Solution Keys to MAT3210 Assignment 1

1. Solution

Form the augmented matrices $[A|b]$.

(a)

$$\begin{pmatrix} 1 & 0 & -1 & 1 \\ 3 & 1 & 1 & 0 \\ -1 & 1 & 2 & 2 \end{pmatrix} \to \begin{pmatrix} 1 & 0 & -1 & 1 \\ 0 & 1 & 4 & -3 \\ 0 & 1 & 1 & 3 \end{pmatrix}$$

$$\to \begin{pmatrix} 1 & 0 & -1 & 1 \\ 0 & 1 & 4 & -3 \\ 0 & 0 & -3 & 6 \end{pmatrix} \to \begin{pmatrix} 1 & 0 & -1 & 1 \\ 0 & 1 & 0 & 5 \\ 0 & 0 & -3 & 6 \end{pmatrix}$$

$$\to \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} -1 \\ 5 \\ -2 \end{pmatrix}$$

(b)

$$\begin{pmatrix} 1 & 2 & -1 & 1 \\ 2 & 1 & 1 & 1 \\ 1 & -1 & 3 & 1 \end{pmatrix} \to \begin{pmatrix} 1 & 2 & -1 & 1 \\ 0 & -3 & 3 & -1 \\ 0 & -3 & 4 & 0 \end{pmatrix}$$

$$\to \begin{pmatrix} 1 & 0 & 1 & \frac{1}{3} \\ 0 & -3 & 3 & -1 \\ 0 & 0 & 1 & 1 \end{pmatrix} \to \begin{pmatrix} 1 & 0 & 0 & -\frac{2}{3} \\ 0 & -3 & 0 & -4 \\ 0 & 0 & 1 & 1 \end{pmatrix}$$

$$\to \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} -\frac{2}{3} \\ \frac{4}{3} \\ 1 \end{pmatrix}$$

2. Solution

Matrix multiplication gives

$$(A\vec{x})_k = \sum_{j=1}^{n} a_{kj} x_j$$

Let k varies from 1 to m, then we have

$$A\vec{x} = \sum_{j=1}^{n} \vec{a}_j x_j$$

3. Solution

(a) $A = \begin{pmatrix} 1 & 2 & -1 \\ 1 & 0 & -1 \end{pmatrix}, b = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$

(i) $B = \begin{pmatrix} 1 & 2 \\ 1 & 0 \end{pmatrix}$

basic variables: x_1, x_2
nonbasic variables: x_3
basic solution: $x = \begin{pmatrix} 0 & \frac{1}{2} & 0 \end{pmatrix}^T$ is degenerate.

(ii) $B = \begin{pmatrix} 2 & -1 \\ 0 & -1 \end{pmatrix}$

basic variables: x_2, x_3
nonbasic variables: x_1
basic solution: $x = \begin{pmatrix} 0 & \frac{1}{2} & 0 \end{pmatrix}^T$ is degenerate.

(b) $A = \begin{pmatrix} 3 & 5 & 1 & 0 \\ 1 & -1 & 0 & 1 \end{pmatrix}, b = \begin{pmatrix} 5 \\ 0 \end{pmatrix}$

(i) $B = \begin{pmatrix} 3 & 5 \\ 1 & -1 \end{pmatrix}$

basic variables: x_1, x_2
nonbasic variables: x_3, x_4
basic solution: $x = \begin{pmatrix} -\frac{5}{2} & \frac{5}{2} & 0 & 0 \end{pmatrix}^T$ is non-degenerate.

(ii) $B = \begin{pmatrix} 3 & 1 \\ 1 & 0 \end{pmatrix}$

basic variables: x_1, x_3
nonbasic variables: x_2, x_4
basic solution: $x = \begin{pmatrix} 0 & 0 & 5 & 0 \end{pmatrix}^T$ is non-degenerate.

(iii) $B = \begin{pmatrix} 3 & 0 \\ 1 & 1 \end{pmatrix}$

basic variables: x_1, x_4
nonbasic variables: x_2, x_3
basic solution: $x = \begin{pmatrix} \frac{5}{3} & 0 & 0 & -\frac{5}{3} \end{pmatrix}^T$ is non-degenerate.

