A NEUMANN PROBLEM WITH CRITICAL EXPONENT IN NON-CONVEX DOMAINS AND LIN-NI'S CONJECTURE

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Abstract. We consider the following nonlinear Neumann problem

$$\begin{cases} -\Delta u + \mu u = u^{\frac{N+2}{N-2}}, \quad u > 0 & \text{in } \Omega, \\ \frac{\partial u}{\partial n} = 0, & \text{on } \partial\Omega, \end{cases}$$

where $\Omega \subset \mathbb{R}^N$ is a smooth and bounded domain, $\mu > 0$ and n denotes the outward unit normal vector of $\partial\Omega$. Lin and Ni ([37]) conjectured that *for μ small, all solutions are constants*. We show that this conjecture is false for *all dimensions* in some (partially symmetric) non-convex domains Ω. Furthermore, we prove that for any fixed μ , there are *infinitely many positive solutions*, whose energy can be made arbitrarily large. This seems to be a *new phenomenon* for elliptic problems in bounded domains.

1. Introduction

In this paper, we consider the nonlinear elliptic Neumann problem

$$\begin{cases} -\Delta u + \mu u - u^q = 0, \quad u > 0 & \text{in} \quad \Omega, \\ \frac{\partial u}{\partial n} = 0, & \text{on} \quad \partial\Omega, \end{cases} \tag{1.1}$$

where $1 < q < +\infty, \mu > 0, n$ denotes the outward unit normal vector of $\partial\Omega$, and Ω is a smooth and bounded domain in $\mathbb{R}^N, N \geq 3$.

Equation (1.1) arises in many branches of the applied science. For example, it can be viewed as a steady-state equation for the shadow system of the Gierer-Meinhardt system in biology pattern formation [24], [43], or for parabolic equations in chemotaxis, e.g. Keller-Segel model [38].

When q is sub-critical, i.e. $q < \frac{N+2}{N-2}$, Lin, Ni and Takagi [38] proved that the only solution, for small μ, is the constant one, whereas nonconstant solutions appear for large μ [38] which blow up, as μ goes to infinity, at one or several points. The least energy solution blows up at a boundary point which maximizes the mean curvature of the boundary [45], [46]. Higher energy solutions exist which blow up at one or several points, located on the boundary [15], [27], [34], [55], [31], in the interior of the domain [8], [14], or some of them

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on the boundary and others in the interior [29]. (A good review can be found in [43].) In the critical case, for large μ, nonconstant solutions exist [1], [54]. As in the subcritical case the least energy solution blows up, as μ goes to infinity, at a unique point which maximizes the mean curvature of the boundary[3], [42]. Higher energy solutions have also been exhibited, blowing up at one [2], [55], [48], [26] or several separated boundary points[41], [37], [56], [57], [62]. For the study of interior blow-ups, we refer to [17], [20], [49], [53] and [63]. Some priori estimates for those solutions are given in [26], [32].

As we mentioned above that in the case of small μ, Lin, Ni and Takagi proved in the subcritical case that problem (1.1) admits only the trivial solution(i.e. $u \equiv \mu^{\frac{1}{p-1}}$). Based on this, Lin and Ni [37] asked:

Lin-Ni's Conjecture: *For μ small and $q = \frac{N+2}{N-2}$, problem (1.1) admits only the constant solution.*

The above conjecture was studied by Adimurthi-Yadava [4], [5] and Budd-Knapp-Peletier [11] in the case $\Omega = B_R(0)$ and u radial. Namely, they considered the following problem:

$$\begin{cases} \Delta u - \mu u + u^{\frac{N+2}{N-2}} = 0 \quad \text{in} \quad B_R(0), \quad u > 0 \quad \text{in} \quad B_R(0), \\ u \text{ is radial}, \quad \frac{\partial u}{\partial n} = 0 \quad \text{on} \quad \partial B_R(0). \end{cases} \tag{1.2}$$

The following results were proved:

Theorem A([4], [5], [6], [11]). *For μ sufficiently small,*
(1) if $N = 3$ or $N \geq 7$, problem (1.2) admits only the constant solution;
(2) if $N = 4, 5$ or 6, problem (1.2) admits a nonconstant solution.

Theorem A reveals that Lin-Ni's conjecture depends very sensitively on the dimension N. A natural question is: what about general dimensions? The proofs of Theorem A use radial symmetry to reduce the problem to an ODE boundary value problem. Consequently, they do not carry over to general domains. In the general three-dimensional domain case, M. Zhu [66] and Wei-Xu [65] proved:

Theorem B ([66] [65]): *The conjecture is true if $N = 3$ $(q = 5)$ and Ω is convex.*

In the case of $N = 5, q = \frac{7}{3}$, Rey and Wei [52] proved that for any smooth bounded domain Ω, problem(1.1) admits a solution, which blow up at K interior points for any

$K \in N^*$, if $\mu > 0$ is small. Therefore, (1.1) has arbitrary number of solutions as $\mu \to 0$. Thus Lin-Ni's conjecture is false in dimension five.

When $N \geq 7$, Druet, Robert and Wei [19] proved the following result:

Theorem C: *Suppose that $N \geq 7$ and $H(x) \neq 0$ for all $x \in \partial\Omega$. Assume that there exists $C > 0$ such that*

$$\int_\Omega u^{\frac{2N}{N-2}} \leq C. \tag{1.3}$$

Then for μ small, $u \equiv$ constant.

The purpose of this paper is to give a *negative answer* to Lin-Ni's conjecture in *all dimensions* for some *non-convex* domain Ω. More precisely, we assume that Ω is a smooth and bounded domain Ω satisfying the following conditions:

Let $y = (y', y'') \in \mathbb{R}^2 \times \mathbb{R}^{N-2}, r = |y'|$, then

- (H_1) $y \in \Omega$ *if and only if* $(y_1, y_2, y_3, \cdots, -y_i, \ldots, y_N) \in \Omega, \quad \forall \quad i = 3, \ldots, N$;
- (H_2) $(r\cos\theta, r\sin\theta, y'') \in \Omega$ *if* $(r, 0, y'') \in \Omega, \quad \forall\, \theta \in (0, 2\pi)$;
- (H_3) *Let* $T := \partial\Omega \cap \{y_3 = \cdots = y_N = 0\}$. *There exists a connected component* Γ *of* T, *such that* $H(x) \equiv \gamma < 0, \quad \forall x \in \Gamma$, *where* $H(x)$ *is the mean curvature of* $\partial\Omega$ *at* $x \in \partial\Omega$.

Note that by the assumption (H_2), Γ is a circle in the plane $y_3 = \cdots = y_N = 0$. Thus, we may assume that $\Gamma = \{y_1^2 + y_2^2 = r_0^2,\ y_3 = \cdots = y_N = 0\}$, where $r_0 > 0$ is a constant. Note also that for $x \in \gamma$, $H(x) = \frac{\sum_{j=1}^{N-1} k_j(x)}{N-1}$ where $k_j(x)$ are the principal curvatures and $k_1(x) = r_0$.

For instance, the domains in Figure 1 satisfy (H_1), (H_2) and (H_3). Note that Ω can be simply connected.

Another example is the annulus: $\Omega = \{a < |x| < b\}$ with $0 < a < b < +\infty$.

For normalization reason, we consider throughout the paper the equation

$$\begin{cases} -\Delta u + \mu u - \alpha_N u^{\frac{N+2}{N-2}} = 0, \quad u > 0 & \text{in} \quad \Omega, \\ \frac{\partial u}{\partial n} = 0, & \text{on} \quad \partial\Omega, \end{cases} \tag{1.4}$$

where $\alpha_N = N(N-2)$. The solutions are identical up to the multiplicative constant $(\alpha_N)^{-\frac{N-2}{4}}$.

Our main result in this paper can be stated as follows:

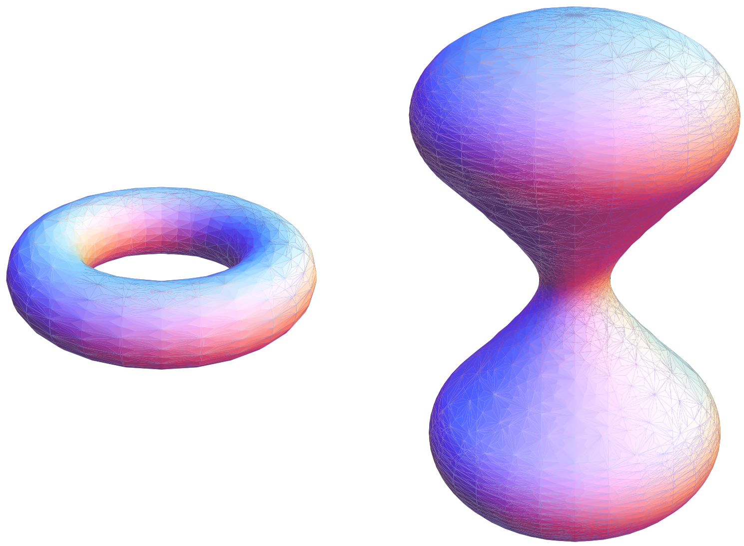

Figure 1

Theorem 1.1. *Suppose that $N \geq 3$ and Ω is a smooth and bounded domain satisfying (H_1), (H_2) and (H_3). Let μ be any fixed positive number. Then problem (1.4) has infinitely many positive solutions, whose energy can be made arbitrarily large.*

We can make $r_0 = 1$ by a suitable change of variable, where r_0 is the radius of the circle in (H_3).

The constant μ in (1.4) is fixed. We obtain *infinitely many positive solutions*. This is a *new phenomenon*. For subcritical problems, by a compactness result of Gidas-Spruck [21], the energy of positive solutions remains uniformly bounded. So this kind of phenomena can only happen for critical exponent problems. On the other hand, the existence of *infinitely many sign-changing radial* solutions for another critical exponent problem with Dirichlet boundary condition has been studied by Cerami-Solimini-Struwe [13] for $N \geq 7$.

Similar phenomenon occurs in the prescribed scalar curvature problem [64]. It is interesting to compare the results in this paper and [64] with recent work of S. Brendle on the non-compactness of Yamabe problem. Consider the Yamabe problem on S^N, which can be reduced the following problem in $\mathbb{R}^N$:

$$\frac{4(N-1)}{N-2}\Delta_g u - R_g u + c u^{\frac{N+2}{N-2}} = 0 \ \text{ in } \mathbb{R}^N \tag{1.5}$$

where Δ_g is the Laplace operator with respect to g, R_g denotes the scalar curvature of g, and the constant c is the scalar curvature of the new metric $u^{\frac{4}{N-2}}g$. R. Schoen conjectured all solutions to (1.5) are compact. This conjecture is proved to be true in dimensions less than 24. See [18], [33], [35], [36] and [39]. In [10], S. Brendle constructed a metric g in dimension $N \geq 52$, with the following properties: (i) $g_{ij} = \delta_{ij}$ for $|x| \geq \frac{1}{2}$, (ii) g is not conformally flat. Then, for this metric, there exists a sequence of positive smooth solutions u_n to (1.5) such that $\sup_{|x|\leq 1} u_n(x) \to +\infty$, and u_n develops exactly one singularity. This disapproves Schoen's conjecture in dimensions $N \geq 52$. On one hand, both problems (1.5) and (1.4) have no parameters but possess infinitely many positive solutions. The proofs are similar: a kind of variational reduction method (we call it *localized energy method*) is used. On the other hand, the solutions constructed by Brendle has a single bubble near the origin, and the energy of the solutions remains uniformly bounded. Here we obtain solutions with *arbitrarily many bubbles*, and the energy of the solutions can be *arbitrarily large*.

We believe that the symmetric condition in Theorem 1.1 is technical. A more general result, as follows, should be true.

