Solution Keys to MAT3210 Assignment 1

1. Solution

Form the augmented matrices $[A|\ b]$.

(a)

$$\begin{pmatrix} 1 & 0 & -1 & 1 \\ 3 & 1 & 1 & 0 \\ -1 & 1 & 2 & 2 \end{pmatrix} \to \begin{pmatrix} 1 & 0 & -1 & 1 \\ 0 & 1 & 4 & -3 \\ 0 & 1 & 1 & 3 \end{pmatrix}$$

$$\to \begin{pmatrix} 1 & 0 & -1 & 1 \\ 0 & 1 & 4 & -3 \\ 0 & 0 & -3 & 6 \end{pmatrix} \to \begin{pmatrix} 1 & 0 & 1 & -1 \\ 0 & 1 & 0 & 5 \\ 0 & 0 & -3 & 6 \end{pmatrix} \to \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} -1 \\ 5 \\ -2 \end{pmatrix}$$

(b)

$$\begin{pmatrix} 1 & 2 & -1 & 1 \\ 2 & 1 & 1 & 1 \\ 1 & -1 & 3 & 1 \end{pmatrix} \to \begin{pmatrix} 1 & 2 & -1 & 1 \\ 0 & -3 & 3 & -1 \\ 0 & -3 & 4 & 0 \end{pmatrix}$$

$$\to \begin{pmatrix} 1 & 0 & 1 & \frac{1}{3} \\ 0 & -3 & 3 & -1 \\ 0 & 0 & 1 & 1 \end{pmatrix} \to \begin{pmatrix} 1 & 0 & 0 & -\frac{2}{3} \\ 0 & -3 & 0 & -4 \\ 0 & 0 & 1 & 1 \end{pmatrix} \to \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} -\frac{2}{3} \\ \frac{4}{3} \\ 1 \end{pmatrix}$$

2. Solution

Matrix multiplication gives

$$(A\vec{x})_k = \sum_{j=1}^{n} a_{kj}x_j.$$

Let k varies from 1 to m we have

$$A\vec{x} = \sum_{j=1}^{n} \vec{a}_j x_j.$$

3. Solution

(a) $A = \begin{pmatrix} 1 & 2 & -1 \\ 1 & 0 & -1 \end{pmatrix}$, $b = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$

(i) $B = \begin{pmatrix} 1 & 2 \\ 1 & 0 \end{pmatrix}$

basic variables: x_1, x_2
nonbasic variables: x_3
basic solution: $x = \begin{pmatrix} 0 & \frac{1}{2} & 0 \end{pmatrix}^T$ is degenerate.

(ii) $B = \begin{pmatrix} 2 & -1 \\ 0 & -1 \end{pmatrix}$

basic variables: x_2, x_3
nonbasic variables: x_1
basic solution: $x = \begin{pmatrix} 0 & \frac{1}{2} & 0 \end{pmatrix}^T$ is degenerate.

(b) $A = \begin{pmatrix} 3 & 5 & 1 & 0 \\ 1 & -1 & 0 & 1 \end{pmatrix}$, $b = \begin{pmatrix} 5 \\ 0 \end{pmatrix}$

(i) $B = \begin{pmatrix} 3 & 5 \\ 1 & -1 \end{pmatrix}$

basic variables: x_1, x_2
nonbasic variables: x_3, x_4
basic solution: $x = \begin{pmatrix} \frac{5}{8} & \frac{5}{8} & 0 & 0 \end{pmatrix}^T$ is non-degenerate.

(ii) $B = \begin{pmatrix} 3 & 1 \\ 1 & 0 \end{pmatrix}$

basic variables: x_1, x_3
nonbasic variables: x_2, x_4
basic solution: $x = \begin{pmatrix} 0 & 0 & 5 & 0 \end{pmatrix}^T$ is degenerate.

(iii) $B = \begin{pmatrix} 3 & 0 \\ 1 & 1 \end{pmatrix}$

basic variables: x_1, x_4
nonbasic variables: x_2, x_3
basic solution: $x = \begin{pmatrix} \frac{5}{3} & 0 & 0 & -\frac{5}{3} \end{pmatrix}^T$ is non-degenerate.