(iv) $B = \begin{pmatrix} 5 & 1 \\ -1 & 0 \end{pmatrix}$

basic variables: x_2, x_3
nonbasic variables: x_1, x_4
basic solution: $x = \begin{pmatrix} 0 & 0 & 5 & 0 \end{pmatrix}^T$ is degenerate.

(v) $B = \begin{pmatrix} 5 & 0 \\ -1 & 1 \end{pmatrix}$

basic variables: x_2, x_4

nonbasic variables: x_1, x_3

basic solution: $x = \begin{pmatrix} 0 & 1 & 0 & -1 \end{pmatrix}^T$ is non-degenerate.

(vi) $B = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

basic variables: x_3, x_4

nonbasic variables: x_1, x_2

basic solution: $x = \begin{pmatrix} 0 & 0 & 5 & 0 \end{pmatrix}^T$ is degenerate.

(c) $A = \begin{pmatrix} 1 & 0 & 0 & -1 \\ 3 & 1 & 2 & 1 \\ -1 & 1 & 2 & 2 \end{pmatrix}, b = \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix}$

(i) $B = \begin{pmatrix} 1 & 0 & -1 \\ 3 & 1 & 1 \\ -1 & 1 & 2 \end{pmatrix}$

basic variables: x_1, x_2, x_4

nonbasic variables: x_3

basic solution: $x = \begin{pmatrix} -1 & 5 & 0 & -2 \end{pmatrix}^T$ is non-degenerate.

(ii) $B = \begin{pmatrix} 1 & 0 & -1 \\ 3 & 2 & 1 \\ -1 & 2 & 2 \end{pmatrix}$

basic variables: x_1, x_3, x_4

nonbasic variables: x_2

basic solution: $x = \begin{pmatrix} -1 & 0 & \frac{5}{2} & -2 \end{pmatrix}^T$ is non-degenerate.

For other cases, the basic solution does not exist.

(d) $A = \begin{pmatrix} 1 & 2 & 0 & 0 \\ 2 & 4 & 3 & 2 \\ 3 & 2 & 5 & -1 \end{pmatrix}, b = \begin{pmatrix} 2 \\ 3 \\ 1 \end{pmatrix}$

(i) $B = \begin{pmatrix} 1 & 2 & 0 \\ 2 & 4 & 3 \\ 3 & 2 & 5 \end{pmatrix}$

basic variables: x_1, x_2, x_3

nonbasic variables: x_4

basic solution: $x = \begin{pmatrix} \frac{1}{3} & \frac{5}{6} & -\frac{1}{3} & 0 \end{pmatrix}^T$ is non-degenerate.

(ii) $B = \begin{pmatrix} 1 & 2 & 0 \\ 2 & 4 & 2 \\ 3 & 2 & -1 \end{pmatrix}$

basic variables: x_1, x_2, x_4

nonbasic variables: x_3

basic solution: $x = \begin{pmatrix} -\frac{3}{4} & \frac{11}{8} & 0 & -\frac{1}{2} \end{pmatrix}^T$ is non-degenerate.

(iii) $B = \begin{pmatrix} 1 & 0 & 0 \\ 2 & 3 & 2 \\ 3 & 5 & -1 \end{pmatrix}$

basic variables: x_1, x_3, x_4

nonbasic variables: x_2

basic solution: $x = \begin{pmatrix} 2 & 0 & -\frac{11}{13} & \frac{10}{13} \end{pmatrix}^T$ is non-degenerate.

(iv) $B = \begin{pmatrix} 2 & 0 & 0 \\ 4 & 3 & 2 \\ 2 & 5 & -1 \end{pmatrix}$

basic variables: x_2, x_3, x_4

nonbasic variables: x_1

basic solution: $x = \begin{pmatrix} 0 & 1 & -\frac{3}{13} & -\frac{2}{13} \end{pmatrix}^T$ is non-degenerate.