Conjecture: *Assume that* $\min_{x\in\partial\Omega} H(x) < 0$ *and that the set* $\{x \in \partial\Omega | H(x) = \min_{x\in\partial\Omega} H(x)\}$ *is a smooth* l*-dimensional sub-manifold on* $\partial\Omega$*, with* $1 \leq l \leq N-1$*. Then there are infinitely many positive solutions to (1.4).*

Recently, we are able to prove that there are convex domains, such that problem (1.2) has infinitely many solutions if $N \geq 4$. Thus, the Lin-Ni's conjecture is false even in a convex domain if $N \geq 4$. By the result of [66, 65], the condition $N \geq 4$ is necessary. The energy of these solutions is unbounded as $\mu \to 0$, which is consistent to the result in [19].

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2. Outline of Proofs

We outline the main idea in the proof of Theorem 1.1.

It is well-known that the functions

$$U_{\lambda,a}(y)=\left(\frac{\lambda}{1+\lambda^2|y-a|^2}\right)^{\frac{N-2}{2}},\ \lambda>0,\ a\in\mathbb{R}^N$$

are the only solutions to the problem

$$-\Delta u=\alpha_N u^{\frac{N+2}{N-2}},\qquad u>0,\quad \text{in}\quad \mathbb{R}^N.$$

Let us fix a positive integer

$$k\ge k_0,$$

where k_0 is a large positive integer which is to be determined later.

Integral estimates (see Appendix A) suggest to make the additional a priori assumption that λ behaves as the following

$$\begin{cases}\lambda=\frac{1}{\Lambda}k^{\frac{N-2}{N-3}} & \text{if}\quad N\ge 4,\\ \lambda=\frac{1}{\Lambda}e^{-\frac{D_3}{D_2\gamma}\beta_k k\ln k}, & \text{if}\quad N=3,\end{cases}$$

where $\delta\le\Lambda\le\frac{1}{\delta}, D_2, D_3$ are some positive constants in Proposition A.4, δ is a small positive constant which is to be determined later, and β_k is the quantity in Proposition A.4 satisfying $\beta_k\to 1$ as $k\to+\infty$.

Fix $a\in\Gamma\subset\partial\Omega$. We introduce a boundary deformation which strengthens the boundary near a. After rotation and translation of the coordinate system, we may assume that $a=0$ and the inward normal to Γ at a is the positive x_N-axis. Denote $x'=(x_1,\dots,x_{N-1})$, and $B(a,\delta)=\{x\in\mathbb{R}^N:|x-a|<\delta'\}$. Then, we can find a constant $\delta'>0$ such that $\Gamma\cap B(a,\delta')$ can be represented by the graph of a smooth function $\rho_a(x')=\frac{1}{2}\sum\limits_{i=1}^{N-1}k_i x_i^2+O(|x'|^3)$, and

$$\Omega\cap B(a,\delta')=\{(x',x_N)\in B(a,\delta'):x_N>\rho_a(x')\}. \tag{2.1}$$

Here k_i, $i=1,\dots,N-1$ are the principal curvatures at a. Furthermore, the average of the principal curvatures of Γ at a is the mean curvature $H(a)=\frac{1}{N-1}\sum_{i=1}^{N-1}k_i\equiv\gamma$ because of (H_3). To avoid clumsy notations we drop the index a in ρ.

On $\Gamma\cap B(a,\delta')$, the outward normal vector $n(x)$ is

$$n(x)=\frac{1}{\sqrt{1+|\nabla'\rho|^2}}(\nabla'\rho,-1).$$

Let $2^*=\frac{2N}{N-2}$. Using the transformation $u(y)\mapsto\varepsilon^{-\frac{N-2}{2}}u\big(\frac{y}{\varepsilon}\big)$, we find that (1.4) becomes

$$(2.2)\qquad \begin{cases} -\Delta u + \mu\varepsilon^2 u = \alpha_N u^{2^*-1}, u > 0, & \text{in} \quad \Omega_\varepsilon, \\ \frac{\partial u}{\partial n} = 0, & \text{on} \quad \partial\Omega_\varepsilon, \end{cases}$$

where

$$(2.3)\qquad \begin{cases} \varepsilon = k^{-\frac{N-2}{N-3}} & \text{if } N \geq 4, \\ \varepsilon = e^{\frac{D_3}{D_2\gamma}\beta_k k \ln k} & \text{if } N = 3 \end{cases}$$

and $\Omega_\varepsilon = \{y | \varepsilon y \in \Omega\}$.

Define

$$H_s = \big\{ u : u \in H^1(\Omega_\varepsilon), u \text{ is even in } y_h, h = 2, \cdots, N,$$
$$u(r\cos\theta, r\sin\theta, y'') = u(r\cos(\theta + \frac{2\pi j}{k}), r\sin(\theta + \frac{2\pi j}{k}), y''), j = 1, \dots, k-1\big\},$$

and

$$x_j = \big(\frac{1}{\varepsilon}\cos\frac{2(j-1)\pi}{k}, \frac{1}{\varepsilon}\sin\frac{2(j-1)\pi}{k}, 0\big), \quad j = 1, \cdots, k,$$

where 0 is the zero vector in $\mathbb{R}^{N-2}$.

We define W_{Λ,x_j} to be the unique solution of

$$(2.4)\qquad \begin{cases} -\Delta u + \mu\varepsilon^2 u = \alpha_N U_{\frac{1}{\Lambda},x_j}^{2^*-1} & \text{in} \quad \Omega_\varepsilon, \\ \frac{\partial u}{\partial n} = 0 & \text{on} \quad \partial\Omega_\varepsilon. \end{cases}$$

Let

$$W(y) = \sum_{j=1}^{k} W_{\Lambda,x_j}.$$

Theorem 1.1 is a direct consequence of the following result:

Theorem 2.1. *Suppose that $N \geq 3$ and Ω is a smooth and bounded domain satisfying (H_1), (H_2) and (H_3). Then there is an integer $k_0 > 0$, such that for any integer $k \geq k_0$,* (2.2) *has a solution u_k of the form*

$$u_k = W(y) + \omega_k,$$

where $\omega_k \in H_s$, and as $k \to +\infty$, $\|\omega_k\|_{L^\infty} \to 0$.

We will use the techniques in the singularly perturbed elliptic problems to prove Theorem 2.1. In all the singularly perturbed problems, some small parameters are present either in the operator or in the nonlinearity or in the boundary condition. Here there is *no parameter*. Instead, we use k, *the number of the bubbles* of the solutions, as the parameter in the construction of bubble solutions for (1.4). This idea is motivated by the recent paper [64], where infinitely many solutions to a prescribed scalar curvature problem were constructed. The difference is that now the location of bubbles is fixed.

The main difficulty in constructing solution with k-bubbles is that we need to obtain a better control of the error terms. Since the number of the bubbles is large, it is very difficult to carry out the reduction procedure by using the standard norm. Noting that the maximum norm will not be affected by the number of the bubbles, we will carry out the reduction procedure in a space with weighted maximum norm. Similar weighted maximum norm has been used in [41],[50]–[52], [64]. But the estimates in the reduction procedure in this paper are much more complicated than those in [41],[50]–[52], because the number of the bubbles is large.

3. Finite-dimensional Reduction

In this section, we perform a finite-dimensional reduction.

Let

$$\|u\|_* = \sup_y \Big(\sum_{j=1}^k \frac{1}{(1+|y-x_j|)^{\frac{N-2}{2}+\tau}} \Big)^{-1} |u(y)|, \tag{3.1}$$

and

$$\|f\|_{**} = \sup_y \Big(\sum_{j=1}^k \frac{1}{(1+|y-x_j|)^{\frac{N+2}{2}+\tau}} \Big)^{-1} |f(y)|, \tag{3.2}$$

where we choose

$$\tau = \frac{N-3}{N-2}. \tag{3.3}$$

For this choice of τ, we have

$$\sum_{j=2}^{k} \frac{1}{|x_j - x_1|^\tau} \le C, \quad \text{if } N \ge 4. \tag{3.4}$$

Let

$$Y_i = \frac{\partial W_{\Lambda,x_i}}{\partial \Lambda}, \qquad Z_i = -\Delta Y_i + \varepsilon^2 \mu Y_i = (2^* - 1) U_{\frac{1}{\Lambda},x_i}^{2^*-2} \frac{\partial U_{\frac{1}{\Lambda},x_i}}{\partial \Lambda}.$$

We consider

$$\begin{cases} -\Delta \phi_k + \mu\varepsilon^2 \phi_k - N(N+2) W^{2^*-2} \phi_k = h + c_1 \sum\limits_{i=1}^{k} Z_i, & \text{in } \Omega_\varepsilon, \\ \frac{\partial \phi_k}{\partial n} = 0, & \text{on } \partial\Omega_\varepsilon, \\ \phi_k \in H_s, \\ < \sum\limits_{i=1}^{k} Z_i, \phi_k >= 0 \end{cases} \tag{3.5}$$

for some number c_1, where $< u, v >= \int_{\Omega_\varepsilon} uv$.

Let us remark that in general we should also include the translational derivatives of W in the right hand side of (3.5). However due to the symmetry assumption $\phi \in H_s$, this part of kernel automatically disappears. This is the main reason for imposing the symmetries.

We recall the following result, whose proof is given in [52].

Lemma 3.1. *Let f satisfy $\|f\|_{**} < \infty$ and let u be the solution of*

$$-\Delta u + \mu\varepsilon^2 u = f \quad in \quad \Omega_\varepsilon, \qquad \frac{\partial u}{\partial n} = 0 \quad on \quad \partial\Omega_\varepsilon.$$

Then we have

$$|u(x)| \le C \int_{\Omega_\varepsilon} \frac{|f(y)|}{|x-y|^{N-2}} dy.$$

Next, we need the following lemma to carry out the reduction.

Lemma 3.2. *Assume that ϕ_k solves (3.5) for $h = h_k$. If $\|h_k\|_{**}$ goes to zero as k goes to infinity, so does $\|\phi_k\|_*$.*

Proof. We argue by contradiction. Suppose that there are $k \to +\infty$, $h = h_k$, $\Lambda_k \in [\delta, \delta^{-1}]$, and ϕ_k solving (3.5) for $h = h_k$, $\Lambda = \Lambda_k$, with $\|h_k\|_{**} \to 0$, and $\|\phi_k\|_* \ge c' > 0$. We may assume that $\|\phi_k\|_* = 1$. For simplicity, we drop the subscript k.

