

Solution Keys to MAT3210 Assignment 2

1. Solution

(i) C is convex. $\forall u, v \in C, \forall \lambda \in (0, 1)$,

$$\lambda u + (1 - \lambda)v = [\lambda u_1 + (1 - \lambda)v_1, \lambda u_2 + (1 - \lambda)v_2]$$

We have

$$\begin{aligned} & [\lambda u_1 + (1 - \lambda)v_1]^2 + [\lambda u_2 + (1 - \lambda)v_2]^2 \\ = & \lambda^2(u_1^2 + u_2^2) + (1 - \lambda)^2(v_1^2 + v_2^2) + 2\lambda(1 - \lambda)(u_1v_1 + u_2v_2) \\ \leq & \lambda^2(u_1^2 + u_2^2) + (1 - \lambda)^2(v_1^2 + v_2^2) + 2\lambda(1 - \lambda)\sqrt{(u_1^2 + u_2^2) + (v_1^2 + v_2^2)} \\ \leq & \lambda^2 + (1 - \lambda)^2 + 2\lambda(1 - \lambda) \\ = & 1 \end{aligned}$$

and

$$\begin{aligned} & [\lambda u_1 + (1 - \lambda)v_1] + [\lambda u_2 + (1 - \lambda)v_2] \\ = & \lambda(u_1 + u_2) + (1 - \lambda)(v_1 + v_2) \\ \geq & 0 \end{aligned}$$

Therefore $\lambda u + (1 - \lambda)v \in C$.

(ii) C is bounded from below, since

$$x_1^2 + x_2^2 \leq 1 \Rightarrow x_1 \geq -\sqrt{1 - x_2^2} \geq -1$$

so is x_2 .

(iii) Rewrite C in polar coordinate:

$$C = \{(r, \theta) | r \in [0, 1], \theta \in [-\frac{\pi}{4}, \frac{3\pi}{4}]\}$$

Then the extreme points of C are those points with $r = 1$. To verify this, let $w = (1, \theta)$ be such a point and $u = (r, \alpha), v = (s, \beta)$ be two points in C s.t.

$$w = \lambda u + (1 - \lambda)v$$

for some $\lambda \in (0, 1)$. Back to Euclidean coordinate we have

$$\begin{aligned} \cos \theta &= \lambda r \cos \alpha + (1 - \lambda)s \cos \beta \\ \sin \theta &= \lambda r \sin \alpha + (1 - \lambda)s \sin \beta \end{aligned}$$

Taking square on both sides,

$$\begin{aligned} 1 &= \cos^2 \theta + \sin^2 \theta \\ &= \lambda^2 r^2 + (1 - \lambda)^2 s^2 + 2\lambda(1 - \lambda) \cos(\alpha - \beta) \\ &\leq \lambda^2 + (1 - \lambda)^2 + 2\lambda(1 - \lambda) = 1 \end{aligned}$$

Since $\lambda \in (0, 1)$, we have $r = s = 1$ and $\alpha = \beta$, i.e. $u = v$. Therefore w is an extreme point.

(iv) Now let $(x_0, y_0) \in C$ with $x_0^2 + y_0^2 = 1$. Then

$$H = \{(x, y) \in \mathbf{R}^2 : xx_0 + yy_0 = 1\}$$

is a supporting hyperplane of C at (x_0, y_0) .

2. Solution

See the solution of assignment 1.

3. Solution

(3.1) The graph of the above set is shown in Figure 1. The extreme points

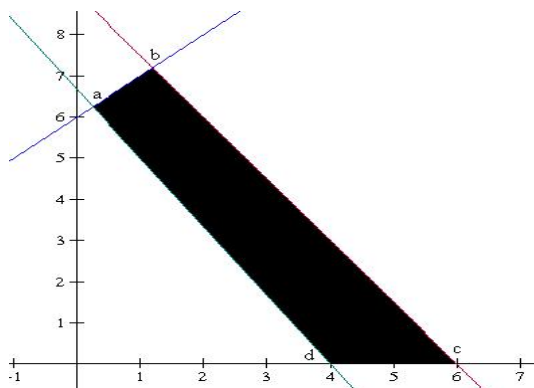


Figure 1: Figure 1: Linear constraints, feasible region and extreme points.

are a, b, c, d . We have

$$a = \left(\frac{1}{4}, \frac{25}{4}\right), b = \left(\frac{6}{5}, \frac{36}{5}\right), c = (6, 0), d = (4, 0).$$

