MAT3210 (09-10) Assignment 6 Solution

1. Solution

   The initial tableau is

	x_1	x_2	x_3	x_4	x_5	x_6	b
x_4	3	-3	5	1	0	0	50
x_5	1	0	1	0	1	0	10
x_6	1	-1	4	0	0	1	2
x_0	-40	-20	-2	0	0	0	0

   1) No positive ratio exists in the x_2 column. Hence the solution space is unbounded in the x_2 solution.

   2) Since $z_2 - c_2 = -20 < 0$, the optimal value is $+\infty$.

2. Solution (two-phase method)

   Using a surplus variable x_3, an artificial variable x_4 and a slack variable x_5, we have the augmented system:

$$\begin{aligned} 3x_1 + 2x_2 - x_3 + x_4 &= 6 \\ 2x_1 + x_2 + x_5 &= 2 \end{aligned}$$

   T1:

	x_1	x_2	x_3	x_4	x_5	b
x_4	3	2	-1	1	0	6
x_5	2^*	1	0	0	1	2
x_0	-3	-2	1	0	0	-6

   T2:

	x_1	x_2	x_3	x_4	x_5	b
x_4	0	$\frac{1}{2}$	-1	1	$-\frac{3}{2}$	3
x_1	1	$\frac{1}{2}$	0	0	$\frac{1}{2}$	1
x_0	0	$-\frac{1}{2}$	1	0	$\frac{3}{2}$	-3

   T3:

	x_1	x_2	x_3	x_4	x_5	b
x_4	-1	0	-1	1	-2	2
x_2	2	1	0	0	1	2
x_0	1	0	1	0	2	-2

   Since all the entries in row 0 are non-negative, we end Phase I. But $x_0^* = -2 < 0$ means that there are no feasible solution to the LPP.

3. Solution

   By eliminating x_1, we have the initial tableau:

	x_1	x_2	x_3	x_4	x_5	b
x_1	1	1	1	0	0	1
	0	1	2	-1	0	2
	0	2	-1	0	1	2
x_0	0	-1	-2	0	0	-1

We save the first row for later use and solve the reduced LPP by two-phase method. The artificial variables a_1, a_2 are introduced.

PI,T0:

	x_2	x_3	x_4	x_5	a_1	a_2	b
	1	2	-1	0	1	0	2
	2	-1	0	1	0	1	2
x_0	0	0	0	0	1	1	0

PI,T1:

	x_2	x_3	x_4	x_5	a_1	a_2	b
	1	2	-1	0	1	0	2
	2	-1	0	1	0	1	2
x_0	-3	-1	1	-1	0	0	-4

PI,T2:

	x_2	x_3	x_4	x_5	a_1	a_2	b
	0	$\frac{5}{2}$	-1	$-\frac{1}{2}$	1	$-\frac{1}{2}$	1
	1	$-\frac{1}{2}$	0	$\frac{1}{2}$	0	$\frac{1}{2}$	1
x_0	0	$-\frac{5}{2}$	1	$\frac{1}{2}$	0	$\frac{3}{2}$	-1

PI,T3:

	x_2	x_3	x_4	x_5	a_1	a_2	b
	0	1	$-\frac{2}{5}$	$-\frac{1}{5}$	$\frac{2}{5}$	$-\frac{1}{5}$	$\frac{2}{5}$
	1	0	$-\frac{1}{5}$	$\frac{2}{5}$	$\frac{1}{5}$	$\frac{2}{5}$	$\frac{6}{5}$
x_0	0	0	0	0	1	1	0

PII,T0:

	x_2	x_3	x_4	x_5	b
	0	1	$-\frac{2}{5}$	$-\frac{1}{5}$	$\frac{2}{5}$
	1	0	$-\frac{1}{5}$	$\frac{2}{5}$	$\frac{6}{5}$
x_0	-1	-2	0	0	-1

PII,T1:

	x_2	x_3	x_4	x_5	b
	0	1	$-\frac{2}{5}$	$-\frac{1}{5}$	$\frac{2}{5}$
	1	0	$-\frac{1}{5}$	$\frac{2}{5}$	$\frac{6}{5}$
x_0	0	0	-1	0	1

We find that the feasible region is unbounded in x_3 direction and the optimal value is $+\infty$.

4. Solution By two phase method, we have

PI,T1:

3	6	3	-4	1			12
2	0	1	0		1		4
3	-6	0	4^*			1	0
-8	0	-4	0				-16

PI,T2:

0	12	3	-8	1		12
0	4^*	1	$-\frac{8}{3}$		1	4
1	-2	0	$\frac{4}{3}$			0
0	-16	-4	$\frac{32}{3}$			16

PI,T3:

0	0	0	0	1	0
0	1	$\frac{1}{4}$	$-\frac{2}{3}$		1
1	0	$\frac{1}{2}$	0		2
0	0	0	0		0

All artificial variables are now 0. We go to phase 2.
PII,T1:

0	1	$\frac{1}{4}$	$-\frac{2}{3}$	1
1	0	$\frac{1}{2}^*$	0	2
0	0	-1	0	8

The feasible region is unbounded in x_4 direction.
PII,T2:

$-\frac{1}{2}$	1	0	$-\frac{2}{3}$	0
2	0	1	0	4
2	0	0	0	12

The optimal BFS is $(0, 0, 4, 0)^T$, which is not unique, e.g. $(0, \frac{2}{3}, 4, 1)^T$.

