Topological Degree For Solutions of Fourth Order Mean Field Equations

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Abstract

We consider the following fourth order mean field equation with Navier boundary condition

\[ \Delta^2 u = \rho \frac{h(x)e^u}{\int_{\Omega} he^u} \text{ in } \Omega, \quad u = \Delta u = 0 \text{ on } \partial \Omega, \]

where \( h \) is a \( C^{2,\beta} \) positive function, \( \Omega \) is a bounded and smooth domain in \( \mathbb{R}^d \). We prove that for \( \rho \in (32m\sigma_3, 32(m + 1)\sigma_3) \) the degree-counting formula for (*) is given by

\[
d(\rho) = \begin{cases} 
\frac{1}{m!}(-\chi(\Omega) + 1) \cdots (-\chi(\Omega) + m) & \text{for } m > 0, \\
1 & \text{for } m = 0 
\end{cases}
\]

where \( \chi(\Omega) \) is the Euler characteristic of \( \Omega \). Similar result is also proved for the corresponding Dirichlet problem

\[ \Delta^2 u = \rho \frac{h(x)e^u}{\int_{\Omega} he^u} \text{ in } \Omega, \quad u = \nabla u = 0 \text{ on } \partial \Omega. \]

1 Introduction

This is a continuation of the previous paper Lin-Wei [18], where they studied the following fourth order mean field equation

\[
\begin{cases}
\Delta^2 u = \rho \frac{he^u}{\int_{\Omega} he^u} & \text{in } \Omega, \\
u = \Delta u = 0 & \text{on } \partial \Omega
\end{cases}
\]

\[ (1.1) \]

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and computed the sharp estimates for the bubbling solutions. Our aim here is to compute the Leray-Schauder degree for solutions of (1.1).

In dimension two, the analogous problem
\[
\begin{align*}
-\Delta u &= \rho \frac{he^u}{\int_{\Omega} he^u} \quad \text{in } \Omega, \\
u &= 0 \quad \text{on } \partial \Omega,
\end{align*}
\]
where \( \Omega \) is a smooth and bounded domain in \( \mathbb{R}^2 \), has been extensively studied by many authors. We summarize the results for (1.2) and identify the difficulty in studying (1.1) now. Let \((u_k, \rho_k)\) be a bubbling sequence to (1.2) with \( \rho_k \leq C, \max_{x \in \Omega} u_k(x) \to +\infty \). Then it has been proved that

(P1) **(no boundary bubbles)** \( u_k \) is uniformly bounded near a neighborhood of \( \partial \Omega \) (Ma-Wei [19]);

(P2) **(bubbles are simple)** \( \rho_k \to 8m\pi \) for some \( m \geq 1 \) and \( u_k(x) \to 8\pi \sum_{j=1}^{m} G_2(\cdot, p_j) \) in \( C^2(\Omega \setminus \{p_1, \ldots, p_m\}) \) (Brezis-Merle [2], Nagasaki-Suzuki [22], Ma-Wei [19]), where \( G_2 \) is the Green function of \(-\Delta\) with Dirichlet boundary condition;

(P3) **(sup + inf estimates)** at each bubble \( p_{k,j} \) where \( u_k(p_{k,j}) = \max_{x \in B_{\delta}(p_j)} u_k(x) \), the following refined estimate holds (Brezis-Li-Shafrir [5], Li [13], Li-Shafrir [14])
\[
|u_k(x) - u_k(p_{k,j}) - \log \frac{1}{(1 + \frac{|x-p_{k,j}|}{\epsilon_{k,j}})^2}| \leq C
\]
where \( u_k(p_{k,j}) - \log(\int_{\Omega} he^{u_k}) = \log \frac{1}{\epsilon_{k,j}} \);

(P4) **(exact bubbling rate)** It holds then (Chen and Lin [7])
\[
\rho_k - 8m\pi = c_0 \sum_{j=1}^{m} h(p_{k,j})^{-1} \Delta \log h(p_{k,j}) \epsilon_{k,j}^2 \log \frac{1}{\epsilon_{k,j}} + O(\sum_{j=1}^{m} \epsilon_{k,j}^2);
\]

(P5) **(Leray-Schauder degree)** The Leray-Schauder degree of (1.2) is given by (Chen and Lin [8])
\[
d(\rho) = \begin{cases} 
\frac{1}{m}(-\chi(\Omega) + 1) \cdots (-\chi(\Omega) + m) & \text{for } m > 0, \\
1 & \text{for } m = 0
\end{cases}
\]
where \( \chi(\Omega) \) is the Euler characteristic of \( \Omega \).

In the previous paper [18], the same program was carried out for equation (1.1) and the properties (P1)-(P4) were established. Namely, we have the following theorem:

**Theorem 1.1** ([18]) Let \( h \) be a positive \( C^{2,\beta} \) function in \( \Omega \) and \( u_k \) be a sequence of blowup solutions of (1.1) with \( \rho = \rho_k \leq C \). Then (after extracting a subsequence), \( \lim_{k \to +\infty} \rho_k = 32\sigma_3 m \) for some positive integer \( m \). Furthermore,
\[
\rho_k - 32\sigma_3 m
\]
\[
= c_0 \sum_{j=1}^{m} ( h(p_{k,j}) )^{1/2} c_{k,j}^2 \left( \frac{1}{32 \sigma_3} \Delta \log h(p_{k,j}) + \Delta R_4(p_{k,j}, p_{k,j}) + \sum_{i \neq j} \Delta G_4(p_{k,j}, p_{k,i}) \right) + o(\sum_{j=1}^{m} c_{k,j}^2)
\]

where \( \sigma_3 \) is the area of the unit sphere in \( \mathbb{R}^3 \), i.e. \( \sigma_3 = 2\pi^2 \), \( c_0 > 0 \) is a generic constant, \( G_4(\cdot, p) \) is the Green function of \( \Delta^2 \) with Navier boundary condition \( G_4(\cdot, p) = \Delta G_4(\cdot, p) = 0 \) on \( \partial \Omega \), \( R_4 \) is the regular part of \( G_4 \), \( p_{k,j} \) are the local maximum points of \( u_k \) on \( B_6(p_j) \), \( \log \frac{384}{c_{k,j}^2} = u_k(p_{k,j}) - \log(\int_{\Omega} he^{uw}) \) and \( p_j \) satisfies

\[
\nabla \left( \frac{1}{32 \sigma_3} \log h(p_j) + R_4(p_j, p_j) + \sum_{i \neq j} G_4(p_i, p_j) \right) = 0, \quad j = 1, \ldots, m.
\]

In this paper, we prove the following theorem, which completes the program (P5):

**Theorem 1.2** Let \( 32m\sigma_3 < \rho < 32(m+1)\sigma_3 \) and \( d(\rho) \) be the Leray-Schauder degree for equation (1.1). Then

\[
d(\rho) = \begin{cases} 
\frac{1}{m}(-\chi(\Omega)+1) \cdots (-\chi(\Omega)+m) & \text{for } m > 0, \\
1 & \text{for } m = 0
\end{cases}
\]

(1.7)

where \( \chi(\Omega) \) is the Euler characteristic of \( \Omega \).

**Remark.** We are informed by Prof. Malchiodi that he obtained a similar degree counting formula for the corresponding prescribing \( Q \)-curvature problem on a four dimensional compact manifold, [21]. He used Morse theory to obtain the formula. We remark that on compact manifolds, one doesn’t need to prove Property (P1). On the other hand, one of the main difficulties in our proof is the property (P1).

As a consequence of Theorem 1.2, equation (1.1) always possesses a solution for \( \rho \neq 32m\sigma_3 \) whenever the Euler characteristic \( \chi(\Omega) \leq 0 \). (Here \( m \) can be made \( \geq 2 \), by results of Lin-Wei [17].) In particular, we have

**Corollary 1.3** If for some integer \( d \geq 1 \) such that \( H^d(\Omega) \neq 0 \), then equation (1.1) always has a solution for \( \rho \notin \{32m\sigma_3, m \geq 2\} \).

Set \( d_+^m = \lim_{\rho \to 32m\sigma_3^+} d(\rho) \) and \( d_-^m = \lim_{\rho \to 32m\sigma_3^-} d(\rho) \). One of the main steps in the proof of Theorem 1.2 is to calculate the gap \( d_+^m - d_-^m \) for any integer \( m \geq 1 \). Once this is known, \( d(\rho) \) can be computed inductively on \( m \). Clearly, the gap of \( d_+^m - d_-^m \) is due to the occurrence of blowup solutions when \( \rho \to 32m\sigma_3 \). Thus an important question is to analyze the blowup behavior of sequence of solutions \( u_k \) to (1.1) and to know the sign of \( \rho_k - 32m\sigma_3 \), which has been done in [18].

**Remark 1.4** Theorems 1.1-1.2 can be extended easily to the following \( n \)-th order mean field type equation

\[
\begin{align*}
(-\Delta)^n u &= \rho \frac{he^u}{\int_{\Omega} he^u} \quad \text{in } \Omega, \\
(-\Delta)^j u &= 0 \quad \text{on } \partial \Omega, \quad j = 0, \ldots, n - 1
\end{align*}
\]

(1.8)

where \( \Omega \) is a smooth and bounded domain in \( \mathbb{R}^{2n} \).
Similar result can also be obtained for the corresponding Dirichlet problem
\[ \begin{align*}
\Delta^2 u &= \rho \frac{h^\alpha}{h^{\alpha}} \quad \text{in } \Omega, \\
\n u &= \nabla u = 0 \quad \text{on } \partial \Omega.
\end{align*} \] (1.9)

We summarize in the following theorem, whose proof will be given in Section 4.

**Theorem 1.5** (1) Let \( h \) be a positive \( C^{2,\beta} \) function in \( \Omega \) and \( u_k \) be a sequence of blowup solutions of (1.9) with \( \rho = \rho_k \leq C \). Then (after extracting a subsequence), \( \lim_{k \to +\infty} \rho_k = 32\sigma_3 m \) for some positive integer \( m \). Furthermore,
\[ \rho_k - 32\sigma_3 m \]
\[ = c_0 \sum_{j=1}^{m} (h(p_{k,j}))^{-\frac{1}{2}e_{k,j}^2} \left( \frac{1}{32\sigma_3} \Delta \log h(p_{k,j}) + \Delta R_4'(p_{k,j}, p_{k,j}) + \sum_{i \neq j} \Delta G'_4(p_{k,j}, p_{k,i}) \right) + o\left( \sum_{j=1}^{m} e_{k,j}^2 \right) \] (1.10)
where \( \sigma_3 \) is the area of the unit sphere in \( \mathbb{R}^4 \), i.e. \( \sigma_3 = 2\pi^2 \), \( c_0 > 0 \) is a generic constant, \( G'_4(\cdot, p) \) is the Green function of \( \Delta^2 \) with Dirichlet boundary condition \( G'_4(\cdot, p) = \nabla G'_4(\cdot, p) = 0 \) on \( \partial \Omega \), \( R_4' \) is the regular part of \( G'_4 \), \( p_{k,j} \) are the local maximum points of \( u_k \) on \( B_{\delta}(p_j) \), \( \log e_{k,j}^2 = u_k(p_{k,j}) - \log(\int_{\Omega} h^\alpha) \) and \( p_j \) satisfies
\[ \nabla \left( \frac{1}{32\sigma_3} \Delta \log h(p_j) + R_4(p_j, p_j) + \sum_{i \neq j} G_4(p_i, p_j) \right) = 0, \quad j = 1, \ldots, m; \]

(2) Let \( 32m\sigma_3 < \rho < 32(m + 1)\sigma_3 \) and \( d(\rho) \) be the Leray-Schauder degree for equation (1.9). Then
\[ d(\rho) = \begin{cases} 
\frac{1}{m!}(-\chi(\Omega) + 1) \cdots (-\chi(\Omega) + m) & \text{for } m > 0, \\
1 & \text{for } m = 0.
\end{cases} \] (1.11)

Semilinear equations involving exponential nonlinearity and fourth order elliptic operator appear naturally in conformal geometry and in particular in prescribing \( Q \)-curvature on 4-dimensional Riemannian manifold \( M \) (see e.g. Chang-Yang [6])
\[ P_g w + 2Q_g = 2\bar{Q}_{g_w} e^{4w} \] (1.12)
where \( P_g \) is the so-called Paneitz operator:
\[ P_g = (\Delta g)^2 + \delta \left( \frac{2}{3} R_g I - 2\text{Ric}_g \right) d, \]
\( g_w = e^{2w} g \), \( Q_g \) is \( Q \)-curvature under the metric \( g \), and \( \bar{Q}_{g_w} \) is the \( Q \)-curvature under the new metric \( g_w \).
Integrating (1.12) over $M$, we obtain

$$k_g := \int_M Q_g = \int_M (\bar{Q}_{g^w}) e^{4w}$$

where $k_g$ is conformally-invariant. Thus, we can write (1.12) as

$$P_g w + 2Q_g = k_g \frac{\bar{Q}_{g^w} e^{4w}}{\int_M Q_{g^w} e^{4w}}$$  \hspace{1cm} (1.13)$$

In the special case, where the manifold is the Euclidean space, $P_g = \Delta^2$, and (1.13) becomes

$$\Delta^2 w = \rho \frac{h(x)e^{4w}}{\int_{\Omega} h(x)e^{4w}}$$  \hspace{1cm} (1.14)$$

There is now an extensive literature about this problem, we refer to Adimurthi-Robert-Struwe [1], Baraket-Dammak-Ouni-Pacard [3], Clapp-Munoz-Musso [9], Druet-Robert [10], Hebey-Robert [11], Hebey-Robert-Wen [12], Malchiodi [20], Robert-Wei [24] and the references therein. In particular, we mention the two papers [3] and [9], where they constructed $m-$point blowing-up solutions for

$$\begin{cases}
\Delta^2 u = e^{4u} & \text{in } \Omega, \\
u = \Delta u = 0 & \text{on } \partial\Omega
\end{cases}$$  \hspace{1cm} (1.15)$$

under either nondegenerate conditions ([3]) or topological nontrivial condition ([9]).