Supporting hyperplanes at the extreme points can be chosen as follows:

$$a, b : x_1 - x_2 = 6; c, d : x_2 = 0$$

(3.2) The standard form is

$$\begin{aligned} 3x_1 + 2x_2 + x_3 &= 18 \\ -x_1 + x_2 + x_4 &= 6 \\ 5x_1 + 3x_2 - x_5 &= 20 \\ x_i &\geq 0, i = 1, 2, 3, 4, 5, \end{aligned}$$

where x_3, x_4, x_5 are slack variables. The BFS correspond to the extreme points:

$$\begin{aligned} a &: \left(\frac{1}{4}, \frac{25}{4}, \frac{19}{4}, 0, 0\right) \\ b &: \left(\frac{6}{5}, \frac{36}{5}, 0, 0, \frac{38}{5}\right) \\ c &: (6, 0, 0, 12, 10) \\ d &: (4, 0, 6, 10, 0) \end{aligned}$$

4. Solution

The standard form is

$$\begin{aligned} \text{maximize } z' &= -10x_1 - 2x_2^+ + 2x_2^- + x_3 \\ \text{subject to } x_1 + x_2^+ - x_2^- + x_4 &= 30 \\ x_1 - x_3 - x_5 &= 1 \\ x_2^+ - x_2^- + x_3 + x_6 &= 3 \\ -x_2^+ + x_2^- - x_3 + x_7 &= 7 \\ x_1 + x_2^+ - x_2^- + x_3 &= 10 \\ x_1, x_2^+, x_2^-, x_3, x_4, x_5, x_6, x_7 &\geq 0, \end{aligned}$$

where

$$x_2^+ = \max(0, x_2), x_2^- = \max(0, -x_2).$$

5. Solution

(a) Set $x_3 = 0$. We have $x_1 = \frac{5}{9}, x_2 = \frac{11}{18}$. Since $\mathbf{a}_1, \mathbf{a}_2$ are linear independent, the solution is a basic feasible solution.

- (b) Set $x_2 = 0$. We have $x_1 = \frac{13}{8}, x_3 = -\frac{11}{8}$. The solution is infeasible.
(c) Set $x_1 = 0$. We have $x_2 = \frac{13}{14}, x_3 = \frac{5}{7}$. Since $\mathbf{a}_2, \mathbf{a}_3$ are linearly independent, the solution is a BFS.

6. Solution

- (a) Set $x_1 = x_2 = 0$. We have $x_3 = 2, x_4 = -1$. The solution is infeasible.
(b) Set $x_1 = x_3 = 0$. We have $x_2 = \frac{1}{2}, x_4 = 0$. Since $\mathbf{a}_2, \mathbf{a}_4$ are linearly independent, the solution is a BFS.
(c) Set $x_1 = x_4 = 0$. We have $x_2 = \frac{1}{2}, x_3 = -1$. We get the same BFS obtained in (b).
(d) Set $x_2 = x_3 = 0$. We have $x_1 = \frac{8}{3}, x_4 = -\frac{7}{3}$. The solution is infeasible.
(e) Set $x_2 = x_4 = 0$. We have $x_1 = -2, x_3 = \frac{7}{2}$. The solution is infeasible.
(f) Set $x_3 = x_4 = 0$. We have $x_1 = 0, x_2 = \frac{1}{2}$. We get the same BFS obtained in (b).

7. Solution

We first express the LP in standard form:

$$\begin{aligned} \text{maximize } z &= x_1 + x_2 - 4x_3^+ + 4x_3^- \\ \text{subject to } &x_2 + 3x_3^+ - 3x_3^- = 3 \\ &x_1 + 3x_2 - 2x_3^+ + 2x_3^- = 2 \\ &x_1, x_2, x_3^+, x_3^- \geq 0 \end{aligned}$$

- (a) Set $x_1 = x_2 = 0$. We have $x_3^+ = 2, x_3^- = 3$. The solution is a BFS. The value of z is 4.
(b) Set $x_1 = x_3^+ = 0$. We have $x_2 = \frac{8}{7}, x_3^- = -\frac{5}{7}$. The solution is infeasible.
(c) Set $x_1 = x_3^- = 0$. We have $x_2 = \frac{12}{13}, x_3^+ = \frac{5}{13}$. The solution is a BFS. The value of z is $-\frac{8}{13}$.
(d) Set $x_2 = x_3^+ = 0$. We have $x_1 = 8, x_3^- = -3$. The solution is infeasible.
(e) Set $x_2 = x_3^- = 0$. We have $x_1 = 4, x_3^+ = 1$. The solution is a BFS. The value of z is 0.
(f) Set $x_3^+ = x_3^- = 0$. We have $x_1 = -\frac{5}{2}, x_2 = \frac{3}{2}$. The solution is infeasible.
The optimal value of the LP is 4. It is attained at $(0, 0, 2, 3)^T$.