According to Lemma 3.1, we have

$$
\begin{aligned}
|\phi(y)| \le & C\int_{\Omega_\varepsilon} \frac{1}{|z-y|^{N-2}} W^{2^*-2}|\phi(z)|\,dz \\
& + C\int_{\Omega_\varepsilon} \frac{1}{|z-y|^{N-2}}\big(|h(z)| + |c_1\sum_{i=1}^k Z_i(z)|\big)\,dz
\end{aligned}
\tag{3.6}
$$

Using Lemma B.4, there is a strictly positive number θ such that

$$
\begin{aligned}
& \Big|\int_{\Omega_\varepsilon} \frac{1}{|z-y|^{N-2}} W^{2^*-2}\phi(z)\,dz\Big| \\
\le & C\|\phi\|_*\Big(\sum_{j=1}^k \frac{1}{(1+|y-x_j|)^{\frac{N-2}{2}+\tau+\theta}} + o(1)\sum_{j=1}^k \frac{1}{(1+|y-x_j|)^{\frac{N-2}{2}+\tau}}\Big).
\end{aligned}
\tag{3.7}
$$

It follows from Lemma B.3 that

$$
\begin{aligned}
& \Big|\int_{\Omega_\varepsilon} \frac{1}{|z-y|^{N-2}} h(z)\,dz\Big| \\
\le & C\|h\|_{**}\int_{\mathbb{R}^N} \frac{1}{|z-y|^{N-2}}\sum_{j=1}^k \frac{1}{(1+|z-x_j|)^{\frac{N+2}{2}+\tau}}\,dz \\
\le & C\|h\|_{**}\sum_{j=1}^k \frac{1}{(1+|y-x_j|)^{\frac{N-2}{2}+\tau}},
\end{aligned}
\tag{3.8}
$$

and

$$
\begin{aligned}
& \Big|\int_{\Omega_\varepsilon} \frac{1}{|z-y|^{N-2}}\sum_{i=1}^k Z_i(z)\,dz\Big| \\
\le & C\sum_{i=1}^k \int_{\mathbb{R}^N} \frac{1}{|z-y|^{N-2}} \frac{1}{(1+|z-x_i|)^{N+2}}\,dz \\
\le & C\sum_{i=1}^k \frac{1}{(1+|y-x_i|)^{\frac{N-2}{2}+\tau}}.
\end{aligned}
\tag{3.9}
$$

Next, we estimate c_1. Multiplying (3.5) by Y_1 and integrating, we see that c_1 satisfies

$$
\Big\langle\sum_{i=1}^k Z_i, Y_1\Big\rangle c_1 = \Big\langle -\Delta\phi + \mu\varepsilon^2\phi - N(N+2)W^{2^*-2}\phi, Y_1\Big\rangle - \Big\langle h, Y_1\Big\rangle.
\tag{3.10}
$$

It follows from Lemma B.2 that

$$\begin{aligned}\big|\big\langle h, Y_1\big\rangle\big| \le & C\|h\|_{**}\int_{\mathbb{R}^N}\frac{1}{(1+|z-x_1|)^{N-2}}\sum_{j=1}^{k}\frac{1}{(1+|z-x_j|)^{\frac{N+2}{2}+\tau}}\,dz\\ \le & C\|h\|_{**}.\end{aligned}$$

On the other hand,

$$\begin{aligned}&\big\langle -\Delta\phi+\mu\varepsilon^2\phi-N(N+2)W^{2^*-2}\phi, Y_1\big\rangle\\ =&\big\langle -\Delta Y_1+\mu\varepsilon^2 Y_1-N(N+2)W^{2^*-2}Y_1,\phi\big\rangle\\ =&N(N+2)\big\langle U^{2^*-2}_{\frac{1}{\Lambda},x_1}\partial_\Lambda U_{\frac{1}{\Lambda},x_j}-W^{2^*-2}Y_1,\phi\big\rangle.\end{aligned}\tag{3.11}$$

By Lemmas B.1,

$$|\phi(y)|\le C\|\phi\|_*.$$

On the other hand, it follows from Lemma A.1,

$$|\varphi_{\Lambda,x_i}(y)|\le\frac{C\varepsilon|\ln\varepsilon|}{(1+|y-x_i|)^{N-3}}\le\frac{C\varepsilon^\sigma|\ln\varepsilon|}{(1+|y-x_i|)^{N-2-\sigma}},$$

since $\varepsilon\le\frac{C}{1+|y-x_i|}$.

We consider the cases $N\ge6$ first. Note that $\frac{4}{N-2}\le1$ for $N\ge6$. Using Lemmas B.2, A.1 and A.2, we obtain

$$\begin{aligned}&\Big|\big\langle U^{2^*-2}_{\frac{1}{\Lambda},x_1}\partial_\Lambda U_{\frac{1}{\Lambda},x_j}-W^{2^*-2}Y_1,\phi\big\rangle\Big|\\ \le&C\|\phi\|_*\int_{\Omega_\varepsilon}\frac{1}{(1+|z-x_1|)^{(N-2)(1-\beta)}}\sum_{i=2}^{k}\frac{1}{(1+|z-x_i|)^{4(1-\beta)}}\,dz\\ &+\|\phi\|_*\int_{\Omega_\varepsilon}\big(U^{2^*-2}_{\frac{1}{\Lambda},x_1}|\partial_\Lambda\varphi_{\Lambda,x_1}|+|Y_1||\varphi_{\Lambda,x_j}|^{2^*-2}\big)\\ \le&C\|\phi\|_*\sum_{j=2}^{k}\frac{1}{|x_1-x_j|^{1+\sigma}}+o(1)\|\phi\|_*=o(1)\|\phi\|_*.\end{aligned}\tag{3.12}$$

For $N=3,4,5$, we have $\frac{4}{N-2}>1$. By Lemmas B.1, B.2, A.1 and A.2,

$$
\begin{aligned}
&\Big|\big\langle U^{2^*-2}_{\frac{1}{\Lambda},x_1}\partial_\Lambda U_{\frac{1}{\Lambda},x_j}-W^{2^*-2}Y_1,\phi\big\rangle\Big|\\
\le& C\int_{\Omega_\varepsilon}U^{2^*-3}_{\frac{1}{\Lambda},x_1}\sum_{j=2}^k U_{\frac{1}{\Lambda},x_j}|Y_1\phi|+C\int_{\Omega_\varepsilon}\big(\sum_{j=2}^k U_{\frac{1}{\Lambda},x_j}\big)^{\frac{4}{N-2}}|Y_1\phi|\\
&+\int_{\Omega_\varepsilon}\Big(U^{2^*-2}_{\frac{1}{\Lambda},x_1}|\partial_\Lambda\varphi_{\Lambda,x_1}|+U^{2^*-3}_{\frac{1}{\Lambda},x_1}|\varphi_{\Lambda,x_1}||Y_1|+|\varphi_{\Lambda,x_1}|^{2^*-2}|Y_1|\Big)|\phi|\\
\le& C\|\phi\|_*\int_{\Omega_\varepsilon}\frac{1}{(1+|z-x_1|)^{4(1-\beta)}}\sum_{j=2}^k\frac{1}{(1+|z-x_j|)^{N-2}} \qquad (3.13)\\
&+C\int_{\Omega_\varepsilon}\big(\sum_{j=2}^k U_{\frac{1}{\Lambda},x_j}\big)^{\frac{4}{N-2}}|Y_1\phi|+o(1)\|\phi\|_*\\
\le& C\|\phi\|_*\int_{\Omega_\varepsilon}\frac{1}{(1+|z-x_1|)^{(N-2)(1-\beta)}}\big(\sum_{j=2}^k U_{\frac{1}{\Lambda},x_j}\big)^{\frac{4}{N-2}}\sum_{i=1}^k\frac{1}{(1+|y-x_i|)^{\frac{N-2}{2}+\tau}}\\
&+o(1)\|\phi\|_*.
\end{aligned}
$$

Let

$$
\Omega_j=\big\{y=(y',y'')\in\Omega_\varepsilon:\ \big\langle\frac{y'}{|y'|},\frac{x_j}{|x_j|}\big\rangle\ge\cos\frac{\pi}{k}\big\}.
$$

If $y\in\Omega_1$, then

$$
\begin{aligned}
\sum_{j=2}^k U_{\frac{1}{\Lambda},x_j}\le&\frac{1}{(1+|y-x_1|)^{N-2-\tau-\theta}}\sum_{j=2}^k\frac{1}{|x_j-x_1|^{\tau+\theta}}\\
=&o(1)\frac{1}{(1+|y-x_1|)^{N-2-\tau-\theta}},
\end{aligned}
$$

and

$$
\sum_{i=1}^k\frac{1}{(1+|y-x_i|)^{\frac{N-2}{2}+\tau}}\le\frac{C}{(1+|y-x_1|)^{\frac{N-2}{2}}}.
$$

So, we obtain

$$
\int_{\Omega_1}\frac{1}{(1+|z-x_1|)^{(N-2)(1-\beta)}}\big(\sum_{j=2}^k U_{\frac{1}{\Lambda},x_j}\big)^{\frac{4}{N-2}}\sum_{i=1}^k\frac{1}{(1+|y-x_i|)^{\frac{N-2}{2}+\tau}}=o(1).
$$

If $y \in \Omega_l$, $l \geq 2$, then

$$\sum_{j=2}^{k} U_{\frac{1}{\Lambda},x_j} \leq \frac{C}{(1+|y-x_l|)^{N-2-\tau}},$$

and

$$\sum_{i=1}^{k} \frac{1}{(1+|y-x_i|)^{\frac{N-2}{2}+\tau}} \leq \frac{C}{(1+|y-x_l|)^{\frac{N-2}{2}}}.$$

As a result,

$$\begin{aligned}
&\int_{\Omega_l} \frac{1}{(1+|z-x_1|)^{(N-2)(1-\beta)}} \Big(\sum_{j=2}^{k} U_{\frac{1}{\Lambda},x_j}\Big)^{\frac{4}{N-2}} \sum_{i=1}^{k} \frac{1}{(1+|y-x_i|)^{\frac{N-2}{2}+\tau}} \\
\leq& C\int_{\Omega_l} \frac{1}{(1+|z-x_1|)^{(N-2)(1-\beta)}} \frac{1}{(1+|y-x_l|)^{4-\frac{4\tau}{N-2}+\frac{N-2}{2}}} \\
\leq& \frac{C}{|x_l-x_1|^{\frac{N+2}{2}-\frac{4\tau}{N-2}-\theta}},
\end{aligned}$$

where $\theta > 0$ is a fixed small constant.

Noting that for $\theta > 0$ small, $\frac{N+2}{2} - \frac{4\tau}{N-2} - \theta > \tau$. Thus

$$\begin{aligned}
&\int_{\Omega_\varepsilon} \frac{1}{(1+|z-x_1|)^{(N-2)(1-\beta)}} \Big(\sum_{j=2}^{k} U_{\frac{1}{\Lambda},x_j}\Big)^{\frac{4}{N-2}} \sum_{i=1}^{k} \frac{1}{(1+|y-x_i|)^{\frac{N-2}{2}+\tau}} \\
\leq& o(1) + C\sum_{l=2}^{k} \frac{1}{|x_l-x_1|^{\frac{N+2}{2}-\frac{4\tau}{N-2}-\theta}} = o(1).
\end{aligned}$$

So, we have proved

$$\Big|\Big\langle U_{\frac{1}{\Lambda},x_1}^{2^*-2}\partial_\Lambda U_{\frac{1}{\Lambda},x_j} - W^{2^*-2}Y_1, \phi\Big\rangle\Big| = o(1)\|\phi\|_*.$$

But there is a constant $\bar{c} > 0$,

$$\Big\langle \sum_{i=1}^{k} Z_i, Y_1\Big\rangle = \bar{c} + o(1).$$

Thus we obtain that

$$c_1 = o(\|\phi\|_*) + O(\|h\|_{**}).$$

So,

$$\|\phi\|_* \le \Big(o(1) + \|h_k\|_{**} + \frac{\sum\limits_{j=1}^{k} \frac{1}{(1+|y-x_j|)^{\frac{N-2}{2}+\tau+\theta}}}{\sum\limits_{j=1}^{k} \frac{1}{(1+|y-x_j|)^{\frac{N-2}{2}+\tau}}}\Big). \tag{3.14}$$

Since $\|\phi\|_* = 1$, we obtain from (3.14) that there is $R > 0$, such that

$$\|\phi(y)\|_{B_R(x_i)} \ge c_0 > 0, \tag{3.15}$$

for some i. But $\bar{\phi}(y) = \phi(y - x_i)$ converges uniformly in any compact set of $\mathbb{R}^N_+$ to a solution u of

$$\Delta u + N(N+2) U_{\frac{1}{\Lambda},0}^{2^*-2} u = 0 \tag{3.16}$$

for some $\Lambda \in [\delta, \delta^{-1}]$, and u is perpendicular to the kernel of (3.16). So, $u = 0$. This is a contradiction to (3.15).

□

From Lemma 3.2, using the same argument as in the proof of Proposition 4.1 in [41], Proposition 3.1 in [52], we can prove the following result :

Proposition 3.3. *There exists $k_0 > 0$ and a constant $C > 0$, independent of k, such that for all $k \ge k_0$ and all $h \in L^\infty(\Omega_\varepsilon)$, problem (3.5) has a unique solution $\phi \equiv L_k(h)$. Besides,*

$$\|L_k(h)\|_* \le C\|h\|_{**}, \qquad |c_1| \le C\|h\|_{**}. \tag{3.17}$$

Moreover, the map $L_k(h)$ is C^1 with respect to Λ.

