MAT 3210——Assignment 1 (on Chapt 1 of Lecture Notes)
No need to turn in

1. Use Gauss-Jordan row operation to solve

$$a.\ \begin{cases} x_1 \quad - x_3 = 1 \\ 3x_1 + x_2 + x_3 = 0 \\ -x_1 + x_2 + 2x_3 = 2 \end{cases} \qquad b.\ \begin{cases} x_1 + 2x_2 - x_3 = 1 \\ 2x_1 + x_2 + x_3 = 1 \\ x_1 - x_2 + 3x_3 = 1 \end{cases}$$

2. Let A be an $m \times n$ matrix and use $\vec{a_j}$ to denote the j- th column of A, i.e., $A = [\vec{a_1}, \cdots, \vec{a_n}]$. Let

$$\vec{x} = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} \text{ and } \vec{b} = \begin{pmatrix} b_1 \\ \vdots \\ b_n \end{pmatrix}$$

show that $A\vec{x} = \sum_{j=1}^{n} x_j \vec{a_j}$.

3. Find out all possible basic variables, the associated nonbasic variables and basic solutions. Determine if the basic solution is degenerate or nondegenerate.

$$a.\ \begin{cases} x_1 + 2x_2 - x_3 = 1 \\ x_1 - x_3 = 0 \end{cases}$$

$$b.\ \begin{cases} 3x_1 + 5x_2 + x_3 = 5 \\ x_1 - x_2 + x_4 = 0 \end{cases}$$

$$c.\ \begin{cases} x_1 \quad - x_4 = 1 \\ 3x_1 + x_2 + 2x_3 + x_4 = 0 \\ -x_1 + x_2 + 2x_3 + 2x_4 = 2 \end{cases}$$

$$d.\ \begin{cases} x_1 + x_2 = 2 \\ 2x_1 + 4x_2 + 3x_3 + 2x_4 = 3 \\ 3x_1 + 2x_2 + 5x_3 - x_4 = 1 \end{cases}$$

4. Show that the following two hyperplanes are parallel. Find the distance between them.

$$\begin{cases} x - 2y - z = 5 \\ 2x - 4y - 2z = 9 \end{cases}$$

5. Show that the set $Q = \{(x_1, x_2) | x_1 \geq 1, x_2 \geq 1, x_1 x_2 \leq 4\}$ is not convex.

6. Determine the extreme points of the convex set $Q = \{x_1, x_2 | x_1 + x_2 \leq 2, x_1 \geq 0, x_2 \geq 0\}$.

7. Graph the following linear constraints, mark the area that satisfies the constraints, and determine the extreme points of this convex set

$$a. \begin{cases} 6x_1 + 4x_2 \leq 24, \\ x_1 + 2x_2 \leq 6, \\ -x_1 + x_2 \leq 1, \\ x_2 \leq 2, \\ x_1, x_2 \geq 0. \end{cases}$$

$$b. \begin{cases} -2x_1 + 5x_2 \leq 10, \\ 2x_1 + x_2 \leq 6, \\ x_1 + 2x_2 \geq 2, \\ -x_1 + 3x_2 \leq 3. \end{cases}$$

8. Let $\mathbf{b} \geq 0$ and

$$C = \{\mathbf{x} | A\mathbf{x} \leq \mathbf{b}, \mathbf{x} \geq 0\}.$$

Show that the point $\mathbf{x} = 0$ is an extreme point of C.

9. Consider the triangle on the plane with vertices $x_1 = (2, 1), x_2 = (3, 4)$ and $x_3 = (4, 2)$.

(1) Show that any point inside the triangle is a convex combination of x_1, x_2 and x_3.

(2) Find out a supporting hyperplane of the triangle at each point x_1, x_2 and x_3.

10. Show that if C_1 and C_2 are closed convex sets, $C_1 \cap C_2$ is also a closed convex set. Is the set $C_1 \cup C_2$ also convex?

11. Consider the following convex set C:

$$C = \{(x_1, x_2) | x_1^2 + x_2^2 \leq 1, x_2 \leq 0\} \cup \{(x_1, x_2) | -1 \leq x_1 \leq 1, 0 \leq x_2 \leq 1\}.$$

(1) Let $y = (2, 0)$. Use Theorem 3 of 1.8 to find a hyperplane $X = \{\mathbf{x} | \mathbf{a}^T\mathbf{x} = z\}$ that contains y and such that $C \subset \{\mathbf{x} | \mathbf{a}^T\mathbf{x} > z\}$.

(2) Let $y = (-1, 0)$. Find a supporting hyperplane of C at y. Then use the method of Theorem 5 of 1.8 to find an extreme point of C.