MAT3210 Quiz 2 Solution

1. Solution

$$A = \begin{bmatrix} 1 & 1 & 1 & 0 \\ 3 & 0 & -1 & 1 \end{bmatrix}, \mathbf{b} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}, \mathbf{c} = \begin{bmatrix} 1 & 0 & 5 & -2 \end{bmatrix}^T.$$

(a) $x_1 = \begin{bmatrix} \frac{2}{3} & \frac{1}{3} & 1 & 0 \end{bmatrix}^T$ is the initial FS. Observe that $\mathbf{a}_1 - 4\mathbf{a}_2 + 3\mathbf{a}_3 = 0$, i.e.

$$\alpha_1 = 1, \alpha_2 = -4, \alpha_3 = 3, \alpha_4 = 0.$$

Since $\frac{x_3}{\alpha_3} = \frac{1}{3} = \min_j\{\frac{x_j}{\alpha_j} : \alpha_j > 0\}$, the vector $\mathbf{x}_2 = \begin{bmatrix} \hat{x}_1 & \hat{x}_2 & \hat{x}_3 & \hat{x}_4 \end{bmatrix}^T$ defined by

$$\hat{x}_1 = x_1 - \alpha_1 \frac{x_3}{\alpha_3} = \frac{2}{3} - 1\frac{1}{3} = \frac{1}{3},$$

$$\hat{x}_2 = x_2 - \alpha_2 \frac{x_3}{\alpha_3} = \frac{1}{3} + 4\frac{1}{3} = \frac{5}{3},$$

$$\hat{x}_3 = x_3 - \alpha_3 \frac{x_2}{\alpha_2} = 1 - 3\frac{1}{3} = 0,$$

$$\hat{x}_4 = x_4 - \alpha_4 \frac{x_2}{\alpha_2} = 0 - 0\frac{1}{3} = 0,$$

is a feasible solution. (4 marks)
Since the columns of the matrix $\begin{bmatrix} \mathbf{a}_1 & \mathbf{a}_2 \end{bmatrix}$ are linearly independent, $\mathbf{x}_2$ is a new BFS. (1 mark)

(b) We have

$$B = \begin{bmatrix} 1 & 1 \\ 3 & 0 \end{bmatrix}, c_B = \begin{bmatrix} 1 \\ 0 \end{bmatrix}.$$

Observe that

$$y_3 = \begin{bmatrix} -\frac{1}{3} \\ \frac{4}{3} \end{bmatrix}, y_4 = \begin{bmatrix} \frac{1}{3} \\ -\frac{1}{3} \end{bmatrix}.$$

Compute

$$z_3 = \mathbf{c}_B^T \mathbf{y}_3 = \frac{1}{3}, z_3 - c_3 = \frac{1}{3} - 5 < 0,$$

$$z_4 = \mathbf{c}_B^T \mathbf{y}_4 = -\frac{1}{3}, z_3 - c_3 = -\frac{1}{3} + 2 > 0.$$

Since $z_3 - c_3 < 0$, we let x_3 be the entering variable. We have no choice but to let x_2 be the leaving variable. Now the matrix B becomes

$$B = \begin{bmatrix} 1 & 1 \\ 3 & -1 \end{bmatrix}.$$

Solve $\mathbf{x}_B = B^{-1}\mathbf{b}$ to get $\mathbf{x}_B = \begin{bmatrix} \frac{3}{4} & \frac{5}{4} \end{bmatrix}^T$. Hence

$$\mathbf{x}_3 = \begin{bmatrix} \frac{3}{4} & 0 & \frac{5}{4} & 0 \end{bmatrix}^T$$

is an improved BFS. (5 marks)

(b) Let $C = \{\mathbf{x} : A\mathbf{x} \leq \mathbf{b}, \mathbf{x} \geq 0\}$ be the feasible region and $z = \mathbf{c}^T\mathbf{x}$, where $\mathbf{c}$ is a non-zero vector, is the objective function. We show that interior points cannot be optimal.

To see why, let $\mathbf{x}$ be any interior point of C. So there exists $\epsilon > 0$ such that the ball $B_\epsilon(\mathbf{x})$ lies in C. (3 marks)

Consider the point

$$\mathbf{x}_1 = \mathbf{x} + \frac{\epsilon}{2}\frac{\mathbf{c}}{\|\mathbf{c}\|}. \text{ (3 marks)}$$

Since $\mathbf{c}$ is non-zero,

$$\mathbf{c}^T\mathbf{x}_1 = \mathbf{c}^T(\mathbf{x} + \frac{\epsilon}{2}\frac{\mathbf{c}}{\|\mathbf{c}\|}) = \mathbf{c}^T\mathbf{x} + \frac{\epsilon}{2}\|\mathbf{c}\| > \mathbf{c}^T\mathbf{x}.$$

Hence the point $\mathbf{x}$ is not optimal. (4 marks)

—— **END** ——