4. Solution

Rewrite two equations as

$$c_1^T u = 5, c_2^T u = 9$$

where

$$c_1 = \begin{pmatrix} 1 & -2 & -1 \end{pmatrix}^T, c_2 = \begin{pmatrix} 2 & -4 & -2 \end{pmatrix}^T, u = \begin{pmatrix} x & y & z \end{pmatrix}^T.$$

SInce $c_2 = 2c_1$, they are parallel vectors. Therefore the two hyperplanes are parallel.

Let u_0 be any point in the first hyperplane and u_1 be a point in the second hyperplane such that $u_1 = u_0 + \alpha c_1$ for some $\alpha \in \mathbb{R}$. Then the distance between the two hyperplanes will just be $|u_1 - u_0|$.

$$c_2^T u_1 = c_2^T u_0 + \alpha c_2^T c_1 \Rightarrow \alpha = -\frac{1}{12}$$

Then

$$|u_1 - u_0| = |\alpha c_1| = \frac{\sqrt{6}}{12}$$

5. Solution

Consider two elements in Q, namely $u = (1, 4), v = (4, 1)$, and let $\lambda = \frac{1}{2}$, then $w = \lambda u + (1 - \lambda)v = (\frac{5}{2}, \frac{5}{2})$ does not lie in Q. Hence Q is not convex.

6. Solution

Now that (0,0),(2,0),(0,2) are extreme points of Q, let $u, v \in Q, \lambda \in (0, 1)$.

(i) Let

$$(0, 0) = (\lambda u_1 + (1 - \lambda)v_1, \lambda u_2 + (1 - \lambda)v_2),$$

then

$$0 \leq u_1 = \frac{\lambda - 1}{\lambda} v_1 \leq 0 \Rightarrow u_1 = 0.$$

Similarly we have $u_2 = 0$, so as v_1, v_2.
Therefore $u = v = 0$, which means (0,0) is an extreme point.

(ii) Let

$$(2, 0) = (\lambda u_1 + (1 - \lambda)v_1, \lambda u_2 + (1 - \lambda)v_2),$$

then

$$\lambda u_2 + (1 - \lambda)v_2 = 0 \Rightarrow u_2 = v_2 = 0$$

$$\lambda u_1 + \lambda v_1 = 2 \Rightarrow u_1 = v_1 = 2$$

Therefore $u = v = (2, 0)$, which means (2,0) is an extreme point.
Similarly, we can show that (0,2) is an extreme point. Moreover, there are no other extreme points of Q. First, if x is an interior point of Q, then we can find an open ball $B_r(x)$ s.t. $B_r(x) \subset Q$. Let

$$x_1 = x + (r, 0), x_2 = x - (r, 0),$$

then

$$x_1, x_2 \in B_r(x) \subset Q$$

and

$$x = \frac{1}{2} x_1 + \frac{1}{2} x_2.$$

Therefore x is not an extreme point. Second, if x is a boundary point other than the three extreme points, then we can use the same method as above to show that x is not an extreme point.

7. Solution

(a) The extreme points are $(0,0),(0,1),(1,2),(2,2),(3,\frac{3}{2}),(4,0)$, see Fig. 1 (left).

(b) The extreme points are $(0,1),(\frac{15}{7},\frac{12}{7}),(\frac{10}{3},-\frac{2}{3})$, see Fig. 1 (right).

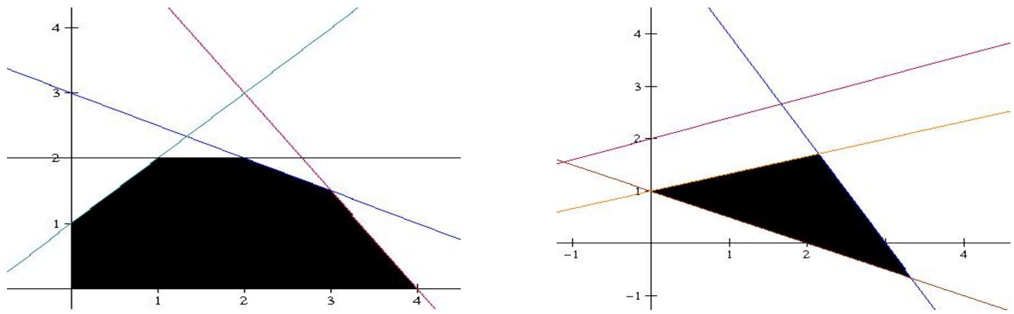

Figure 1: the constrained areas marked by shadow

8. Solution

Clearly, $x=0\in C$. Let $u,v\in C$ and

$$0=\lambda u+(1-\lambda)v, 0<\lambda<1$$

$$u,v\le 0\Rightarrow u_i,v_i\ge 0,\forall i.$$

Then

$$0=\lambda u_i+(1-\lambda)v_i$$
$$\Rightarrow u_i=v_i=0,\forall i$$
$$\Rightarrow u=v=0$$