(iv) $B = \begin{pmatrix} 5 & 1 \\ -1 & 0 \end{pmatrix}$

basic variables: x_2, x_3
nonbasic variables: x_1, x_4
basic solution: $x = \begin{pmatrix} 0 & 0 & 5 & 0 \end{pmatrix}^T$ is degenerate.

(v) $B = \begin{pmatrix} 5 & 0 \\ -1 & -1 \end{pmatrix}$

basic variables: x_2, x_4
nonbasic variables: x_1, x_3
basic solution: $x = \begin{pmatrix} 0 & 1 & 0 & -1 \end{pmatrix}^T$ is non-degenerate.

(vi) $B = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

basic variables: x_3, x_4
nonbasic variables: x_1, x_2
basic solution: $x = \begin{pmatrix} 0 & 0 & 5 & 0 \end{pmatrix}^T$ is degenerate.

(c) $A = \begin{pmatrix} 1 & 0 & 0 & -1 \\ 3 & 1 & 2 & 1 \\ -1 & 1 & 2 & 2 \end{pmatrix}$, $b = \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix}$

(i) $B = \begin{pmatrix} 1 & 0 & -1 \\ 3 & 1 & 1 \\ -1 & 1 & 2 \end{pmatrix}$

basic variables: x_1, x_2, x_4
nonbasic variable: x_3
basic solution: $x = \begin{pmatrix} -1 & 5 & 0 & -2 \end{pmatrix}^T$ is non-degenerate.

(ii) $B = \begin{pmatrix} 1 & 0 & -1 \\ 3 & 2 & 1 \\ -1 & 2 & 2 \end{pmatrix}$

basic variables: x_1, x_3, x_4
nonbasic variable: x_2
basic solution: $x = \begin{pmatrix} -1 & 0 & \frac{5}{2} & -2 \end{pmatrix}^T$ is non-degenerate.

(d) $A = \begin{pmatrix} 1 & 2 & 0 & 0 \\ 2 & 4 & 3 & 2 \\ 3 & 2 & 5 & -1 \end{pmatrix}$, $b = \begin{pmatrix} 2 \\ 3 \\ 1 \end{pmatrix}$

(i) $B = \begin{pmatrix} 1 & 2 & 0 \\ 2 & 4 & 3 \\ 3 & 2 & 5 \end{pmatrix}$

basic variables: x_1, x_2, x_3
nonbasic variable: x_4
basic solution: $x = \begin{pmatrix} \frac{1}{3} & \frac{5}{6} & -\frac{1}{3} & 0 \end{pmatrix}^T$ is non-degenerate.

(ii) $B = \begin{pmatrix} 1 & 2 & 0 \\ 2 & 4 & 2 \\ 3 & 2 & -1 \end{pmatrix}$

basic variables: x_1, x_2, x_4
nonbasic variable: x_3
basic solution: $x = \begin{pmatrix} -\frac{3}{4} & \frac{11}{8} & 0 & -\frac{1}{2} \end{pmatrix}^T$ is non-degenerate.

(iii) $B = \begin{pmatrix} 1 & 0 & 0 \\ 2 & 3 & 2 \\ 3 & 5 & -1 \end{pmatrix}$

basic variables: x_1, x_3, x_4
nonbasic variable: x_2
basic solution: $x = \begin{pmatrix} 2 & 0 & -\frac{11}{13} & \frac{10}{13} \end{pmatrix}^T$ is non-degenerate.

(iv) $B = \begin{pmatrix} 2 & 0 & 0 \\ 4 & 3 & 2 \\ 2 & 5 & -1 \end{pmatrix}$
basic variables: x_2, x_3, x_4
nonbasic variable: x_1
basic solution: $x = \begin{pmatrix} 0 & 1 & -\frac{3}{13} & -\frac{2}{13} \end{pmatrix}^T$ is non-degenerate.