The organization of this paper is as follows: In Section 2, important preliminaries are presented. In Section 3, we prove Theorem 1.2 and the proof of a key lemma is in Appendix. Finally in Section 4, we give an outline of proof of Theorem 1.5.

Throughout this paper, unless otherwise stated, the letter $C$ will always denote various generic constants which are independent of $k \geq 1$.

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## 2 Preliminaries

We state several results in this section, which will be used for the proof of Theorem 1.2 and Theorem 1.5.

Let $G_4$ denote the Green’s function of $\Delta^2$ under the Navier boundary condition, that is

$$\Delta^2 G_4(x, y) = \delta(x - y), \quad G_4|_{\partial\Omega} = \Delta G_4|_{\partial\Omega} = 0.$$  \hspace{1cm} (2.1)$$

We decompose

$$G_4(x, y) = \frac{1}{4\sigma_3} \log \frac{1}{|x - y|} + R_4(x, y).$$  \hspace{1cm} (2.2)$$
It is easy to see that
\[ \Delta_2 G_4(x, y) < 0, \Delta_2 R_4(x, y) > 0. \] (2.3)

In [18], we have proved the following lemmas.

**Lemma 2.1** Let \( u_k \) be a bubbling sequence of (1.1) with \( \rho_k \leq C \). Then (after extracting a subsequence), \( \rho_k \to 32\sigma_3 m \) and \( u_k(x) \to 32\sigma_3 \sum_{j=1}^{m} G_4(\cdot, p_j) \) in \( C^4_{\text{loc}}(\Omega\setminus\{p_1, \ldots, p_m\}) \), where \((p_1, \ldots, p_m)\) satisfies
\[ \nabla \left( \frac{1}{32\sigma_3} \log h(p_i) + R_4(p_i, p) + \sum_{j \neq i} G_4(p_i, p_j) \right) = 0, \quad i = 1, \ldots, m. \] (2.4)

Let \( \delta_0 \) be a fixed small positive constant and \( u_k(p_{k,j}) = \max_{x \in B_{\delta_0}(p_j)} u_k(x) \) and
\[ e^{-c_k} = \frac{1}{\int_{\Omega} h(x)e^{u_k}}. \] (2.5)

Define
\[ l_{k,j} = u_k(p_{k,j}) - c_k, \quad e^{-l_{k,j}} = \frac{\epsilon_{k,j}}{\alpha_4}, \quad \epsilon_k = \min_{1 \leq j \leq m} \epsilon_{k,j}. \] (2.6)

In fact, we can refine the Estimate A and Estimate B in [18]. That is, let
\[ w_k(x) = u_k(x) - \sum_{i=1}^{m} \rho_{k,i} G_4(x, p_{k,i}) \text{ on } \Omega \setminus \bigcup_{j=1}^{m} B_{\frac{\delta_0}{2}}(p_{k,j}) \] (2.7)

and
\[ G^*_j(x) = \rho_{k,j} R_4(x, p_{k,j}) + \sum_{i \neq j} \rho_{k,i} G_4(x, p_{k,i}), \] (2.8)

where \( \rho_{k,j} = \frac{\rho_k}{\int_{B_{\delta_0}(p_{k,j})} h(x)e^{u_k}}. \)

**Lemma 2.2**
\[ |w_k(x)| + |\partial^\alpha w_k(x)| = O(\varepsilon_k^2) \text{ for } |\alpha| \leq 3 \text{ in } \Omega \setminus \bigcup_{j=1}^{m} B_{\frac{\delta_0}{2}}(p_{k,j}), \]

and
\[ |\nabla (\log h(x) + G^*_j(x))| = O(\varepsilon_k^2) \text{ at } x = p_{k,j}. \]

Using the Estimate C-F in [18], the proof is just the same as Lemma 5.3, Lemma 5.4 in [7]. We omit it here.

It has been proved ([16], [25]) that the solution to the following problem
\[ \begin{cases} \Delta^2 U = e^U & \text{in } \mathbb{R}^d, \\ \int_{\mathbb{R}^d} e^U < +\infty, \end{cases} \] (2.9)
is given by
\begin{equation}
U_{e,a}(x) := \log \frac{\alpha_4 \epsilon^4}{(\epsilon^2 + |x - a|^2)^t},
\end{equation}
for any $\epsilon > 0, a \in \mathbb{R}^4, \alpha_4 = 384$, provided that
\begin{equation}
U(x) = o(|x|^2) \quad \text{as} \quad |x| \to +\infty.
\end{equation}

Let $U = \log \frac{Fa}{(1+|x|^2)^\tau}$ and $\tau \in (0, 1)$ be a fixed constant. We have the following lemma which proves the nondegeneracy of $U$:

**Lemma 2.3** The solutions to the following linearized problem
\begin{equation}
\Delta^2 \phi = e^U \phi, \quad |\phi(y)| \leq C(1 + |y|)^\tau
\end{equation}
is given by $\phi = \sum_{j=0}^4 c_j \psi_j$ where
\begin{equation}
\psi_0 = \frac{1 - |y|^2}{1 + |y|^2}, \quad \psi_j = \frac{y_j}{1 + |y|^2}, \quad j = 1, ..., 4.
\end{equation}

Recall that $G'_{4}(x, y)$ is defined in Theorem 1.5. In general, $G'_4$ is not positive. We collect the property of $G'_4$ in the following lemma which has been proved in [24].

**Lemma 2.4** Let $d(x) = d(x, \partial \Omega)$. Then we have
\begin{equation}
|G'_{4}(x, y)| \leq C \log \left(1 + \frac{d(x)d(y)}{|x - y|^2}\right),
\end{equation}
and
\begin{equation}
|\nabla^i G'_{4}(x, y)| \leq C |x - y|^{-i} \min(1, \frac{d(y)}{|x - y|^2}), \quad i \geq 1.
\end{equation}

We also need to recall the well-known Pohozaev’s identity for solutions of fourth-order equation
\begin{equation}
\Delta^2 u = h(x)e^u \quad \text{in} \quad D.
\end{equation}

**Lemma 2.5** Let $u$ satisfy $\Delta^2 u = h(x)e^u$ in $D$, where $D$ is a smooth and bounded domain in $\mathbb{R}^4$. Then we have
\begin{equation}
\int_D (Ah + \langle x - \xi, \nabla h \rangle)e^u = \int_{\partial D} \langle x - \xi, \nu \rangle h(x)e^u + \int_{\partial D} \left[ \frac{1}{2} |\Delta u|^2 \langle x - \xi, \nu \rangle - 2 \frac{\partial u}{\partial \nu} \Delta u \right. \\
- \langle x - \xi, \nabla \rangle \frac{\partial u}{\partial \nu} - \langle x - \xi, \nabla \Delta u \rangle \frac{\partial u}{\partial \nu} + \langle x - \xi, \nu \rangle \langle \nabla u, \nabla \Delta u \rangle \right],
\end{equation}
for any $\xi \in \mathbb{R}^4$.

**Proof:** In fact, multiplying $\Delta^2 u = h(x)e^u$ by $(x - \xi) \cdot \nabla u$ and integrating by parts, we obtain the lemma. \(\square\)

In the rest of the paper, we denote $H^2(\Omega)$ be the usual Sobolev space and $\mathcal{X} = H^2(\Omega) \cap \{u = \Delta u = 0 \text{ on } \partial \Omega\}$. On $\mathcal{X}$, we use the following inner product:
\begin{equation}
(u, v) = \int_{\Omega} (\Delta u \Delta v + uv).
\end{equation}
3 The proof of Theorem 1.2

The purpose of this section is to prove Theorem 1.2. We follow the main steps used in [8]. Note that the notations $c_k, p_{k,j}, l_{k,j}, \epsilon_{k,j}$ given in (2.5), (2.6) have no relation with our definitions in this section and $\rho \in (32m\sigma_3, 32(m + 1)\sigma_3)$. And from now on, we assume $\rho$ is sufficiently close to $32m\sigma_3$.

Let $P = (p_1, \ldots, p_m) \in \Omega^m, \Lambda = (\lambda_1, \ldots, \lambda_m) \in \mathbb{R}^m, \varepsilon_j^4 = \alpha_4 e^{-\lambda_j}$ and $A = (a_1, \ldots, a_m) \in \mathbb{R}^m$. Set

$$v_j(x) = \log \left( e^{\lambda_j} \left( 1 + \frac{\rho \lambda_j}{\alpha_4} \right) \right)^{1/2},$$

then $v_j(x)$ satisfies the equation $\Delta^2 v_j(x) = \rho h(p_j) \varepsilon_j^4$. Let

$$H_j(x) = \exp \left\{ \log \frac{h(x)}{h(p_j)} + G^*_j(x) - G^*_j(p_j) \right\} - 1$$

on $B_{2\delta_0}(p_j)$, and

$$s_j = -\lambda_j - 2 \log \frac{\rho h(p_j)}{\alpha_4} - 64\pi^2 R_4(p_j, p_j).$$

We assume $\lambda_j$ large enough.

Let $\tau : \mathbb{R} \rightarrow \mathbb{R}$ be a smooth cutoff function satisfying $0 \leq \tau \leq 1$ and $\tau(t) = 1$ for $|t| \leq \delta_0$, $\tau(t) = 0$ for $|t| \geq 2\delta_0$. Set $\tau_j(x) = \tau(|x - p_j|)$ and let $\gamma_j(x)$ satisfy

$$\Delta^2 \gamma_j(x) = \rho h(p_j) \varepsilon_j^4 \left( \gamma_j + (H_j(x) - \nabla H_j(p_j) \cdot (x - p_j)) \tau_j \right) \quad \text{on} \quad \mathbb{R}^4 \quad \gamma_j(p_j) = 0, \quad \Delta \gamma_j(p_j) = 0, \quad \gamma_j(x) \rightarrow 0 \quad \text{as} \quad |x| \rightarrow +\infty.$$  

As [8], $\gamma_j$ is solvable and $|\gamma_j|, |\nabla \gamma_j|, |\partial \gamma_j|, |\partial \gamma_j|, |\nabla x \partial \gamma_j|, |\nabla x \partial \gamma_j|$ are of the order $O(\varepsilon_j^2)$. In fact, by Lemma 2.3, we know that $\gamma_j(x)$ is bounded in $\mathbb{R}^4 \setminus B_{2\delta_0}(p_j)$ and hence is bounded in $\mathbb{R}^4$. Now we consider zero mode $\gamma_{j,0}$ of $\gamma_j$ since the remaining terms are of small order with respect to $\gamma_{j,0}$.

