] G_{D_r}(x, x_1)$$

$$+ \left[2cs_1(x_2)t_2\epsilon(\int_R w\phi_2\,dy) + c\eta_1(x_2)t_2^2\epsilon \int_R w^2\,dy\right] G_{D_r}(x, x_2). \tag{4.4}$$

Similarly, using G_{D_s}, we compute

$$\eta_1(x) \sim \epsilon G_{D_s}(x, x_1) \int_R \phi_1\,dy, \quad \eta_2(x) \sim \epsilon G_{D_s}(x, x_2) \int_R \phi_2\,dy. \tag{4.5}$$

Recalling from (2.5) and (2.9) that

$$s_1(x) \sim \epsilon t_1(\int_R w\,dy) G_{D_s}(x, x_1), \quad s_2(x) \sim \epsilon t_2(\int_R w\,dy) G_{D_s}(x, x_2),$$

we get from (4.4)

$$\psi(x) \sim \left[2ct_1t_2\epsilon^2(\int_R w\,dy)(\int_R w\phi_1\,dy) + ct_1^2\epsilon^2(\int_R w\,dy)\int_R \phi_2\,dy\right] G_{D_s}(x_1, x_2) G_{D_r}(x, x_1)$$

$$+ \left[2ct_1t_2\epsilon^2(\int_R w\,dy)(\int_R w\phi_2\,dy) + ct_2^2\epsilon^2(\int_R w\,dy)\int_R \phi_1\,dy\right] G_{D_s}(x_1, x_2) G_{D_r}(x, x_2). \tag{4.6}$$

Further, recall from (2.10) that

$$r(x) = ct_1t_2\epsilon^2(\int_R w\,dy)^2 G_{D_s}(x_1,x_2)(t_1G_{D_r}(x,x_1)+t_2G_{D_r}(x,x_2)) + O(\epsilon^3).$$

Substituting into (4.2), we get for the coefficient of $\int_R \phi_1\,dy$ on the r.h.s.

$$-\frac{cs_2(x_1)t_1^2w_1^2}{r^2(x_1)}c\epsilon^2(\int_R w\,dy)t_2^2G_{D_s}(x_1,x_2)G_{D_r}(x_1,x_2)+O(\epsilon^2)$$

$$= -\frac{w_1^2}{s_2(x_1)}\epsilon^2(\int_R w\,dy)t_2^2G_{D_s}(x_1,x_2)G_{D_r}(x_1,x_2)+O(\epsilon^2) = -\epsilon t_2 w_1^2 G_{D_r}(x_1,x_2)+O(\epsilon^2).$$

Similarly, the coefficient for $\int_R \phi_2\,dy$ is calculated as

$$-\frac{cs_2(x_1)t_1^2w_1^2}{r^2(x_1)}c\epsilon^2(\int_R w^2\,dy)t_1^2G_{D_s}(x_1,x_2)G_{D_r}(x_1,x_1)+\frac{c\epsilon G_{D_s}(x_1,x_2)t_1^2w_1^2}{r(x_1)}+O(\epsilon^2)$$

$$= -\frac{w_1^2}{s_2(x_1)}\epsilon^2(\int_R w^2\,dy)t_1^2G_{D_s}(x_1,x_2)G_{D_r}(x_1,x_1)+\frac{w_1^2}{s_2(x_1)}\epsilon t_1 G_{D_s}(x_1,x_2)+O(\epsilon^2)$$

$$= -\frac{\epsilon t_1^2 w_1^2}{t_2}G_{D_r}(x_1,x_1)+\frac{t_1}{t_2\int_R w\,dy}w_1^2+O(\epsilon^2) = \epsilon t_1 w_1^2 G_{D_r}(x_1,x_2)+O(\epsilon^2).$$

Here we have used (2.14). Then (4.2) gives the nonlocal eigenvalue problem (NLEP)

$$L_0\phi_1 - \frac{1}{\int_R w\,dy}\left[2\hat{t}_1G_{D_r}(x_1,x_1)\left(\int_R w\phi_1\,dy\right)+2\hat{t}_1G_{D_r}(x_1,x_2)\left(\int_R w\phi_2\,dy\right)\right.$$

$$\left.+\hat{t}_2G_{D_r}(x_1,x_2)\left(\int_R \phi_1\,dy\right)-\hat{t}_1G_{D_r}(x_1,x_2)\left(\int_R \phi_2\,dy\right)\right] = \lambda\phi_1, \tag{4.7}$$

where $L_0\phi = \epsilon^2\phi_{yy}-\phi+2w\phi$ and $\hat{t}_i$ has been defined in (3.7). In the same way, for (4.3) we obtain

$$L_0\phi_2 - \frac{1}{\int_R w\,dy}\left[2\hat{t}_2G_{D_r}(x_2,x_2)\left(\int_R w\phi_2\,dy\right)+2\hat{t}_2G_{D_r}(x_1,x_2)\left(\int_R w\phi_1\,dy\right)\right.$$

$$\left.+\hat{t}_1G_{D_r}(x_1,x_2)\left(\int_R \phi_2\,dy\right)-\hat{t}_2G_{D_r}(x_1,x_2)\left(\int_R \phi_1\,dy\right)\right] = \lambda\phi_2, \tag{4.8}$$

where $\phi_1,\ \phi_2 \in H^2(R)$. Set $\phi=(\phi_1,\phi_2)$ and denote by $L\phi$ the left-hand sides of (4.7) and (4.8), respectively.

Then, writing (4.7), (4.8) in matrix notation, we have following the vectorial NLEP:

$$L\phi = \Delta\phi-\phi+2w\phi-\left[\mathcal{B}\int_R\phi\,dy+2\mathcal{C}\left(\int_R w\phi\,dy\right)\right]\left(\int_R w\,dy\right)^{-1}w^2,$$

where

$$\mathcal{B} = G_{D_r}(x_1,x_2)\begin{pmatrix}\hat{t}_2 & -\hat{t}_1\\ -\hat{t}_2 & \hat{t}_1\end{pmatrix} = \frac{G_{D_r}(x_1,x_2)}{G_{D_r}(x_1,x_1)+G_{D_r}(x_1,x_2)}\begin{pmatrix}1 & -1\\ -1 & 1\end{pmatrix} \tag{4.9}$$

and

$$\mathcal{C} = \begin{pmatrix}\hat{t}_1G_{D_r}(x_1,x_1) & \hat{t}_1G_{D_r}(x_1,x_2)\\ \hat{t}_2G_{D_r}(x_1,x_2) & \hat{t}_2G_{D_r}(x_2,x_2)\end{pmatrix} = \frac{1}{G_{D_r}(x_1,x_1)+G_{D_r}(x_1,x_2)}\begin{pmatrix}G_{D_r}(x_1,x_1) & G_{D_r}(x_1,x_2)\\ G_{D_r}(x_1,x_2) & G_{D_r}(x_2,x_2)\end{pmatrix}. \tag{4.10}$$

Here we have used that (2.14), (2.15) imply

$$\hat{t}_1G_{D_r}(x_1,x_1)+\hat{t}_2G_{D_r}(x_1,x_2)=1,\quad \hat{t}_1G_{D_r}(x_1,x_2)+\hat{t}_2G_{D_r}(x_2,x_2)=1 \tag{4.11}$$

and therefore

$$\hat{t}_i = \frac{G_{D_r}(x_{3-i}, x_{3-i}) - G_{D_r}(x_1, x_2)}{G_{D_r}(x_1, x_1)G_{D_r}(x_2, x_2) - (G_{D_r}(x_1, x_2))^2}, \quad i = 1, 2. \tag{4.12}$$

In the special case when $G_{D_r}(x_1, x_1) = G_{D_r}(x_2, x_2)$ we have

$$\hat{t}_1 = \hat{t}_2 = \frac{1}{G_{D_r}(x_1, x_1) + G_{D_r}(x_1, x_2)}. \tag{4.13}$$

Now, adding (4.7) and (4.8), we obtain

$$L_0(\phi_1 + \phi_2) - w^2 \left(\frac{2\int_R w(\phi_1 + \phi_2)\, dy}{\int_R w^2\, dy} \right) = \lambda(\phi_1 + \phi_2)$$

which implies by Theorem 1.4 of [15] that $\phi_1 + \phi_2 = 0$ if $\mathrm{Re}(\lambda_0) \geq 0$. So we set $\phi_2 = -\phi_1 = -\phi$.

From (4.7), we obtain a scalar NLEP for ϕ

$$L_0\phi - \frac{w^2}{\int_R w^2\, dy} \left[c_0 \int_R w\phi\, dy + d_0 \int_R \phi\, dy \right] = \lambda\phi, \tag{4.14}$$

where

$$c_0 = \frac{2(G_{D_r}(x_1, x_1) - G_{D_r}(x_1, x_2))}{G_{D_r}(x_1, x_1) + G_{D_r}(x_1, x_2)}, \quad d_0 = \frac{2G_{D_r}(x_1, x_2)}{G_{D_r}(x_1, x_1) + G_{D_r}(x_1, x_2)}. \tag{4.15}$$

Note that $0 < c_0 < 2$ and $0 < d_0 < 1$.

In the following section we study the NLEP (4.14). It determines the stability or instability of the large eigenvalues of (4.1) if $0 < \epsilon < \epsilon_0$ for a suitably chosen ϵ_0. By our analysis instabilities for small $\epsilon > 0$ imply instabilities for $\epsilon = 0$. On the other hand, by an argument of Dancer [2], an instability for $\epsilon = 0$ also gives an instability for small $\epsilon > 0$.

Note that the NLEP problem here is quite different from those studied in [4], [5], [14] and [15].

In the next section we study this eigenvalue problem and complete the investigation of $O(1)$ eigenvalues for (4.1).

5. Stability II: A Nonlocal Eigenvalue Problem. In this section, we study the NLEP (4.14) to determine if or if not there are large eigenvalues, i.e. eigenvalues of the order $O(1)$ as $\epsilon \to 0$, which destabilize the mutually exclusive spiky pattern. Integrating (4.14), we have

$$\int_R \phi\, dy = \frac{2 - c_0}{\lambda + 1 + d_0} \int_R w\phi\, dy.$$

Substituting this back into (4.14), we can eliminate the term $\int_R \phi\, dy$. This gives

$$L_0\phi - \mu(\lambda) \frac{w^2}{\int_R w^2\, dy} \int_R w\phi\, dy = \lambda\phi, \quad \text{where} \quad \mu(\lambda) = \frac{c_0\lambda + 2}{\lambda + 2 - c_0/2}. \tag{5.1}$$

Here we have used that $c_0 + 2d_0 = 2$. Applying inequality (2.22) of [18], we get

$$\frac{\int_R w^3\, dy}{\int_R w^2\, dy} |\mu(\lambda_0) - 1|^2 + \mathrm{Re}(\overline{\lambda_0}(\mu(\lambda_0) - 1)) \leq 0 \quad \text{if } \mathrm{Re}(\lambda_0) \geq 0. \tag{5.2}$$

Observe that after multiplying (2.1) by w and by w', respectively, and integrating we get

$$\int_R w^3\, dy = \frac{6}{5} \int_R w^2\, dy.$$

So, assuming without loss of generality that $\lambda_0 = +\sqrt{-1}\lambda_I$, we get for the l.h.s. in (5.2)

$$\frac{6}{5} \left| \frac{c_0\lambda_0 + 2}{\lambda_0 + 1 + d_0} - 1 \right|^2 + \mathrm{Re}\left(\overline{\lambda_0}\left(\frac{c_0\lambda_0 + 2}{\lambda_0 + 1 + d_0} - 1\right) \right)$$

$$= \frac{6}{5} \frac{(c_0 - 1)^2|\lambda_0|^2 + (1 - d_0)^2}{|\lambda_0 + 1 + d_0|^2} + \mathrm{Re}\left(\frac{(c_0|\lambda_0|^2 + 2\overline{\lambda_0})(\overline{\lambda_0} + 1 + d_0)}{|\lambda_0 + 1 + d_0|^2} \right)$$

$$= \frac{|\lambda_0|^2[1.2(1 - c_0)^2 + (1 + d_0)c_0 - 2] + 1.2(1 - d_0)^2}{|\lambda_0 + 1 + d_0|^2}.$$

Thus if $1.2(1-c_0)^2+(1+d_0)c_0-2>0$, we have stability by (5.2). Using $c_0+2d_0=2$, we calculate that this is equivalent to $7c_0^2-4c_0-8>0$ which is true if $c_0>\frac{2}{7}(1+\sqrt{15})\approx 1.3923$.

We compute, using (2.6),

$$c_0=\frac{2(\cosh\theta_r(L+x_1)-\cosh\theta_r(L-x_1))}{\cosh\theta_r(L+x_1)+\cosh\theta_r(L-x_1)},\quad d_0=\frac{2\cosh\theta_r(L-x_1)}{\cosh\theta_r(L+x_1)+\cosh\theta_r(L-x_1)}.$$

Note that for $L=\infty$ we have

$$c_0=\frac{2(e^{2\theta_r|x_1|}-1)}{e^{2\theta_r|x_1|}+1},\quad d_0=\frac{2}{e^{2\theta_r|x_1|}+1}.$$

By (3.17), this implies $\sqrt{\frac{D_s}{D_r}}-1=e^{2\theta_r|x_1|}>5.5822$ and $\frac{D_s}{D_r}>43.33$. If the last condition is valid, we have stability.