4. Solution
Rewrite the two equations as

$$\mathrm{c}_1^{\mathrm{T}}\mathrm{x} = 5, \ \mathrm{c}_2^{\mathrm{T}}\mathrm{x} = 9$$

where

$$\mathrm{c}_1 = \begin{pmatrix} 1 & -2 & -1 \end{pmatrix}^T, \ \mathrm{c}_2 = \begin{pmatrix} 2 & -4 & -2 \end{pmatrix}^T, \ \mathrm{x} = \begin{pmatrix} x & y & z \end{pmatrix}^T.$$

Since $\mathrm{c}_2 = 2\mathrm{c}_1$, they are two parallel vectors. Therefore the two hyperplanes are parallel.
As c_1 and c_2 are in the same direction, we may use the formula mentioned in class:

$$\left| \frac{z_1}{\|c_1\|} - \frac{z_2}{\|c_2\|} \right| = \frac{1}{2\sqrt{6}}$$

5. Solution

Consider two elements in Q

$$u = (1, 4), \ v = (4, 1).$$

Let $\lambda = \frac{1}{2}$, then

$$w = \lambda u + (1 - \lambda)v = (\frac{5}{2}, \frac{5}{2})$$

is obviously not a point in Q.
Therefore Q is not a convex set.

6. Solution
$(0, 0), (2, 0), (0, 2)$ are extreme points of Q.
Let $u, v \in Q, \ \lambda \in (0, 1)$.

(i) Let

$$(0,0) = (\lambda u_1 + (1-\lambda)v_1, \lambda u_2 + (1-\lambda)v_2).$$

Then

$$0 \leq u_1 = \frac{\lambda - 1}{\lambda} v_1 \leq 0 \Rightarrow u_1 = 0.$$

Similarly we have $u_2 = 0$, so as v_1, v_2.
Therefore $u = v = 0$, which means $(0,0)$ is an extreme point.

(ii) Let

$$(2,0) = (\lambda u_1 + (1-\lambda)v_1, \lambda u_2 + (1-\lambda)v_2).$$

Then

$$\lambda u_2 + (1-\lambda)v_2 = 0 \;\Rightarrow\; u_2 = v_2 = 0.$$

$$\lambda u_1 + \lambda v_1 = 2 \;\Rightarrow\; u_1 = v_1 = 2.$$

Therefore $u = v = (2,0)$, which means $(2,0)$ is an extreme point.
Similarly, we can show that $(0,2)$ is an extreme point.

There are no other extreme points of Q.

First, if x is an interior point of Q, then we can find an open ball $B_r(x)$ such that $B_r(x) \subset Q$.
Let

$$x_1 = x + (r,0), \;x_2 = x - (r,0).$$

Then

$$x_1, x_2 \in B_r(x) \subset Q.$$

and

$$x = \frac{1}{2}x_1 + \frac{1}{2}x_2.$$

Therefore x is not an extreme point of Q.

Second, if x is a boundary point other than the three extreme points, then we can use the same method as above to show that x is not an extreme point.

7. Solution

(a) The extreme points are $(0,0), (0,1), (1,2), (2,2), (3, \frac{3}{2}), (4,0)$(Fig. 1).

(b) The extreme points are $(0,1), (\frac{15}{7}, \frac{12}{7}, (\frac{10}{3}, -\frac{2}{3})$(Fig. 2).

Figure 1 and 2 are on the last page.

8. Solution

Clearly, $x = 0 \in C$.
Let $u, v \in C$ and

$$0 = \lambda u + (1-\lambda)v,\ 0 < \lambda < 1.$$

$$u, v \geq 0 \ \Rightarrow \ u_i, v_i \geq 0,\ \forall i.$$

Then

$$\begin{aligned} & 0 = \lambda u_i + (1-\lambda)v_i \\ \Rightarrow\ & u_i = v_i = 0,\ \forall i \\ \Rightarrow\ & u = v = 0 \end{aligned}$$

Therefore $x = 0$ is an extreme point of C.

9. Solution

(1) Consider any point x inside the triangle. Let y be the intersection of the line segment x_1x_2 and the extension of the line segment x_3y. Then

$$y = \lambda_1 x_1 + (1-\lambda_1)x_2$$

with $\lambda_1 = \frac{|x_1y|}{|x_1x_2|}$.

$$x = \lambda_2 x_3 + (1-\lambda_2)y$$

with $\lambda_2 = \frac{|x_3x|}{|x_3y|}$.
Then

$$x = \lambda_2 x_3 + (1-\lambda_2)\lambda_1 x_1 + (1-\lambda_2)x_2$$

Since

$$\lambda_2 + (1-\lambda_2)\lambda_1 + (1-\lambda_2)(1-\lambda_1) = 1$$

we conclude that x is a convex combination of x_1, x_2, x_3.