Let

$$\Psi(x) = \frac{\varepsilon_j^2 - \sqrt{\rho h(p_j)} |x - p_j|^2}{\varepsilon_j^2 + \sqrt{\rho h(p_j)} |x - p_j|^2}.$$  

Then it is easy to see that $\Psi(x)$ satisfies

$$\Delta^2 \Psi(x) = \rho h(p_j) \varepsilon_j^4 \Psi(x).$$

Using (3.4) and (3.5), we obtain

$$\int_{B_{\delta_0}(p_j)} [\Delta^2 \gamma_{j,0}(x) \Psi(x) - \Delta^2 \Psi(x) \gamma_{j,0}(x)] = \int_{B_{\delta_0}(p_j)} \rho h(p_j) \varepsilon_j^4 H_{j,0} \Psi(x),$$

where $H_{j,0}$ is zero mode of $H_j(x) - \nabla H_j(p_j) \cdot (x - p_j) = O(|x - p_j|^2)$ and $B_{\delta_0}(p_j) \subset \Omega$.  

The left-hand-side of (3.6) equals
\[
\int_{\partial B_{\varepsilon_0}(p_j)} \left[ \frac{\partial \Delta \gamma_{j,0}}{\partial \nu} \Psi(x) - \Delta \gamma_{j,0} \frac{\partial \Psi}{\partial \nu} + \Delta \Psi \frac{\partial \gamma_{j,0}}{\partial \nu} - \gamma_{j,0} \frac{\partial \Delta \Psi}{\partial \nu} \right]
= \int_{\partial B_{\varepsilon_0}(p_j)} \frac{\partial \Delta \gamma_{j,0}}{\partial \nu} (\Psi(x) + 1) - \int_{\partial B_{\varepsilon_0}(p_j)} \frac{\partial \Delta \gamma_{j,0}}{\partial \nu} + O(\varepsilon_j^2)
= \int_{\partial B_{\varepsilon_0}(p_j)} \frac{\partial \Delta \gamma_{j,0}}{\partial \nu} \varepsilon_j^2 + \sqrt{\rho h(p_j)} \delta_0^2 - \int_{\partial B_{\varepsilon_0}(p_j)} \frac{\partial \Delta \gamma_{j,0}}{\partial \nu} + O(\varepsilon_j^2)
= - \int_{\partial B_{\varepsilon_0}(p_j)} \frac{\partial \Delta \gamma_{j,0}}{\partial \nu} + O(\varepsilon_j^2)
\]

The right-hand-side of (3.6) equals \(O(\varepsilon_j^2)\) since after scaling the integration is finite. Thus
\[
\int_{\partial B_{\varepsilon_0}(p_j)} \frac{\partial \Delta \gamma_{j,0}}{\partial \nu} = O(\varepsilon_j^2).
\]

On the other hand, taking integration of (3.4) under zero mode and using Green’s Formula, we have
\[
\int_{\partial B_{\varepsilon_0}(p_j)} \frac{\partial \Delta \gamma_{j,0}}{\partial \nu} = \int_{B_{\varepsilon_0}(p_j)} \rho h(p_j) e^{\gamma_j} \gamma_{j,0} + O(\varepsilon_j^2).
\]

Hence \(\gamma_{j,0} = O(\varepsilon_j^2)\), so does \(\gamma_j\). By elliptic estimates we get that \(|\partial^a \gamma_j| = O(\varepsilon_j^2)\) for \(|a| \leq 3\).

Now we can refine the estimate of the left hand side of (3.6). That is,
\[
\int_{B_{\varepsilon_0}(p_j)} [\Delta^2 \gamma_j(x) \Psi(x) - \Delta^2 \Psi(x) \gamma_j(x)] = \int_{B_{\varepsilon_0}(p_j)} \rho h(p_j) e^{\gamma_j} (H_j(x) - \nabla H_j(p_j) \cdot (x - p_j)) \Psi(x).
\]

The left-hand-side of (3.7) equals
\[
= \int_{\partial B_{\varepsilon_0}(p_j)} \left[ \frac{\partial \Delta \gamma_j}{\partial \nu} \Psi(x) - \Delta \gamma_j \frac{\partial \Psi}{\partial \nu} + \Delta \Psi \frac{\partial \gamma_j}{\partial \nu} - \gamma_j \frac{\partial \Delta \Psi}{\partial \nu} \right]
= \int_{\partial B_{\varepsilon_0}(p_j)} \frac{\partial \Delta \gamma_j}{\partial \nu} \varepsilon_j^2 + \sqrt{\rho h(p_j)} \delta_0^2 - \int_{\partial B_{\varepsilon_0}(p_j)} \frac{\partial \Delta \gamma_j}{\partial \nu} + O(\varepsilon_j^4)
= - \int_{\partial B_{\varepsilon_0}(p_j)} \frac{\partial \Delta \gamma_j}{\partial \nu} + O(\varepsilon_j^4).
\]

Since
\[
H_j(x) - \nabla H_j(p_j) \cdot (x - p_j) = \sum_{i,m} \frac{\partial^2}{\partial x_i \partial x_m} [G_j^* + \log h](x_i - p_{ji})(x_m - p_{jm}) + O(|x - p_j|^3),
\]

9
the right-hand-side of (3.7) equals

\[
\begin{aligned}
&= \frac{\alpha_4}{\sqrt{\rho_k h(p_j)}} \int_{\mathbb{R}^4} \frac{1}{(1 + |z|^2)^4} \left[ \varepsilon_j^2 \sum_{i,m} \frac{\partial^2}{\partial x_i \partial x_m} |G_i^* + \log h|z_i z_m + O(\varepsilon_j^3 |z|^3) \right] \frac{1 - |z|^2}{1 + |z|^2} dz \\
&= \frac{\alpha_4}{\sqrt{\rho_k h(p_j)}} \varepsilon_j^2 \Delta (G_j^* + \log h)(p_j) \frac{1}{4} \int_{\mathbb{R}^4} \frac{|z|^2}{(1 + |z|^2)^4} \frac{1 - |z|^2}{1 + |z|^2} dz + o(\varepsilon_j^2),
\end{aligned}
\]

and

\[
\frac{\alpha_4}{4} \int_{\mathbb{R}^4} \frac{|z|^2}{(1 + |z|^2)^4} \frac{1 - |z|^2}{1 + |z|^2} dz = \frac{\alpha_4}{4} \int_{\mathbb{R}^4} \frac{|z|^2 - 1}{(1 + |z|^2)^4} \frac{1 - |z|^2}{1 + |z|^2} dz < 0. \tag{3.8}
\]

Thus, we obtain

\[
\int_{\partial B_\delta_0(p_j)} \frac{\partial \Delta \gamma_j}{\partial v} = l_j(Q) \varepsilon_j^2 + o(\varepsilon_j^2), \tag{3.9}
\]

where \(l_j(Q)\) is defined in (3.13).

Define

\[
\begin{aligned}
&\{ u_j = [v_j + \gamma_j + 32 \sigma_3 (R_4(x, p_j) - R_4(p_j, p_j)) - s_j] \tau_j + 32 \sigma_3 G_4(x, p_j)(1 - \tau_j), \\
&u_{P, A}(x) = \sum_{j=1}^m a_j u_j. \tag{3.10}
\end{aligned}
\]

We will see that when \(\lambda_j, p_j, a_j\) are suitably chosen, \(u_{P, A}\) is considered an approximate solution to equation (1.1).

Claim: Letting \(\xi_j(x) = u_j - 32 \sigma_3 G_4(x, p_j)\), then \(\nabla_{p_j} \xi_j(x), \partial \gamma_j \xi_j(x), \nabla_x \xi_j(x), \Delta_2 \xi_j(x)\) are of the order \(O(\varepsilon_j^2)\) in \(B_{2\delta_0}(p_j) \setminus B_{\delta_0}(p_j)\).

Proof: First, note that \(u_j - 32 \sigma_3 G_4(x, p_j) = 0\) on \(\Omega \setminus B_{2\delta_0}(p_j)\). On \(B_{2\delta_0}(p_j) \setminus B_{\delta_0}(p_j)\), we have

\[
\begin{aligned}
\quad \quad u_j - 32 \sigma_3 G_4(x, p_j) &= \left( v_j - s_j + \gamma_j + 64 \pi^2 (R_4(x, p_j) - R_4(p_j, p_j) - G_4(x, p_j)) \right) \tau_j \\
&= \left[ 2\lambda_j - 4 \log \left( 1 + \frac{\rho h(p_j)}{\alpha_4} \right) + 2 \log \frac{\rho h(p_j)}{\alpha_4} \right] \tau_j + 64 \pi^2 (R_4(x, p_j) - G_4(x, p_j)) \tau_j + O(\varepsilon_j^2) \\
&= \left( -8 \log |x - p_j| + 64 \pi^2 (R_4(x, p_j) - G_4(x, p_j)) \right) \tau_j + O(\varepsilon_j^2) \\
&= \left( -8 \log |x - p_j| + 64 \pi^2 (R_4(x, p_j) - G_4(x, p_j)) \right) \tau_j \\
&+ O \left( \varepsilon_j^2 + \frac{1}{1 + \frac{\rho h(p_j)}{\alpha_4} \varepsilon_j^2 \delta_0^2} \right) \\
&= O(\varepsilon_j^2).
\end{aligned}
\]
It is not hard to see that all derivatives of $\xi_j$ is bounded by $\varepsilon_j^2$. \[\square\]

Now we start to define $S_\rho(Q)$ such that $u_{P,\Lambda,A}$ is a good approximate solution to our problem.

Let $\Gamma_m$ be a subset of $\Omega^m$ defined by

$$\Gamma_m = \{(x_1, \ldots, x_m) \in \Omega^m : x_i = x_j \text{ for some } i \neq j\}.$$  

For any function $h(x)$ on $\Omega$, we set $f_h$ to be a function on $\Omega^m \setminus \Gamma_m$:

$$f_h(x_1, \ldots, x_m) = \sum_{j=1}^m \left( \log h(x_j) + 16\sigma_3 R_4(x_j, x_j) + 32\sigma_3 \sum_{i \neq j} G_4(x_i, x_j) \right).$$

(3.11)

Clearly,

$$\nabla H_j(p_j) = 0 \text{ if and only if } \nabla x_j f_h(p_1, \ldots, p_m) = 0.$$  

(3.12)

For any critical $Q = (q_1, \ldots, q_m)$ of $f_h$, we set

$$\begin{align*}
l_j(Q) &= c_0 h(p_j)^{-\frac{1}{2}} \left( \frac{1}{32\sigma_3} \Delta \log h(q_j) + \Delta R_4(q_j, q_j) + \sum_{i \neq j} G_4(q_i, q_j) \right), \\
l(Q) &= \sum_{j=1}^m l_j(Q) h(p_j) e^{\frac{-G_4(p_j)}{2}},
\end{align*}$$

(3.13)

where $c_0$ is defined in Theorem 1.1.

From now on, $h$ is assumed to satisfy the following two conditions:

(i) The function $f_h$ is a Morse function on $\Omega^m \setminus \Gamma_m$ with critical points $Q_1, \ldots, Q_N$;

(ii) The quantity $l(Q)$ doesn’t vanish for any critical point of $f_h$.

It is not difficult to construct a function $h$ on $\Omega$ such that both conditions (i) and (ii) hold. For $\rho \neq 32m\sigma_3$ and each critical point of $f_h$, we set $\lambda_j(Q)$, $j = 1, \ldots, m$ to satisfy

$$\rho - 32\sigma_3 m = \frac{1}{2} \left( \frac{l(Q)}{h(p_j)} e^{-\frac{G_4(p_j)}{2}} \right)^{-\frac{1}{2}} \lambda_j(Q),$$

(3.14)

which implies that $\lambda_j(Q) \to +\infty$ as $\rho \to 32m\sigma_3$.

Next we will decompose a solution $u$ as $u = u_{P,\Lambda,A} + w$ for some $P, \Lambda, A$ and $w \in O_{P,\Lambda}$, where

$$O_{P,\Lambda} = \left\{ w \in X : \int_\Omega \Delta w \Delta u_j = \int_\Omega \Delta w \Delta \partial_{x_j} u_j = \int_\Omega \Delta w \Delta \partial_{p_j} u_j = 0, \quad j = 1, \ldots, m \right\}. $$

(3.15)

The triplet $(P, \Lambda, A)$ is chosen according to each critical point $Q$ of $f_h$, we define

$$S_\rho(Q) = \left\{ u = u_{P,\Lambda,A} + w : |p_j - q_j| \leq C\varepsilon_j^2(Q), \quad |\lambda_j - \lambda_1(Q)| \leq C, \quad |t_j - t_1| \leq C\varepsilon_j^2(Q), \quad \text{for } 2 \leq j \leq m, \quad |a_j - 1| \leq C\varepsilon_j^2(Q), \quad \|w\|_X \leq C\varepsilon_j^2(Q), \quad w \in O_{P,\Lambda} \right\},$$

(3.16)
where \( t_j \) is defined by

\[
t_j = \lambda_j + G_j^\rho(q_j) + 2 \log \frac{\rho h(q_j)}{\alpha_4},
\]

and the constant \( C \) is large.