We summarize the stability result for the $O(1)$ eigenvalues as follows:

THEOREM 5.1. *The mutually exclusive, spiky steady state given in Theorem 3.2 is linearly stable with respect to large eigenvalues $\lambda_\epsilon=O(1)$ for $\tau\geq 0$ and $\epsilon>0$ small enough if*

$$\frac{\cosh\theta_r(L+x_1)-\cosh\theta_r(L-x_1)}{\cosh\theta_r(L+x_1)+\cosh_r\theta_r(L-x_1)}>\frac{1}{7}(1+\sqrt{15}). \tag{5.3}$$

For $L=\infty$, this corresponds to

$$\frac{D_s}{D_r}>43.33.$$

Now the study of the large eigenvalues is completed. In the next section we study the small eigenvalues.

6. Stability III: The Small Eigenvalues. Now we study the small eigenvalues for (6.3), namely those with $\lambda_\epsilon\to 0$ as $\epsilon\to 0$. In this section we summarize the main steps and results in several lemmas. Their proofs are rather technical and we therefore delay them to the appendices.

For given $f\in L^2(\Omega)$, let $T_r[f]$ be the unique solution in $H_N^2(\Omega)$ of the problem

$$D_r\Delta(T_r[f])-T_r[f]+\alpha_\epsilon f=0. \tag{6.1}$$

In the same way, the operator T_s is defined with D_r replaced by D_s.

Let

$$\bar g_{\epsilon,1}=\hat t_1 w_{\epsilon,x_1^\epsilon}+\phi_{\epsilon,x_1^\epsilon},\quad \bar g_2=\hat t_2 w_{\epsilon,x_2^\epsilon}+\phi_{\epsilon,x_2^\epsilon},$$

$$\bar r_\epsilon=cT_r[T_s[\bar g_{\epsilon,2}]\bar g_{\epsilon,1}^2]+T_s[\bar g_{\epsilon,1}]\bar g_{\epsilon,2}^2],\quad \bar s_{\epsilon,1}=T_s[\bar g_{\epsilon,2}],\quad \bar s_{\epsilon,2}=T_s[\bar g_{\epsilon,1}], \tag{6.2}$$

where $\hat t_i$ has been defined in (3.7) After re-scaling, the eigenvalue problem (4.1) becomes

$$\begin{cases}\lambda_\epsilon\phi_{\epsilon,1}=\epsilon^2\Delta\phi_{\epsilon,1}-\phi_{\epsilon,1}+\frac{c\eta_{\epsilon,2}\bar g_{\epsilon,1}^2}{\bar r_\epsilon}+\frac{2c\bar s_{\epsilon,2}\bar g_{\epsilon,1}\phi_{\epsilon,1}}{\bar r_\epsilon}-\frac{c\bar s_{\epsilon,2}\bar g_{\epsilon,1}^2\psi_\epsilon}{\bar r_\epsilon^2},\\ \lambda_\epsilon\phi_{\epsilon,2}=\epsilon^2\Delta\phi_{\epsilon,2}-\phi_{\epsilon,2}+\frac{c\eta_{\epsilon,1}\bar g_{\epsilon,2}^2}{\bar r_\epsilon}+\frac{2c\bar s_{\epsilon,1}\bar g_{\epsilon,2}\phi_{\epsilon,2}}{\bar r_\epsilon}-\frac{c\bar s_{\epsilon,1}\bar g_{\epsilon,2}^2\psi_\epsilon}{\bar r_\epsilon^2},\\ \tau\lambda_\epsilon\psi_\epsilon=D_r\Delta\psi_\epsilon-\psi_\epsilon+c\alpha_\epsilon\eta_{\epsilon,2}\bar g_{\epsilon,1}^2+2c\alpha_\epsilon\bar s_{\epsilon,2}\bar g_{\epsilon,1}\phi_{\epsilon,1}+c\alpha_\epsilon\eta_{\epsilon,1}\bar g_{\epsilon,2}^2+2c\alpha_\epsilon\bar s_{\epsilon,1}\bar g_{\epsilon,2}\phi_{\epsilon,2},\\ \tau\lambda_\epsilon\eta_{\epsilon,1}=D_s\Delta\eta_{\epsilon,1}-\eta_{\epsilon,1}+\alpha_\epsilon\phi_{\epsilon,1},\\ \tau\lambda_\epsilon\eta_{\epsilon,2}=D_s\Delta\eta_{\epsilon,2}-\eta_{\epsilon,2}+\alpha_\epsilon\phi_{\epsilon,2},\end{cases} \tag{6.3}$$

where all functions are in $H_N^2(\Omega)$, and α_ϵ has been defined in (3.7).

For simplicity, we set $\tau=0$. Since $\tau\lambda_\epsilon<<1$ the results in this section are also valid for τ finite. The case of general $\tau>0$ can be treated as in [18]. We will see that the small eigenvalues are of the order $O(\epsilon^2)$. To compute them, we will need to expand the eigenfunction up to the order $O(\epsilon)$ term.

Let us define

$$\tilde g_{\epsilon,j}(x)=\chi\left(\frac{x-x_j^\epsilon}{r_0}\right)\bar g_{\epsilon,j}(x),\quad j=1,2, \tag{6.4}$$

where $\chi(x)$ is a smooth cut-off function such that $\chi(x) = 1$ for $|x| < 1$ and $\chi(x) = 0$ for $|x| > 2$. Further,

$$r_0 = \frac{1}{10}\left(1 + x_2,\, 1 - x_1,\, \frac{1}{2}|x_1 - x_2|\right). \tag{6.5}$$

In a similar way as in Section 3, we define approximate kernel and co-kernel, but in contrast now we can use the exact solution given in Theorem 1:

$$\mathcal{K}^{new}_{\epsilon,\mathbf{x}^\epsilon} := \text{span}\left\{\epsilon\frac{d}{dx}\tilde{g}_{\epsilon,1}\right\} \oplus \text{span}\left\{\epsilon\frac{d}{dx}\tilde{g}_{\epsilon,2}\right\} \subset (H^2_N(\Omega_\epsilon))^2,$$

$$\mathcal{C}^{new}_{\epsilon,\mathbf{x}^\epsilon} := \text{span}\left\{\epsilon\frac{d}{dx}\tilde{g}_{\epsilon,1}\right\} \oplus \text{span}\left\{\epsilon\frac{d}{dx}\tilde{g}_{\epsilon,2}\right\} \subset (L^2(\Omega_\epsilon))^2,$$

where $\mathbf{x}^\epsilon = (x_1^\epsilon, x_2^\epsilon)$ and $\Omega_\epsilon = \left(-\frac{L}{\epsilon}, \frac{L}{\epsilon}\right)$.

Then it is easy to see that

$$\bar{g}_i(x) = \tilde{g}_{\epsilon,i}(x) + e.s.t., \quad i = 1,\, 2. \tag{6.6}$$

Note that, by Theorem 1, $\tilde{g}_{\epsilon,j}(x) \sim \hat{t}_j w\left(\frac{x - x_j^\epsilon}{\epsilon}\right)$ in $H^2_{loc}(\Omega_\epsilon)$ and $\tilde{g}_{\epsilon,j}$ satisfies

$$\epsilon^2\Delta\tilde{g}_{\epsilon,j} - \tilde{g}_{\epsilon,j} + \frac{(\tilde{g}_{\epsilon,j})^2\bar{s}_{\epsilon,3-j}}{\bar{r}_\epsilon} + e.s.t. = 0, \quad j = 1,\, 2.$$

Thus $\tilde{g}'_{\epsilon,j} := \frac{d\tilde{g}_{\epsilon,j}}{dx}$ satisfies

$$\epsilon^2\Delta\tilde{g}'_{\epsilon,j} - \tilde{g}'_{\epsilon,j} + \frac{2c\tilde{g}_{\epsilon,j}\bar{s}_{\epsilon,3-j}}{\bar{r}_\epsilon}\tilde{g}'_{\epsilon,j} + \frac{c\tilde{g}^2_{\epsilon,j}}{\bar{r}_\epsilon}\bar{s}'_{\epsilon,3-j} - \frac{c\tilde{g}^2_{\epsilon,j}\bar{s}_{\epsilon,3-j}}{(\bar{r}_\epsilon)^2}\bar{r}'_\epsilon + e.s.t. = 0. \tag{6.7}$$

Let us now decompose

$$\phi_{\epsilon,j} = \epsilon a_j^\epsilon \tilde{g}'_{\epsilon,j} + \phi^\perp_{\epsilon,j}, \quad j = 1,\, 2, \tag{6.8}$$

with complex numbers a_j^ϵ, where the factor ϵ is for scaling purposes, to achieve that a_j^ϵ is of order $O(1)$, and

$$\phi^\perp_\epsilon = (\phi^\perp_{\epsilon,1}, \phi^\perp_{\epsilon,2}) \in (\mathcal{K}^{new}_{\epsilon,\mathbf{x}^\epsilon})^\perp,$$

where orthogonality is taken for the scalar product of the product space $(L^2(\Omega_\epsilon))^2$. Note that, by definition,

$$\phi_\epsilon = (\phi_{\epsilon,1}, \phi_{\epsilon,2}) \in \mathcal{K}^{new}_{\epsilon,\mathbf{x}^\epsilon}.$$

Suppose that $\|\phi_\epsilon\|_{H^2(\Omega_\epsilon)} = 1$. Then we need to have $|a_j^\epsilon| \le C$.

Similarly, we decompose

$$\psi_\epsilon = \epsilon\sum_{j=1}^{2} a_j^\epsilon \psi_{\epsilon,j} + \psi^\perp_\epsilon, \quad \eta_{\epsilon,j} = \epsilon a_j^\epsilon \eta^0_{\epsilon,j} + \eta^\perp_{\epsilon,j}, \quad j = 1,\, 2, \tag{6.9}$$

where $\psi_{\epsilon,j}$ satisfies

$$D_r\Delta\psi_{\epsilon,j} - \psi_{\epsilon,j} + 2\alpha_\epsilon c\tilde{g}_{\epsilon,j}\tilde{g}'_{\epsilon,j}\bar{s}_{\epsilon,3-j} + \alpha_\epsilon c\tilde{g}^2_{\epsilon,3-j}\eta^0_{\epsilon,j} = 0, \tag{6.10}$$

$\eta^0_{\epsilon,i}$ is given by

$$D_s\Delta\eta^0_{\epsilon,i} - \eta^0_{\epsilon,i} + \alpha_\epsilon\tilde{g}'_{\epsilon,i} = 0, \tag{6.11}$$

$\psi^\perp_\epsilon$ satisfies

$$D_r\Delta\psi^\perp_\epsilon - \psi^\perp_\epsilon + 2\alpha_\epsilon\, c\tilde{g}_{\epsilon,1}\bar{s}_{\epsilon,2}\phi^\perp_{\epsilon,1} + \alpha_\epsilon\, c\tilde{g}^2_{\epsilon,1}\eta^\perp_{\epsilon,2} + 2\alpha_\epsilon\, c\tilde{g}_{\epsilon,2}\bar{s}_{\epsilon,1}\phi^\perp_{\epsilon,2} + \alpha_\epsilon\, c\tilde{g}^2_{\epsilon,2}\eta^\perp_{\epsilon,1} = 0, \tag{6.12}$$

and finally $\eta^\perp_i$ is given by

$$D_s\Delta\eta^\perp_{\epsilon,i} - \eta^\perp_{\epsilon,i} + \alpha_\epsilon\phi^\perp_{\epsilon,i} = 0. \tag{6.13}$$

Substituting the decompositions of $\phi_{\epsilon,i}$, ψ_ϵ and $\eta_{\epsilon,i}$ into (6.3) we have

$$\epsilon c\left(a_j^\epsilon\frac{(\tilde g_{\epsilon,j})^2\bar s_{\epsilon,3-j}}{\bar r_\epsilon^2}\bar r_\epsilon^{'}-\sum_{k=1}^2 a_k^\epsilon\frac{(\tilde g_{\epsilon,j})^2\bar s_{\epsilon,3-j}}{\bar r_\epsilon^2}\psi_{\epsilon,k}\right)-\epsilon c\left(a_j^\epsilon\frac{(\tilde g_{\epsilon,j})^2}{\bar r_\epsilon}\bar s_{\epsilon,3-j}^{'}-a_{3-j}^\epsilon\frac{(\tilde g_{\epsilon,j})^2}{\bar r_\epsilon}\eta_{\epsilon,3-j}^0\right)$$

$$+\epsilon^2\Delta\phi_{\epsilon,j}^\perp-\phi_{\epsilon,j}^\perp+\frac{2c\tilde g_{\epsilon,j}\bar s_{\epsilon,3-j}}{\bar r_\epsilon}\phi_{\epsilon,j}^\perp-\frac{c\tilde g_{\epsilon,j}^2\bar s_{\epsilon,3-j}}{\bar r_\epsilon^2}\psi_\epsilon^\perp+\frac{c\tilde g_{\epsilon,j}^2}{\bar r_\epsilon}\eta_{\epsilon,3-j}^\perp-\lambda_\epsilon\phi_{\epsilon,j}^\perp+e.s.t.$$

$$=\lambda_\epsilon\epsilon a_j^\epsilon\tilde g_{\epsilon,j}^{'},\quad j=1,\,2, \tag{6.14}$$

since

$$\epsilon^2\Delta\tilde g_{\epsilon,j}^{'}-\tilde g_{\epsilon,j}^{'}+\frac{2c\tilde g_{\epsilon,j}\bar s_{3-j,\epsilon}}{\bar r_\epsilon}\tilde g_{\epsilon,j}^{'}+\text{e.s.t.}=0.$$