(2) The obvious supporting hyperplane at x_1 is the line through x_1 and which is parallel to the line segment x_2x_3. i.e.

$$2x + y = 5.$$

Similarly for x_2 and x_3 we have

$$x - 2y = -5,\ 3x - y = 10.$$

10. Solution

(i) $C_1 \cap C_2$ is convex.

For all $x_1, x_2 \in C_1 \cap C_2$, we have $x_1, x_2 \in C_\alpha$, $\alpha = 1, 2$. The for all $\lambda \in (0, 1)$ we have

$$\begin{aligned} & x = \lambda x_1 + (1 - \lambda x_2) \in C_\alpha,\ \alpha = 1, 2 \\ \Rightarrow\ & x \in C_1 \cap C_2. \end{aligned}$$

Therefore $C_1 \cap C_2$ is convex.

(ii) $C_1 \cap C_2$ is closed.

Want to show $\partial(C_1 \cap C_2) \subset (C_1 \cap C_2)$.

Let $x \in \partial(C_1 \cap C_2)$. Then every open ball $B_r(x)$ contains both a point in $C_1 \cap C_2$ and a point out of it. If x is not in C_1 which is closed, then we can find an open ball containing no point in C_1, and therefore containing no point in $C_1 \cap C_2$. Contradiction. Therefore $x \in C_1$. Similarly we can show that $x \in C_2$. Together we have $x \in C_1 \cap C_2$. Therefore we have shown $\partial(C_1 \cap C_2) \subset (C_1 \cap C_2)$.

$C_1 \cup C_2$ is not necessarily convex. For example, let $C_1 = [1, 2]$, $C_2 = [3, 4]$. Then both C_1 and C_2 are closed convex sets, but $C_1 \cup C_2$ is certainly not convex.

11. Solution

(1) We have

$$\inf_{x \in C} |x - y| = |x_0 - y|, \text{ with } x_0 = (1, 0).$$

then

$$a = x_0 - y = (-1, 0),\ z = a^T y = -2.$$

By Theorem 3 of 1.8, $X = \{a^T x = z\}$ is a hyperplane that contains y and such that $C \subset \{x | a^T x > z\}$.

(2) $X = \{x_1 = -1\}$ is a supporting hyperplane of C at y.

Let

$$T = X \cap C = \{(x_1, x_2) | x_1 = -1,\ 0 \le x_2 \le 1\}.$$

Let $t^1 = [t_1^1, t_2^1] \in T$, where

$$t_1^1 = \inf\{t_1 | (t_1, t_2) \in T\} = -1,\ t_2^1 = \inf\{t_2 | (t_1, t_2) \in T\} = 0.$$

By Theorem 5 of 1.8, $t^1 = (-1, 0)$ is an extreme point of C.

— END —

This solution was written by WANG Yuliang.

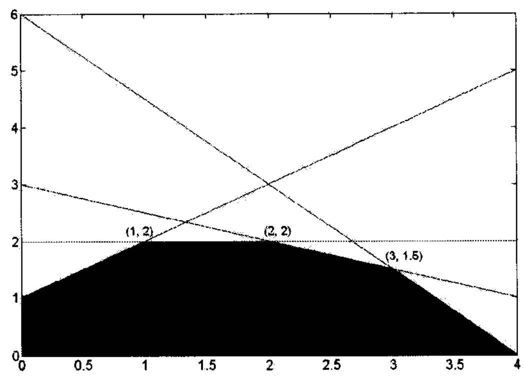

Figure 1: The shadowed area satisfies the constraints.

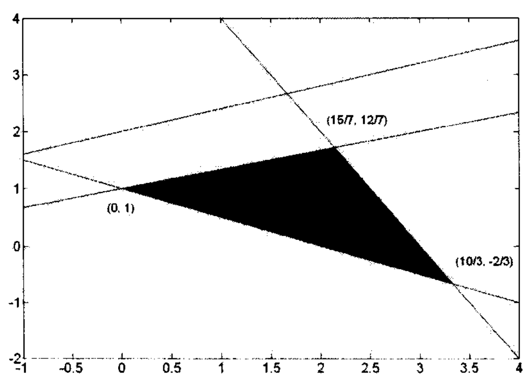

Figure 2: The shadowed area satisfies the constraints.