Solution Keys to MAT3210 Assignment 2

1. Solution

   i. C is convex.
   $\forall u, v \in C$, $\forall \lambda \in (0,1)$

   $$\lambda u + (1-\lambda)v = [\lambda u_1 + (1-\lambda)v_1,\ \lambda u_2 + (1-\lambda)v_2].$$

   we have

   $$\begin{aligned} & [\lambda u_1 + (1-\lambda)v_1]^2 + [\lambda u_2 + (1-\lambda)v_2]^2 \\ = \ & \lambda^2(u_1^2+u_2^2) + (1-\lambda)^2(v_1^2+v_2^2) + 2\lambda(1-\lambda)(u_1v_1+u_2v_2) \\ \leq \ & \lambda^2(u_1^2+u_2^2) + (1-\lambda)^2(v_1^2+v_2^2) + 2\lambda(1-\lambda)\sqrt{(u_1^2+u_2^2)(v_1^2+v_2^2)} \\ \leq \ & \lambda^2 + (1-\lambda)^2 + 2\lambda(1-\lambda) = 1. \end{aligned}$$

   and

   $$\begin{aligned} & [\lambda u_1 + (1-\lambda)v_1] + [\lambda u_2 + (1-\lambda)v_2] \\ = \ & \lambda(u_1+u_2) + (1-\lambda)(v_1+v_2) \\ \geq \ & 0. \end{aligned}$$

   Therefore $\lambda u + (1-\lambda)v \in C$.

   ii. C is bounded from below, since

   $$x_1^2 + x_2^2 \leq 1 \ \Rightarrow\ x_1 \geq -\sqrt{1-x_2^2} \geq -1.$$

   so is x_2.

   iii. The extreme points of C $= \{(x_1, x_2) \mid x_1^2 + x_2^2 = 1, x_1 + x_2 \geq 0\}$.
   Let a=(r,s) be an extreme point of C. Then the tangent line of C at a is a supporting hyperplane at a: $\frac{y-s}{x-r} = -\frac{r}{s}$

2. Solution

   1. The extreme points are a, b, c, d where

   $$a = (\frac{1}{4},\ \frac{25}{4}),\ b = (\frac{6}{5},\ \frac{36}{5}),\ c = (6,\ 0),\ d = (4,\ 0).$$

   supporting hyperplanes at the extreme points can be chosen as follows

   $$a, b : x_1 - x_2 = 6;\ c, d : x_2 = 0$$

2. The standard form is

$$\begin{aligned} 3x_1 + 2x_2 + x_3 &= 18 \\ -x_1 + x_2 + x_4 &= 6 \\ 5x_1 + 3x_2 - x_5 &= 20 \\ x_i &\geq 0, \quad i = 1,2,3,4,5. \end{aligned}$$

where x_3, x_4, x_5 are slack variables.
The BFS correspond to the extreme point:

$$\begin{aligned} \text{a} &: \quad (\frac{1}{4}, \frac{25}{4}, \frac{19}{4}, 0, 0) \\ \text{b} &: \quad (\frac{6}{5}, \frac{36}{5}, 0, 0, \frac{38}{5}) \\ \text{c} &: \quad (6, 0, 0, 12, 10) \\ \text{d} &: \quad (4, 0, 6, 10, 0) \end{aligned}$$

3. Solution

The standard form is

$$\text{maximize} \quad z' = -10x_1 - 2x_2^+ + 2x_2^- + x_3$$

$$\begin{aligned} \text{subject to} \quad x_1 + x_2^+ - x_2^- + x_4 &= 30 \\ x_1 - x_3 - x_5 &= 1 \\ x_2^+ - x_2^- + x_3 + x_6 &= 3 \\ -x_2^+ + x_2^- - x_3 + x_7 &= 7 \\ x_1 + x_2^+ - x_2^- + x_3 &= 10 \\ x_1, x_2^+, x_2^-, x_3, x_4, x_5, x_6, x_7 &\geq 0 \end{aligned}$$

where

$$x_2^+ = \max(0, x_2), \; x_2^- = \max(0, -x_2).$$

4. Solution

1. Set $x_3 = 0$, we have $x_1 = \frac{5}{9}$ and $x_2 = \frac{11}{18}$. Since a_1,a_2 are linearly independent, the solution is a basic feasible solution.

2. Set $x_2 = 0$, we have $x_1 = \frac{13}{8}$ and $x_3 = -\frac{11}{8}$. The solution is infeasible.

3. Set $x_1 = 0$, we have $x_2 = \frac{13}{14}$ and $x_3 = \frac{5}{7}$. Since a_2, a_3 are linearly independent, the solution is a BFS.

5. Solution

1. Set $x_1 = x_2 = 0$, we have $x_3 = 2$, $x_4 = -1$. The solution is infeasible.

2. Set $x_1 = x_3 = 0$, we have $x_2 = \frac{1}{2}$, $x_4 = 0$. Since a_2, a_4 are linearly independent, the solution is a BFS.