Now, we consider

$$\begin{cases} -\Delta\big(W + \phi\big) + \mu\varepsilon^2 (W + \phi) = \alpha_N \big(W + \phi\big)^{2^*-1} + c_1 \sum\limits_{i=1}^{k} Z_i, & \text{in } \Omega_\varepsilon, \\ \frac{\partial \phi}{\partial n} = 0, & \text{on } \partial\Omega_\varepsilon, \\ \phi \in H_s, \\ < \sum\limits_{i=1}^{k} Z_i, \phi >= 0. \end{cases} \tag{3.18}$$

We have

Proposition 3.4. *There is an integer $k_0 > 0$, such that for each $k \geq k_0$, $\delta \leq \Lambda \leq \delta^{-1}$, where δ is a fixed small constant,* (3.18) *has a unique solution ϕ, satisfying*

$$\|\phi\|_* \leq C\varepsilon^{\frac{1}{2}+\sigma},$$

where $\sigma > 0$ is a fixed small constant. Moreover, $\Lambda \to \phi(\Lambda)$ is C^1.

Rewrite (3.18) as

$$\begin{cases} -\Delta\phi + \mu\varepsilon^2\phi - N(N+2)W^{2^*-2}\phi = N(\phi) + l_k + c_1\sum\limits_{i=1}^{k} Z_i, & \text{in } \Omega_\varepsilon, \\ \frac{\partial\phi}{\partial n} = 0, & \text{on } \partial\Omega_\varepsilon, \\ \phi \in H_s, \\ < \sum\limits_{i=1}^{k} Z_i, \phi >= 0 \end{cases} \tag{3.19}$$

where

$$N(\phi) = \alpha_N\Big(\big(W+\phi\big)^{2^*-1} - W^{2^*-1} - (2^*-1)W^{2^*-2}\phi\Big),$$

and

$$l_k = \alpha_N\Big(W^{2^*-1} - \sum_{j=1}^{k} U^{2^*-1}_{\frac{1}{\Lambda},x_j}\Big).$$

In order to use the contraction mapping theorem to prove that (3.19) is uniquely solvable in the set that $\|\phi\|_*$ is small, we need to estimate $N(\phi)$ and l_k.

In the following, we always assume that $\|\phi\|_* \leq \varepsilon|\ln\varepsilon|$.

Lemma 3.5. *We have*

$$\|N(\phi)\|_{**} \leq C\|\phi\|_*^{\min(2^*-1,2)}.$$

Proof. We have

$$|N(\phi)| \leq \begin{cases} C|\phi|^{2^*-1}, & N \geq 6; \\ C\big(W^{\frac{6-N}{N-2}}\phi^2 + |\phi|^{2^*-1}\big), & N = 3, 4, 5. \end{cases}$$

Firstly, we consider $N \geq 6$. We have

$$\begin{aligned}|N(\phi)| \leq & C\|\phi\|_*^{2^*-1}\Big(\sum_{j=1}^k \frac{1}{(1+|y-x_j|)^{\frac{N-2}{2}+\tau}}\Big)^{2^*-1} \\ \leq & C\|\phi\|_*^{2^*-1}\sum_{j=1}^k \frac{1}{(1+|y-x_j|)^{\frac{N+2}{2}+\tau}}\Big(\sum_{j=1}^k \frac{1}{(1+|y-x_j|)^{\tau}}\Big)^{\frac{4}{N-2}}\end{aligned} \tag{3.20}$$

where we use the inequality

$$\sum_{j=1}^k a_j b_j \leq \Big(\sum_{j=1}^k a_j^p\Big)^{\frac{1}{p}}\Big(\sum_{j=1}^k b_j^q\Big)^{\frac{1}{q}}, \quad \frac{1}{p}+\frac{1}{q}=1, a_j, b_j \geq 0, j=1,\dots,k.$$

By Lemma B.1 and (3.3), we find,

$$\sum_{j=1}^k \frac{1}{(1+|y-x_j|)^{\tau}} \leq C + \sum_{j=2}^k \frac{C}{|x_1-x_j|^{\tau}}. \leq C.$$

Thus,

$$|N(\phi)| \leq C\|\phi\|_*^{2^*-1}\sum_{j=1}^k \frac{1}{(1+|y-x_j|)^{\frac{N+2}{2}+\tau}}.$$

For $N=4,5$, similarly to the case $N \geq 6$, we have

$$\begin{aligned}&|N(\phi)| \\ \leq & C\|\phi\|_*^2\Big(\sum_{j=1}^k \frac{1}{(1+|y-x_j|)^{(N-2)(1-\beta)}}\Big)^{\frac{6-N}{N-2}}\Big(\sum_{j=1}^k \frac{1}{(1+|y-x_j|)^{\frac{N-2}{2}+\tau}}\Big)^2 \\ & + C\|\phi\|_*^{2^*-1}\sum_{j=1}^k \frac{1}{(1+|y-x_j|)^{\frac{N+2}{2}+\tau}} \\ \leq & C\|\phi\|_*^2\Big(\sum_{j=1}^k \frac{1}{(1+|y-x_j|)^{\frac{N-2}{2}+\tau}}\Big)^{2^*-1} + C\|\phi\|_*^{2^*-1}\sum_{j=1}^k \frac{1}{(1+|y-x_j|)^{\frac{N+2}{2}+\tau}} \\ \leq & C\|\phi\|_*^2\sum_{j=1}^k \frac{1}{(1+|y-x_j|)^{\frac{N+2}{2}+\tau}}.\end{aligned} \tag{3.21}$$

Now, we discuss the case $N=3$. Without loss of generality, we assume $y \in \Omega_1$, where

$$y \in \Omega_j = \Big\{y=(y',y'') \in \mathbb{R}^2 \times \mathbb{R}^{N-2} : \Big\langle \frac{y'}{|y'|}, \frac{x_j}{|x_j|}\Big\rangle \geq \cos\frac{\pi}{k}\Big\}.$$

Then for any small $\alpha > \beta > 0$,

$$\begin{aligned}\sum_{j=2}^{k} \frac{1}{(1+|y-x_j|)^{1-\beta}} &\leq \frac{C}{(1+|y-x_1|)^{1-\alpha}} \sum_{j=2}^{k} \frac{1}{(1+|y-x_j|)^{\alpha-\beta}} \\ &\leq \sum_{j=2}^{k} \frac{C}{|x_1-x_j|^{\alpha-\beta}} \Big(\frac{1}{(1+|y-x_j|)^{1-\alpha}} + \frac{1}{(1+|y-x_1|)^{1-\alpha}} \Big) \\ &\leq \frac{C}{(1+|y-x_1|)^{1-\alpha}}\end{aligned}$$

since $\varepsilon = e^{\frac{D_3}{D_2 \gamma} \beta_k k \ln k}$.

Similarly,

$$\sum_{j=2}^{k} \frac{1}{(1+|y-x_j|)^{\frac{1-\beta}{2}}} \leq \frac{C}{(1+|y-x_1|)^{\frac{1}{2}-\alpha}}.$$

Thus

$$\begin{aligned}|N(\phi)| &\leq \|\phi\|_*^2 \frac{C}{(1+|y-x_1|)^{3+1-5\alpha}} + \|\phi\|_*^5 \frac{C}{(1+|y-x_1|)^{\frac{5}{2}-5\alpha}} \\ &\leq \|\phi\|_*^2 \frac{C}{(1+|y-x_1|)^{\frac{5}{2}}}, \qquad y \in \Omega_1\end{aligned}$$

since $\alpha > \beta$ can be made as small as desired, and

$$\|\phi\|_*^3 \leq C\varepsilon^{\frac{3}{2}} |\ln \varepsilon|^3 \leq \frac{C}{(1+|y-x_1|)^{5\alpha}}.$$

Thus

$$\|N(\phi)\|_{**} \leq C \|\phi\|_*^{\min(2^*-1,2)}.$$

□

Next, we estimate l_k.

Lemma 3.6. *We have*

$$\|l_k\|_{**} \leq C\varepsilon^{\frac{1}{2}+\sigma},$$

where $\sigma > 0$ is a fixed small constant.

Proof. Recall

$$\Omega_j = \Big\{ y = (y', y'') \in \mathbb{R}^2 \times \mathbb{R}^{N-2} : \Big\langle \frac{y'}{|y'|}, \frac{x_j}{|x_j|} \Big\rangle \geq \cos \frac{\pi}{k} \Big\}.$$

By the symmetry, we can assume that $y \in \Omega_1$. Then,

$$|y - x_j| \ge |y - x_1|, \quad \forall\, y \in \Omega_1.$$

Thus, for $y \in \Omega_1$,

$$\begin{aligned} |l_k| \le & C\frac{1}{(1+|y-x_1|)^{4(1-\beta)}}\sum_{j=2}^k \frac{1}{(1+|y-x_j|)^{(N-2)(1-\beta)}} \\ & + C\Big(\sum_{j=2}^k \frac{1}{(1+|y-x_j|)^{(N-2)(1-\beta)}}\Big)^{2^*-1} + C\sum_{j=1}^k \frac{1}{(1+|y-x_j|)^4}|\varphi_{\Lambda,x_j}|. \end{aligned} \tag{3.22}$$

Let us estimate the first term of (3.22). Using Lemma B.2, we obtain

$$\begin{aligned} & \frac{1}{(1+|y-x_1|)^{4(1-\beta)}}\frac{1}{(1+|y-x_j|)^{(N-2)(1-\beta)}} \\ \le & C\Big(\frac{1}{(1+|y-x_1|)^{\frac{N+2}{2}+\tau}} + \frac{1}{(1+|y-x_j|)^{\frac{N+2}{2}+\tau}}\Big)\frac{1}{|x_j-x_1|^{\frac{N+2}{2}-\tau-(N+2)\beta}} \\ \le & C\frac{1}{(1+|y-x_1|)^{\frac{N+2}{2}+\tau}}\frac{1}{|x_j-x_1|^{\frac{N+2}{2}-\tau-(N+2)\beta}}, \quad j>1. \end{aligned} \tag{3.23}$$

Since $\frac{N+2}{2} - \tau > 1$, we find that for $\beta > 0$ small,

$$\begin{aligned} & \frac{1}{(1+|y-x_1|)^{4(1-\beta)}}\sum_{j=2}^k \frac{1}{(1+|y-x_j|)^{(N-2)(1-\beta)}} \\ \le & C\frac{1}{(1+|y-x_1|)^{\frac{N+2}{2}+\tau}}\big(k\varepsilon\big)^{\frac{N+2}{2}-\tau-(N+2)\beta} = C\varepsilon^{\frac{1}{2}+\sigma}\frac{1}{(1+|y-x_1|)^{\frac{N+2}{2}+\tau}}. \end{aligned} \tag{3.24}$$

Now, we estimate the second term of (3.22).