Multiplying both sides of (6.14) for $j=1,2$ by $\tilde g_{\epsilon,l}^{'}$ for $l=1,2$ and integrating over $(-L,L)$, we obtain

$$\text{r.h.s. of (6.14)}=\lambda_\epsilon a_j^\epsilon\epsilon\int_{-L}^L\tilde g_{\epsilon,j}^{'}\tilde g_{\epsilon,l}^{'}\,dx=\lambda_\epsilon\delta_{jl}a_l^\epsilon(\hat t_l)^2\int_R(w^{'}(y))^2\,dy\,(1+o(1)) \tag{6.15}$$

and

$$\text{l.h.s. of (6.14)}=c\epsilon\sum_{k=1}^2 a_k^\epsilon\delta_{jl}\int_{-L}^L\frac{(\tilde g_{\epsilon,j})^2\bar s_{\epsilon,3-j}}{\bar r_\epsilon^2}\left(\delta_{jk}\bar r_\epsilon^{'}-\psi_{\epsilon,k}\right)\tilde g_{\epsilon,l}^{'}\,dx$$

$$+c\epsilon\sum_{k=1}^2 a_k^\epsilon\delta_{jl}\int_{-L}^L\frac{(\tilde g_{\epsilon,j})^2}{\bar r_\epsilon}\left(\delta_{j,3-k}\eta_{\epsilon,3-j}^0-\delta_{j,k}\bar s_{\epsilon,3-j}^{'}\right)\tilde g_{\epsilon,l}^{'}\,dx$$

$$+c\delta_{jl}\int_{-L}^L\frac{(\tilde g_{\epsilon,l})^2\bar s_{\epsilon,3-j}}{\bar r_\epsilon}\left(\frac{\bar r_\epsilon^{'}}{\bar r_\epsilon}-\frac{\bar s_{\epsilon,3-j}^{'}}{\bar s_{\epsilon,3-j}}\right)\phi_{\epsilon,j}^\perp\,dx$$

$$+c\delta_{jl}\int_{-L}^L\frac{(\tilde g_{\epsilon,j})^2\bar s_{\epsilon,3-j}}{\bar r_\epsilon}\left(\frac{\eta_{\epsilon,3-j}^\perp}{\bar s_{\epsilon,3-j}}-\frac{\psi_\epsilon^\perp}{\bar r_\epsilon}\right)\tilde g_{\epsilon,l}^{'}\,dx+o(\epsilon^2)$$

$$=J_{1,l}+J_{2,l}+J_{3,l}+J_{4,l}:=J_l, \tag{6.16}$$

where $J_{i,l}$, $i=1,...,4$ are defined by the last equality. The following is the key lemma for the asymptotic behavior of the small eigenvalues:

LEMMA 6.1. *We have*

$$J_l=-\epsilon^2\left(\int_R\frac{1}{3}w^3\,dy\right)\sum_{k=1}^2 a_k^\epsilon\Bigg\{\Big\{-\hat t_l\nabla_{x_l^\epsilon}\nabla_{x_k^\epsilon}\left(H_{D_r}(x_l^\epsilon,x_l^\epsilon)\right)+\hat t_{3-l}\nabla_{x_l^\epsilon}\nabla_{x_k^\epsilon}(G_{D_r}(x_l^\epsilon,x_{3-l}^\epsilon))\Big\}$$

$$-\nabla_{x_l^\epsilon}\left(\frac{\delta_{k,3-l}\nabla_{x_{3-l}^\epsilon}G_{D_s}(x_l^\epsilon,x_{3-l}^\epsilon)}{G_{D_s}(x_l^\epsilon,x_{3-l}^\epsilon)}\right)$$

$$+\Big\{(\nabla_{x_k^\epsilon}\hat t_l(x_1^\epsilon,x_2^\epsilon))\nabla_{x_l^\epsilon}G_{D_r}(x_l^\epsilon,x_l^\epsilon)+(\nabla_{x_k^\epsilon}\hat t_{3-l}(x_1^\epsilon,x_2^\epsilon))\nabla_{x_l^\epsilon}G_{D_r}(x_l^\epsilon,x_{3-l}^\epsilon)\Big\}\Bigg\}+o(\epsilon^2). \tag{6.17}$$

.3cm

Lemma 6.1 follows from the following series of lemmas:

LEMMA 6.2. *We have*

$$\eta_{\epsilon,k}^0(x_{3-k}^\epsilon)=\hat t_k\nabla_{x_k^\epsilon}G_{D_s}(x_{3-k}^\epsilon,x_k^\epsilon)+O(\epsilon). \tag{6.18}$$

LEMMA 6.3. *We have*

$$\bar{s}'_{\epsilon,k}(x^\epsilon_{3-k}) = \hat{t}_k \nabla_{x^\epsilon_{3-k}} G_{D_s}(x^\epsilon_{3-k}, x^\epsilon_k) + O(\epsilon). \tag{6.19}$$

LEMMA 6.4. *For $k, l = 1, 2$ we have*

$$\left(\delta_{kl}\bar{r}'_\epsilon - \psi_{\epsilon,k}\right)(x^\epsilon_l) = c\hat{t}_1\hat{t}_2\Big\{ - \hat{t}_l \nabla_{x^\epsilon_k}\left(H_{D_r}(x^\epsilon_l, x^\epsilon_l) G_{D_s}(x^\epsilon_l, x^\epsilon_{3-l})\right) + \hat{t}_{3-l}\nabla_{x^\epsilon_k}\left(G_{D_r}(x^\epsilon_l, x^\epsilon_{3-l}) G_{D_s}(x^\epsilon_{3-l}, x^\epsilon_l)\right)$$

$$+ \frac{1}{2\sqrt{D_r}}\hat{t}_l \nabla_{x^\epsilon_k} G_{D_s}(x^\epsilon_l, x^\epsilon_{3-l})\Big\} + O(\epsilon). \tag{6.20}$$

Similar to Lemma 6.4, we get

LEMMA 6.5. *For $k, l = 1, 2$ we have*

$$\left(\delta_{kl}\bar{r}'_\epsilon - \psi_{\epsilon,k}\right)(x^\epsilon_l + \epsilon y) - \left(\delta_{kl}\bar{r}'_\epsilon - \psi_{\epsilon,k}\right)(x^\epsilon_l) = \epsilon y c\hat{t}_1\hat{t}_2\Big\{ - \hat{t}_l \nabla_{x^\epsilon_l}\nabla_{x^\epsilon_k}\left(H_{D_r}(x^\epsilon_l, x^\epsilon_l) G_{D_s}(x^\epsilon_l, x^\epsilon_{3-l})\right)$$

$$+ \hat{t}_{3-l}\nabla_{x^\epsilon_l}\nabla_{x^\epsilon_k}\left(G_{D_r}(x^\epsilon_l, x^\epsilon_{3-l}) G_{D_s}(x^\epsilon_{3-l}, x^\epsilon_l)\right) + \frac{1}{2\sqrt{D_r}}\hat{t}_l \nabla_{x^\epsilon_l}\nabla_{x^\epsilon_k} G_{D_s}(x^\epsilon_l, x^\epsilon_{3-l})\Big\} + O(\epsilon^2). \tag{6.21}$$

Lemma 6.1 will be shown in Appendix A, proving Lemmas 6.2 – 6.5 first.

After obtaining the asymptotic behavior of the small eigenvalues, our next goal is to study their stability.

Combining Lemma 6.1 with (6.15) and (6.16), the small eigenvalues λ^ϵ are given by the following two-dimensional eigenvalue problem, where $(a^\epsilon_1, a^\epsilon_2)$ are the corresponding eigenvectors:

$$-\epsilon^2\hat{t}_l\left(\int_R \frac{1}{3}w^3\,dy\right)\sum_{k=1}^{2} a^\epsilon_k \Big\{\Big\{ - \hat{t}_l \nabla_{x^\epsilon_l}\nabla_{x^\epsilon_k}\left(H_{D_r}(x^\epsilon_l, x^\epsilon_l)\right) + \hat{t}_{3-l}\nabla_{x^\epsilon_l}\nabla_{x^\epsilon_k}\left(G_{D_r}(x^\epsilon_l, x^\epsilon_{3-l})\right)\Big\}$$

$$-\nabla_{x^\epsilon_l}\left(\frac{\delta_{k,3-l}\nabla_{x^\epsilon_{3-l}} G_{D_s}(x^\epsilon_l, x^\epsilon_{3-l})}{G_{D_s}(x^\epsilon_l, x^\epsilon_{3-l})}\right)$$

$$+\Big\{(\nabla_{x^\epsilon_k}\hat{t}_l(x^\epsilon_1, x^\epsilon_2))\nabla_{x^\epsilon_l} G_{D_r}(x^\epsilon_l, x^\epsilon_l) + (\nabla_{x^\epsilon_k}\hat{t}_{3-l}(x^\epsilon_1, x^\epsilon_2))\nabla_{x^\epsilon_l} G_{D_r}(x^\epsilon_l, x^\epsilon_{3-l})\Big\}\Big\} + o(\epsilon^2).$$

$$= \lambda_\epsilon \delta_{jl} a^\epsilon_l (\hat{t}_l)^2 \int_R (w'(y))^2\,dy\,(1 + o(1)). \tag{6.22}$$

From (6.22) it follows that the eigenvectors $(a^0_1, a^0_2) = \lim_{\epsilon\to 0}(a^\epsilon_1, a^\epsilon_2)$ satisfy $(a^0_1, a^0_2) = (1, -1)$ or $(a^0_1, a^0_2) = (1, 1)$, up to a constant factor.

For the eigenvector $(a^0_1, a^0_2) = (1, -1)$, the computations of the eigenvalue λ^ϵ_1 are similar to those given in Section 3. We get

$$\lambda^\epsilon_1 = C_3\epsilon^2 M'(x^\epsilon_1) + o(\epsilon^2),$$

where

$$M(x) = -2\theta_s \tanh\theta_s(L - x) + \theta_r \tanh\theta_r(L - x) + \theta_r \frac{\sinh\theta_r(L - x) - \sinh\theta_r(L + x)}{\cosh\theta_r(L - x) + \cosh\theta_r(L + x)}$$

and

$$C_3 = \frac{1}{3\hat{t}_l}\frac{\int_R w^3\,dy}{\int_R (w')^2\,dy} > 0. \tag{6.23}$$

This implies

$$M'(x) = \frac{2\theta_s^2}{\cosh^2\theta_s(L - x)} - \frac{\theta_r^2}{\cosh^2\theta_r(L - x)} - \theta_r^2\left(1 - \frac{[\sinh\theta_r(L - x) - \sinh\theta_r(L + x)]^2}{[\cosh\theta_r(L - x) - \cosh\theta_r(L + x)]^2}\right).$$

Obviously, $M'(x) < 0$ if $\theta_s = 0$ or if θ_s is small compared to θ_r. A simple sufficient condition is obtained by taking into account the first two terms of $M'(x)$ which has been derived in Section 3 and is given by (3.13). Recall that (3.13) holds if $D_s/D_r > 4$.

If $D_s/D_r > 4$, the eigenvalue λ_1^ϵ has negative real part.

Now we consider the eigenvalue λ_2^ϵ with eigenvector such that $\lim_{\epsilon\to 0}(a_1^\epsilon, a_2^\epsilon) = (1,1)$. We have

LEMMA 6.6. *Suppose λ_2^ϵ is the eigenvalue with eigenvector $\lim_{\epsilon\to 0}(a_1^\epsilon, a_2^\epsilon) = (1,1)$. Then we have*

$$\lambda_2^\epsilon = C_3\epsilon^2 P(x_1^\epsilon, x_2^\epsilon) + o(\epsilon^2), \qquad \text{where } C_3 > 0 \text{ has been defined in (6.23)}, \tag{6.24}$$

and

$$P(x_1^\epsilon, x_2^\epsilon) = (\nabla_{x_1^\epsilon} + \nabla_{x_2^\epsilon})\Bigg\{ \frac{(\nabla_{x_1^\epsilon} - \nabla_{x_2^\epsilon}) G_{D_s}(x_1^\epsilon, x_2^\epsilon)}{G_{D_s}(x_1^\epsilon, x_2^\epsilon)}$$

$$-\hat{t}_1^\epsilon(x_1^\epsilon, x_2^\epsilon)(\nabla_{x_1^\epsilon} - \nabla_{x_2^\epsilon}) H_{D_r}(x_1^\epsilon, x_1^\epsilon) - \hat{t}_2^\epsilon(x_1^\epsilon, x_2^\epsilon)(\nabla_{x_1^\epsilon} - \nabla_{x_2^\epsilon}) H_{D_r}(x_1^\epsilon, x_1^\epsilon) \Bigg\}.$$

We have $P(x_1^\epsilon, x_2^\epsilon) \le 0$ with equality if and only if $x_1^\epsilon = x_2^\epsilon = 0$.

Lemma 6.6 will be proved in Appendix B.