Similarly as [8], we obtain that any blowup solution must be contained in \( S_\rho(Q) \) for some critical point \( Q \) of \( f_h \) provided that \( \rho \) is sufficiently close to \( 32 m \sigma_3 \). The proof is omitted here.

Set

\[
T(\rho) = \rho \Delta \int_\Omega h e^u
\]

Since each solution in \( S_\rho(Q) \) has a representation \( (P, \Lambda, A, w) \), the nonlinear operator \( u + T(\rho) u \) of (1.1) can be split according to this representation. Thus, our problem of counting degree can be reduced to a finite-dimensional problem. At each point \( p_j \), the nonlinear term \( \rho h e^u \) is linearly expressed, up to higher order, in terms of \( a_j - 1, \gamma_j, H_j \) and \( w \). More precisely, we let

\[
\hat{\beta}_j = \lambda_j [a_j - 1] + \sum_{i \neq j} |a_i - 1| + |\gamma_j| + |H_j| + |w|.
\]

Then on \( B_{\delta_0}(p_j) \),

\[
\rho h e^u = \rho h e^{u - w + w} = \rho h e^{u - w}(1 + w) + (e^w - 1 - w)\rho h e^{u - w}
\]

\[
= \rho h(p_j) e^{v_j + t_j}[1 + (a_j - 1)(v_j - s_j) + \sum_{i \neq j} 32 \sigma_3 (a_i - 1) G(p_i, p_j) + H_j(x) + \gamma_j(x) + w + \sum_{i=1}^n (a_i - 1)O(|x - p_j|)] + \bar{E},
\]

where

\[
\bar{E} = (e^w - 1 - w)\rho h e^{u - w} + \rho h(p_j) e^{v_j + t_j}[O(\hat{\beta}_j^2) + O(w^2)].
\]

Using the expression for \( \rho h e^u \) above, we obtain an estimate for \( \int_\Omega \rho h e^u \).

**Lemma 3.1** Let \( u = u_{P, \Lambda, A} + w \in S_\rho(Q) \). Then as \( \rho \to 32 m \sigma_3 \),

\[
\int_\Omega \rho h e^u = \sum_{j=1}^m \left( 64 \pi^2 e^{\lambda_j} + 128 \pi^2 (a_j - 1) \lambda_j e^{\lambda_j} + \alpha_4 \frac{t_j(Q)}{4} \right)
\]

\[
+ O(1 + \sum_{i=1}^m |a_i - 1| e^{\lambda_i}) + o(e^{\lambda_j - \lambda_j}).
\]

**Proof:** Note that \( t_j = \lambda_j + O(1) \) and \( v_j - s_j = 2 \lambda_j - 4 \log(1 + \sqrt{\frac{\rho h(p_j)}{\alpha_4}} e^{\lambda_j} |x - p_j|^2) + O(1) \). By (3.17),

\[
\int_\Omega \rho h e^u = \sum_{j=1}^m \int_{B_{\delta_0}(p_j)} \rho h e^u + \int_{\Omega \setminus \bigcup_{j=1}^m B_{\delta_0}(p_j)} \rho h e^u
\]

\[
= \sum_{j=1}^m \int_{B_{\delta_0}(p_j)} \rho h(p_j) e^{v_j + t_j}(1 + \ldots) + \bar{E} + O(1).
\]

12
By the explicit expression of $v_j$, we have
\[ \int_{B_{\delta_0}(p_j)} \rho_h(p_j) e^{v_j+t_j} = 64\pi^2 e^{t_j} + O(1), \] (3.20)
\[ \int_{B_{\delta_0}(p_j)} \rho_h(p_j) e^{v_j+t_j} (a_j - 1)(v_j - s_j) = 128\pi^2(a_j - 1)\lambda_j e^{t_j} + O(|a_j - 1|e^{t_j}), \] (3.21)
\[ \int_{B_{\delta_0}(p_j)} \rho_h(p_j) e^{v_j+t_j} \left( \sum_{l \neq j} (a_l - 1)32\sigma_3 G_4(p_l, p_j) + \sum_{l=1}^{m} |a_l - 1|O(|x - p_j|) \right) = O\left( \sum_{l=1}^{m} |a_l - 1|e^{t_j} \right). \] (3.22)
Since these computations are straightforward, we skip the details. By (3.9) and the fact \( \nabla H_j(p_j) \cdot (x - p_j) \) is an odd function in $B_{\delta_0}(p_j)$, we have
\[ \int_{B_{\delta_0}(p_j)} \rho_h(p_j) e^{v_j+t_j} (\gamma_j + H_j) = \int_{B_{\delta_0}(p_j)} \rho_h(p_j) e^{v_j+t_j} (\gamma_j + H_j - \nabla H_j(p_j) \cdot (x - p_j)) \]
\[ = \int_{B_{\delta_0}(p_j)} e^{t_j} \Delta^2 \gamma_j = e^{t_j} \int_{\partial B_{\delta_0}(p_j)} \frac{\partial(\Delta \gamma_j)}{\partial n} \]
\[ = \alpha_j^2 e^{t_j} - \frac{\lambda_j}{2} l_j(Q) + o(e^{t_j} - \frac{\lambda_j}{2}). \] (3.23)

To estimate the terms involving $w$, we use (3.10) for $u_j$ at each $p_j$. Then
\[ \int_{B_{\delta_0}(p_j)} \rho_h(p_j) e^{v_j+t_j} w = e^{t_j} \left( \int_{\Omega} w \Delta^2 u_j - \int_{B_{\delta_0}(p_j)} w \Delta^2 \gamma_j - \int_{\Omega \setminus B_{\delta_0}(p_j)} w \Delta^2 (u_j - 32\sigma_3 G_4(x, p_j)) \right) \]
\[ = e^{t_j} \left( \int_{\Omega} \Delta w \Delta u_j - \int_{B_{\delta_0}(p_j)} w \Delta^2 \gamma_j - \int_{\Omega \setminus B_{\delta_0}(p_j)} w \Delta^2 (u_j - 32\sigma_3 G_4(x, p_j)) \right) \]
\[ \leq C e^{t_j} \varepsilon_j^2 \| w \|_{H_0^2(\Omega)} \] (3.24)
since $w \in O_p$. A.

As to $\int_{B_{\delta_0}(p_j)} \rho_h(p_j) e^{v_j+t_j} \tilde{E}$, using the same method in [8], we can get
\[ \int_{B_{\delta_0}(p_j)} \rho_h(p_j) e^{v_j+t_j} |\tilde{E}| = O(1). \] (3.25)

By (3.20)-(3.25), we finish the proof of Lemma 3.1. □

13
Now, we want to express \( u + T(\rho)u \) in a formula similar to (3.17). For simplicity, we use \( T \) to denote \( T(\rho) \). We then have by (3.17) that
\[
\frac{\rho he^u}{\int_{\Omega} he^u} = (1 + w)\frac{\rho e^{u,\rho,\lambda,a}}{\int_{\Omega} he^u} + (e^w - 1 - w)\frac{\rho e^{u,\rho,\lambda,a}}{\int_{\Omega} he^u}
\]
\[
= \rho h(p_j)e^{v_j} \left[1 + \left(\frac{e^{t_j}}{\int_{\Omega} he^u} - 1\right) + (a_{i,j} - 1)(v_j - s_j) + \sum_{i \neq j} 32\sigma_3(a_i - 1)G_4(p_j, p_i) + \sum_{i=1}^m (a_i - 1)O(|x - p_j|) + H_j(x) + \gamma_j + w + O(\beta_j^2)\right] + E,
\]
where
\[
\beta_j = \beta_j - \left|\frac{e^{t_j}}{\int_{\Omega} he^u} - 1\right|.
\]
Thus, in \( B_{\delta_0}(p_j) \), we have
\[
\Delta^2(u + Tu) = \Delta^2 u - \frac{\rho he^u}{\int_{\Omega} he^u} = a_j(\Delta^2 v_j + \Delta^2 \gamma_j) + \Delta^2 w - \frac{\rho he^u}{\int_{\Omega} he^u}
\]
\[
= a_j \rho h(p_j)e^{v_j} \left(1 + \gamma_j + H_j - \nabla H_j(p_j) \cdot (x - p_j)\right) + \Delta^2 w - \frac{\rho he^u}{\int_{\Omega} he^u}
\]
\[
= \Delta^2 w - \rho h(p_j)e^{v_j} \left[(a_{i,j} - 1)(v_j - s_j - 1) + a_j \nabla H_j(p_j) \cdot (x - p_j) + \left(\frac{e^{t_j}}{\int_{\Omega} he^u} - 1\right)\right]
\]
\[
+ \sum_{i \neq j} 32\sigma_3(a_i - 1)G_4(p_i, p_j) + \sum_{i=1}^m (a_i - 1)O(|x - p_j|) + w\right] - E,
\]
where
\[
E = (e^w - 1 - w)\frac{\rho e^{u,\rho,\lambda,a}}{\int_{\Omega} he^u} + \rho h(p_j)e^{v_j} \left(O(w^2) + O(\beta_j^2)\right).
\]
On \( B_{2\delta_0}(p_j)\setminus B_{\delta_0}(p_j) \), since \( u_j - 32\sigma_3G_4(x, p_j) \) is small, we write \( \Delta^2(u + Tu) \) as
\[
\Delta^2(u + Tu) = \Delta^2 w + a_j \Delta^2(u_j - 32\sigma_3G_4(x, p_j))
\]
\[
= \frac{\rho h}{\int_{\Omega} he^u} e^{a_j(u_j - 32\sigma_3G_4(x, p_j)) + \sum_{i=1}^m a_i 32\sigma_3G(x, p_i) + w}
\]
and on \( \Omega\setminus \bigcup_{j=1}^m B_{2\delta_0}(p_j) \), we have
\[
\Delta^2(u + Tu) = \Delta^2 w - \frac{\rho h}{\int_{\Omega} he^u} e^{\sum_{i=1}^m a_i 32\sigma_3G(x, p_i) + w}.
\]
According to the above expression of $\Delta^2 (u + Tu)$ in different regions, the dominate terms of $u + Tu \in S_p(Q)$ can be obtained as follows:

**Lemma 3.2** Let $u = u_{P, \lambda, A} + w \in S_p(Q)$, then as $\rho \to 32m\sigma_3$,

(1) $\langle \Delta(u + Tu), \Delta w_1 \rangle = B(w, w_1) + O(\varepsilon^2 \|w_1\|_{H_0^2}),$

where

$$B(w, w_1) := \int_\Omega \Delta w \Delta w_1 - \sum_{j=1}^m \int_{B_{\rho_j}(p_j)} \rho_h(p_j) \mathbf{e}^{\nu_j} \Delta w \mathbf{u}_w$$

is a positive, symmetric, bilinear form satisfying $B(w, w) \geq c\|w\|_{H_0^2}$ for some constant $c > 0$;

(2) $\langle \Delta(u + Tu), \Delta \partial_{p_j} u_j \rangle = 64\pi^2 \nabla H_j(p_j) + O\left(|a_j - 1|\lambda_j + \left|\frac{\mathbf{e}^{t_j}}{\int_\Omega h e^\mathbf{u}} - 1\right| + \varepsilon^2\right)$;

(3) $\langle \Delta(u + Tu), \Delta \partial_{\lambda_j} u_j \rangle = -(a_j - 1)\left(128\pi^2 \lambda_j - \frac{640}{3}\pi^2 + 128\pi^2 \log \frac{\rho_h(p_j)}{a_4} + (64\pi^2)^2 R_4(p_j, p_j)\right)
- \sum_{i \neq j}(a_i - 1)(64\pi^2)^2 G_4(p_i, p_j) - 64\pi^2\left(\frac{\mathbf{e}^{t_j}}{\int_\Omega h e^\mathbf{u}} - 1\right) + O(\varepsilon^2);