Suppose that $N \ge 5$. Then $\frac{N-2}{2} - \frac{N-2}{N+2}\tau > 1$. Using Lemma B.2 again, we find for $y \in \Omega_1$,

$$
\begin{aligned}
&\frac{1}{(1+|y-x_j|)^{(N-2)(1-\beta)}} \le \frac{1}{(1+|y-x_1|)^{\frac{N-2}{2}(1-\beta)}}\frac{1}{(1+|y-x_j|)^{\frac{N-2}{2}(1-\beta)}}\\
\le&\frac{C}{|x_j-x_1|^{\frac{N-2}{2}-\frac{N-2}{N+2}\tau-(N-2)\beta}}\Big(\frac{1}{(1+|y-x_1|)^{\frac{N-2}{2}+\frac{N-2}{N+2}\tau}}+\frac{1}{(1+|y-x_j|)^{\frac{N-2}{2}+\frac{N-2}{N+2}\tau}}\Big)\\
\le&\frac{C}{|x_j-x_1|^{\frac{N-2}{2}-\frac{N-2}{N+2}\tau-(N-2)\beta}}\frac{1}{(1+|y-x_1|)^{\frac{N-2}{2}+\frac{N-2}{N+2}\tau}}\\
\le&C\big(k\varepsilon\big)^{\frac{N-2}{2}-\frac{N-2}{N+2}\tau-(N-2)\beta}\frac{1}{(1+|y-x_1|)^{\frac{N-2}{2}+\frac{N-2}{N+2}\tau}}
\end{aligned}
$$

which, gives for $y\in\Omega_1$

$$
\begin{aligned}
&\Big(\sum_{j=2}^{k}\frac{1}{(1+|y-x_j|)^{N-2}}\Big)^{2^*-1}\\
\le&C\big(k\varepsilon\big)^{\frac{N+2}{2}-\tau-(N+2)\beta}\frac{1}{(1+|y-x_1|)^{\frac{N+2}{2}+\tau}}=C\varepsilon^{\frac12+\sigma}\frac{1}{(1+|y-x_1|)^{\frac{N+2}{2}+\tau}}.
\end{aligned}
$$

If $N=4$, by the same computation we get

$$
\begin{aligned}
\sum_{j=2}^{k}\frac{1}{(1+|y-x_j|)^{2(1-\beta)}}&\le\sum_{j=2}^{k}\frac{C}{|x_1-x_j|^{1-\frac13\tau-2\beta}}\frac{1}{(1+|y-x_1|)^{1+\frac13\tau}}\\
&\le\frac{Ck\varepsilon^{1-\frac13\tau-2\beta}}{(1+|y-x_1|)^{1+\frac13\tau}}=\frac{C\varepsilon^{\frac12-\frac13\tau-2\beta}}{(1+|y-x_1|)^{1+\frac13\tau}},\qquad y\in\Omega_1.
\end{aligned}
$$

Hence

$$
\Big(\sum_{j=2}^{k}\frac{1}{(1+|y-x_j|)^{2(1-\beta)}}\Big)^3\le\sum_{i=1}^{k}\frac{C\varepsilon^{\frac32-\tau-6\beta}}{(1+|y-x_i|)^{3+\tau}}.
$$

For $N=3$, noting $\varepsilon=e^{\frac{D_3}{D_2\gamma}\beta_k k\ln k}$, by the similar computation we can get that for $y\in\Omega_1$,

$$
\sum_{j=2}^{k}\frac{1}{(1+|y-x_j|)^{1-\beta}}\le\frac{C}{(1+|y-x_1|)^{\frac12}}\sum_{j=2}^{k}\frac{1}{|x_j-x_1|^{\frac12-\beta}}\le\frac{C\varepsilon^{\frac12-2\beta}}{(1+|y-x_1|)^{\frac12}},
$$

and thus

$$
\Big(\sum_{j=2}^{k}\frac{1}{1+|y-x_j|}\Big)^5\le\frac{C\varepsilon^{\frac12+\sigma}}{(1+|y-x_1|)^{\frac52}}.
$$

Finally, we estimate the last term of (3.22). From Lemma A.1, we can check

$$\sum_{j=1}^{k}\frac{1}{(1+|y-x_j|)^4}|\varphi_{\Lambda,x_j}|\le C\varepsilon^{\frac{1}{2}+\sigma}\sum_{j=1}^{k}\frac{1}{(1+|y-x_j|)^{\frac{N+2}{2}+\tau}}.$$

Combining all the above estimates, we obtain the result.

□

Now, we are ready to prove Proposition 3.4.

Proof of Proposition 3.4. Let us recall that

$$\varepsilon=k^{-\frac{N-2}{N-3}},\ \text{ if } N\ge 4;\quad \varepsilon=e^{\frac{D_3}{D_2\gamma}\beta_k k\ln k},\ \text{ if } N=3.$$

Let

$$E_N=\big\{u: u\in C(\Omega_\varepsilon), \|u\|_*\le \varepsilon^{\frac{1}{2}}, \int_{\Omega_\varepsilon}\sum_{i=1}^{k}Z_i\phi=0\big\}$$

if $N\ge 4$. and

$$E_3=\big\{u: u\in C(\Omega_\lambda), \|u\|_*\le \varepsilon^{\frac{1}{2}}\ln\frac{1}{\varepsilon}, \int_{\Omega_\varepsilon}\sum_{i=1}^{k}Z_i\phi=0.\big\}$$

Then, (3.19) is equivalent to

$$\phi=A(\phi)=: L(N(\phi))+L(l_k).$$

Now we prove that A is a contraction map from E_N to E_N. Using Lemma 3.5, we have

$$\begin{aligned}\|A\phi\|_*\le C\|N(\phi)\|_{**}+C\|l_k\|_{**}\le C\|\phi\|_*^{\min(2^*-1,2)}+C\|l_k\|_{**}\\ \le C\varepsilon^{\frac{1}{2}\min(2^*-1,2)}+C\|l_k\|_{**}\\ \le C\varepsilon^{\frac{1}{2}+\sigma}+C\|l_k\|_{**}.\end{aligned}\tag{3.25}$$

Thus, by Lemma 3.6, we find that A maps E_N to E_N.

Next, we show that A is a contraction map.

$$\|A(\phi_1)-A(\phi_2)\|_*=\|L(N(\phi_1))-L(N(\phi_2))\|_*\le C\|N(\phi_1)-N(\phi_2)\|_{**}.$$

If $N\ge 6$, then

$$|N'(t)|\le C|t|^{2^*-2}.$$

As a result, we have

$$
\begin{aligned}
&|N(\phi_1)-N(\phi_2)| \le C\big(|\phi_1|^{2^*-2}+|\phi_2|^{2^*-2}\big)|\phi_1-\phi_2|\\
\le &C\big(\|\phi_1\|_*^{2^*-2}+\|\phi_2\|_*^{2^*-2}\big)\|\phi_1-\phi_2\|_*\Big(\sum_{j=1}^k \frac{1}{(1+|y-x_j|)^{\frac{N-2}{2}+\tau}}\Big)^{2^*-1}.
\end{aligned}
$$

As in the proof of Lemma 3.5, we have

$$
\Big(\sum_{j=1}^k \frac{1}{(1+|y-x_j|)^{\frac{N-2}{2}+\tau}}\Big)^{2^*-1} \le C\sum_{j=1}^k \frac{1}{(1+|y-x_j|)^{\frac{N+2}{2}+\tau}}.
$$

So,

$$
\begin{aligned}
&\|A(\phi_1)-A(\phi_2)\|_* \le C\|N(\phi_1)-N(\phi_2)\|_{**}\\
\le &C\big(\|\phi_1\|_*^{2^*-2}+\|\phi_2\|_*^{2^*-2}\big)\|\phi_1-\phi_2\|_* \le \frac{1}{2}\|\phi_1-\phi_2\|_*.
\end{aligned}
$$

Thus, A is a contraction map if $N\ge 6$.

If $N=3,4,5$, then

$$
|N'(\phi)| \le C\big(W^{\frac{6-N}{N-2}}|\phi|+|\phi|^{2^*-2}\big).
$$

Hence, similar to the proof of Lemma 3.5, we have

$$
\begin{aligned}
&|N(\phi_1)-N(\phi_2)|\\
\le &C\big(W^{\frac{6-N}{N-2}}\big(|\phi_1|+|\phi_2|\big)+|\phi_1|^{2^*-2}+|\phi_2|^{2^*-2}\big)|\phi_1-\phi_2|\\
\le &C\big(\|\phi_1\|_*+\|\phi_2\|_*\big)\|\phi_1-\phi_2\|_* W^{\frac{6-N}{N-2}}\Big(\sum_{j=1}^k \frac{1}{(1+|y-x_j|)^{\frac{N-2}{2}+\tau}}\Big)^2\\
&+C\big(\|\phi_1\|_*^{2^*-2}+\|\phi_2\|_*^{2^*-2}\big)\|\phi_1-\phi_2\|_*\Big(\sum_{j=1}^k \frac{1}{(1+|y-x_j|)^{\frac{N-2}{2}+\tau}}\Big)^{2^*-1}\\
\le &C\big(\|\phi_1\|_*+\|\phi_2\|_*\big)\|\phi_1-\phi_2\|_*\sum_{j=1}^k \frac{1}{(1+|y-x_j|)^{\frac{N+2}{2}+\tau}}.
\end{aligned}
$$

So,

$$
\begin{aligned}
&\|A(\phi_1)-A(\phi_2)\|_* \le C\|N(\phi_1)-N(\phi_2)\|_{**}\\
\le &C\big(\|\phi_1\|_*+\|\phi_2\|_*\big)\|\phi_1-\phi_2\|_* \le \frac{1}{2}\|\phi_1-\phi_2\|_*.
\end{aligned}
$$

Thus, we have proved that A is a contraction map.

It follows from the contraction mapping theorem that there is a unique $\phi \in E_N$, such that

$$\phi = A(\phi).$$

Moreover, it follows from (3.25) that

$$\|\phi\|_* \le C\varepsilon^{\frac{1}{2}+\sigma} + C\|l_k\|_{**}.$$

So, the estimate for $\|\phi\|_*$ follows from Lemma 3.6.

□

4. Proof of Theorem 2.1

Let

$$F(\Lambda) = I\big(W + \phi\big),$$

where ϕ is the function obtained in Proposition 3.4, and let

$$I(u) = \frac{1}{2}\int_{\Omega_\varepsilon} (|Du|^2 + \mu\varepsilon^2 u^2) - \frac{(N-2)^2}{2}\int_{\Omega_\varepsilon} |u|^{2^*}.$$

Using the symmetry, we can check that if Λ is a critical point of $F(\Lambda)$, then $W + \phi$ is a solution of (1.4).

Proposition 4.1. *For $N \ge 4$, we have*

$$F(\Lambda) = k\big((A_0 - A_1\gamma\Lambda\varepsilon - A_2\Lambda^{N-2}\varepsilon + o(\varepsilon)\big),$$

where the constant $A_i > 0, i = 0, 1, 2$ are positive constants, which are given in Proposition A.3.

For $N = 3$, we have

$$F(\Lambda) = k\Big(D_1 - D_2\gamma\varepsilon\Lambda\ln\frac{1}{\Lambda\varepsilon} - D_3\varepsilon\Lambda k\ln k + O(\varepsilon)\Big),$$

where the constants $D_i, i = 1, 2, 3$ are strictly positive numbers, which are given in Proposition A.4.

