Solution Keys to MAT3210 Quiz 1

1. Solution

The expression $\mathbf{x}_B = B^{-1}(\mathbf{b} - R\mathbf{x})$ serves well as a formulation of basic solution but it is not an efficient way of doing the calculations. Observe that the augmented matrix $[A|\mathbf{b}]$

$$\begin{pmatrix} 1 & 1 & 1 & 0 & 0 & 2 \\ 2 & 3 & 0 & 1 & 0 & 3 \\ -1 & 1 & 0 & 0 & 1 & 1 \end{pmatrix}$$

is already in lower-echelon form if we move the columns corresponding to x_3, x_4 and x_5 to the left. Thanks to the special structure (the identity!) of A, this allows us to start backward substitution directly for 7 out of the 10 cases (i.e. when at least one of x_3, x_4, x_5 is a non-basic variable). To illustrate, we calculate a few cases explicitly:

Case 1: $x_1 = x_2 = 0$
We are left with the identity matrix, so $\mathbf{x} = (0, 0, 2, 3, 1)$. The basic solution is non-degenerate and feasible. (1 point)

Case 2: $x_1 = x_3 = 0$
From the first row, we get $x_2 = 2$ immediately. Using the third row, $x_5 = 1 - x_2 = -1$. Finally, the second row gives $x_4 = 3 - 3x_2 = -3$. So $\mathbf{x} = (0, 2, 0, -3, -1)$. The basic solution is non-degenerate and infeasible. (1 point)

Case 3: $x_1 = x_4 = 0$
From the second row, $x_2 = 1$. The first row then gives $x_3 = 2 - x_2 = 1$, and the third row gives $x_5 = 1 - x_2 = 0$. So $\mathbf{x} = (0, 1, 1, 0, 0)$. The basic solution is degenerate and feasible. (1 point)

We continue up to case 7 without further comments:

Case 4: $x_1 = x_5 = 0$
The basic solution $\mathbf{x} = (0, \frac{1}{2}, \frac{1}{2}, \frac{5}{2}, 0)$ is non-degenerate and feasible. (1 point)

Case 5: $x_2 = x_3 = 0$
The basic solution $\mathbf{x} = (2, 0, 0, -1, 3)$ is non-degenerate and infeasible. (1

point)

Case 6: $x_2 = x_4 = 0$
The basic solution $x = (\frac{3}{2}, 0, \frac{1}{2}, 0, \frac{5}{2})$ is non-degenerate and feasible. (1 point)

Case 7: $x_2 = x_5 = 0$
The basic solution $\mathbf{x} = (-1, 0, 3, 5, 0)$ is non-degenerate and infeasible. (1 point)

To tackle the remaining cases, we use Gaussian elimination to transform $[A|\mathbf{b}]$ to the form

$$\begin{pmatrix} 1 & 0 & 0 & 0.2 & -0.6 & 0 \\ 0 & 1 & 0 & 0.2 & 0.4 & 1 \\ 0 & 0 & 1 & -0.4 & 0.2 & 1 \end{pmatrix}.$$

We may now use the same method to continue

Case 8: $x_3 = x_4 = 0$
The basic solution $\mathbf{x} = (3, -1, 0, 0, 5)$ is non-degenerate and infeasible. (1 point)

Case 9: $x_3 = x_5 = 0$
The basic solution $\mathbf{x} = (\frac{1}{2}, \frac{3}{2}, -\frac{5}{2})$ is non-degenerate and infeasible. (1 point)

Case 10: $x_4 = x_5 = 0$
The basic solution $\mathbf{x} = (0, 1, 1, 0, 0)$ is degenerate and feasible. (1 point)

2. Solution
(i) Each of the sets

$$\{\mathbf{x} : x_2 - x_1 \leq 1\}, \{\mathbf{x} : x_1 + x_2 \leq 1\}, \{\mathbf{x} : x_1^2 + x_2^2 \leq 1\}$$

is closed and convex (see Example 1.3, 1.6 and Lemma 1.1(b) of Section 1.7). Since

$$Q = \{\mathbf{x} : x_2 - x_1 \leq 1\} \cap \{\mathbf{x} : x_1 + x_2 \leq 1\} \cap \{\mathbf{x} : x_1^2 + x_2^2 \leq 1\}$$

is the intersection of three closed convex sets, Q is closed and convex (see Exercise 10 of Assignment 1). (2 points)

(For completeness, we show that $\overline{B_1(\mathbf{0})} = \{\mathbf{x} : x_1^2 + x_2^2 \leq 1\}$ is closed using the definition given on Page 7, Section 1.7. It is enough to show that $\partial B = \{\mathbf{x} : x_1^2 + x_2^2 = 1\}$ (which clearly lies in $\overline{B_1(\mathbf{0})}$).

If $|\mathbf{x}| < 1$, let $r = 1 - |\mathbf{x}| > 0$. Then $B_r(\mathbf{x}) \subset \overline{B_1(\mathbf{0})}$ and so $x \notin \partial\overline{B_1(\mathbf{0})}$. If $|\mathbf{x}| > 1$, let $r = |\mathbf{x}| - 1 > 0$. Then $B_r(\mathbf{x}) \subset \overline{B_1(\mathbf{0})}^c$ and so $x \notin \partial\overline{B_1(\mathbf{0})}$. (Use triangle inequality.) Now let $|\mathbf{x}| = 1$ and $r > 0$. Then $\mathbf{x} + r\frac{\mathbf{x}}{2|\mathbf{x}|} \in B_r(\mathbf{x}) \cap \overline{B_1(\mathbf{0})}^c$ and $\mathbf{x} - \min(r,1)\frac{\mathbf{x}}{2|\mathbf{x}|} \in B_r(\mathbf{x}) \cap \overline{B_1(\mathbf{0})}$. So $\partial B = \{\mathbf{x} : x_1^2 + x_2^2 = 1\}$.)

(ii) Q is bounded from below because $(x_1, x_2) \in Q$ implies that $x_1, x_2 \geq -1$. To show this, it suffices to observe that Q is a subset of $\overline{B_1(\mathbf{0})}$. (2 points)

(iii) The boundary of Q is

$$\partial Q = \{\mathbf{x} : x_2 - x_1 = 1, -1 \leq x_1 \leq 0\} \cup$$
$$\{\mathbf{x} : x_1 + x_2 = 1, 0 < x_1 \leq 1\} \cup \{\mathbf{x} : x_1^2 + x_2^2 = 1, x_2 < 0\}$$

where the union is disjoint. On the first set, a supporting hyperplane is

$$\{\mathbf{x} : x_2 - x_1 = 1\}.$$

The extreme points on this hyperplane are $(-1, 0)$ and $(0, 1)$. (1 point + 1 point)

On the second set, a supporting hyperplane is

$$\{\mathbf{x} : x_1 + x_2 = 1\}.$$

An extreme point on this hyperplane is $(1, 0)$. (1 point)

Finally, let $(\bar{x}_1, \bar{x}_2)$ be a point on the third set. Then

$$\{\mathbf{x} : x_1\bar{x}_1 + x_2\bar{x}_2 = 1\}$$

is the supporting hyperplane at $(\bar{x}_1, \bar{x}_2)$. By Theorem 1.5 of Section 1.8, $(\bar{x}_1, \bar{x}_2)$ is an extreme point. (3 points)

—— **END** ——