(4) $\langle \Delta(u + Tu), \Delta u_j \rangle = \left(2\lambda_j - \frac{10}{3} + 2 \log \frac{\rho_h(p_j)}{a_4} + 64\pi^2 R_4(p_j, p_j)\right)\langle \Delta(u + Tu), \Delta \partial_{\lambda_j} u_j \rangle
+ 64\pi^2 \sum_{i \neq j} G_4(p_i, p_j) \langle \Delta(u + Tu), \Delta \partial_{\lambda_j} u_j \rangle + 128\pi^2(a_j - 1)\lambda_j + O(\varepsilon^2).$

We prove Lemma 3.2 in Appendix because its proof involves a lot of computations. Using Lemma 3.2, we can deform $T$ to a simpler operator on $S_p(Q)$. For $u = u_{P, \lambda, A} + w \in S_p(Q)$ and $0 \leq t \leq 1$, define $T_t$ by the following inner products:

$$\langle \Delta(u + T_t u), \Delta w_1 \rangle = t\langle \Delta(u + Tu), \Delta w_1 \rangle + (1 - t)\langle \Delta u, \Delta w_1 \rangle$$

for $w_1 \in S_p(Q),$ }
\[
\langle \Delta (u + T_i u), \Delta u_j \rangle = t \left( 2\lambda_j - \frac{10}{3} + 2 \log \frac{p_i(p_j)}{\alpha_4} + 64\pi^2 R_i(p_j, p_j) \right) \langle \Delta (u + T u), \Delta \partial_{\lambda_j} u_j \rangle \\
+ 64\pi^2 \sum_{i \neq j} G_i(p_i, p_j) \langle \Delta (u + T u), \Delta \partial_{\lambda_j} u_j \rangle + t \left( O(\varepsilon_j^2) \right) \\
+(1 - t)128\pi^2(a_j - 1)\lambda_j.
\]

Then \( T_0 \) is simpler than \( T \), and it is trivial in the direction \( w_1 \in O_{P,\Lambda} \). Then, we have the following:

**Lemma 3.3** Assume \( (\rho - 32m\sigma_3)(Q) > 0 \). There is \( \varepsilon_1 > 0 \) such that if \( |\rho - 32m\sigma_3| < \varepsilon_1 \), then \( u + T_i u \neq 0 \) for any \( u \in \partial S_{\rho}(Q) \) and \( 0 \leq t \leq 1 \).

Although the proof is very similar as the proof of Lemma 4.3 in [8], here we prove it again for completeness.

**Proof:** Assume \( u = u_{P,\Lambda, A} + w \in \tilde{S}_{\rho}(Q) \) and \( u + T_i u = 0 \) for some \( 0 \leq t \leq 1 \). We will show \( u \notin \partial S_{\rho}(Q) \).

From \( \langle \Delta (u + T_i u), \Delta w \rangle = 0 \), we have by Lemma 3.2

\[
\|w\|^2 \leq O(\varepsilon_j^2)\|w\|_{H_0^2},
\]

which implies that

\[
\|w\|_X = O(\varepsilon_j^2) < C\varepsilon_j^2
\]

provided that \( C \) is large.

Using \( \langle \Delta (u + T_i u), \Delta \partial_{\lambda_j} u_j \rangle = 0, \langle \Delta (u + T_i u), \Delta u_j \rangle = 0, \) \( (3.31) \) and part (iv) of Lemma 3.2 yields

\[128\pi^2\lambda_j(a_j - 1) = O(e^{-\frac{\lambda_j}{2}}) \quad \text{for} \quad j = 1, \ldots, m,\]

that is, when \( \rho \) is close to \( 32m\sigma_3 \),

\[
|a_j - 1| = O(\lambda_j^{-1}e^{-\frac{\lambda_j}{2}}) < Ce^{-\frac{h(Q)}{4}} \quad \text{for} \quad 1 \leq j \leq m.
\]

Next we estimate the term \( \frac{e^{ij}}{\int_{\Omega} h e^u} - 1 \). Since \( \rho \) is close to \( 32m\sigma_3 \), we have that \( \lambda_j \) is large,

\[
|\lambda_j - \lambda_j(Q)| = O(1) \quad \text{and} \quad \lambda_1 = \lambda_j + O(1).
\]

\[
e^{-\frac{\lambda_j}{2}} = e^{-\frac{\tau_j}{2}} \frac{p_i(p_j)}{\alpha_4} e^{\frac{c^*_j(p_j)}{2}} = e^{-\frac{\tau_j}{2}} \frac{p_i(p_j)}{\alpha_4} e^{\frac{c^*_j(p_j)}{2}} + O(\varepsilon_j^2)
\]

\[
= e^{-\frac{\lambda_j}{2}} \frac{h(p_j)}{h(p_1)} e^{\frac{c^*_j(p_j) - c^*_j(p_1)}{2}} + O(\varepsilon_j^2).
\]

16
By \( \langle \Delta (u + T_i u), \Delta \partial_{x_j} u_j \rangle = 0 \), (iii) of Lemma 3.2 and (3.32) yield

\[
e^{t_j} \left( \int_\Omega h e^u \right)^{-1} - 1 = O(e^{-\frac{\lambda_1(Q)}{2}}),
\]

(3.34)

\[
e^{t_j} \left( \int_\Omega h e^u \right)^{-1} - 1 + 2 \lambda_j(a_j - 1) = O(e^{-\frac{\lambda_1(Q)}{2}}).
\]

(3.35)

By \( \langle \Delta (u + T_i u), \Delta \partial_{x_j} u_j \rangle = 0 \) and (3.32), (ii) of Lemma 3.2 yields

\[
\nabla H_j(p_j) = O(\lambda_j |a_j - 1| + e^{t_j} \left( \int_\Omega h e^u \right)^{-1} - 1 + e^{-\frac{\lambda_1}{2}})
\]

(3.36)

\[
\leq O(1)e^{-\frac{\lambda_1(Q)}{2}}.
\]

Also we have

\[
|P - Q| \leq c |\nabla H_j(p_j)| \leq O(1)e^{-\frac{\lambda_1(Q)}{2}} < C e^{-\frac{\lambda_1(Q)}{2}}.
\]

(3.37)

It remains to estimate \( t_j - t_1 \) and \( \lambda_1 - \lambda_1(Q) \). By \( |t_j - t_1| \leq C e^{-\frac{\lambda_1(Q)}{2}} \),

\[
e^{t_i-t_j} = 1 + (t_i - t_j) + O(C^2 e^{-\lambda_1(Q)})
\]

\[
= 1 + (t_i - t_j) + O(e^{-\frac{\lambda_1(Q)}{2}}).
\]

By Lemma 3.1, we have

\[
e^{-t_j} \int_\Omega h e^u = \sum_{i=1}^m \left( 64\pi^2 + 128\pi^2(a_i - 1)\lambda_i + \alpha_4 l(Q)e^{-\frac{\lambda_1}{2}} \right)e^{t_i-t_j}
\]

\[
+ O \left( \sum_{i=1}^m |a_i - 1|e^{t_i-t_j} \right) + o \left( \sum_{i=1}^m e^{t_i-t_j-\frac{\lambda_1}{2}} \right)
\]

(3.38)

\[
= \sum_{i=1}^m \left( 64\pi^2(1 + t_i - t_j) + 128\pi^2(a_i - 1)\lambda_i + \alpha_4 l(Q)e^{-\frac{\lambda_1}{2}} \right)
\]

\[
+ O \left( \sum_{i=1}^m |a_i - 1| \right) + o \left( \sum_{i=1}^m e^{-\frac{\lambda_1}{2}} \right)
\]

and

\[
\frac{e^{t_j}}{\rho} \int_\Omega h e^u = \frac{1}{64\pi^2 m} \left( 1 + o \left( \sum_{i=1}^m \varepsilon_i \right) \right).
\]

(3.39)
Thus

\[
\frac{e^{t_j}}{\int_\Omega he^u} - 1 = \frac{e^{t_j}}{\rho \int_\Omega he^u} (\rho - \rho \frac{\int_\Omega he^u}{e^{t_j}})
\]

\[
= \frac{1}{64\pi^2 m} \left\{ \rho - 64\pi^2 m - \sum_{i=1}^m \left[ 64\pi^2 (t_i - t_j) + 128\pi^2 \lambda_i (a_i - 1) + \alpha_i^\perp l_i(Q) e^{-\frac{\lambda_i}{2}} \right] \right\}
\]

\[+ O(\sum_{i=1}^m |a_i - 1|) + o(\sum_{i=1}^m e^{\frac{\lambda_i}{2}}). \tag{3.40}\]

Now taking the summation of (3.40) from \( j = 1, \ldots, m \) and by (3.35), we obtain

\[
O(1) e^{-\frac{\lambda_1(Q)}{2}} = \sum_j \left[ \frac{e^{t_j}}{\int_\Omega he^u} - 1 \right] + 2\lambda_j (a_j - 1)
\]

\[
= \frac{1}{64\pi^2} \left( \rho - 32m\sigma_3 - \sum_i \alpha_i^\perp l_i(Q) e^{-\frac{\lambda_i}{2}} \right).
\]

Hence

\[
O(e^{-\frac{\lambda_1(Q)}{2}}) = \frac{\alpha_1^\perp}{64\pi^2} \frac{l(Q)}{h(p_1)} e^{-\frac{\sigma_1(p_1)}{2}} \left( e^{-\frac{\lambda_1(Q)}{2}} - e^{-\frac{\lambda_j}{2}} \right),
\]

which implies that

\[
|\lambda_1(Q) - \lambda_1| = O(1) < C. \tag{3.41}\]

To obtain estimates for \( \lambda_j - \lambda_1, j \geq 2 \), or equivalently for \( t_j - t_1, j \geq 2 \), we have by part (iii) of Lemma 3.2 and (3.40)

\[
0 = \langle \Delta (u + T_i u), \Delta \partial_{t_j} u_j \rangle = \langle e^{-\frac{\lambda_j}{2}} \rangle - 64\pi^2 \left( \frac{e^{t_j}}{\int_\Omega he^u} - 1 \right)
\]

\[
= O(e^{-\frac{\lambda_j}{2}}) - \frac{1}{m} (\rho - 32m\sigma_3) - \frac{64\pi^2}{m} \sum_i (t_i - t_j).
\]

Therefore,

\[
|t_j - \frac{1}{m} \sum_i t_i| = O(e^{-\frac{\lambda_j}{2}})
\]

and

\[
|t_j - t_1| \leq |t_j - \frac{1}{m} \sum_i t_i| + \frac{1}{m} \sum_i t_i - t_1 \leq O(e^{-\frac{\lambda_j}{2}}) < C e^{-\frac{\lambda_1(Q)}{2}} \text{ for } j \geq 2. \tag{3.42}\]
From (3.31), (3.32), (3.37), (3.41) and (3.42), we obtain \( u \notin \partial S_{\rho}(Q) \). The proof is complete.

Next we start to derive our degree counting formulas. We denote

\[ B_R = \{ u \in \mathcal{X} : \| u \|_\mathcal{X} < R \}. \]

Then as in [8], for \( \rho > 32m\sigma_3 \), the degree \( d(\rho) \) of the nonlinear map \( u + Tu \) can be counted by

\[
d^+_m - d^-_m = \frac{1}{m!} \sum_{i(Q_i) > 0} \deg(u + Tu; S_{\rho}(Q_i), 0) - \frac{1}{m!} \sum_{i(Q_i) < 0} \deg(u + Tu; S_{\rho}(Q_i), 0),
\]

where the summation is over all the critical points \( Q_i \) of \( f_h \) such that \( l(Q_i) > 0 \) (\( l(Q_i) < 0 \)).

To compute \( \deg(u + Tu; S_{\rho}(Q_i), 0) \), we set \( \Phi_i := (\Phi_{i,1}, \Phi_{i,2}, \Phi_{i,3}) : S^*(Q_i) \to \mathbb{R}^{4m} \times \mathbb{R}^m \times \mathbb{R}^m \) by

\[
\Phi_{i,1}^{(j)} = \langle \Delta(u + T_0u), \Delta \partial_{p_j} u_j \rangle = 64\pi^2 \nabla H_j(p_j),
\]

\[
\Phi_{i,2}^{(j)} = \langle \Delta(u + T_0u), \Delta \partial_{\lambda_j} u_j \rangle,
\]

\[
\Phi_{i,3}^{(j)} = \langle \Delta(u + T_0u), \Delta u_j \rangle = 128\pi^2(a_j - 1)\lambda_j,
\]

where \( S^*(Q_i) = \{(P, \Lambda, A) : u_{P,\Lambda, A} + w \in S_{\rho}(Q_i)\} \) and \( \Phi_{i,l}^{(j)} ; j = 1, \ldots, m \) are components of \( \Phi_{i,l} ; l = 1, 2, 3 \). Clearly,

\[
\deg(u + Tu; S_{\rho}(Q_i), 0) = \deg(\Phi_i ; S^*(Q_i), 0). \tag{3.43}
\]

Now we state the degree-counting formula for (3.43).