By the argument of Dancer [2] the eigenvalue problem (6.22) captures all converging sequences of small eigenvalues λ^ϵ and so λ_1^ϵ and λ_2^ϵ are all $o(1)$ eigenvalues for ϵ small enough. Therefore we have the following main result on $o(1)$ eigenvalues:

THEOREM 6.7. *Suppose $D_s/D_r > 4$ and $\lim_{\epsilon\to 0} x_1^\epsilon = x_1 \neq 0$. The mutually exclusive, spiky steady state given in Theorem 3.2 is linearly stable with respect to small eigenvalues $\lambda_\epsilon = o(1)$ if $\tau \ge 0$ and $\epsilon > 0$ are both small enough. More precisely, we have $Re(\lambda_\epsilon) \le c\epsilon^2$ for some $c > 0$ independent of ϵ and τ.*

7. Numerical Simulations. For the simulations we use the domain $\Omega = (-1,1)$ and Neumann boundary conditions for all components. The constants in the five-component Meinhardt-Gierer system are chosen as follows:

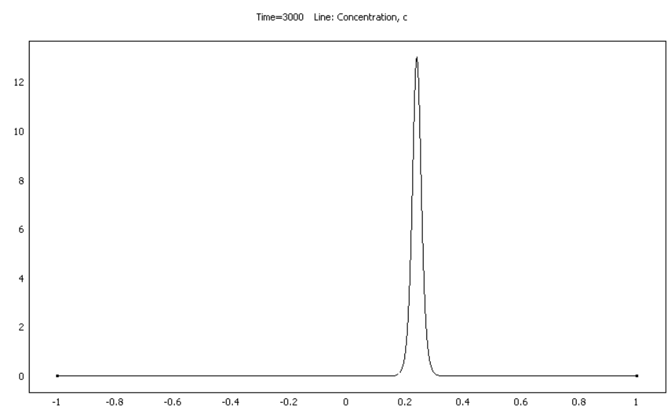
$$\epsilon^2 = .001,\ D_r = .1,\ D_s = 1,\ c = 1,\ \tau = 1.$$

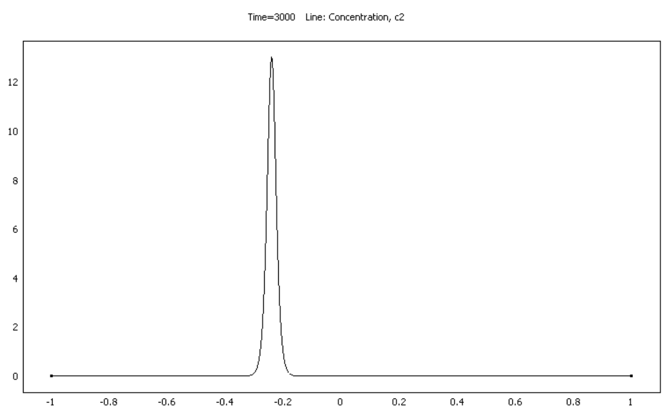
The pictures show the numerically obtained long-term limit of the five components g_1, g_2, r, s_1, s_2, i.e. the state at $t = 3{,}000$. After that the solution is numerically stable and does not change anymore. This confirms the analytical result that the steady state with two mutually exclusive spikes for the two activators which are located in different positions is stable.

Our simulations support the conjecture that the spikes are not only linearly stable as steady states but that, at least locally, they are also dynamically stable for the parabolic reaction-diffusion system.

The choice of constants for the numerical simulations has been motivated by the analysis. In particular, D_r has to be rather small compared to D_s by the stability result in Section 4. On the other hand, D_r cannot be too small since otherwise by the results in Section 3 the distance between the spikes becomes very large and there is no such solution on the interval $(-1,1)$. So the parameters have to be chosen very carefully, and without any analytical results it would be very hard to find the parameter range for which stable mutually exclusive spikes exist.

The pictures show that the inhibitor r has two peaks which are near the peaks of the local activators g_1 and g_2. The profile of the peaks of r is "smoother" than for those of the local activators. The lateral activator s_i has a peak near the peak of g_i and its profile again is smoother than the latter.





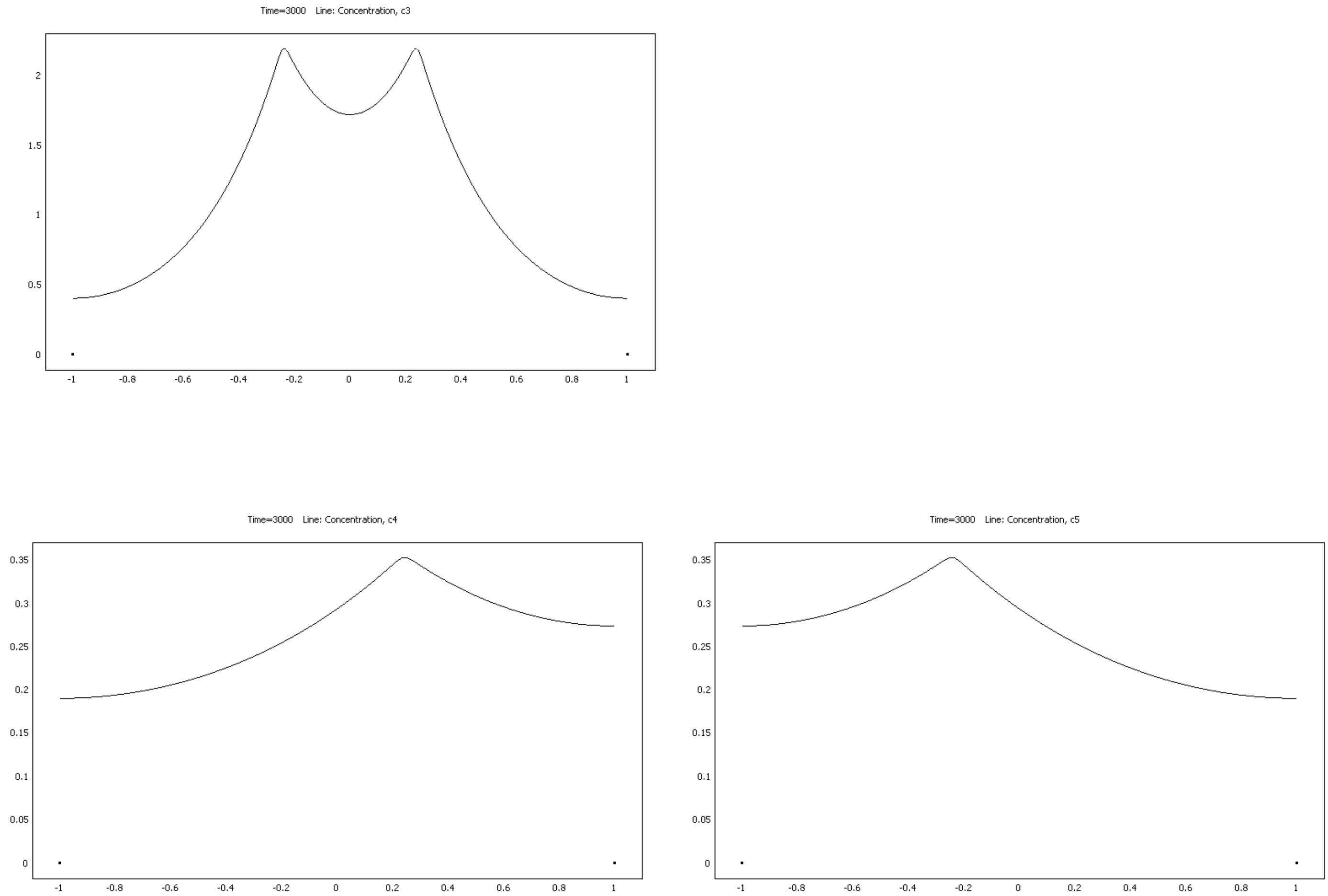


Figure 1. The stable, mutually exclusive, two-spike steady state. All five components have been plotted to highlight the interactions between them.

We expect Hopf bifurcation and oscillating spikes to occur for sufficiently large tau. We analyzed only the case $\tau = 0$ and did not observe oscillations numerically for $\tau = 1$. The instabilities of the spikes which we encountered in the numerical calculations were (i) disappearance of spikes when their amplitudes becomes unstable (related to large eigenvalues) – this happens if the ratio of the diffusion constants $\frac{D_s}{D_r}$ is too small (ii) movement of the spikes to the boundary or towards each other when their positions became unstable (related to small eigenvalues) – this occurs if D_r is too small.

For numerical simulations with very large τ we expect oscillations to occur.

8. Appendix A: Proof of Lemma 6.1. In this appendix we prove Lemma 6.1 in a sequence of lemmas. First we introduce some notation.

Using the notation (3.7), we introduce matrix notation

$$e = (1,1)^T, \quad t = \left(\hat{t}_1, \hat{t}_2\right)^T, \nabla_{x_i}\hat{t} = \left(\nabla_{x_i}\hat{t}_1, \nabla_{x_i}\hat{t}_2\right)^T, \quad i = 1,2,$$

$$\mathcal{G}_{ij} = \left(\ G(x_i, x_j)\ \right), \quad i,j = 1,2, \quad \nabla_{x_i}\mathcal{G}_{kl} = \left(\ \nabla_{x_i} G(x_k, x_l)\ \right), \quad i,j,k = 1,2,$$

we get

$$\left\{\begin{array}{l} e = \mathcal{G}\hat{t}, \\ 0 = \left(\nabla_{x_1}\mathcal{G}\right)\hat{t} + \mathcal{G}\left(\nabla_{x_1}\hat{t}\right), \\ 0 = \left(\nabla_{x_2}\mathcal{G}\right)\hat{t} + \mathcal{G}\left(\nabla_{x_2}\hat{t}\right). \end{array}\right. \tag{8.1}$$

The system (8.1) has a unique solution $(\hat{t}, \nabla_{x_1}\hat{t}, \nabla_{x_2}\hat{t})$ since $\det(\mathcal{G}) \neq 0$ which can be written as follows:

$$\hat{t} = \mathcal{G}^{-1}e, \quad \nabla_{x_i}\hat{t} = -\mathcal{G}^{-1}\left(\nabla_{x_i}\mathcal{G}\right)\mathcal{G}^{-1}e, \quad i = 1,2. \tag{8.2}$$

Let us put

$$\tilde{L}_{\epsilon,j}\phi_\epsilon^\perp := \epsilon^2\Delta\phi_{\epsilon,j}^\perp - \phi_{\epsilon,j}^\perp + \frac{2c\tilde{g}_{\epsilon,j}\bar{s}_{\epsilon,3-j}}{\bar{r}_\epsilon}\phi_{\epsilon,j}^\perp - \frac{c\tilde{g}_{\epsilon,j}^2\bar{s}_{\epsilon,3-j}}{\bar{r}_\epsilon^2}\psi_\epsilon^\perp + \frac{c\tilde{g}_{\epsilon,j}^2}{\bar{r}_\epsilon}\eta_{\epsilon,3-j}^\perp \tag{8.3}$$

and $\mathbf{a}_\epsilon := (a_1^\epsilon, a_2^\epsilon)^T$.

We now prove the key lemma, Lemma 6.1, in a sequence of lemmas.

Proof of Lemma 6.2: Note that for $k = 3-l$ we have

$$\eta_{\epsilon,k}^0(x_l^\epsilon) = \alpha_\epsilon \int_{-L}^{L} G_{D_s}(x_l^\epsilon, z)\tilde{g}_{\epsilon,k}^{'}(z)\,dz + O(\epsilon) = \alpha_\epsilon \hat{t}_k \nabla_{x_k^\epsilon} G_{D_s}(x_l^\epsilon, x_k^\epsilon)\int_{-L}^{L} zw'\left(\frac{z-x_k}{\epsilon}\right)(z)\,dz$$

$$= -\hat{t}_k\nabla_{x_k^\epsilon}G_{D_s}(x_l^\epsilon, x_k^\epsilon)\alpha_\epsilon\left(\epsilon\int_{-\infty}^{\infty} w(y)\,dy\right) + O(\epsilon) = -\hat{t}_k\nabla_{x_k^\epsilon}G_{D_s}(x_l^\epsilon, x_k^\epsilon) + O(\epsilon). \tag{8.4}$$

.3cm

Proof of Lemma 6.3: Note that for $k = 3-l$ we have

$$\bar{s}_{\epsilon,k}^{'}(x_l^\epsilon) = \alpha_\epsilon \nabla_{x_l^\epsilon}\int_{-L}^{L} G_{D_s}(x_l^\epsilon, z)\tilde{g}_{\epsilon,k}(z)\,dz = \alpha_\epsilon\nabla_{x_l^\epsilon}G_{D_s}(x_l^\epsilon, x_k^\epsilon)\int_{-L}^{L}\hat{t}_k w\left(\frac{z-x_k}{\epsilon}\right)(z)\,dz + O(\epsilon)$$

$$= \alpha_\epsilon\hat{t}_k\nabla_{x_l^\epsilon}G_{D_s}(x_l^\epsilon, x_k^\epsilon)\left(\epsilon\int_{-\infty}^{\infty} w(y)\,dy\right) + O(\epsilon) = \hat{t}_k\nabla_{x_l^\epsilon}G_{D_s}(x_l^\epsilon, x_k^\epsilon) + O(\epsilon). \tag{8.5}$$