Proof. There is $t \in (0,1)$, such that

$$
\begin{aligned}
&F(\Lambda) = I(W) + \big\langle I'\big(W\big), \phi\big\rangle + \frac{1}{2} D^2 I\big(W + t\phi\big)(\phi, \phi) \\
=&I(W) - \int_{\Omega_\varepsilon} l_k \phi + \int_{\Omega_\varepsilon} \big(|D\phi|^2 + \varepsilon^2 \mu \phi^2 - N(N+2)\big(W + t\phi\big)^{2^*-2} \phi^2\big) \\
=&I(W) - N(N+2) \int_{\Omega_\varepsilon} \Big(\big(W + t\phi\big)^{2^*-2} - W^{2^*-2}\Big) \phi^2 + \int_{\Omega_\varepsilon} N(\phi)\phi \\
=&I(W) - N(N+2) \int_{\Omega_\varepsilon} \Big(\big(W + t\phi\big)^{2^*-2} - W^{2^*-2}\Big) \phi^2 + O\Big(\int_{\Omega_\varepsilon} |N(\phi)||\phi|\Big).
\end{aligned}
$$

But

$$
\begin{aligned}
&\int_{\Omega_\varepsilon} |N(\phi)||\phi| \\
\le& C\|N(\phi)\|_{**}\|\phi\|_* \int_{\Omega_\varepsilon} \sum_{j=1}^k \frac{1}{(1+|y-x_j|)^{\frac{N+2}{2}+\tau}} \sum_{i=1}^k \frac{1}{(1+|y-x_i|)^{\frac{N-2}{2}+\tau}}.
\end{aligned}
$$

Using Lemma B.2, we find that if $N \ge 4$,

$$
\begin{aligned}
&\sum_{j=1}^k \frac{1}{(1+|y-x_j|)^{\frac{N+2}{2}+\tau}} \sum_{i=1}^k \frac{1}{(1+|y-x_i|)^{\frac{N-2}{2}+\tau}} \\
=&\sum_{j=1}^k \frac{1}{(1+|y-x_j|)^{N+2\tau}} + \sum_{j=1}^k \sum_{i\ne j} \frac{1}{(1+|y-x_j|)^{\frac{N+2}{2}+\tau}} \frac{1}{(1+|y-x_i|)^{\frac{N-2}{2}+\tau}} \\
\le&\sum_{j=1}^k \frac{1}{(1+|y-x_j|)^{N+2\tau}} + C\sum_{j=1}^k \frac{1}{(1+|y-x_j|)^{N+\frac{1}{2}\tau}} \sum_{i=2}^k \frac{1}{|x_i - x_1|^{\frac{3}{2}\tau}} \\
\le&C\sum_{j=1}^k \frac{1}{(1+|y-x_j|)^{N+\frac{1}{2}\tau}}.
\end{aligned}
$$

Thus, we obtain that for $N \ge 4$,

$$
\int_{\Omega_\varepsilon} |N(\phi)||\phi| \le Ck\|N(\phi)\|_{**}\|\phi\|_* \le Ck\|\phi\|_*^2 \le Ck\varepsilon^{1+\sigma}.
$$

Now we consider the case $N=3$. In this case, $\tau=0$. Let $\eta>0$ be a small constant. Then it holds

$$
\begin{aligned}
&\sum_{j=1}^k \frac{1}{(1+|y-x_j|)^{\frac52}}\sum_{i=1}^k \frac{1}{(1+|y-x_i|)^{\frac12}}\\
=&\sum_{j=1}^k \frac{1}{(1+|y-x_j|)^{3}}+\sum_{j=1}^k\sum_{i\neq j} \frac{1}{(1+|y-x_j|)^{\frac52}}\frac{1}{(1+|y-x_i|)^{\frac12}}\\
\le&\sum_{j=1}^k \frac{1}{(1+|y-x_j|)^{3}}+C\varepsilon^{\eta}k\sum_{j=1}^k \frac{1}{(1+|y-x_j|)^{3-\eta}}.
\end{aligned}
$$

Thus,

$$
\begin{aligned}
&\int_{\Omega_\varepsilon}|N(\phi)||\phi|\le C\big(k\ln\frac{1}{\varepsilon}+k^2\big)\|N(\phi)\|_{**}\|\phi\|_*\\
\le& C\big(k\ln\frac{1}{\varepsilon}+k^2\big)\|\phi\|_*^3\le Ck\varepsilon^{1+\sigma}.
\end{aligned}
$$

Thus, we obtain

$$
F(\Lambda)=I(W))-N(N+2)\int_{\Omega_\varepsilon}\Big(\big(W+t\phi\big)^{2^*-2}-W^{2^*-2}\Big)\phi^2+O\big(\varepsilon^{1+\sigma}\big).
$$

Now

$$
\big(W+t\phi\big)^{2^*-2}-W^{2^*-2}=\begin{cases} O\big(|\phi|^{2^*-2}\big), & N\ge 6;\\ O\big(W^{\frac{6-N}{N-2}}|\phi|+|\phi|^{2^*-2}\big), & N=3,4,5.\end{cases}
$$

Thus, we have

$$
\begin{aligned}
&\Big|-N(N+2)\int_{\Omega_\varepsilon}\Big(\big(W+t\phi\big)^{2^*-2}-W^{2^*-2}\Big)\phi^2\Big|\\
\le& C\|\phi\|_*^{2^*}\int_{\Omega_\varepsilon}\Big(\sum_{j=1}^k\frac{1}{(1+|y-x_j|)^{\frac{N-2}{2}+\tau}}\Big)^{2^*},
\end{aligned}
$$

if $N\ge 6$. If $N=3,4,5$, noting that $N-2\ge\frac{N-2}{2}+\tau$, we obtain

$$\begin{aligned}&\Big|-N(N+2)\int_{\Omega_\varepsilon}\Big(\big(W+t\phi\big)^{2^*-2}-W^{2^*-2}\Big)\phi^2\Big|\\ \le& C\int_{\Omega_\varepsilon}W^{\frac{6-N}{N-2}}|\phi|^3+C\int_{\Omega_\varepsilon}|\phi|^{2^*}\le\|\phi\|_*^3\int_{\Omega_\varepsilon}\Big(\sum_{j=1}^k\frac{1}{(1+|y-x_j|)^{\frac{N-2}{2}+\tau}}\Big)^{2^*}.\end{aligned}$$

Suppose that $N\ge 4$. Let $\bar\eta>0$ small. Using Lemma B.2, if $y\in\Omega_1$, then

$$\begin{aligned}&\sum_{j=2}^k\frac{1}{(1+|y-x_j|)^{\frac{N-2}{2}+\tau}}\\ \le&\sum_{j=2}^k\frac{1}{(1+|y-x_1|)^{\frac{N-2}{4}+\frac12\tau}}\frac{1}{(1+|y-x_j|)^{\frac{N-2}{4}+\frac12\tau}}\\ \le& C\frac{1}{(1+|y-x_1|)^{\frac{N-2}{2}+\frac12\bar\eta}}\sum_{j=2}^k\frac{1}{|x_j-x_1|^{\tau-\frac12\bar\eta}}\le C\varepsilon^{-\bar\eta}\frac{1}{(1+|y-x_1|)^{\frac{N-2}{2}+\frac12\bar\eta}}.\end{aligned}$$

As a result,

$$\Big(\sum_{j=1}^k\frac{1}{(1+|y-x_j|)^{\frac{N-2}{2}+\tau}}\Big)^{2^*}\le C\varepsilon^{-2^*\bar\eta}\frac{1}{(1+|y-x_1|)^{N+2^*\frac12\bar\eta}},\quad y\in\Omega_1.$$

Thus

$$\int_{\Omega_\varepsilon}\Big(\sum_{j=1}^k\frac{1}{(1+|y-x_j|)^{\frac{N-2}{2}+\tau}}\Big)^{2^*}\le Ck\varepsilon^{-2^*\bar\eta}.$$

So, we have proved that for $N\ge 4$,

$$\begin{aligned}&\Big|-N(N+2)\int_{\Omega_\varepsilon}\Big(\big(W+t\phi\big)^{2^*-2}-W^{2^*-2}\Big)\phi^2\Big|\\ \le& Ck\varepsilon^{-2^*\bar\eta}\|\phi\|_*^{\min(3,2^*)}\le Ck\varepsilon^{1+\sigma}.\end{aligned}$$

For $N=3$, we have

$$\begin{aligned}&\Big|-15\int_{\Omega_\varepsilon}\Big(\big(W+t\phi\big)^4-W^4\Big)\phi^2\Big|\le C\|\phi\|_*^3\int_{\Omega_\varepsilon}\Big(\sum_{j=1}^k\frac{1}{(1+|y-x_j|)^{\frac12}}\Big)^6\\ \le& C\sum_{j=1}^k\|\phi\|_*^3\int_{\Omega_j}\Big(\frac{k}{(1+|y-x_j|)^{\frac12}}\Big)^6\le Ck^7\ln\frac1\varepsilon\|\phi\|_*^3\le Ck\varepsilon^{1+\sigma}.\end{aligned}$$

So, we have proved

$$F(\Lambda) = I(W) + O\big(k\varepsilon^{1+\sigma}\big).$$

□

Proof of Theorem 2.1: We just need to prove that $F(\Lambda)$ has a critical point.

For $N \geq 4$, since $\gamma < 0$, the function

$$-A_1\gamma\Lambda - A_2\Lambda^{N-2}$$

has a maximum point at $\Lambda_0 = \left(\frac{-A_1\gamma}{A_2(N-2)}\right)^{\frac{1}{N-3}}$. Thus, $F(\Lambda)$ attains its maximum in the interior of $[\delta, \delta^{-1}]$ if $\delta > 0$ is small. As a result, $F(\Lambda)$ has a critical point in $[\delta, \delta^{-1}]$.

Suppose $N = 3$. Then

$$\begin{aligned}\bar F(\Lambda) :=& -D_2\gamma\varepsilon\Lambda\ln\frac{1}{\Lambda\varepsilon} - D_3\varepsilon\Lambda\beta_k k\ln k + O\big(\varepsilon\Lambda\big)\\ =&\varepsilon\Big(-D_2\gamma\Lambda\ln\frac{1}{\Lambda} + O(\Lambda)\Big).\end{aligned}$$

Since

$$-D_2\gamma\Lambda\ln\frac{1}{\Lambda} + O(\Lambda) \to -\infty, \quad \text{as } \Lambda \to +\infty.$$

and

$$-D_2\gamma\Lambda\ln\frac{1}{\Lambda} + O(\Lambda) \geq \Lambda, \quad \text{as } \Lambda \to +0,$$

we see that $\bar F(\Lambda)$ has a maximum point in (δ, δ^{-1}), if $\delta > 0$ is small. As a result, $F(\Lambda)$ has a critical point in $[\delta, \delta^{-1}]$.

□

Appendix A. Energy Expansion

In all of the appendixes, we always assume that

$$x_j = \Big(\frac{1}{\varepsilon}\cos\frac{2(j-1)\pi}{k}, \frac{1}{\varepsilon}\sin\frac{2(j-1)\pi}{k}, 0\Big), \quad j = 1, \cdots, k,$$

where 0 is the zero vector in R^{N-2} and

$$\varepsilon = k^{-\frac{N-2}{N-3}}, \ \text{ if } N \geq 4, \quad \varepsilon = e^{\frac{D_3}{D_2\gamma}\beta_k k\ln k}, \ \text{ if } N = 3.$$

In this section, we will estimate the energy of W. Recall that

$$I(u)=\frac{1}{2}\int_{\Omega_\varepsilon}(|Du|^2+\mu\varepsilon^2|u|^2)-\frac{\alpha_N}{2^*}\int_{\Omega_\varepsilon}|u|^{2^*},$$

$$U_{\frac{1}{\Lambda},x_j}(y)=\frac{(\frac{1}{\Lambda})^{\frac{N-2}{2}}}{(1+\frac{1}{\Lambda^2}|y-x_j|^2)^{\frac{N-2}{2}}},$$

and

$$W(y)=\sum_{j=1}^{k}W_{\Lambda,x_j}(y),$$

where W_{Λ,x_j} is the solution of (2.4).