3. Set $x_1 = x_4 = 0$, we have $x_2 = \frac{1}{2}$, $x_3 = 0$. We get the same BFS as that in (b).

4. Set $x_2 = x_3 = 0$, we have $x_1 = \frac{8}{3}$, $x_4 = -\frac{7}{3}$. The solution is infeasible.

5. Set $x_2 = x_4 = 0$, we have $x_1 = -2$, $x_3 = \frac{7}{2}$. The solution is infeasible.

6. Set $x_3 = x_4 = 0$, we have $x_1 = 0$, $x_2 = \frac{1}{2}$. We get the same BFS as that in (b).

To summary, $(0,\ \frac{1}{2},\ 0,\ 0)^T$ is the only BFS.

6. Solution

Substitute x_0 into the system of equations we find it is a FS. But x_0 is certainly not a BFS since it contains only one 0 element, while a BFS of the above LPP should contains at least two 0 elements. To find BFS from x_0, we first observe that $0a_1 + 2a_2 - a_3 = 0$, we have

$$\alpha_1 = 0,\ \alpha_2 = 2,\ \alpha_3 = -1, \alpha_4 = 0.$$

$$\frac{x_2}{\alpha_2} = 1 = \min\{\frac{x_j}{\alpha_j}\ :\ \alpha_j > 0,\ j = 1, 2, 3, 4\}$$

Hence

$$\begin{aligned} \hat{x}_1 &= x_1 - \alpha_1 \cdot \frac{x_2}{\alpha_2} = 3 \\ \hat{x}_2 &= 0 \\ \hat{x}_3 &= x_3 - \alpha_3 \cdot \frac{x_2}{\alpha_2} = 2 \\ \hat{x}_4 &= x_4 - \alpha_4 \cdot \frac{x_2}{\alpha_2} = 1 \end{aligned}$$

Therefore the new FS is $x_1 = (3,\ 0,\ 2,\ 1)^T$. Since $2a_1 - a_4 = 0$, x_1 is not basic. Do a similar procedure as above:

$$\beta_1 = 2,\ \beta_2 = 0,\ \beta_3 = 0, \beta_4 = -1.$$

$$\frac{\hat{x}_1}{\beta_1} = \frac{3}{2} = \min\{\frac{\hat{x}_j}{\beta_j}\ :\ \beta_j > 0,\ j = 1, 2, 3, 4\}$$

Hence

$$\begin{aligned} \bar{x}_1 &= 0 \\ \bar{x}_2 &= 0 \\ \bar{x}_3 &= \hat{x}_3 - \beta_3 \cdot \frac{\hat{x}_1}{\beta_1} = 2 \\ \bar{x}_4 &= \hat{x}_4 - \beta_4 \cdot \frac{\hat{x}_1}{\beta_1} = \frac{5}{2} \end{aligned}$$

Since a_3 and a_4 are linearly independent, x=(0,0,2,$\frac{5}{2}$)is a BFS.

7. Solution

(a) Just substitute the vector into the system of equations and we can see it is a FS.

(b) Since $\text{rank}[a_1,\ a_2,\ a_3,\ a_4] = 3$, the solution is not basic. Since

$$-a_1 - 5a_2 + 2a_3 - 4a_4 = 0$$

we have

$$\alpha = 1,\ \alpha_2 = -5,\ \alpha_3 = 2,\ \alpha_4 = -4$$

$$\frac{x_3}{\alpha_3} = \frac{1}{2} = \min\{\frac{x_j}{\alpha_j}\ :\ \alpha_j > 0,\ j = 1, 2, 3, 4\}$$

Hence

$$\begin{aligned}
\hat{x}_1 &= x_1 - x_3 \cdot \frac{\alpha_1}{\alpha_3} = \frac{5}{2} \\
\hat{x}_2 &= x_2 - x_3 \cdot \frac{\alpha_2}{\alpha_3} = \frac{7}{2} \\
\hat{x}_3 &= 0 \\
\hat{x}_4 &= x_4 - x_3 \cdot \frac{\alpha_4}{\alpha_3} = 5
\end{aligned}$$

Therefore the new FS is $x_1 = (\frac{5}{2},\ \frac{7}{2},\ 0,\ 5)^T$. Since $a_1,\ a_2,\ a_4$ are linearly independent, the solution is also a BFS.

8. Solution

After drawing the graphs of the linear constraints and the feasible region, you will find that a, b, c, d are the extreme points where

$$a = (0,\ 1),\ b = (\frac{4}{3},\ \frac{7}{3}),\ c = (3,\ \frac{3}{2}),\ d = (4,\ 0).$$

Now compute the values of z at the extreme points:

$$z(a) = -5,\ z(b) = -\frac{19}{3},\ z(c) = \frac{9}{2},\ z(d) = 16.$$

Therefore

$$\min(z) = -\frac{19}{3} \text{ at } x_1 = \frac{4}{3}, x_2 = \frac{7}{3}.$$

—END—

This modified solution was originally written by WANG Yuqiang.