.3cm

Proof of Lemma 6.4: We first consider the case $k = l$ and compute $\psi_{\epsilon,l}(x_l^\epsilon)$ as follows:

$$\psi_{\epsilon,l}(x_l^\epsilon) = c\alpha_\epsilon\int_{-L}^{L} G_{D_r}(x_l^\epsilon, z)\left(2\tilde{g}_{\epsilon,l}^{'}\tilde{g}_{\epsilon,l}\bar{s}_{\epsilon,3-l} + \tilde{g}_{\epsilon,3-l}^2\eta_{\epsilon,l}^0\right)(z)\,dz + O(\epsilon)$$

$$= c(\alpha_\epsilon)^2\int_{-\infty}^{\infty} K_{D_r}(|z|)\left(2\tilde{g}_{\epsilon,l}(x_l^\epsilon+z)\tilde{g}_{\epsilon,l}^{'}(x_l^\epsilon+z)\right)\int_{-L}^{L} G_{D_s}(x_l^\epsilon+z, y)\tilde{g}_{\epsilon,3-l}(y)\,dy\,dz$$

$$-c(\alpha_\epsilon)^2\int_{-L}^{L} H_{D_r}(x_l^\epsilon, z)\left(\frac{d}{dz}(\tilde{g}_{\epsilon,l}(z))^2\right)\int_{-L}^{L} G_{D_s}(z, y)\tilde{g}_{\epsilon,3-l}(y)\,dy\,dz$$

$$+c(\alpha_\epsilon)^2\int_{-L}^{L} G_{D_r}(x_l^\epsilon, z)\left(\tilde{g}_{\epsilon,3-l}(z)\right)^2\int_{-L}^{L} G_{D_s}(z, y)\tilde{g}_{\epsilon,l}^{'}(y)\,dy\,dz + O(\epsilon)$$

$$= c(\alpha_\epsilon)^2\int_{-\infty}^{\infty} K_{D_r}(|z|)\left(2\tilde{g}_{\epsilon,l}(x_l^\epsilon+z)\tilde{g}_{\epsilon,l}^{'}(x_l^\epsilon+z)\right)\int_{-L}^{L} G_{D_s}(x_l^\epsilon+z, y)\tilde{g}_{\epsilon,3-l}(y)\,dy\,dz$$

$$+\frac{c}{2}\hat{t}_1\hat{t}_2\hat{t}_l\left(\left(\nabla_{x_l^\epsilon}H_{D_r}(x_l^\epsilon, x_l^\epsilon)\right)G_{D_s}(x_l^\epsilon, x_{3-l}^\epsilon)\right)$$

$$+c\hat{t}_1\hat{t}_2\hat{t}_l\left(H_{D_r}(x_l^\epsilon, x_l^\epsilon)\nabla_{x_l^\epsilon}G_{D_s}(x_l^\epsilon, x_{3-l}^\epsilon)\right)$$

$$-c\hat{t}_1\hat{t}_2\hat{t}_{3-l}\left(G_{D_r}(x_l^\epsilon, x_{3-l}^\epsilon)\nabla_{x_l^\epsilon}G_{D_s}(x_{3-l}^\epsilon, x_l^\epsilon)\right) + O(\epsilon). \tag{8.6}$$

Next we consider the case $k = 3-l$ and compute $\psi_{\epsilon,3-l}(x_l^\epsilon)$ as follows:

$$\psi_{\epsilon,3-l}(x_l^\epsilon) = c\alpha_\epsilon\int_{-L}^{L} G_{D_r}(x_l^\epsilon, z)\left(2\tilde{g}_{\epsilon,3-l}^{'}\tilde{g}_{\epsilon,3-l}\bar{s}_{\epsilon,l} + \tilde{g}_{\epsilon,l}^2\eta_{\epsilon,3-l}\right)(z)\,dz + O(\epsilon)$$

$$= c(\alpha_\epsilon)^2\int_{-\infty}^{\infty} K_{D_r}(|z|)\left(\tilde{g}_{\epsilon,l}(x_l^\epsilon+z)\right)^2\int_{-L}^{L} G_{D_s}(x_l^\epsilon+z, y)\tilde{g}_{\epsilon,3-l}^{'}(y)\,dy\,dz$$

$$+c(\alpha_\epsilon)^2\int_{-L}^{L}G_{D_r}(x_l^\epsilon,z)\left(\frac{d}{dz}(\tilde g_{\epsilon,3-l}(z))^2\right)\int_{-L}^{L}G_{D_s}(z,y)\tilde g_{\epsilon,l}(y)\,dy\,dz$$

$$-c(\alpha_\epsilon)^2\int_{-L}^{L}H_{D_r}(x_l^\epsilon,z)\,(\tilde g_{\epsilon,l}(z))^2\int_{-L}^{L}G_{D_s}(z,y)\tilde g_{\epsilon,3-l}^{'}(y)\,dy\,dz+O(\epsilon)$$

$$=c(\alpha_\epsilon)^2\int_{-\infty}^{\infty}K_{D_r}(|z|)\,(\tilde g_{\epsilon,l}(x_l^\epsilon+z))^2\int_{-L}^{L}G_{D_s}(x_l^\epsilon+z,y)\tilde g_{\epsilon,3-l}^{'}(y)\,dy\,dz$$

$$+c\hat t_1\hat t_2\hat t_l\left(H_{D_r}(x_l^\epsilon,x_l^\epsilon)\nabla_{x_{3-l}^\epsilon}G_{D_s}(x_l^\epsilon,x_{3-l}^\epsilon)\right)-c\hat t_1\hat t_2\hat t_{3-l}\left(\left(\nabla_{x_{3-l}^\epsilon}G_{D_r}(x_l^\epsilon,x_{3-l}^\epsilon)\right)G_{D_s}(x_{3-l}^\epsilon,x_l^\epsilon)\right)$$

$$-c\hat t_1\hat t_2\hat t_{3-l}\left(G_{D_r}(x_l^\epsilon,x_{3-l}^\epsilon)\nabla_{x_{3-l}^\epsilon}G_{D_s}(x_{3-l}^\epsilon,x_l^\epsilon)\right)+O(\epsilon). \tag{8.7}$$

Next we compute $\bar r_\epsilon(x_l^\epsilon)$:

$$\bar r_\epsilon(x_l^\epsilon)=\alpha_\epsilon c\int_{-L}^{L}G_{D_r}(x_l^\epsilon,z)\left(\tilde g_{\epsilon,1}^2\bar s_{\epsilon,2}+\tilde g_{\epsilon,2}^2\bar s_{\epsilon,1}\right)(z)\,dz+O(\epsilon)$$

$$=(\alpha_\epsilon)^2c\int_{-\infty}^{\infty}K_{D_r}(|z|)\left\{(\tilde g_{\epsilon,l}(x_l^\epsilon+z))^2\int_{-L}^{L}G_{D_s}(x_l^\epsilon+z,y)\tilde g_{\epsilon,3-l}(y)\,dy\right\}dz$$

$$-(\alpha_\epsilon)^2c\int_{-L}^{L}H_{D_r}(x_l^\epsilon,z)\left\{(\tilde g_{\epsilon,l}(z))^2\int_{-L}^{L}G_{D_s}(z,y)\tilde g_{\epsilon,3-l}(y)\,dy\right\}dz$$

$$+(\alpha_\epsilon)^2c\int_{-L}^{L}G_{D_r}(x_l^\epsilon,z)\left\{(\tilde g_{\epsilon,3-l}(z))^2\int_{-L}^{L}G_{D_s}(z,y)\tilde g_{\epsilon,l}(y)\,dy\right\}dz+O(\epsilon).$$

So we have

$$\bar r_\epsilon^{'}(x_l^\epsilon)=(\alpha_\epsilon)^2c\int_{-\infty}^{\infty}K_{D_r}(|z|)\left\{\left(2\tilde g_{\epsilon,l}(x_l^\epsilon+z)\tilde g_{\epsilon,l}^{'}(x_l^\epsilon+z)\right)\int_{-L}^{L}G_{D_s}(x_l^\epsilon+z,y)\tilde g_{\epsilon,3-l}(y)\,dy\right.$$

$$\left.+(\tilde g_{\epsilon,l}(x_l^\epsilon+z))^2\int_{-L}^{L}\nabla_{x_l^\epsilon}G_{D_s}(x_l^\epsilon+z,y)\tilde g_{\epsilon,3-l}(y)\,dy\right\}dz$$

$$-(\alpha_\epsilon)^2c\int_{-L}^{L}\nabla_{x_l^\epsilon}H_{D_r}(x_l^\epsilon,z)\,(\tilde g_{\epsilon,l}(z))^2\int_{-L}^{L}G_{D_s}(z,y)\tilde g_{\epsilon,3-l}(y)\,dy\,dz$$

$$+(\alpha_\epsilon)^2c\int_{-L}^{L}\nabla_{x_l^\epsilon}G_{D_r}(x_l^\epsilon,z)\,(\tilde g_{\epsilon,3-l}(z))^2\int_{-L}^{L}G_{D_s}(z,y)\tilde g_{\epsilon,l}(y)\,dy+O(\epsilon)$$

$$=(\alpha_\epsilon)^2c\int_{-\infty}^{\infty}K_{D_r}(|z|)\left\{\left(2\tilde g_{\epsilon,l}(x_l^\epsilon+z)\tilde g_{\epsilon,l}^{'}(x_l^\epsilon+z)\right)\int_{-L}^{L}G_{D_s}(x_l^\epsilon+z,y)\tilde g_{\epsilon,3-l}(y)\,dy\right.$$

$$\left.+(\tilde g_{\epsilon,l}(x_l^\epsilon+z))^2\int_{-L}^{L}\nabla_{x_l^\epsilon}G_{D_s}(x_l^\epsilon+z,y)\tilde g_{\epsilon,3-l}(y)\,dy\right\}dz$$

$$-\frac{c}{2}\hat t_1\hat t_2\hat t_l\left((\nabla_{x_l^\epsilon}H_{D_r}(x_l^\epsilon,x_l^\epsilon))G_{D_s}(x_l^\epsilon,x_{3-l}^\epsilon)\right)+c\hat t_1\hat t_2\hat t_{3-l}\left((\nabla_{x_l^\epsilon}G_{D_r}(x_l^\epsilon,x_{3-l}^\epsilon))G_{D_s}(x_{3-l}^\epsilon,x_l^\epsilon)\right)+O(\epsilon). \tag{8.8}$$

Now we compute $\left(\delta_{kl}\bar r_\epsilon^{'}-\psi_{\epsilon,k}\right)(x_l^\epsilon)$. Again we consider the two cases $k=l$ and $k\neq l$ separately.

First, for $k=l$, we get

$$\left(\bar{r}_\epsilon^{'}-\psi_{\epsilon,l}\right)(x_l^\epsilon)=-c\hat{t}_1\hat{t}_2\hat{t}_l\nabla_{x_l^\epsilon}\left(H_{D_r}(x_l^\epsilon,x_l^\epsilon)G_{D_s}(x_l^\epsilon,x_{3-l}^\epsilon)\right)$$

$$+c\hat{t}_1\hat{t}_2\hat{t}_{3-l}\nabla_{x_l^\epsilon}\left(G_{D_r}(x_l^\epsilon,x_{3-l}^\epsilon)G_{D_s}(x_{3-l}^\epsilon,x_l^\epsilon)\right)$$

$$+(\alpha_\epsilon)^2c\int_{-\infty}^{\infty}K_{D_r}(|z|)\left(\tilde{g}_{\epsilon,l}(x_l^\epsilon+z)\right)^2\int_{-L}^{L}\nabla_{x_l^\epsilon}G_{D_s}(x_l^\epsilon+z,y)\tilde{g}_{\epsilon,3-l}(y)\,dy\,dz+O(\epsilon)$$

$$=c\hat{t}_1\hat{t}_2\Big\{-\hat{t}_l\nabla_{x_l^\epsilon}\left(H_{D_r}(x_l^\epsilon,x_l^\epsilon)G_{D_s}(x_l^\epsilon,x_{3-l}^\epsilon)\right)+\hat{t}_{3-l}\nabla_{x_l^\epsilon}\left(G_{D_r}(x_l^\epsilon,x_{3-l}^\epsilon)G_{D_s}(x_{3-l}^\epsilon,x_l^\epsilon)\right)$$

$$+\frac{1}{2\sqrt{D_r}}\hat{t}_l\nabla_{x_l^\epsilon}G_{D_s}(x_l^\epsilon,x_{3-l}^\epsilon)\Big\}+O(\epsilon).$$

Next we consider the case $k=3-l$ and get

$$-\psi_{\epsilon,3-l}(x_l^\epsilon)=-c\hat{t}_1\hat{t}_2\hat{t}_l\nabla_{x_{3-l}^\epsilon}\left(H_{D_r}(x_l^\epsilon,x_l^\epsilon)G_{D_s}(x_l^\epsilon,x_{3-l}^\epsilon)\right)$$

$$+\hat{t}_1\hat{t}_2\hat{t}_{3-l}\nabla_{x_{3-l}^\epsilon}\left(G_{D_r}(x_l^\epsilon,x_{3-l}^\epsilon)G_{D_s}(x_{3-l}^\epsilon,x_l^\epsilon)\right)$$

$$+(\alpha_\epsilon)^2c\int_{-\infty}^{\infty}K_{D_r}(|z|)\left(\tilde{g}_{\epsilon,l}(x_l^\epsilon+z)\right)^2\int_{-L}^{L}\nabla_{x_l^\epsilon}G_{D_s}(x_l^\epsilon+z,y)\tilde{g}_{\epsilon,3-l}(y)\,dy\,dz+O(\epsilon)$$

$$=c\hat{t}_1\hat{t}_2\Big\{-\hat{t}_l\nabla_{x_{3-l}^\epsilon}\left(H_{D_r}(x_l^\epsilon,x_l^\epsilon)G_{D_s}(x_l^\epsilon,x_{3-l}^\epsilon)\right)+\hat{t}_{3-l}\nabla_{x_{3-l}^\epsilon}\left(G_{D_r}(x_l^\epsilon,x_{3-l}^\epsilon)G_{D_s}(x_{3-l}^\epsilon,x_l^\epsilon)\right)$$

$$+\frac{1}{2\sqrt{D_r}}\hat{t}_l\nabla_{x_{3-l}^\epsilon}G_{D_s}(x_l^\epsilon,x_{3-l}^\epsilon)\Big\}+O(\epsilon).$$

This implies (6.20). The proof of Lemma 6.4 is finished.
.3cm
Remark: Note that Lemma 6.4 can be written in the simpler way

$$\left(\delta_{kl}\bar{r}_\epsilon^{'}-\psi_{\epsilon,k}\right)(x_l^\epsilon)$$

$$=c\hat{t}_1\hat{t}_2\Big\{\hat{t}_l\nabla_{x_k^\epsilon}\left(G_{D_r}(x_l^\epsilon,x_l^\epsilon)G_{D_s}(x_l^\epsilon,x_{3-l}^\epsilon)\right)+\hat{t}_{3-l}\nabla_{x_k^\epsilon}\left(G_{D_r}(x_l^\epsilon,x_{3-l}^\epsilon)G_{D_s}(x_{3-l}^\epsilon,x_l^\epsilon)\right)\Big\}+O(\epsilon) \tag{8.9}$$

with the understanding that at jump discontinuities the derivative is defined as the arithmetic mean of its left hand and right hand derivatives.