Let

$$\varphi_{\Lambda,x_j}(y)=U_{\frac{1}{\Lambda},x_j}(y)-W_{\Lambda,x_j}(y). \tag{A.1}$$

Then, φ_{Λ,x_j} satisfies

$$\begin{cases}-\Delta\varphi_{\Lambda,x_j}+\mu\varepsilon^2\varphi_{\Lambda,x_j}=\mu\varepsilon^2U_{\frac{1}{\Lambda},x_j}(y), & \text{in } \Omega_\varepsilon,\\ \frac{\partial\varphi_{\Lambda,x_j}}{\partial n}=\frac{\partial}{\partial n}U_{\frac{1}{\Lambda},x_j}, & \text{on } \partial\Omega_\varepsilon.\end{cases} \tag{A.2}$$

We need to estimate φ_{Λ,x_j}. Write $\varphi_{\Lambda,x_j}=\varphi_1+\varphi_2$, where φ_1 is the solution of

$$\begin{cases}-\Delta\varphi_1+\mu\varepsilon^2\varphi_1=\mu\varepsilon^2U_{\frac{1}{\Lambda},x_j}(y), & \text{in } \Omega_\varepsilon,\\ \frac{\partial\varphi_{\Lambda,x_j}}{\partial n}=0, & \text{on } \partial\Omega_\varepsilon,\end{cases} \tag{A.3}$$

and φ_2 is the solution of

$$\begin{cases}-\Delta\varphi_2+\mu\varepsilon^2\varphi_2=0, & \text{in } \Omega_\varepsilon,\\ \frac{\partial\varphi_2}{\partial n}=\frac{\partial}{\partial n}U_{\frac{1}{\Lambda},x_j}, & \text{on } \partial\Omega_\varepsilon.\end{cases} \tag{A.4}$$

Using Lemma 3.1, we find

$$
\begin{aligned}
|\varphi_1(y)| \le & C\varepsilon^2 \int_{\Omega_\varepsilon} \frac{U_{\frac{1}{\Lambda},x_j}(z)}{|y-z|^{N-2}}\,dz \\
\le & C\varepsilon^2 \int_{\Omega_\varepsilon} \frac{1}{(1+|z-x_j|)^{N-2}|y-z|^{N-2}}\,dz \\
\le & \begin{cases} \frac{C\varepsilon^2}{(1+|y-x_j|)^{N-4}}, & N\ge 5; \\ C\varepsilon^2 \ln\frac{1}{\varepsilon}, & N=4; \\ C\varepsilon, & N=3. \end{cases}
\end{aligned} \tag{A.5}
$$

Next, we estimate φ_2. Let $\lambda = \frac{1}{\varepsilon\Lambda}$, $\tilde{x}_j = \varepsilon x_j$, and $\tilde{\varphi}_2(y) = \varepsilon^{-\frac{N-2}{2}}\varphi_2\big(\frac{1}{\varepsilon}y\big)$. Then

$$
\begin{cases} -\Delta\tilde{\varphi}_2 + \mu\tilde{\varphi}_2 = 0, & \text{in } \Omega, \\ \frac{\partial\varphi_2}{\partial n} = \frac{\partial U_{\lambda,\tilde{x}_j}}{\partial n}, & \text{on } \partial\Omega. \end{cases} \tag{A.6}
$$

Let $G(z,y)$ be the Green function of $-\Delta + \mu I$ in Ω with the Neumann boundary condition. We have

$$
\begin{aligned}
\tilde{\varphi}_2(y) = & \int_{\partial\Omega} G(z,y)\frac{\partial U_{\lambda,\tilde{x}_j}(z)}{\partial n}\,dz \\
= & \int_{\partial\Omega\cap B_{\frac{\delta}{2}}(\tilde{x}_j)} G(z,y)\frac{\partial U_{\lambda,\tilde{x}_j}(z)}{\partial n}\,dz + \int_{\partial\Omega\setminus B_{\frac{\delta}{2}}(\tilde{x}_j)} G(z,y)\frac{\partial U_{\lambda,\tilde{x}_j}(z)}{\partial n}\,dz \\
= & \int_{\partial\Omega\cap B_{\frac{\delta}{2}}(\tilde{x}_j)} G(z,y)\frac{\partial U_{\lambda,\tilde{x}_j}(z)}{\partial n}\,dz + O\big(\varepsilon^{\frac{N-2}{2}}\big).
\end{aligned} \tag{A.7}
$$

If $y \notin B_\delta(\tilde{x}_j)$, then $|G(z,y)| \le C$ for all $z \in B_{\frac{\delta}{2}}(\tilde{x}_j)$, which, together with (A.7), give

$$
\tilde{\varphi}_2(y) = O\Big(\varepsilon^{\frac{N-2}{2}} \int_{\partial\Omega\cap B_{\frac{\delta}{2}}(\tilde{x}_j)} \frac{1}{|z-\tilde{x}_j|^{N-2}} + \varepsilon^{\frac{N-2}{2}}\Big) = O\big(\varepsilon^{\frac{N-2}{2}}\big), \quad y \notin B_\delta(\tilde{x}_j). \tag{A.8}
$$

Thus, it remains to estimate $\tilde{\varphi}_2(y)$ for $y \in B_\delta(\tilde{x}_j)$.

Let $K(|z-y|)$ and $H(z,y)$ be the singular part and the regular part of $G(z,y)$ respectively. For $y \in B_\delta(\tilde{x}_j)$, we have

$$
H(z,y) = -K(|z-\bar{y}|)\big(1+O(d)\big),
$$

where $\bar{y}$ is the reflection point of y with respect to $\partial\Omega$, and $d = d(y,\partial\Omega)$. It is easy to see

$$d(y,\partial\Omega)\le C|y-\tilde{x}_j|\quad \text{if } y\in B_\delta(\tilde{x}_j).$$

Noting that

$$\frac{\partial U_{\lambda,\tilde{x}_j}(z)}{\partial n}=-\frac{(N-2)\lambda^{\frac{N-2}{2}}\lambda^2\langle z-\tilde{x}_j,n\rangle}{(1+\lambda^2|z-\tilde{x}_j|^2)^{\frac{N}{2}}},$$

we find

$$\begin{aligned}&\int_{\partial\Omega\cap B_{\frac{\delta}{2}}(\tilde{x}_j)}G(z,y)\frac{\partial U_{\lambda,\tilde{x}_j}(z)}{\partial n}\,dz\\ &=-\,\varepsilon^{\frac{N}{2}}\frac{1}{\Lambda^{\frac{N+2}{2}}}\int_{\partial\Omega_\varepsilon\cap B_{\frac{\delta}{2\varepsilon}}(x_j)}G(\varepsilon z,y)\frac{(N-2)\varepsilon^{-1}\langle z-x_j,n\rangle}{(1+\frac{1}{\Lambda^2}|z-x_j|^2)^{\frac{N}{2}}}\,dz.\end{aligned}\tag{A.9}$$

If $N\ge 4$, noting that

$$d\le C|y-\tilde{x}_j|=C\varepsilon|\varepsilon^{-1}y-x_j|,$$

we can check (see also [51]) that

$$\begin{aligned}&\int_{\partial\Omega\cap B_{\frac{\delta}{2}}(\tilde{x}_j)}G(z,y)\frac{\partial U_{\lambda,\tilde{x}_j}(z)}{\partial n}\,dz\\ &=-\,(\Lambda\varepsilon)^{-\frac{N-4}{2}}\int_{\mathbb{R}^{N-1}}\Big(\frac{1}{|z-\frac{\varepsilon^{-1}y-x_j}{\Lambda}|^{N-2}}+\frac{1}{|z-\frac{\overline{\varepsilon^{-1}y-x_j}}{\Lambda}|^{N-2}}\Big)\frac{N-2}{2}\frac{\sum_{i=1}^{N-1}k_iz_i^2}{(1+|z|^2)^{\frac{N}{2}}}\,dz\\ &\quad+(\Lambda\varepsilon)^{-\frac{N-4}{2}}O\Big((d+\varepsilon)\int_{\mathbb{R}^{N-1}}\Big(\frac{1}{|z-\frac{\varepsilon^{-1}y-x_j}{\Lambda}|^{N-2}}+\frac{1}{|z-\frac{\overline{\varepsilon^{-1}y-x_j}}{\Lambda}|^{N-2}}\Big)\frac{1}{(1+|z|)^{N-2}}\,dz\Big)\\ &=(\Lambda\varepsilon)^{-\frac{N-4}{2}}\Big(\varphi_0\big(\frac{\varepsilon^{-1}y-x_j}{\Lambda}\big)+O\big(\frac{\varepsilon}{(1+|\varepsilon^{-1}y-x_j|)^{N-4}}\big)\Big),\end{aligned}\tag{A.10}$$

where $\bar{z}$ is the reflection point of z with respect to $z_N=0$, and φ_0 solving the following linear problem

$$\begin{cases}-\Delta\varphi_0=0, & \text{in } \mathbb{R}^N_+=\{(x',x_N),x_N>0\},\\ \frac{\partial\varphi_0}{\partial n}=-\frac{N-2}{2}\frac{\sum_{i=1}^{N-1}k_ix_i^2}{(1+|x'|^2)^{\frac{N}{2}}}, & \text{on } \partial\mathbb{R}^N_+,\\ \varphi_0(x)\to 0, & \text{as } |x|\to+\infty.\end{cases}\tag{A.11}$$

So, we obtain from (A.7), (A.8) and (A.10),

$$\varphi_2(y)=\varepsilon^{\frac{N-2}{2}}\tilde\varphi_2(\varepsilon y)=\varepsilon\Lambda^{\frac{4-N}{2}}\varphi_0(\frac{y-x_j}{\Lambda})+O\Big(\frac{\varepsilon^2}{(1+|y-x_j|)^{N-4}}+\varepsilon^{N-2}\Big). \tag{A.12}$$

Combining (A.5) and (A.12), we obtain

$$\varphi_{\Lambda,x_j}(y)=\varepsilon\Lambda^{\frac{4-N}{2}}\varphi_0(\frac{y-x_j}{\Lambda})+O\Big(\frac{\varepsilon^2|\ln\varepsilon|^m}{(1+|y-x_j|)^{N-4}}+\varepsilon^{N-2}\Big),\quad N\ge 4, \tag{A.13}$$

with $m=1$ for $N=4, m=0$ for $N\ge 5$.

Now we study the case $N=3$. In this case, (A.10) becomes

(A.14)

$$\begin{aligned}&\int_{\partial\Omega\cap B_{\frac{\delta}{2}}(\tilde x_j)}G(z,y)\frac{\partial U_{\lambda,\tilde x_j}(z)}{\partial n}\,dz\\ =&-(\Lambda\varepsilon)^{\frac12}\Big(\int_{\mathbb{R}^2\cap B_{\frac{\delta}{2\varepsilon}}(0)}\big(\frac{1}{|z-\frac{\varepsilon^{-1}y-x_j}{\Lambda}|}+\frac{1+O(|y-\tilde x_j|)}{|z-\overline{\frac{\varepsilon^{-1}y-x_j}{\Lambda}}|}\big)\frac12\frac{\sum_{i=1}^2k_iz_i^2}{(1+|z|^2)^{\frac32}}\,dz+O(\varepsilon|\ln\varepsilon|)\Big).\end{aligned}$$

So, we obtain

$$\begin{aligned}\varphi_2(y)=&\varepsilon^{\frac12}\tilde\varphi_2(\varepsilon y)\\ =&-\varepsilon\Lambda^{\frac12}\int_{\mathbb{R}^2\cap B_{\frac{\delta}{2\varepsilon}}(0)}\big(\frac{1}{|z-\frac{y-x_j}{\Lambda}|}+\frac{1+\varepsilon O(|y-x_j|)}{|z-\overline{\frac{y-x_j}{\Lambda}}|}\big)\frac12\frac{\sum_{i=1}^2k_iz_i^2}{(1+|z|^2)^{\frac32}}\,dz+O(\varepsilon^2|\ln\varepsilon|).\end{aligned} \tag{A.15}$$

Denote $y^*=\frac{y-x_j}{\Lambda}$ and $d^*=\frac1L|y^*|$, for some large $L>0$. Then

$$\begin{aligned}&\int_{B_{d^*}(0)}\big(\frac{1}{|z-y|}+\frac{1}{|z-\bar{y^*}|}\big)\frac12\frac{\sum_{i=1}^2k_iz_i^2}{(1+|z|^2)^{\frac32}}\,dz\\ \le&\frac{C}{d^*}\int_{B_{d^*}(0)}\frac{1}{|z|}\,dz\le C,\end{aligned}$$

and

$$\int_{B_{d^*}(y^*)}\big(\frac{1}{|z-y^*|}+\frac{1}{|z-\bar{y^*}|}\big)\frac12\frac{\sum_{i=1}^2k_iz_i^2}{(1+|z|^2)^{\frac32}}\,dz\le C.$$