**Lemma 3.4** Assume \( h \) satisfy the conditions (i) and (ii). Then

\[
\deg(\Phi_i ; S^*(Q_i), 0) = sgn(\rho - 32m\sigma_3)(-1)^m \text{ind}(Q_i),
\]

where

\[
sgn(\rho - 32m\sigma_3) = \begin{cases} 
1, & \text{if } \rho > 32m\sigma_3, \\
-1, & \text{if } \rho < 32m\sigma_3, \\
0, & \text{if } \rho = 32m\sigma_3.
\end{cases}
\]

Note that the sign for \( \rho - 32m\sigma_3 \) at a critical point \( Q \) is completely determined by the quantity \( l(Q) \).

**Proof:** Let

\[
\theta_j = \frac{1}{64\pi^2m} \left\{ \rho - 64\pi^2m - 64\pi^2 \sum_{l=1}^{m} \left[ (t_i - t_j) + 2\lambda_l(a_l - 1) \right] - \sum_{l=1}^{m} \alpha_i^{1/2} l(Q) e^{-\frac{\lambda}{2}} \right\}. \tag{3.44}
\]
We note that in $\langle \Delta (u + T_0 u), \Delta \partial_{\lambda_j} u_j \rangle$, the term
\[
\frac{\epsilon^j}{\int_{\Omega} h e^u} - 1 = \theta_j + o(\theta_j)
\]
can be further deformed such that $\frac{\epsilon^j}{\int_{\Omega} h e^u} - 1$ is replaced by $\theta_j$. For simplicity, we still denote our new map by $\Phi_i$.

To compute the degree, we can simplify $\Phi_i$ further by setting $\hat{\Phi}_{i,1} = \Phi_{i,1}$, $\hat{\Phi}_{i,3} = \Phi_{i,3}$, and
\[
\hat{\Phi}_{i,2}^{(j)} = \Phi_{i,2}^{(j)} + \left(1 - \frac{5}{3\lambda_j} + \frac{1}{\lambda_j} \log \frac{\rho h(p_j)}{\alpha_4} + \frac{32\pi^2}{\lambda_j} R_4(p_j, p_j) \right) \hat{\Phi}_{i,3}^{(j)} + \sum_{l \neq j} \frac{32\pi^2}{\lambda_l} G_4(p_l, p_j) \hat{\Phi}_{i,3}^{(l)} - \frac{1}{m} \sum_{l=1}^{m} \hat{\Phi}_{i,3}^{(l)}
\]
\[
= -\frac{1}{m} \left( \rho - 32m\sigma_3 - 64\pi^2 \sum_{l=1}^{m} (t_l - t_j) - \alpha_4^2 \sum_{l=1}^{m} l_i(Q) e^{-\frac{\lambda_j}{2}} \right).
\]
Clearly, we have
\[
\frac{\partial \hat{\Phi}_{i,1}}{\partial \Lambda} = \frac{\partial \hat{\Phi}_{i,1}}{\partial \Lambda} = \frac{\partial \hat{\Phi}_{i,2}}{\partial \Lambda} = 0,
\]
and
\[
\Phi_i(P, \Lambda, A) = 0 \quad \text{if and only if} \quad \hat{\Phi}_i(P, \Lambda, A) = 0,
\]
\[
\deg(\Phi_i; S^*(Q_i), 0) = \deg(\hat{\Phi}_i; S^*(Q_i), 0).
\]
Note that $\hat{\Phi}_{i,1} = 64\pi^2 \nabla H_j(p_j)$. Therefore $\hat{\Phi}_i(P, \Lambda, A) = 0$ if and only if
\[
P = Q_i, \quad A = (1, \ldots, 1) \quad \text{and}
\]
\[
\begin{cases}
\rho - 32m\sigma_3 = \alpha_4^2 \sum_{l=1}^{m} l_i(Q) e^{-\frac{\lambda_j}{2}}, \\
t_1 = \cdots = t_m.
\end{cases}
\]
(3.47)
It is not difficult to see that if $|\rho - 32m\sigma_3|$ is sufficiently small, equation (3.47) possesses a unique solution $(\lambda_1, \ldots, \lambda_m)$ up to permutation. Let $\Lambda_i(\rho) = (\lambda_1, \ldots, \lambda_m)$ denote the solution of (3.47). Hence $(Q_i, \Lambda_i(\rho), A_1)$ is the unique solution of $\hat{\Phi}_i = 0$, where $A_1 = (1, \ldots, 1)$. By (3.46), the degree of $\hat{\Phi}_i$ at $(Q_i, \Lambda_i(\rho), A_1)$ is the sign of the product of $\det \frac{\partial \hat{\Phi}_{i,1}}{\partial P} \cdot \det \frac{\partial \hat{\Phi}_{i,2}}{\partial \Lambda} \cdot \det \frac{\partial \hat{\Phi}_{i,3}}{\partial \Lambda}$. Thus,
\[
\deg(\hat{\Phi}_i; S^*(Q_i), 0) = (-1)^{\text{ind}(Q_i)} \text{sgn} \det \left( \frac{\partial \hat{\Phi}_{i,2}}{\partial \Lambda} \right).
\]
20
By (3.45), we have
\[
\frac{1}{64\pi^2} \frac{\partial \hat{\Phi}^{(j)}_{i,2}}{\partial \lambda_j} = -\frac{m-1}{m} \frac{\partial t_j}{\partial \lambda_j} - \frac{1}{128m\pi^2} l_j(Q_i) \varepsilon_j^2 + o(\varepsilon_j^2),
\]
and for \( l \neq j \),
\[
\frac{1}{64\pi^2} \frac{\partial \hat{\Phi}^{(j)}_{i,2}}{\partial \lambda_l} = \frac{1}{m} \frac{\partial t_l}{\partial \lambda_l} - \frac{1}{128m\pi^2} l_i(Q_l) \varepsilon_i^2 + o(\varepsilon_i^2).
\]
Thus,
\[
\frac{1}{64\pi^2} \sum_{l=1}^{m} \frac{\partial \hat{\Phi}^{(j)}_{i,2}}{\partial \lambda_l} = -\frac{m-1}{m} \sum_{l=1}^{m} \frac{\partial t_l}{\partial \lambda_j} - \frac{1}{128m\pi^2} \sum_{l=1}^{m} l_i(Q_l) \varepsilon_i^2 + o(\sum \varepsilon_i^2)
\]
and
\[
\frac{1}{64\pi^2} \sum_{j=1}^{m} \sum_{l=1}^{m} \frac{\partial \hat{\Phi}^{(j)}_{i,2}}{\partial \lambda_l} = -\frac{1}{128\pi^2} \sum_{l=1}^{m} l_i(Q_l) \varepsilon_i^2 + o(\sum \varepsilon_i^2).
\]
Therefore,
\[
\det \left[ \frac{\partial \hat{\Phi}_{i,2}}{\partial \Lambda} \right] = \det \left[ \begin{pmatrix}
-\frac{m-1}{m} \frac{\partial t_j}{\partial \lambda_j} & 1 & \ldots & 1 & a_{1m} \\
1 & -\frac{m-1}{m} \frac{\partial t_l}{\partial \lambda_l} & \ldots & 1 & a_{2m} \\
1 & 1 & \ldots & -m+1 & a_{m-1,m} \\
am_{11} & a_{m1} & \ldots & \ldots & -\frac{1}{128\pi^2} \sum_{l=1}^{m} l_i(Q_l) \varepsilon_i^2
\end{pmatrix} + o(\sum \varepsilon_i^2) \right]
\]
\[
= (-1)^m \tau_0 \sum_{l=1}^{m} l_i(Q_l) \varepsilon_i^2 + o(\sum \varepsilon_i^2)
\]
\[
= (-1)^m \tau_0 (\rho - 32m\sigma_3) + o(\sum \varepsilon_i^2),
\]
where
\[
\tau_0 = \frac{1}{128\pi^2} \det \left( \begin{pmatrix}
m-1 & 1 & \ldots & 1 \\
1 & m-1 & \ldots & 1 \\
\vdots & \ddots & \vdots & \vdots \\
1 & 1 & \ldots & m-1
\end{pmatrix}_{(m-1)\times(m-1)} \right) > 0
\]
and \( a_{ij} = O(\sum \varepsilon_i^2) \).

Thus we have
\[
\text{sgn} \det \left( \frac{\partial \hat{\Phi}_{i,2}}{\partial \Lambda} \right) = (-1)^m \text{sgn}(\rho - 32m\sigma_3).
\]
This proves Lemma 3.4. \( \Box \)

Finally we are ready to complete the proof of Theorem 1.2.
Completion of the proof of Theorem 1.2: By Lemma 3.4 and the Hopf theorem, we get by hand that
\[
d_m^+ - d_m^- = \frac{(-1)^m}{m!} \sum_{i=1}^N (-1)^{\text{ind}(Q_i)}
\]
\[
= \frac{1}{m!} \left( -\chi(\Omega)(-\chi(\Omega) + 1) \cdots (-\chi(\Omega) + m - 1) \right)
\]
where \(Q_1, \ldots, Q_N\) are all critical points of \(f_h\) and \(\chi(\Omega)\) is the Euler characteristic of \(\Omega\). Then for any \(\rho \in (32m\sigma_3, 32(m + 1)\sigma_3)\), we have
\[
d(\rho) = 1 + \sum_{j=1}^m (d_j^+ - d_j^-) = \frac{1}{m!} (-\chi(\Omega) + 1) \cdots (-\chi(\Omega) + m).
\]
The detail can be founded in the proof of Lemma 5.2 in [8].

4 Proof of Theorem 1.5

In [24], Robert and Wei have given a complete proof of \((P1)-(P3)\) when \(h(x) = 1\) for problem (1.9). Now let \(h(x)\) be a positive \(C^{2,\beta}\) function. Since the proof is very similar, we only give an outline of proof of Theorem 1.5.

Let \(u_k\) be a sequence of solution of (1.9). We assume that
\[
\max_{x \in \Omega} u_k(x) \to +\infty \quad \text{as} \quad k \to +\infty.
\]
Let \(\alpha_k = \log \int_{\Omega} h(x) e^{u_k}\) and
\[
\hat{u}_k = u_k - \alpha_k.
\]
Define
\[
u_k(p_k) = \max_{x \in \Omega} u_k(x), \quad \frac{\mu_k}{\alpha_k} = e^{-\frac{1}{\alpha_k} \hat{u}_k(p_k)}.
\]
Then using Lemma 2.4 and Green’s representation, along the line of [24], we can prove the following claims step by step. The proof is omitted here.

Claim 1: If \(\max_{x \in \Omega} u_k(x) \to +\infty\), then \(\alpha_k \geq C\) and \(\lim_{k \to +\infty} \frac{d(p_k, \partial \Omega)}{\mu_k} = +\infty\).

Claim 2: \(\lim_{k \to +\infty} u_k(p_k + \mu_k x) - u_k(p_k) = \log \frac{1}{(1 + \sqrt{\rho_k h(p_k)} |x|^2)^4}\) in \(C_{\text{loc}}^4(\mathbb{R}^d)\).

Claim 3: Assume that \(\alpha_k \to +\infty\). Since \(\rho_k \leq C\), there exists \(S = \{p_1, \ldots, p_m\} \subset \tilde{\Omega}\) for some positive integer \(m\) and \(a_1, \ldots, a_m \geq 32\sigma_3\) such that \(\lim_{k \to +\infty} u_k(x) = \sum_{j=1}^m a_j G_4(\cdot, p_j)\) in \(C_{\text{loc}}^4(\tilde{\Omega})\).
Next we prove (P1), that is, we exclude the boundary blow-ups.

**Claim 4:** Assume that \( \alpha_k \to +\infty \). Let \( x_0 \in \partial \Omega \). Then \( \lim_{r \to 0} \lim_{k \to +\infty} \int_{B_r(x_0) \cap \Omega} \rho_k h(x)e^{\bar{u}_k}dx = 0 \). In particular, \( S \cap \partial \Omega = \emptyset \).