5. Solution

   (a) Since no positive ratio exists in the x_6 column, the feasible region is unbounded in the x_6 direction.

   (b) Since all the entries in the row 0 are non-negative, we have reached an optimal solution. However, the reduced cost coefficient for the non-basic variable x_2 is 0 and the minimum of the ratios in the x_2 column is positive. Hence the optimal BFS is not unique. The original BFS is $x_B = [0, 0, 6, 0, 0, 0, 0, 2]^T$. Pivot a the 3 in the x_2 column we get a new optimal BFS $\hat{x}_B = [\frac{2}{3}, \frac{2}{3}, \frac{14}{3}, 0, 0, 0, 0, 0]^T$. Then

$$\frac{1}{2}x_B + \frac{1}{2}\hat{x}_B = [\frac{1}{3}, \frac{1}{3}, \frac{16}{3}, 0, 0, 0, 0, 0, 1]^T$$

   is another non-basic optimal solution.

6. Solution

   T1:

	x_1	x_2	x_3	x_4	x_5	x_6	b
x_4	1	2	0	1	0	0	5
x_5	1^*	1	-1	0	1	0	2
x_6	7	3	-5	0	0	1	20
x_0	-3	-1	0	0	0	0	0

T2:

	x_1	x_2	x_3	x_4	x_5	x_6	b
x_4	0	1	1^*	1	-1	0	3
x_1	1	1	-1	0	1	0	2
x_6	0	-4	2	0	-7	1	6
x_0	0	2	-3	0	3	0	6

T3:

	x_1	x_2	x_3	x_4	x_5	x_6	b
x_4	0	1	1	1	-1	0	3
x_1	1	2	0	1	0	0	5
x_6	0	-6	0	-2	-5	1	0
x_0	0	5	0	3	0	0	15

We have reached a degenerate optimal BFS $[5,0,3,0,0,0]^T$. All the entries in the x_5 column are non-positive and the reduced cost coefficient of x_5 is zero. By Theorem 4.1 in the lecture notes we know $[5,0,3+\theta,0,\theta,5\theta]^T$ is a non-basic optimal feasible solution of the original LPP, where θ is an arbitrary positive number.

7. Solution

Using a surplus variable x_4, an artificial variable x_5 and a slack variable x_6, the augmented system is

$$\begin{aligned} 3x_1 + 4x_2 + 2x_3 - x_4 + x_5 &= 8 \\ 2x_1 + x_2 + x_3 + x_6 &= 2 \end{aligned}$$

Then we use two-phase method.

PI,T1:

	x_1	x_2	x_3	x_4	x_5	x_6	b
x_5	3	4	2	-1	1	0	8
x_6	2	1	1	0	0	1	2
x_0	-3	-4	-2	1	0	0	-8

PI,T2:

	x_1	x_2	x_3	x_4	x_5	x_6	b
x_5	-5	0	-2	-1	1	-4	0
x_2	2	1	1	0	0	1	2
x_0	5	0	2	1	0	4	0

PI,T3:

	x_1	x_2	x_3	x_4	x_5	x_6	b
x_5	-5	0	-2	-1	1	-4	0
x_6	2	1	1	0	0	1	2
x_0	5	0	2	1	0	4	0

With alternating rule, we start Phase II. PII,T0:

	x_1	x_2	x_3	x_4	x_5	x_6	b
x_5	-5	0	-2	-1	1	-4	0
x_6	2	1	1	0	0	1	2
x_0	-3	-2	-3	0	0	0	0

PII,T1:

	x_1	x_2	x_3	x_4	x_5	x_6	b
x_5	-5	0	-2	-1	1	-4	0
x_6	2	1	1	0	0	1	2
x_0	-3	-2	-3	0	0	0	0

PII,T2:

	x_1	x_2	x_3	x_4	x_5	x_6	b
x_5	-5^*	0	-2	-1	1	-4	0
x_6	2	1	1	0	0	1	2
x_0	1	0	-1	0	0	2	4

PII,T3:

	x_1	x_2	x_3	x_4	x_6	b
x_1	1	0	$\frac{2}{5}$	$\frac{1}{5}$	$\frac{4}{5}$	0
x_2	0	1	$\frac{1}{5}$	$-\frac{2}{5}$	$-\frac{3}{5}$	2
x_0	0	0	$-\frac{7}{5}$	$-\frac{1}{5}$	$\frac{6}{5}$	4

PII,T4:

	x_1	x_2	x_3	x_4	x_6	b
x_3	$\frac{5}{2}$	0	1	$\frac{1}{2}$	2	0
x_2	$-\frac{1}{2}$	1	0	$-\frac{1}{2}$	-1	2
x_0	$\frac{7}{2}$	0	0	$\frac{1}{2}$	4	4

Therefore, $x^* = (0, 2, 0)^T, z^* = 4$.

8. Solution

   (a) True. Since $Ax + I_m x_a = b$, we must have $x_a = 0$ to get $Ax = b$.

   (b) False. At the end of Phase I, if $x_0^* = 0$ and one or more artificial variables appear in the basis at zero level, we have also found a degenerate BFS to the original problem.

   (c) True.

   (d) False. If $c_j - z_j = 0$ but $\min\{\frac{x_{B_i}}{y_{ij}} : y_ij > 0\} = 0$, then the optimal solution is still unique.

   (e) False.

   (f) False.

   (g) True.