Proof of Lemma 6.5: The proof of Lemma 6.5 follows along the same lines as that for Lemma 6.4 and is therefore omitted.
.3cm
Before we can complete the proof of Lemma 6.1, we need to study the asymptotic expansion of $\phi_\epsilon^\perp$ as $\epsilon\to 0$ first. Let us denote

$$\phi_\epsilon^1=\begin{pmatrix}\phi_{\epsilon,1}^1\\ \phi_{\epsilon,2}^1\end{pmatrix}:=\epsilon a_1^\epsilon\begin{pmatrix}(\nabla_{x_1}t_1)w_1\\ (\nabla_{x_1}t_2)w_2\end{pmatrix}+\epsilon a_2^\epsilon\begin{pmatrix}(\nabla_{x_2}t_1)w_1\\ (\nabla_{x_2}t_2)w_2\end{pmatrix}+\epsilon\frac{\mathcal{G}^{-1}\mathcal{W}\mathcal{A}_\epsilon^0\nabla G_{D_s}(x_1,x_2)}{G_{D_s}(x_1,x_2)}, \tag{8.10}$$

where w_i, $i=1,2$ have been defined in (2.3) and

$$\mathcal{A}_\epsilon^0=\begin{pmatrix}0 & a_2^\epsilon\\ a_1^\epsilon & 0\end{pmatrix},\quad \mathcal{W}=\begin{pmatrix}w_1 & 0\\ 0 & w_2\end{pmatrix}.$$

Then we have the following estimate.

LEMMA 8.1. *For ϵ sufficiently small enough, it holds that*

$$\|\phi_\epsilon^\perp - \phi_\epsilon^1\|_{(H^2(\Omega_\epsilon))^2} = O(\epsilon^2). \tag{8.11}$$

Proof: To prove Lemma 8.1, we first need to derive a relation between $\phi_{\epsilon,j}^\perp$, $\eta_{\epsilon,j}^\perp$ and $\psi_{\epsilon,j}^\perp$. Note that similarly to the proof of Proposition 3.1 in Section 3 it follows that $\tilde{L}_\epsilon$ is uniformly invertible from $\left(\mathcal{K}_{\epsilon,\mathbf{x}^\epsilon}^{new}\right)^\perp$ to $\left(\mathcal{C}_{\epsilon,\mathbf{x}^\epsilon}^{new}\right)^\perp$. By this uniform invertibility, we deduce that

$$\|\phi_\epsilon^\perp\|_{(H^2(\Omega_\epsilon))^2} = O(\epsilon), \quad \text{where } \phi_\epsilon^\perp = \left(\phi_{\epsilon,1}^\perp, \phi_{\epsilon,2}^\perp\right)^T \in (\mathcal{K}_{\epsilon,\mathbf{x}^\epsilon}^{new})^\perp. \tag{8.12}$$

Let us cut off and re-scale $\phi_{\epsilon,j}^\perp$ as follows $\tilde{\phi}_{\epsilon,j} = \frac{\phi_{\epsilon,j}^\perp}{\epsilon}\chi\left(\frac{x-x_j^\epsilon}{r_0}\right)$. Then $\phi_{\epsilon,j}^\perp = \epsilon\tilde{\phi}_{\epsilon,j} + e.s.t.$

Choose $\phi_{\epsilon,j}$ such that $\|\tilde{\phi}_{\epsilon,j}\|_{H^1(R)} = 1$. Then we have, possibly for a subsequence, that $\tilde{\phi}_{\epsilon,j} \to \phi_j$ in $H^1_{loc}(R)$.

By (6.12) and (6.13), $\psi_\epsilon^\perp$ can be represented as follows (the proof is similar that of Lemma 6.4):

$$\psi_\epsilon^\perp(x_j^\epsilon) = \epsilon(\alpha_\epsilon)^2 c \sum_{k=1}^2 \int_{-L}^L G_{D_r}(x_j^\epsilon, z)$$

$$\left\{2\tilde{g}_{\epsilon,k}(z)\tilde{\phi}_{\epsilon,k}(z)\int_{-L}^L G_{D_s}(z,y)\tilde{g}_{\epsilon,3-k}(y)\,dy + \tilde{g}_{\epsilon,k}^2(z)\int_{-L}^L G_{D_s}(z,y)\tilde{\phi}_{\epsilon,3-k}(y)\,dy\right\} dz$$

$$= \epsilon\alpha_\epsilon c\sum_{k=1}^2 G_{D_r}(x_j^\epsilon, x_k^\epsilon)G_{D_s}(x_k^\epsilon, x_{3-k}^\epsilon)\left(2\hat{t}_{3-k}\int_{-L}^L \tilde{g}_{\epsilon,k}\tilde{\phi}_{\epsilon,k}\,dx + (\hat{t}_k)^2\int_{-L}^L \tilde{\phi}_{\epsilon,3-k}\,dx\right) + o(\epsilon)$$

$$= \epsilon c\sum_{k=1}^2 \hat{t}_k G_{D_r}(x_j^\epsilon, x_k^\epsilon)G_{D_s}(x_k^\epsilon, x_{3-k}^\epsilon)\left(2\hat{t}_{3-k}\frac{\int_R w\phi_k\,dy}{\int_R w^2\,dy} + \hat{t}_k\frac{\int_R \phi_{3-k}\,dy}{\int_R w\,dy}\right) + o(\epsilon)$$

$$= \frac{\epsilon c}{G_{D_r}(x_1^\epsilon, x_1^\epsilon) + G_{D_r}(x_1^\epsilon, x_2^\epsilon)}\left\{G_{D_r}(x_j^\epsilon, x_j^\epsilon)G_{D_s}(x_j^\epsilon, x_{3-j}^\epsilon)\left(2\hat{t}_{3-j}\frac{\int_R w\phi_j\,dy}{\int_R w^2\,dy} + \hat{t}_j\frac{\int_R \phi_{3-j}\,dy}{\int_R w\,dy}\right)\right.$$

$$\left. + G_{D_r}(x_j^\epsilon, x_{3-j}^\epsilon)G_{D_s}(x_{3-j}^\epsilon, x_j^\epsilon)\left(2\hat{t}_j\frac{\int_R w\phi_{3-j}\,dy}{\int_R w^2\,dy} + \hat{t}_{3-j}\frac{\int_R \phi_j\,dy}{\int_R w\,dy}\right)\right\} + o(\epsilon). \tag{8.13}$$

In the same way, we calculate

$$\eta_{\epsilon,3-j}^\perp(x_j^\epsilon) = \epsilon\alpha_\epsilon\int_{-L}^L G_{D_s}(x_j^\epsilon, z)\tilde{\phi}_{\epsilon,3-j}(z)\,dz = \epsilon\alpha_\epsilon G_{D_s}(x_j^\epsilon, x_{3-j}^\epsilon)\int_{-L}^L \tilde{\phi}_{\epsilon,3-j}\,dx + O(\epsilon^2)$$

$$= \epsilon G_{D_s}(x_j^\epsilon, x_{3-j}^\epsilon)\frac{\int_R \phi_{3-j}\,dy}{\int_R w\,dy} + o(\epsilon) \tag{8.14}$$

and

$$\eta_{\epsilon,j}^\perp(x_j^\epsilon) = o(\epsilon). \tag{8.15}$$

Substituting (6.18), (6.19), (6.20), (8.13), (8.14) into (6.14) and calculating the limit $\epsilon \to 0$ as we have done in Section 4, it follows that $\phi = (\phi_1, \phi_2)^T$ satisfies

$$L\phi = \Delta\phi - \phi + 2w\phi - \left[\mathcal{B}\int\phi + 2\mathcal{C}\left(\int_R w\phi\right)\right]\left(\int_R w\right)^{-1} w^2 = \hat{t}_1(\mathbf{a}\cdot\nabla\mathcal{G})\mathcal{G}^{-1}ew^2 - \frac{\hat{t}_1\mathcal{A}^0\nabla G_{D_s}(x_1, x_2)}{G_{D_s}(x_1, x_2)}w^2. \tag{8.16}$$

In the previous calculation we have used (4.9), (4.10), (8.2), the notations

$$\mathbf{a} = (a_1, a_2)^T = \lim_{\epsilon\to 0}(a_1^\epsilon, a_2^\epsilon)^T, \quad \mathbf{a}\cdot\nabla = a_1\nabla_{x_1} + a_2\nabla_{x_2}, \quad x_j = \lim_{\epsilon\to 0} x_j^\epsilon,\ j = 1, 2,$$

$$\mathcal{A}^0 = \begin{pmatrix} 0 & a_2 \\ a_1 & 0 \end{pmatrix}$$

and (compare Section 2)

$$\bar{r}_\epsilon(x_j^\epsilon) = c\hat{t}_1\hat{t}_2 G_{D_s}(x_j^\epsilon, x_{3-j}^\epsilon) + O(\epsilon), \quad j = 1, 2, \tag{8.17}$$

$$\bar{s}_{\epsilon,3-j}(x_j^\epsilon) = \hat{t}_{3-j} G_{D_s}(x_j^\epsilon, x_{3-j}^\epsilon) + O(\epsilon), \quad j = 1, 2. \tag{8.18}$$

We compute

$$\text{Id} - \mathcal{B} - 2\mathcal{C} = -\frac{1}{G_{D_r}(x_1,x_1) + G_{D_r}(x_1,x_2)} \begin{pmatrix} G_{D_r}(x_1,x_1) & G_{D_r}(x_1,x_2) \\ G_{D_r}(x_1,x_2) & G_{D_r}(x_2,x_2) \end{pmatrix} = -\hat{t}_1 \mathcal{G}.$$

By the Fredholm alternative and since $\det(\mathcal{G}) \neq 0$, equation (8.16) has a unique solution ϕ which is given by

$$\phi = -\mathcal{G}^{-1}(\mathbf{a} \cdot \nabla \mathcal{G})\mathcal{G}^{-1} e w + \frac{\mathcal{G}^{-1}\mathcal{A}^0 \nabla G_{D_s}(x_1,x_2)}{G_{D_s}(x_1,x_2)} w. \tag{8.19}$$

Now we compare ϕ with ϕ_ϵ^1. By definition and using (8.2), we get

$$\phi_\epsilon^1 = \left(\epsilon\left(a_1^\epsilon \nabla_{x_1^\epsilon}\hat{t}_1 + a_2^\epsilon \nabla_{x_2^\epsilon}\hat{t}_1\right)\tilde{g}_{\epsilon,1},\, \epsilon\left(a_1^\epsilon \nabla_{x_1^\epsilon}\hat{t}_2 + a_2^\epsilon \nabla_{x_2^\epsilon}\hat{t}_2\right)\tilde{g}_{\epsilon,2}\right)^T + \epsilon\frac{\mathcal{G}^{-1}\mathcal{W}\mathcal{A}^0 \nabla G_{D_s}(x_1,x_2)}{G_{D_s}(x_1,x_2)}$$

$$= \epsilon(\mathbf{a}^\epsilon \cdot \nabla_{x^\epsilon}\hat{t}) w + \epsilon\frac{\mathcal{G}^{-1}\mathcal{A}^0 \nabla G_{D_s}(x_1,x_2)}{G_{D_s}(x_1,x_2)} w + o(\epsilon)$$

$$= -\epsilon\mathcal{G}^{-1}(\mathbf{a} \cdot \nabla \mathcal{G})\mathcal{G}^{-1} e w + \epsilon\frac{\mathcal{G}^{-1}\mathcal{A}^0 \nabla G_{D_s}(x_1,x_2)}{G_{D_s}(x_1,x_2)} w + o(\epsilon). \tag{8.20}$$

On the other hand, using (8.19) gives

$$\phi_\epsilon^\perp = \epsilon\left(\tilde{\phi}_{\epsilon,1}, \tilde{\phi}_{\epsilon,2}\right)^T + e.s.t. = \epsilon\left(\phi_j\left(\frac{x - t_j^\epsilon}{\epsilon}\right)\right)_{j=1,2} + o(\epsilon)$$

$$= -\epsilon\mathcal{G}^{-1}(\mathbf{a} \cdot \nabla \mathcal{G})\mathcal{G}^{-1} e w + \epsilon\frac{\mathcal{G}^{-1}\mathcal{A}^0 \nabla G_{D_s}(x_1,x_2)}{G_{D_s}(x_1,x_2)} w + o(\epsilon). \tag{8.21}$$

From (8.20) and (8.21), it follows that $\phi_\epsilon = \phi_\epsilon^1 + o(1)$.
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Finally, we complete the proof of the key lemma – Lemma 6.1.