Since we use different form of Pohozaev’s identity in the proof with respect to [24], we give the proof of this claim for completeness.

**Proof:** Let \( x_0 \in \partial \Omega \cap S \). Then \( \lim_{r \to 0} \lim_{k \to +\infty} \int_{B_r(x_0) \cap \Omega} \rho_k h(x)e^{\bar{u}_k}dx = 32\sigma_3 \). Thus there exists \( \delta_0 > 0 \) small such that for any \( \delta \leq \delta_0 \), \( \int_{B_{\delta}(x_0) \cap \Omega} \rho_k h(x)e^{\bar{u}_k}dx \geq 16\sigma_3 \). Furthermore, we may assume that \( S \cap B_{\delta_0}(x_0) = \{x_0\} \).

Let \( y_k = x_0 + \rho_{k,r} \nu(x_0) \) with

\[
\rho_{k,r} = \frac{\int_{\partial \Omega \cap B_r(x_0)} (x - x_0, \nu)(\Delta u_k) \, dx}{\int_{\partial \Omega \cap B_r(x_0)} (\nu(x_0) \cdot \nu)(\Delta u_k) \, dx},
\]

where \( r < \delta_0 \) such that \( \frac{1}{2} \leq \nu(x_0) \cdot \nu \leq 1 \) for \( x \in B_r(x_0) \cap \partial \Omega \). Then it is easy to see that \( |\rho_{k,r}| \leq 2r \) and

\[
\int_{\partial \Omega \cap B_r(x_0)} (x - y_k, \nu)(\Delta u_k) \, dx = 0.
\]

Now applying the Pohozaev’s identity in \( \Omega \cap B_r(x_0) \) with \( \xi = y_k \) and using Dirichlet boundary condition and (4.2), we obtain that

\[
\int_{\Omega \cap B_r(x_0)} \rho_k (4h + \langle x - y_k, \nabla h \rangle) e^{\bar{u}_k} = \int_{\Omega \cap \partial B_r(x_0)} \rho_k (x - y_k, \nu) h(x)e^{u_k - \alpha_k} - 2 \int_{\Omega \cap \partial B_r(x_0)} \frac{\partial u_k}{\partial \nu} \Delta u_k
\]

\[
+ \int_{\Omega \cap \partial B_r(x_0)} \left[ \frac{1}{2}(\Delta u_k) \langle x - y_k, \nu \rangle + \langle x - y_k, \nabla u_k \rangle \frac{\partial (-\Delta u_k)}{\partial \nu} \right]
\]

\[
+ \int_{\Omega \cap \partial B_r(x_0)} \left[ -\langle x - y_k, \nabla \Delta u_k \rangle \frac{\partial u_k}{\partial \nu} + \langle x - y_k, \nu \rangle \langle \nabla u_k, \nabla \Delta u_k \rangle \right].
\]

Note that \( u_k \to \sum_{j=1}^m a_j G_j(x, p_j) \) in \( C^3(\bar{\Omega} \setminus S) \), where \( G_j(x, x_0) = 0 \). Thus we obtain that all the terms in the last three integrals are of the term

\[
\int_{\Omega \cap \partial B_r(x_0)} O(1) = O(r^3)
\]

while

\[
\int_{\Omega \cap \partial B_r(x_0)} \langle x - y_k, \nu \rangle \rho_k h(x)e^{u_k - \alpha_k} \, d\sigma = O(r^4).
\]

On the other hand,

\[
\int_{\Omega \cap B_r(x_0)} \rho_k (x - y_k, \nabla h(x)) e^{\bar{u}_k} \, dx = \int_{\Omega \cap \partial B_r(x_0)} \rho_k (x - y_k, \nabla \log h(x)) h(x)e^{\bar{u}_k} \, dx = O(r).
\]
Thus we obtain that
\[
| \int_{\Omega \cap B_{r_0}(x_0)} 4\rho_k h(x)e^{iu}dx | \leq C r. \tag{4.3}
\]

Claim 5: We show that \( \alpha_k \to +\infty \). This is exactly the same as Claim 12 of [24].

Claim 6: \( a_j = 32\sigma_3, j = 1, \ldots, m \) and \( p_j \) satisfies
\[
\nabla \left( \log h(p_j) + 32\sigma_3 R'_j(p_j, p_j) + 32\sigma_3 \sum_{i \neq j} G'_j(p_i, p_j) \right) = 0, \quad j = 1, \ldots, m. \tag{4.4}
\]

Once we prove that there are no boundary bubbles for \( \beta \), to prove Theorem 1.5, we just need to change the Navier boundary condition to Dirichlet boundary condition. By almost the same computation in the proof of Theorem 1.1 and Theorem 1.2, we finish the proof of Theorem 1.5.

5 Appendix : Proof of Lemma 3.2

This final section is devoted to the proof of Lemma 3.2. According to Appendix B in [18], we get part (i) and the readers can also refer to Section 6 in [8].

Next we prove part (iii) at first. On \( B_{2\delta_0}(p_j) \), we have
\[
\nabla_j u_j = \nabla_j (v_j - s_j) \sigma_j + O(\varepsilon_j^2) = \frac{2}{1 + \rho h(p_j) e^{\frac{\lambda_j}{\alpha_4} |x - p_j|^2}} \sigma_j + O(\varepsilon_j^2) \tag{5.1}
\]
and on \( \Omega \setminus B_{2\delta_0}(p_j) \),
\[
\nabla_j u_j = 0.
\]

We compute \( \langle \Delta (u + Tu), \Delta \nabla_j u_j \rangle = \langle \Delta^2 (u + Tu), \nabla_j u_j \rangle \) by using (3.28)-(3.30).

Since \( w \in O_{\bar{\rho}, \Lambda} \), we have
\[
\int_{\Omega} \Delta w \Delta \nabla_j u_j = 0. \tag{5.2}
\]

By (5.1),
\[
\int_{B_{\delta_0}(p_j)} \rho h(p_j) e^{v_j} \nabla_j u_j = \alpha_4 \int_{\mathbb{R}^4} \frac{2}{(1 + |z|^2)^5} dz + O(\varepsilon_j^2) = 64\pi^2 + O(\varepsilon_j^2) \tag{5.3}
\]
and
\[
\int_{B_{\delta_0}(p_j)} \rho h(p_j) e^{v_j} \left[ -4 \log \left( 1 + \frac{\rho h(p_j) e^{\frac{\lambda_j}{\alpha_4} |x - p_j|^2}}{\alpha_4} \right) \right] \nabla_j u_j
\]
\[
= \alpha_4 \int_{\mathbb{R}^4} \frac{8}{(1 + |z|^2)^5} \left[ - \log (1 + |z|^2) \right] dz + O(\varepsilon_j^2)
\]
\[
= -\frac{448}{3} \pi^2 + O(\varepsilon_j^2). \tag{5.4}
\]
Using $v_j - s_j = 2\lambda_j - 4\log\left(1 + \frac{\rho h(p_j)}{\alpha_4} e^{\frac{\lambda_j}{4}} |x - p_j|^2\right) + 2\log\frac{\rho h(p_j)}{\alpha_4} + 64\pi^2 R_4(p_j, p_j)$, we have

$$-\int_{B_{\delta_0}(p_j)} \rho h(p_j) e^{v_j} (a_j - 1)(v_j - s_j - 1) \partial_\lambda_j u_j$$

$$= (a_j - 1) \left[-128\pi^2 \lambda_j + \frac{640}{3} \pi^2 - 128\pi^2 \log \frac{\rho h(p_j)}{\alpha_4} - (64\pi^2)^2 R_4(p_j, p_j) + O(\varepsilon_j^2) \right].$$

and

$$-\int_{B_{\delta_0}(p_j)} \rho h(p_j) e^{v_j} \sum_{i \neq j} 64\pi^2 (a_i - 1) G_4(p_i, p_j) \partial_\lambda_j u_j$$

$$= - \sum_{i \neq j} (64\pi^2)^2 (a_i - 1) G_4(p_i, p_j) + O\left(\sum_{i \neq j} |a_i - 1| \varepsilon_j^2\right).$$

Since $\nabla H_j(p_j) \cdot (x - p_j)$ is an odd function in $B_{\delta_0}(p_j)$,

$$\int_{B_{\delta_0}(p_j)} \rho h(p_j) e^{v_j} \nabla H_j(p_j) \cdot (x - p_j) \partial_\lambda_j u_j$$

$$= \int_{B_{\delta_0}(p_j)} \rho h(p_j) e^{v_j} \nabla H_j(p_j) \cdot (x - p_j) \left(\frac{2}{1 + \frac{\rho h(p_j)}{\alpha_4} e^{\frac{\lambda_j}{4}} |x - p_j|^2} \sigma_j + O(\varepsilon_j^2)\right)$$

$$= O(\varepsilon_j^3).$$

To estimate the term with $w \partial_\lambda_j u_j$, we use the condition $w \in O_{P, \Lambda}$, that is

$$0 = \int_{\Omega} \Delta w \Delta \partial_\lambda_j u_j = \int_{\Omega} w \Delta^2 \partial_\lambda_j u_j$$

$$= \int_{B_{\delta_0}(p_j)} w \Delta^2 \partial_\lambda_j u_j + O(\varepsilon_j^2 w \| w \|_{H^6_0(\Omega)})$$

$$= \int_{B_{\delta_0}(p_j)} w \Delta^2 \partial_\lambda_j v_j + O(\varepsilon_j^4)$$

$$= \int_{B_{\delta_0}(p_j)} w \rho h(p_j) e^{v_j} \partial_\lambda_j v_j + O(\varepsilon_j^4).$$

Hence, by (3.24), we have

$$\int_{B_{\delta_0}(p_j)} w \rho h(p_j) e^{v_j} \partial_\lambda_j u_j = \int_{B_{\delta_0}(p_j)} w \rho h(p_j) e^{v_j} \left(1 + \partial_\lambda_j v_j + O(\varepsilon_j^2)\right)$$

$$= O(\varepsilon_j^4).$$

25
Just by the similar computation in [8], we can obtain that

\[ \int_{B_{\delta_0}(p_j)} E \partial_{\lambda_j} u_j = O(\varepsilon_j^3). \]

On \( \Omega \setminus \cup_{j=1}^m B_{\delta_0}(p_j) \),

\[ e^u = O(e^w), \quad \frac{\rho h e^u}{\int_{\Omega} h e^u} = O(e^{-\lambda_j})e^w, \quad \partial_{\lambda_j} u_j = O(\varepsilon_j^2). \]

Thus,

\[ \int_{\Omega \setminus \cup_{j=1}^m B_{\delta_0}(p_j)} \frac{\rho h e^u}{\int_{\Omega} h e^u} \partial_{\lambda_j} u_j = o(\varepsilon_j^4) \quad (5.10) \]

and

\[ \int_{B_{2\delta_0}(p_j) \setminus B_{\delta_0}(p_j)} \Delta^2 u \partial_{\lambda_j} u_j = \int_{B_{2\delta_0}(p_j) \setminus B_{\delta_0}(p_j)} \Delta^2(u - 32\sigma_3 G_4(x, p_j)) \partial_{\lambda_j} u_j = O(\varepsilon_j^2). \quad (5.11) \]

Hence

\[ \langle \Delta(u + Tu), \Delta \partial_{\lambda_j} u_j \rangle = -(a_j - 1) \left( 128\pi^2 \lambda_j - \frac{640}{3} \pi^2 + 128\pi^2 \log \frac{\rho h(p_j)}{\alpha_4} + (64\pi^2)^2 R_4(p_j, p_j) \right) \]

\[ -\sum_{i \neq j} (64\pi^2)^2 (a_i - 1) G_4(p_i, p_j) - 64\pi^2 \left( \frac{e^{t_j}}{\int_{\Omega} h e^u} - 1 \right) + O(\varepsilon_j^3). \quad (5.12) \]

This proves part (iii).