8. Solution

A direct check shows that x_0 is an FS. But x_0 is not a BFS since it contains only one zero element, while a BFS of the above LPP should contain

at least two zero elements. To find a BFS from x_0 , we first observe that $0\mathbf{a}_1 + 2\mathbf{a}_2 - \mathbf{a}_3 = 0$. We have, using the notation of the lecture notes,

$$\alpha_1 = 0, \alpha_2 = 2, \alpha_3 = 1,$$

$$\frac{x_2}{\alpha_2} = 1 = \min\left\{\frac{x_j}{\alpha_j} : \alpha_j > 0, j = 1, 2, 3\right\}.$$

Hence

$$\hat{x}_1 = x_1 - x_2 \cdot \frac{\alpha_1}{\alpha_2} = 3$$

$$\hat{x}_2 = 0$$

$$\hat{x}_3 = x_3 - x_2 \cdot \frac{\alpha_3}{\alpha_2} = 2$$

Therefore the new FS is $\hat{x} = (3, 0, 2)^T$. Since $\mathbf{a}_1, \mathbf{a}_3$ are linearly independent, the solution is also a BFS.

9. Solution

(a) Just substitute the vector into the system of equations and we can see that it is an FS.

(b) Since $\text{rank}[\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3, \mathbf{a}_4] = 3$, the solution is not basic. Since

$$-a_1 - 5a_2 + 2a_3 - 4a_4 = 0,$$

we have

$$\alpha_1 = 1, \alpha_2 = -5, \alpha_3 = 2, \alpha_4 = -4,$$

$$\frac{x_3}{\alpha_3} = \frac{1}{2} = \min\left\{\frac{x_j}{\alpha_j} : \alpha_j > 0, j = 1, 2, 3, 4\right\}.$$

Hence

$$\hat{x}_1 = x_1 - x_3 \cdot \frac{\alpha_1}{\alpha_3} = \frac{5}{2}$$

$$\hat{x}_2 = x_2 - x_3 \cdot \frac{\alpha_2}{\alpha_3} = \frac{7}{2}$$

$$\hat{x}_3 = 0$$

$$\hat{x}_4 = x_4 - x_3 \cdot \frac{\alpha_4}{\alpha_3} = 5$$

Therefore the new FS is $\hat{x} = (\frac{5}{2}, \frac{7}{2}, 0, 5)^T$. Since $\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_4$ are linearly independent, the solution is also a BFS.

10. Solution

Figure 2 shows the graph of the linear constraints and the feasible region (shaded area). a, b, c, d are the extreme points. We have

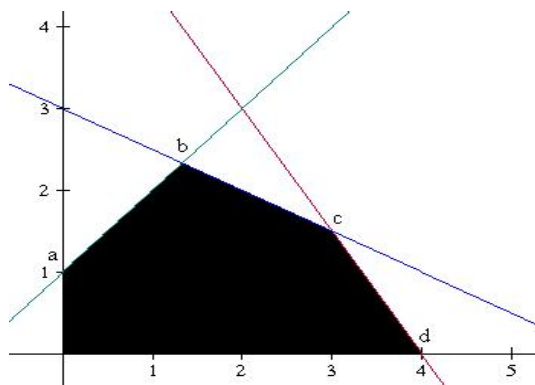


Figure 2: Figure 2: Linear constraints, feasible region and extreme points.

$$a = (0, 1), b = \left(\frac{4}{3}, \frac{7}{3}\right), c = \left(3, \frac{3}{2}\right), d = (4, 0).$$

Compute the values of z at the extreme points:

$$z(a) = -5, z(b) = -\frac{19}{3}, z(c) = \frac{9}{2}, z(d) = 16.$$

Therefore

$$\min z = -\frac{19}{3} \text{ at } (x_1, x_2) = \left(\frac{4}{3}, \frac{7}{3}\right).$$

— **END** —