Therefore $x=0$ is an extreme point of C.

9. Solution

(1) Consider any point x inside the triangle. Let y be the intersection of the line segment $\overline{x_1x_2}$ and the extension of the line segment $\overline{x_3y}$. Then

$$y=\lambda_1x_1+(1-\lambda_1)x_2, \text{ where } \lambda_1=\frac{|x_1y|}{|x_1x_2|}$$

and

$$x = \lambda_2 x_3 + (1 - \lambda_2)y, \text{ where } \lambda_2 = \frac{|x_3 x|}{|x_3 y|}$$

Then

$$x = \lambda_2 x_3 + (1 - \lambda_2)\lambda_1 x_1 + (1 - \lambda_2)x_2$$

Since

$$\lambda_2 + (1 - \lambda_2)\lambda_1 + (1 - \lambda_2)(1 - \lambda_1) = 1$$

we conclude that x is a convex combination of x_1, x_2, x_3.

(2) The obvious supporting hyperplane at x_1 is the line across x_1 and parallel to $\overline{x_2 x_3}$, i.e.

$$2x + y = 5$$

SImilarly for x_2 and x_3 we have

$$x - 2y = -5, 3x - y = 10.$$

10. Solution

(i) $C_1 \cap C_2$ is convex.

For all $x_1, x_2 \in C_1 \cap C_2$, we have $x_1, x_2 \in C_\alpha, \alpha = 1, 2$. Then $\forall \lambda \in (0, 1)$, we have

$$x = \lambda x_1 + (1 - \lambda x_2) \in C\alpha, \alpha = 1, 2 \Rightarrow x \in C_1 \cap C_2$$

Therefore $C_1 \cap C_2$ is convex.

(ii) $C_1 \cap C_2$ is closed.

Next we show $\partial(C_1 \cap C_2) \subset (C_1 \cap C_2)$.

Let $x \in \partial(C_1 \cap C_2)$. Then every open ball $B_r(x)$ contains both a point in and another out of $C_1 \cap C_2$. If x is not in C_1 which is closed, then we can find an open ball containing no point in C_1, and therefore containg no point in $C_1 \cap C_2$. Contradiction! Therefore $x \in C_1$. Similarly $x \in C_2$. Together we have $x \in C_1 \cap C_2$. Therefore $\partial(C_1 \cap C_2) \subset (C_1 \cap C_2)$.

(iii) $C_1 \cup C_2$ is not necessarily convex.

For example, let $C_1 = [1, 2], C_2 = [3, 4]$. Then both C_1 and C_2 are closed convex sets, but $C_1 \cup C_2$ is not convex for this case.

11. Solution

(1) We have

$$\inf_{x\in C}|x-y| = |x_0-y|, \text{ with } x_0=(1,0),$$

then

$$a = x_0 - y = (-1,0), z = a^T y = -2.$$

By Theorem 3 of 1.8, $X=\{a^T x = z\}$ is a hyperplane that contains y and s.t. $C \subset \{x|a^T x > z\}$.

(2) $X = \{x_1 = -1\}$ is a sopporting hyperplane of C at y. Let

$$T = X \cap C = \{(x_1,x_2)|x_1=-1, 0\le x_2 \le 1\}$$

and $t^1 = (t_1^1, t_2^1) \in T$, where

$$t_1^1 = \inf\{t_1|(t_1,t_2)\in T\} = -1, t_2^1 = \inf\{t_2|(t_1,t_2)\in T\} = 0.$$

By Theorem 5 of 1.8, $t^1 = (-1,0)$ is an extreme point of C.

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