Proof of Lemma 6.1: The computation of J_1 follows from the Lemmas 6.4 and 6.5 and the equations (8.17), (8.18). We get

$$J_{1,l} = c\epsilon \sum_{k=1}^2 a_k^\epsilon \delta_{jl} \int_{-L}^{L} \frac{c(\tilde{g}_{\epsilon,j})^2 \bar{s}_{\epsilon,3-j}}{\bar{r}_\epsilon} \left(\delta_{kl}\frac{\bar{r}_\epsilon'}{\bar{r}_\epsilon} - \frac{\psi_{\epsilon,k}}{\bar{r}_\epsilon}\right) \tilde{g}_{\epsilon,l}' \, dx$$

$$= \epsilon \sum_{k=1}^2 a_k^\epsilon \delta_{jl} \int_{-L}^{L} c(\tilde{g}_{\epsilon,j})^2 \frac{\bar{s}_{\epsilon,3-j}}{\bar{r}_\epsilon}(x_l^\epsilon) \left(\delta_{kl}\frac{\bar{r}_\epsilon'}{\bar{r}_\epsilon} - \frac{\psi_{\epsilon,k}}{\bar{r}_\epsilon}\right) \tilde{g}_{\epsilon,l}' \, dx$$

$$+\epsilon \sum_{k=1}^2 a_k^\epsilon \delta_{jl} \int_{-L}^{L} \frac{c(\tilde{g}_{\epsilon,j})^2 \bar{s}_{\epsilon,3-j}}{\bar{r}_\epsilon} \left[\left(\delta_{kl}\frac{\bar{r}_\epsilon'}{\bar{r}_\epsilon} - \frac{\psi_{\epsilon,k}}{\bar{r}_\epsilon}\right)(x_l^\epsilon)\right] \tilde{g}_{\epsilon,l}' \, dx + o(\epsilon^2)$$

$$= -\epsilon^2 \hat{t}_l \left(\int_R \frac{1}{3} w^3 \, dy\right) \sum_{k=1}^2 a_k^\epsilon \Bigg\{ \nabla_{x_l^\epsilon} \Bigg\{ -\hat{t}_l \nabla_{x_k^\epsilon}\left(H_{D_r}(x_l^\epsilon, x_l^\epsilon) G_{D_s}(x_l^\epsilon, x_{3-l}^\epsilon)\right)$$

$$+\hat{t}_{3-l}\nabla_{x_k^\epsilon}\left(G_{D_r}(x_l^\epsilon, x_{3-l}^\epsilon) G_{D_s}(x_{3-l}^\epsilon, x_l^\epsilon)\right) + \frac{1}{2\sqrt{D_r}} \hat{t}_l \nabla_{x_k^\epsilon} G_{D_s}(x_l^\epsilon, x_{3-l}^\epsilon)\Bigg\}$$

$$\left\{ \left(\hat{t}_l G_{D_r}(x_l^\epsilon, x_l^\epsilon) + \hat{t}_{3-l} G_{D_r}(x_l^\epsilon, x_{3-l}^\epsilon)\right) G_{D_s}(x_l^\epsilon, x_{3-l}^\epsilon)\right\}^{-1}$$

$$-\left\{ - \hat{t}_l \nabla_{x_k^\epsilon} \left(H_{D_r}(x_l^\epsilon, x_l^\epsilon) G_{D_s}(x_l^\epsilon, x_{3-l}^\epsilon)\right)\right.$$

$$\left. +\hat{t}_{3-l} \nabla_{x_k^\epsilon} \left(G_{D_r}(x_l^\epsilon, x_{3-l}^\epsilon) G_{D_s}(x_{3-l}^\epsilon, x_l^\epsilon)\right) + \frac{1}{2\sqrt{D_r}} \hat{t}_l \nabla_{x_k^\epsilon} G_{D_s}(x_l^\epsilon, x_{3-l}^\epsilon)\right\}$$

$$\left.\times \left\{ \nabla_{x_l^\epsilon} G_{D_s}(x_l^\epsilon, x_{3-l}^\epsilon)\right\}\left\{ G_{D_s}(x_l^\epsilon, x_{3-l}^\epsilon)\right\}^{-2}\right\} + o(\epsilon^2)$$

$$= -\epsilon^2 \hat{t}_l \left(\int_R \frac{1}{3} w^3\, dy\right) \sum_{k=1}^2 a_k^\epsilon \left\{\left\{ - \hat{t}_l \nabla_{x_l^\epsilon} \nabla_{x_k^\epsilon} H_{D_r}(x_l^\epsilon, x_l^\epsilon) + \hat{t}_{3-l} \nabla_{x_l^\epsilon} \nabla_{x_k^\epsilon} G_{D_r}(x_l^\epsilon, x_{3-l}^\epsilon)\right\}\right.$$

$$+\nabla_{x_l^\epsilon} \left(\frac{\nabla_{x_k^\epsilon} G_{D_s}(x_l^\epsilon, x_{3-l}^\epsilon)}{G_{D_s}(x_l^\epsilon, x_{3-l}^\epsilon)}\right)$$

$$-\left\{ - \hat{t}_l \nabla_{x_k^\epsilon} H_{D_r}(x_l^\epsilon, x_l^\epsilon) + \hat{t}_{3-l} \nabla_{x_k^\epsilon} G_{D_r}(x_l^\epsilon, x_{3-l}^\epsilon)\right\}$$

$$\left.\times \left\{ - \hat{t}_l \nabla_{x_l^\epsilon} H_{D_r}(x_l^\epsilon, x_l^\epsilon) + \hat{t}_{3-l} \nabla_{x_l^\epsilon} G_{D_r}(x_l^\epsilon, x_{3-l}^\epsilon)\right\}\right\} + o(\epsilon^2).$$

In the previous computation of $J_{1,l}$ we have used the condition for the positions of the spikes given in the derivation of Theorem 3.2 which implies that $\frac{\bar{s}_{\epsilon,3-j}}{\bar{r}_\epsilon}(x_j^\epsilon) = O(\epsilon)$. More precisely, this condition implies that the second line in the previous computation has only a contribution which was included into the error terms. We will use the same condition in the computation of the other $J_{i,l}$ without explicitly mentioning it again.

Similarly, we compute $J_{2,l}$. We get

$$J_{2,l} = \epsilon \sum_{k=1}^2 a_k^\epsilon \int_{-L}^L \frac{c(\tilde{g}_{\epsilon,j})^2 \bar{s}_{\epsilon,3-j}}{\bar{r}_\epsilon} \left(\delta_{3-j,k} \frac{\eta_{\epsilon,3-j}^0}{\bar{s}_{\epsilon,3-j}} - \delta_{jk} \frac{\bar{s}_{\epsilon,3-j}'}{\bar{s}_{\epsilon,3-j}}\right) \tilde{g}_{\epsilon,l}'\, dx$$

$$= -\epsilon \sum_{k=1}^2 a_k^\epsilon \int_{-L}^L \frac{c(\tilde{g}_{\epsilon,j})^2 \bar{s}_{\epsilon,3-j}}{\bar{r}_\epsilon} \left(\frac{(\delta_{jk} \nabla_{x_j^\epsilon} + \delta_{3-j,k} \nabla_{x_{3-j}^\epsilon}) G_{D_s}(x_j^\epsilon, x_{3-j}^\epsilon)}{G_{D_s}(x_j^\epsilon, x_{3-j}^\epsilon)}\right) \tilde{g}_{\epsilon,l}'\, dx + o(\epsilon^2)$$

$$= \epsilon^2 \hat{t}_l \left(\int_R \frac{1}{3} w^3(y)\, dy\right) \sum_{k=1}^2 a_k^\epsilon \nabla_{x_l^\epsilon} \left(\frac{(\delta_{kl} \nabla_{x_l^\epsilon} + \delta_{k,3-l} \nabla_{x_{3-l}^\epsilon}) G_{D_s}(x_l^\epsilon, x_{3-l}^\epsilon)}{G_{D_s}(x_l^\epsilon, x_{3-l}^\epsilon)}\right) + o(\epsilon^2).$$

Note that we need to have $k = 3 - j$ and $j = l$; otherwise $J_{2,l}$ is of the order $o(\epsilon^2)$.

The estimate $J_{3,l} = o(\epsilon^2)$ follows by the fact that $\phi_{\epsilon,j}^\perp \perp \tilde{g}_{\epsilon,j}$.

Next we determine $J_{4,l}$. We compute, using (8.13), (8.14) and Lemma 7, that

$$J_{4,l} = c\delta_{jl} \int_{-L}^L \frac{(\tilde{g}_{\epsilon,j})^2 \bar{s}_{\epsilon,3-j}}{\bar{r}_\epsilon} \left(\frac{\eta_{\epsilon,3-j}^\perp}{\bar{s}_{\epsilon,3-j}} - \frac{\psi_\epsilon^\perp}{\bar{r}_\epsilon}\right) \tilde{g}_{\epsilon,l}'\, dx$$

$$= -\epsilon^2 \hat{t}_l \left(\int_R \frac{1}{3} w^3\, dy\right) \sum_{k=1}^2 a_k^\epsilon \left\{\left\{ (\nabla_{x_k^\epsilon} \hat{t}_l(x_1^\epsilon, x_2^\epsilon)) \nabla_{x_l^\epsilon} G_{D_r}(x_l^\epsilon, x_l^\epsilon) + (\nabla_{x_k^\epsilon} \hat{t}_{3-l}(x_1^\epsilon, x_2^\epsilon)) \nabla_{x_l^\epsilon} G_{D_r}(x_l^\epsilon, x_{3-l}^\epsilon)\right\}\right.$$

$$-\left\{ (\nabla_{x_k^\epsilon} \hat{t}_l(x_1^\epsilon, x_2^\epsilon)) G_{D_r}(x_l^\epsilon, x_l^\epsilon) + (\nabla_{x_k^\epsilon} \hat{t}_{3-l}(x_1^\epsilon, x_2^\epsilon)) G_{D_r}(x_l^\epsilon, x_{3-l}^\epsilon)\right\}$$

$$\times\Big\{-\hat{t}_l\nabla_{x_l^\epsilon}H_{D_r}(x_l^\epsilon,x_l^\epsilon)+\hat{t}_{3-l}\nabla_{x_l^\epsilon}G_{D_r}(x_l^\epsilon,x_{3-l}^\epsilon)\Big\}\Big\}+o(\epsilon^2).$$

Here we have used the relation

$$\int_{-L}^{L}\frac{c\tilde{g}_{\epsilon,j}^2\bar{s}_{\epsilon,3-j}}{\bar{r}_\epsilon}\frac{\eta_{\epsilon,3-j}^\perp}{\bar{s}_{\epsilon,3-j}}\epsilon\tilde{g}_{\epsilon,j}'\,dx=o(\epsilon^2)$$

which follows from the trivial identity

$$\nabla_{x_l^\epsilon}\left(\frac{G_{D_s}(x_j^\epsilon,x_{3-j}^\epsilon)}{G_{D_s}(x_j^\epsilon,x_{3-j}^\epsilon)}\right)=0.$$

In a similar way, using the identity

$$\nabla_{x_l^\epsilon}\left(\frac{\hat{t}_jG_{D_r}(x_j^\epsilon,x_j^\epsilon)+\hat{t}_{3-j}G_{D_r}(x_j^\epsilon,x_{3-j}^\epsilon)}{\hat{t}_jG_{D_r}(x_j^\epsilon,x_j^\epsilon)+\hat{t}_{3-j}G_{D_r}(x_j^\epsilon,x_{3-j}^\epsilon)}\right)=0,$$

it can be seen that the contribution of the term $-\epsilon\frac{\mathcal{G}^{-1}\mathcal{W}\mathcal{A}^0\nabla G_{D_s}(x_1^\epsilon,x_2^\epsilon)}{G_{D_s}(x_1^\epsilon,x_2^\epsilon)}$ in $\psi_\epsilon^\perp$ to $J_{4,l}$ is of the order $o(\epsilon^2)$.