For the proof of part (iv), since \( \int_{\Omega} h e^u \sim e^{\lambda_j} \), we write

\[ \langle \Delta(u + Tu), \Delta u_j \rangle = \langle \Delta^2(u + Tu), u_j \rangle + \int_{\partial\Omega} \Delta(u + Tu) \nabla u_j \]

\[ = \langle \Delta^2(u + Tu), u_j \rangle + \int_{\partial\Omega} \Delta T u \nabla u_j \]

\[ = \langle \Delta^2(u + Tu), u_j \rangle + O(\varepsilon_j^3). \]

Since \( w \in S_p(Q) \),

\[ \int_{\Omega} \Delta w \Delta u_j = 0. \]

By (3.10), on \( B_{\delta_0}(p_j) \) we have

\[ u_j = 2\lambda_j + 2 \log \frac{\rho h(p_j)}{\alpha_4} + 64\pi^2 R_4(p_j, p_j) - 4 \log \left( 1 + \sqrt{\frac{\rho h(p_j)}{\alpha_4} e^{\lambda_j} |x - p_j|^2} \right) + O(\varepsilon_j^2) \]
and
\[ v_j - s_j - 1 = 2\lambda_j + 2 \log \frac{\rho (p_j)}{\alpha_4} + 64\pi^2 R_4 (p_j, p_j) - 4 \log \left( 1 + \frac{\rho (p_j)}{\alpha_4} e^{\frac{\lambda_j}{2} |x - p_j|^2} \right) - 1. \]

By direct computation, we have
\[ \int_{B_\delta_0 (p_j)} \rho (p_j) e^{v_j} = 64\pi^2 + O(\varepsilon_j^4) \]
and
\[ \int_{B_\delta_0 (p_j)} \rho (p_j) e^{v_j} \log \left( 1 + \frac{\rho (p_j)}{\alpha_4} e^{\frac{\lambda_j}{2} |x - p_j|^2} \right) - 1 \]
\[ = \int_{B_\delta_0 (p_j)} \rho (p_j) e^{v_j} \left\{ 4\lambda_j^2 + 2 \lambda_j \left[ 4 \log \frac{\rho (p_j)}{\alpha_4} + 128\pi^2 R_4 (p_j, p_j) ight.ight. \\
\[ - 8 \log (1 + \frac{\rho (p_j)}{\alpha_4} e^{\frac{\lambda_j}{2} |x - p_j|^2}) - 1 \right] + O(1) \right\} \]
\[ = 256\pi^2 \lambda_j^3 + 512\pi^2 \lambda_j \left( \log \frac{\rho (p_j)}{\alpha_4} + 32\pi^2 R_4 (p_j, p_j) - \frac{23}{12} \right) + O(1). \]

Similarly,
\[ \int_{B_\delta_0 (p_j)} \rho (p_j) e^{v_j} u_j = 128\pi^2 \lambda_j + 128\pi^2 \left( \log \frac{\rho (p_j)}{\alpha_4} + 32\pi^2 R_4 (p_j, p_j) - \frac{5}{3} \right) + O(\varepsilon_j^2) \] and
\[ \int_{B_\delta_0 (p_j)} \rho (p_j) e^{v_j} \nabla H_j (p_j) \cdot (x - p_j) u_j \]
\[ = \int_{B_\delta_0 (p_j)} \rho (p_j) e^{v_j} \left| \nabla H_j (p_j) \cdot (x - p_j) \right| \times O(\varepsilon_j^2) \]
\[ = O(\varepsilon_j^3) \]
since \( \nabla H_j (p_j) \cdot (x - p_j) \) is an odd function in \( B_\delta_0 (p_j) \).
\[ \int_{B_\delta_0 (p_j)} \rho (p_j) e^{v_j} \sum_{i=1}^{m} (a_i - 1) O(|x - p_j|) u_j = O(\sum_{i=1}^{m} |a_i - 1| \lambda_j \varepsilon_j). \]
Using (3.24) and Hölder’s inequality, we have

\[
\int_{B_{\delta_0}(p_j)} \rho h(p_j)e^{\epsilon_j} w u_j
\]

\[
= \int_{B_{\delta_0}(p_j)} \rho h(p_j)e^{\epsilon_j} w \left[ \lambda_j - s_j - 4 \log \left( 1 + \frac{\rho h(p_j)}{\alpha_4} e^{\frac{\lambda_j}{\alpha} |x - p_j|^2} \right) + O(\epsilon_j^2) \right]
\]

\[
\leq O(\lambda_j \epsilon_j^2 \|w\|_{H^2_0(\Omega)}) + C \left( \int_{B_{\delta_0}(p_j)} \rho h(p_j)e^{\epsilon_j} w^2 \right)^{\frac{1}{2}}
\]

\[
\times \left\{ \int_{B_{\delta_0}(p_j)} \rho h(p_j)e^{\epsilon_j} \left[ -4 \log \left( 1 + \frac{\rho h(p_j)}{\alpha_4} e^{\frac{\lambda_j}{\alpha} |x - p_j|^2} \right) + O(\epsilon_j^2) \right] \right\}^{\frac{1}{2}}
\]

\[
= O(1) \|w\|_{H^2_0(\Omega)}.
\]

By a similar argument as in [8], we can get

\[
\int_{B_{\delta_0}(p_j)} |E| u_j = o(\epsilon_j^2).
\]  

(5.18)

For the integration over \( B_{\delta_0}(p_l), l \neq j \), the dominated term is

\[
- \int_{B_{\delta_0}(p_l)} \rho h(p_l)e^{\epsilon_l} \left[ (a_l - 1)(v_l - s_l - 1) + \left( \frac{e^{\epsilon_l}}{\int_{\Omega} h e^u} - 1 \right) \right] u_j
\]

\[
= -(64\pi^2)^2 G_4(p_l, p_j) \left[ 2(a_l - 1)\lambda_l + \left( \frac{e^{\epsilon_l}}{\int_{\Omega} h e^u} - 1 \right) \right] + O(\epsilon_j^2)
\]  

(5.19)

\[
= 64\pi^2 \sum_{l \neq j} G_4(p_l, p_j) \langle \Delta(u + Tu), \Delta \partial \lambda_j u_j \rangle + O(\epsilon_j^2).
\]

It is easy to see that the other terms are bounded by \( O(\epsilon_j^2) \).

For the integration outside of \( \bigcup_{j=1}^m B_{\delta_0}(p_j) \), we have that \( \int_{\Omega} h e^u = O(e^{4\epsilon}) \) and \( u_j = O(1) \). Then

\[
\int_{B_{2\delta_0}(p_j) \setminus B_{\delta_0}(p_j)} \frac{\rho h e^u}{\int_{\Omega} h e^u} u_j = O(e^{-\epsilon_j}) = O(\epsilon_j^4)
\]  

(5.20)

and

\[
\int_{B_{2\delta_0}(p_j) \setminus B_{\delta_0}(p_j)} u_j \Delta^2 u = \int_{B_{2\delta_0}(p_j) \setminus B_{\delta_0}(p_j)} \Delta^2 (u - 32\sigma_3 G_4(x, p_j)) u_j = O(\epsilon_j^2).
\]  

(5.21)

Similarly,

\[
\int_{\Omega \setminus \bigcup_{j=1}^m B_{2\delta_0}(p_j)} \frac{\rho h e^u}{\int_{\Omega} h e^u} u_j = O(\epsilon_j^4)
\]  

(5.22)
\[
\int_{\Omega \setminus \bigcup_{j=1}^{n} B_{\delta_0}(p_j)} u_j \Delta^2 u = O(\varepsilon_j^2). \tag{5.23}
\]

So, according to the above estimates we have

\[
\langle \Delta(u + Tu), \Delta u_j \rangle = - (a_j - 1) \left[ 256 \pi^2 \lambda_j^2 + 512 \pi^2 \lambda_j \left( \log \frac{\rho h(p_j)}{\alpha_4} + 32 \pi^2 R_4(p_j, p_j) - \frac{23}{12} \right) \right]
\]

\[
-2(64 \pi^2)^2 \sum_{i \neq j} (a_i - 1) G_4(p_i, p_j) \lambda_i - 128 \pi^2 \left( \lambda_j + \log \frac{\rho h(p_j)}{\alpha_4} + 32 \pi^2 R_4(p_j, p_j) - \frac{5}{3} \right)
\]

\[
\times \left( e^{\frac{\lambda_j}{\Omega}} - 1 \right) + 64 \pi^2 \sum_{i \neq j} G_4(p_i, p_j) \langle \Delta(u + Tu), \Delta \partial_{\lambda_j} u_j \rangle + O(\varepsilon_j^2)
\]

\[
= \left[ 2 \lambda_j - \frac{10}{3} + 2 \log \frac{\rho h(p_j)}{\alpha_4} + 64 \pi^2 R_4(p_j, p_j) \right] \langle \Delta(u + Tu), \Delta \partial_{\lambda_j} u_j \rangle
\]

\[
+ 64 \pi^2 \sum_{i \neq j} G_4(p_i, p_j) \langle \Delta(u + Tu), \Delta \partial_{\lambda_j} u_j \rangle + 128 \pi^2 (a_j - 1) \lambda_j + O(\varepsilon_j^2).
\]

Finally we prove part (ii). From \( w \in O_{F, \Lambda} \), we have

\[
\int_{\Omega} \Delta w \Delta \partial_{p_j} u_j = 0. \tag{5.24}
\]

On \( B_{\delta_0}(p_j) \),

\[
\partial_{p_j} u_j = - \partial_x v_j + \frac{\partial_{p_j} h(p_j)}{h(p_j)} \partial_{\lambda_j} (v_j - s_j) + O(1) \tag{5.25}
\]

Since \( \partial_x v_j \) is an odd function,

\[
\int_{B_{\delta_0}(p_j)} \rho h(p_j) e^{v_j} (v_j - s_j - 1) (\partial_x v_j + O(1)) = O(\lambda_j), \tag{5.26}
\]

\[
\int_{B_{\delta_0}(p_j)} \rho h(p_j) e^{v_j} (\partial_x v_j + O(1)) = O(1), \tag{5.27}
\]

29
and
\[
\int_{B_{\delta_0}(p_j)} \rho h(p_j) e^{v_j} \nabla H_j(p_j) \cdot (x - p_j) \partial_x v_j
\]
\[
= 384 \nabla H_j(p_j) \int_{|y| \leq \delta_0} \frac{\rho h(p_j)}{a^4} e^{\lambda_j} \frac{2|y|^2}{(1 + |y|^2)^5} dy
\]
\[
= 64\pi^2 \nabla H_j(p_j) + O(\varepsilon_j^4).
\]
Thus
\[
\int_{B_{\delta_0}(p_j)} \rho h(p_j) e^{v_j} \nabla H_j(p_j) \cdot (x - p_j) \partial_p_j u_j
\]
\[
= (64\pi^2 + O(\varepsilon_j)) \nabla H_j(p_j) + O(\varepsilon_j^4)
\]
\[
= 64\pi^2 \nabla H_j(p_j) + O(\varepsilon_j^3)
\]
since \( \nabla H_j(p_j) = \nabla H_j(p_j) + O(|p_j - q_j|) = O(|p_j - q_j|) = O(\varepsilon_j^2) \).

Using (3.24), by the similar computation as (5.8) and (5.9), we have
\[
\int_{B_{\delta_0}(p_j)} \rho h(p_j) e^{v_j} w \partial_p_j u_j = \int_{B_{\delta_0}(p_j)} \rho h(p_j) e^{v_j} w \left( \partial_p_j v_j + 2 \frac{\partial p_j h(p_j)}{h(p_j)} + 128\pi^2 \partial_x R_4(x, p_j)_{|x = p_j} \right)
\]
\[
= O(\varepsilon_j^2 \|w\|_{H_0^1(\Omega)}) = O(\varepsilon_j^4).
\]
Also we have
\[
|\int_{B_{\delta_0}(p_j)} E \partial_p_j u_j| = O(\varepsilon_j^3).
\]
On \( \Omega \setminus \cup_{j=1}^n B_{\delta_0}(p_j) \), \( \partial_p_j u_j = O(1) \). Hence,
\[
\int_{\Omega \setminus B_{\delta_0}(p_j)} \Delta^2(u - 64\pi^2 G_4(x, p_j)) \partial_p_j u_j = O(\varepsilon_j^2)
\]
and by \( \int_{\Omega} he^u \sim e^{\lambda_j} \),
\[
\int_{\Omega \setminus B_{\delta_0}(p_j)} \frac{\rho h e^u}{\int_{\Omega} h e^u} \partial_p_j u_j = O(e^{-\lambda_j} \int_{\Omega} e^u) = O(\varepsilon_j^4).
\]
Combining (5.24)-(5.33), we get part (ii).
Hence, we finish the proof of Lemma 3.2.
References