Adding $J_{1,l}$, $J_{2,l}$ and $J_{4,l}$ we get

$$J_l=-\epsilon^2\hat{t}_l\left(\int_R\frac{1}{3}w^3\,dy\right)\sum_{k=1}^2 a_k^\epsilon\Big\{\Big\{-\hat{t}_l\nabla_{x_l^\epsilon}\nabla_{x_k^\epsilon}H_{D_r}(x_l^\epsilon,x_l^\epsilon)+\hat{t}_{3-l}\nabla_{x_l^\epsilon}\nabla_{x_k^\epsilon}G_{D_r}(x_l^\epsilon,x_{3-l}^\epsilon)\Big\}$$

$$+\nabla_{x_l^\epsilon}\left(\frac{\delta_{kl}\nabla_{x_l^\epsilon}G_{D_s}(x_l^\epsilon,x_{3-l}^\epsilon)}{G_{D_s}(x_l^\epsilon,x_{3-l}^\epsilon)}\right)$$

$$-\Big\{-\hat{t}_l\nabla_{x_k^\epsilon}H_{D_r}(x_l^\epsilon,x_l^\epsilon)+\hat{t}_{3-l}\nabla_{x_k^\epsilon}G_{D_r}(x_l^\epsilon,x_{3-l}^\epsilon)\Big\}\Big\{-\hat{t}_l\nabla_{x_l^\epsilon}H_{D_r}(x_l^\epsilon,x_l^\epsilon)+\hat{t}_{3-l}\nabla_{x_l^\epsilon}G_{D_r}(x_l^\epsilon,x_{3-l}^\epsilon)\Big\}$$

$$-\nabla_{x_l^\epsilon}\left(\frac{(\delta_{kl}\nabla_{x_l^\epsilon}+\delta_{k,3-l}\nabla_{x_{3-l}^\epsilon})G_{D_s}(x_l^\epsilon,x_{3-l}^\epsilon)}{G_{D_s}(x_l^\epsilon,x_{3-l}^\epsilon)}\right)$$

$$+\Big\{\left(\nabla_{x_k^\epsilon}\hat{t}_l(x_1^\epsilon,x_2^\epsilon)\right)\nabla_{x_l^\epsilon}G_{D_r}(x_l^\epsilon,x_l^\epsilon)+\left(\nabla_{x_k^\epsilon}\hat{t}_{3-l}(x_1^\epsilon,x_2^\epsilon)\right)\nabla_{x_l^\epsilon}G_{D_r}(x_l^\epsilon,x_{3-l}^\epsilon)\Big\}$$

$$-\Big\{\left(\nabla_{x_k^\epsilon}\hat{t}_l(x_1^\epsilon,x_2^\epsilon)\right)G_{D_r}(x_l^\epsilon,x_l^\epsilon)+\left(\nabla_{x_k^\epsilon}\hat{t}_{3-l}(x_1^\epsilon,x_2^\epsilon)\right)G_{D_r}(x_l^\epsilon,x_{3-l}^\epsilon)\Big\}$$

$$\times\Big\{-\hat{t}_l\nabla_{x_l^\epsilon}H_{D_r}(x_l^\epsilon,x_l^\epsilon)+\hat{t}_{3-l}\nabla_{x_l^\epsilon}G_{D_r}(x_l^\epsilon,x_{3-l}^\epsilon)\Big\}\Big\}+o(\epsilon^2).$$

This expression consists of 3+1+2=6 parts, which are given in one line each, with the exception of the last part which is given in the last two lines. Part 3 is minus Part 6 (up to $o(\epsilon^2)$) by (8.1) and they cancel. Part 2 and Part 4 cancel partially.

Making these simplifications, we finally get

$$J_l=-\epsilon^2\hat{t}_l\left(\int_R\frac{1}{3}w^3\,dy\right)\sum_{k=1}^2 a_k^\epsilon\Big\{\Big\{-\hat{t}_l\nabla_{x_l^\epsilon}\nabla_{x_k^\epsilon}\left(H_{D_r}(x_l^\epsilon,x_l^\epsilon)\right)+\hat{t}_{3-l}\nabla_{x_l^\epsilon}\nabla_{x_k^\epsilon}\left(G_{D_r}(x_l^\epsilon,x_{3-l}^\epsilon)\right)\Big\}$$

$$-\nabla_{x_l^\epsilon}\left(\frac{\delta_{k,3-l}\nabla_{x_{3-l}^\epsilon}G_{D_s}(x_l^\epsilon,x_{3-l}^\epsilon)}{G_{D_s}(x_l^\epsilon,x_{3-l}^\epsilon)}\right)$$

$$+\Big\{(\nabla_{x_k^\epsilon}\hat{t}_l(x_1^\epsilon,x_2^\epsilon))\nabla_{x_l^\epsilon}G_{D_r}(x_l^\epsilon,x_l^\epsilon)+(\nabla_{x_k^\epsilon}\hat{t}_{3-l}(x_1^\epsilon,x_2^\epsilon))\nabla_{x_l^\epsilon}G_{D_r}(x_l^\epsilon,x_{3-l}^\epsilon)\Big\}\Big\}+o(\epsilon^2).$$

This finishes the proof of Lemma 6.1.

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9. Appendix B: Proof of Lemma 6.6. Proof of Lemma 6.6:
We show that

$$P(x_1^\epsilon, x_2^\epsilon) = (\nabla_{x_1^\epsilon} + \nabla_{x_2^\epsilon})\Bigg\{ \frac{(\nabla_{x_1^\epsilon} - \nabla_{x_2^\epsilon}) G_{D_s}(x_1^\epsilon, x_2^\epsilon)}{G_{D_s}(x_1^\epsilon, x_2^\epsilon)}$$

$$-\hat{t}_1^\epsilon(x_1^\epsilon, x_2^\epsilon)(\nabla_{x_1^\epsilon} - \nabla_{x_2^\epsilon}) H_{D_r}(x_1^\epsilon, x_1^\epsilon) - \hat{t}_2^\epsilon(x_1^\epsilon, x_2^\epsilon)(\nabla_{x_1^\epsilon} - \nabla_{x_2^\epsilon}) H_{D_r}(x_1^\epsilon, x_1^\epsilon) \Bigg\} < 0.$$

We compute

$$(\nabla_{x_1^\epsilon} + \nabla_{x_2^\epsilon}) G_{D_s}(x_1^\epsilon, x_2^\epsilon) = 0,$$

and

$$(\nabla_{x_1^\epsilon} + \nabla_{x_2^\epsilon})(\nabla_{x_1^\epsilon} - \nabla_{x_2^\epsilon}) G_{D_s}(x_1^\epsilon, x_2^\epsilon) = ((\nabla_{x_1^\epsilon})^2 - (\nabla_{x_2^\epsilon})^2) G_{D_s}(x_1^\epsilon, x_2^\epsilon) = 0.$$

Therefore, the first term coming from G_{D_s} gives no contribution at all.

Further, we get

$$(\nabla_{x_1^\epsilon} + \nabla_{x_2^\epsilon}) \hat{t}_1^\epsilon(x_1^\epsilon, x_2^\epsilon) = \frac{\nabla_{x_2^\epsilon} G_{D_r}(x_2^\epsilon, x_2^\epsilon)}{\det \mathcal{G}}.$$

To simplify the previous expression, we use the identity

$$(\nabla_{x_1^\epsilon} + \nabla_{x_2^\epsilon})(\det \mathcal{G}) = 0. \tag{9.1}$$

which is easy to derive.

Using (9.1), we get

$$(\nabla_{x_1^\epsilon} + \nabla_{x_2^\epsilon}) \hat{t}_1^\epsilon(x_1^\epsilon, x_2^\epsilon) = \frac{\nabla_{x_2^\epsilon} G_{D_r}(x_2^\epsilon, x_2^\epsilon)}{\det \mathcal{G}} \tag{9.2}$$

which gives

$$-[(\nabla_{x_1^\epsilon} + \nabla_{x_2^\epsilon}) \hat{t}_1^\epsilon(x_1^\epsilon, x_2^\epsilon)](\nabla_{x_1^\epsilon} - \nabla_{x_2^\epsilon}) G_{D_r}(x_1^\epsilon, x_1^\epsilon) = -\frac{\nabla_{x_2^\epsilon} G_{D_r}(x_2^\epsilon, x_2^\epsilon)}{\det \mathcal{G}} \nabla_{x_1^\epsilon} G_{D_r}(x_1^\epsilon, x_1 \epsilon).$$

$$= \frac{\nabla_{x_1^\epsilon} G_{D_r}(x_1^\epsilon, x_1^\epsilon)}{\det \mathcal{G}} \nabla_{x_1^\epsilon} G_{D_r}(x_1^\epsilon, x_1^\epsilon). \tag{9.3}$$

In analogy to (9.2), we get

$$(\nabla_{x_1^\epsilon} + \nabla_{x_2^\epsilon}) \hat{t}_2^\epsilon(x_1^\epsilon, x_2^\epsilon) = \frac{\nabla_{x_1^\epsilon} G_{D_r}(x_1^\epsilon, x_1^\epsilon)}{\det \mathcal{G}} \tag{9.4}$$

which implies

$$-[(\nabla_{x_1^\epsilon} + \nabla_{x_2^\epsilon}) \hat{t}_2^\epsilon(x_1^\epsilon, x_2^\epsilon)](\nabla_{x_1^\epsilon} - \nabla_{x_2^\epsilon}) G_{D_r}(x_1^\epsilon, x_2^\epsilon) = -\frac{\nabla_{x_1^\epsilon} G_{D_r}(x_1^\epsilon, x_1^\epsilon)}{\det \mathcal{G}} 2\nabla_{x_1^\epsilon} G_{D_r}(x_1^\epsilon, x_2^\epsilon). \tag{9.5}$$

Finally, we compute

$$-\hat{t}_1^\epsilon(x_1^\epsilon, x_2^\epsilon)(\nabla_{x_1^\epsilon} + \nabla_{x_2^\epsilon})(\nabla_{x_1^\epsilon} - \nabla_{x_2^\epsilon}) G_{D_r}(x_1^\epsilon, x_1^\epsilon) = -\hat{t}_1^\epsilon(x_1^\epsilon, x_2^\epsilon) \nabla^2_{x_1^\epsilon} G_{D_r}(x_1^\epsilon, x_1^\epsilon)$$

$$= -\frac{G_{D_r}(x_2^\epsilon, x_2^\epsilon) - G_{D_r}(x_1^\epsilon, x_2^\epsilon)}{\det \mathcal{G}} \nabla^2_{x_1^\epsilon} G_{D_r}(x_1^\epsilon, x_1^\epsilon). \tag{9.6}$$

Now $P(x_1^\epsilon, x_2^\epsilon)$ is given by the sum of (9.3), (9.5) and (9.6).

Using the explicit expression of the Green's function (2.6), we get for the sum of (9.3) and (9.5):

$$\frac{\nabla_{x_1^\epsilon} G_{D_r}(x_1^\epsilon, x_1^\epsilon)}{\det \mathcal{G}} \left[\nabla_{x_1^\epsilon} G_{D_r}(x_1^\epsilon, x_1^\epsilon) - 2\nabla_{x_1^\epsilon} G_{D_r}(x_1^\epsilon, x_2^\epsilon) \right]$$

$$= \frac{\theta_r^4}{\sinh^2 2\theta_r L \det \mathcal{G}} \sinh(2\theta_r - x_1^\epsilon) \left[\sinh 2\theta_r x_1^\epsilon + \sinh 2\theta_r (L - x_1^\epsilon)\right].$$

For (9.6), we get

$$-\frac{G_{D_r}(x_2^\epsilon, x_2^\epsilon) - G_{D_r}(x_1^\epsilon, x_2^\epsilon)}{\det \mathcal{G}} \nabla^2_{x_1^\epsilon} G_{D_r}(x_1^\epsilon, x_1^\epsilon)$$

$$= -\frac{\theta_r^4}{\sinh^2 2\theta_r L \det \mathcal{G}} \cosh 2\theta_r (L + x_2^\epsilon) \left[\cosh \theta_r (L - x_2^\epsilon) - \cosh \theta_r (L - x_1^\epsilon)\right] 2 \cosh 2\theta_r x_1^\epsilon.$$

Adding all up, we get

$$P(x_1^\epsilon, x_2^\epsilon) = \frac{\theta_r^4}{\sinh^2 2\theta_r L \det \mathcal{G}} \left\{ -2 \cosh 2\theta_r (L + x_2^\epsilon) \left[\cosh \theta_r (L - x_2^\epsilon) - \cosh \theta_r (L - x_1^\epsilon)\right] \cosh 2\theta_r x_1^\epsilon \right.$$

$$\left. + \sinh 2\theta_r x_1^\epsilon \left[\sinh 2\theta_r x_1^\epsilon + \sinh 2\theta_r (L - x_1^\epsilon)\right] \right\}$$

$$= \frac{\theta_r^4}{\sinh^2 2\theta_r L \det \mathcal{G}} \left\{ \cosh 2\theta_r L \, \cdot \left[1 - \cosh 2\theta_r x_1^\epsilon\right] \right\}.$$

Note that for $x_1 = \lim_{\epsilon \to 0} x_1^\epsilon$ we have

$$\cosh 2\theta_r L \, \cdot \left[1 - \cosh 2\theta_r x_1\right] \leq 0$$

and

$$\cosh 2\theta_r L \cdot \left[1 - \cosh 2\theta_r x_1\right] = 0 \quad \text{ if and only if } x_1 = 0.$$

Therefore, if $x_1 \neq 0$, then for ϵ small enough we have $P(x_1^\epsilon, x_2^\epsilon) < 0$.

This concludes the proof of Lemma 6.6.

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