Suppose that $z\in B_{\frac{\delta}{2\varepsilon}}(0)\setminus(B_{d^*}(0)\cup B_{d^*}(y^*))$. Then,

$$\frac{1}{|z-y^*|}=\frac{1}{|z|}\Big(1+O(\frac{|y^*|}{|z|})\Big),$$

and

$$\frac{1}{|z-\bar{y^*}|}=\frac{1}{|z|}\Big(1+O(\frac{|\bar{y^*}|}{|z|})\Big).$$

But

$$(|y^*|+|\bar{y^*}|)\int_{B_{\frac{\delta}{2\varepsilon}}(0)\setminus(B_{d^*}(0)\cup B_{d^*}(y^*))}\frac{1}{(1+|z|)^3}\le(|y^*|+|\bar{y^*}|)\frac{C}{1+d^*}\le C.$$

So, we find

$$\begin{aligned}&\int_{B_{\frac{\delta}{2\varepsilon}}(0)\setminus(B_{d^*}(0)\cup B_{d^*}(y^*))}\Big(\frac{1}{|z-y|}+\frac{1+\varepsilon O(|y^*|)}{|z-\bar{y^*}|}\Big)\frac{1}{2}\frac{\sum_{i=1}^2 k_iz_i^2}{(1+|z|^2)^{\frac{3}{2}}}\,dz\\ =&\int_{B_{\frac{\delta}{2\varepsilon}}(0)\setminus(B_{d^*}(0)\cup B_{d^*}(y^*))}\frac{1+\varepsilon O(|y^*|)}{|z|}\frac{\sum_{i=1}^2 k_iz_i^2}{(1+|z|^2)^{\frac{3}{2}}}\,dz+O(1)\\ =&A\gamma\ln\frac{1}{\varepsilon|y^*|}+O\big(1+\varepsilon|y^*|\ln\frac{1}{\varepsilon|y^*|}\big)=A\gamma\ln\frac{1}{\varepsilon|y^*|}+O(1),\end{aligned}$$

where $A>0$ is a constant. Here we have used $\varepsilon|y^*|\le C$. Thus, we have proved that

$$\varphi_{\Lambda,x_j}(y)=\varphi_2(y)+O(\varepsilon)=-\varepsilon\Lambda^{\frac{1}{2}}A\gamma\ln\frac{1}{\varepsilon\frac{|y-x_j|}{\Lambda}}+O(\varepsilon),\quad N=3. \tag{A.16}$$

Combining (A.13) and (A.16), we obtain

Lemma A.1. *We have*

$$\varphi_{\Lambda,x_j}(y)=\varepsilon\Lambda^{\frac{4-N}{2}}\varphi_0(\frac{y-x_j}{\Lambda})+O\big(\frac{\varepsilon^2|\ln\varepsilon|^m}{(1+|y-x_j|)^{N-4}}+\varepsilon^{N-2}\big),\quad N\ge 4,$$

with $m=1$ for $N=4, m=0$ for $N\ge 5$, where φ_0 is the solution of (A.11)*, while*

$$\varphi_{\Lambda,x_j}(y)=\varphi_2(y)+O(\varepsilon)=-\varepsilon\Lambda^{\frac{1}{2}}A\gamma\ln\frac{1}{\varepsilon\frac{|y-x_j|}{\Lambda}}+O(\varepsilon),\quad N=3,$$

for some constant $A>0$.

As a direct consequence of Lemma A.1, we have

Lemma A.2. *There is a constant $C > 0$, such that*

$$|\partial_\Lambda \varphi_{\Lambda,x_j}| \le \begin{cases} \frac{C\varepsilon|\ln\varepsilon|}{(1+|y-x_j|)^{N-3}}, & N \ge 4; \\ C\varepsilon|\ln\varepsilon|, & N = 3. \end{cases} \tag{A.17}$$

Moreover, for any fixed small $\beta > 0$, there is a constant $C' > 0$, depending on β, such that

$$|W_{\lambda,x_j}| \le C' U^{1-\beta}_{\frac{1}{\Lambda},x_j}, \quad |\partial_\Lambda W_{\lambda,x_j}| \le C' U^{1-\beta}_{\frac{1}{\Lambda},x_j}.$$

Proof. Differentiating (A.2) with respect to Λ, we can repeat the same estimates as in Lemma A.1 to obtain (A.17).

On the other hand, noting that $\varepsilon \le \frac{C}{1+|y-x_j|}$, the other two estimates follow from Lemma A.1.

□

The following estimate is well known, whose calculations are quite standard (see [51])

$$\alpha_N \int_{\Omega_\varepsilon} U^{2^*}_{\frac{1}{\Lambda},x_j} = \bar{A}_0 - \bar{A}_1 \gamma \Lambda \varepsilon + O(\varepsilon^{1+\sigma}), \tag{A.18}$$

where $\bar{A}_0$ and $\bar{A}_1$ are some positive constants, and $\sigma > 0$ is a small constant.

Using Lemma A.1, we find

$$\begin{aligned} &\alpha_N \int_{\Omega_\varepsilon} U^{2^*-1}_{\frac{1}{\Lambda},x_j} \varphi_{\Lambda,x_j} = -\alpha_N \int_{\Omega_\varepsilon} U^{2^*-1}_{\frac{1}{\Lambda},x_j} \varepsilon \Lambda^{\frac{1}{2}} A \gamma \ln \frac{1}{\varepsilon \frac{|y-x_j|}{\Lambda}} + O(\varepsilon) \\ =& -\bar{A}_3 \gamma \Lambda \varepsilon \ln \frac{1}{\varepsilon} + O(\varepsilon), \quad N = 3, \end{aligned} \tag{A.19}$$

for some $\bar{A}_3 > 0$. As a result,

$$\begin{aligned} &\int_{\Omega_\varepsilon} \left(|DW_{\Lambda,x_j}|^2 + \varepsilon^2 \mu W^2_{\Lambda,x_j} \right) \\ =&\alpha_N \int_{\Omega_\varepsilon} U^{2^*}_{\frac{1}{\Lambda},x_j} - \alpha_N \int_{\Omega_\varepsilon} U^{2^*-1}_{\frac{1}{\Lambda},x_j} \varphi_{\Lambda,x_j} = \bar{A}_0 + \bar{A}_3 \gamma \Lambda \varepsilon \ln \frac{1}{\varepsilon} + O(\varepsilon), \quad N = 3, \end{aligned} \tag{A.20}$$

and

$$
\begin{aligned}
&\frac{1}{2^*}\alpha_N\int_{\Omega_\varepsilon} W_{\Lambda,x_j}^{2^*}\\
=&\frac{1}{2^*}\alpha_N\int_{\Omega_\varepsilon} U_{\frac{1}{\Lambda},x_j}^{2^*}-\alpha_N\int_{\Omega_\varepsilon} U_{\frac{1}{\Lambda},x_j}^{2^*-1}\varphi_{\Lambda,x_j}+O\Big(\int_{\Omega_\varepsilon} U_{\frac{1}{\Lambda},x_j}^{2^*-2}\varphi_{\Lambda,x_j}^2\Big)\\
=&\frac{1}{2^*}\bar{A}_0+\bar{A}_3\gamma\Lambda\varepsilon\ln\frac{1}{\varepsilon}+O(\varepsilon),\quad N=3.
\end{aligned}
\tag{A.21}
$$

Similarly, we can prove by using Lemma A.1 that

$$
\alpha_N\int_{\Omega_\varepsilon} U_{\frac{1}{\Lambda},x_j}^{2^*-1}\varphi_{\Lambda,x_j}=-\bar{A}_3\gamma\Lambda\varepsilon+O(\varepsilon^{1+\sigma}),\quad N\geq 4,
\tag{A.22}
$$

for some $\bar{A}_3>0$,

$$
\int_{\Omega_\varepsilon}\big(|DW_{\Lambda,x_j}|^2+\varepsilon^2\mu W_{\Lambda,x_j}^2\big)=\bar{A}_0+\big(\bar{A}_3-\bar{A}_1\big)\gamma\Lambda\varepsilon+O(\varepsilon^{1+\sigma}),\quad N\geq 4,
\tag{A.23}
$$

and

$$
\frac{1}{2^*}\alpha_N\int_{\Omega_\varepsilon} W_{\Lambda,x_j}^{2^*}=\frac{1}{2^*}\bar{A}_0+\big(\bar{A}_3-\frac{1}{2^*}\bar{A}_1\big)\gamma\Lambda\varepsilon+O(\varepsilon^{1+\sigma}),\quad N\geq 4.
\tag{A.24}
$$

The readers can refer to [51] for details for the cases $N\geq 4$.

Next, we discuss the interaction between bubbles.

Define $\lambda=\frac{1}{\varepsilon\Lambda}$ and $\bar{x}_j=\varepsilon x_j$, $j=1,\cdots,k$. Then, we have for $i\neq j$,

$$
\alpha_N\int_{\Omega_\varepsilon} U_{\frac{1}{\Lambda},x_i}^{2^*-1}U_{\frac{1}{\Lambda},x_j}=\frac{B_1\Lambda^{N-2}}{|x_i-x_j|^{N-2}}+O\Big(\frac{1}{|x_i-x_j|^{N-2+\sigma}}\Big),
\tag{A.25}
$$

where $B_1>0$ is a constant, and $\sigma>0$ is a fixed small constant.

On the other hand, using Lemma A.1

$$
\alpha_N\int_{\Omega_\varepsilon} U_{\frac{1}{\Lambda},x_i}^{2^*-1}\varphi_{\frac{1}{\Lambda},x_j}=O\Big(\varepsilon\ln\frac{1}{|\bar{x}_i-\bar{x}_j|}\Big)=O\Big(\frac{\varepsilon}{|\bar{x}_i-\bar{x}_j|^{\frac{1}{2}}}\Big),\quad N=3,\ i\neq j.
\tag{A.26}
$$

As a result,

$$
\begin{aligned}
&\int_{\Omega_\varepsilon}\Big(DW_{\Lambda,x_i}DW_{\Lambda,x_j}+\varepsilon^2\mu W_{\Lambda,x_i}W_{\Lambda,x_j}\Big)\\
=&\alpha_N\int_{\Omega_\varepsilon}U_{\frac{1}{\Lambda},x_i}^{2^*-1}U_{\frac{1}{\Lambda},x_j}-\alpha_N\int_{\Omega_\varepsilon}U_{\frac{1}{\Lambda},x_i}^{2^*-1}\varphi_{\frac{1}{\Lambda},x_j}\\
=&\frac{B_1\Lambda}{|x_i-x_j|}+O\Big(\frac{1}{|x_i-x_j|^{1+\sigma}}+\frac{\varepsilon^{\frac12}}{|x_i-x_j|^{\frac12}}\Big),\quad N=3.
\end{aligned}
\tag{A.27}
$$

For $N\geq 4$, using

$$
|\varphi_0(y)|\leq\frac{C}{(1+|y|)^{N-3}},
$$

we also have

$$
\alpha_N\int_{\Omega_\varepsilon}U_{\frac{1}{\Lambda},x_i}^{2^*-1}\varphi_{\frac{1}{\Lambda},x_j}=O\Big(\frac{\varepsilon}{|x_i-x_j|^{N-3}}\Big),\quad N\geq 4,\ i\neq j,
\tag{A.28}
$$

and

$$
\begin{aligned}
&\int_{\Omega_\varepsilon}\Big(DW_{\Lambda,x_i}DW_{\Lambda,x_j}+\varepsilon^2\mu W_{\Lambda,x_i}W_{\Lambda,x_j}\Big)\\
=&\frac{B_1\Lambda}{|x_i-x_j|^{N-2}}+O\Big(\frac{1}{|x_i-x_j|^{N-2+\sigma}}+\frac{\varepsilon}{|x_i-x_j|^{N-3}}\Big),\quad N\geq 4.
\end{aligned}
\tag{A.29}
$$

We are now ready to compute the energy $I(W)$.

Proposition A.3. *For $N\geq 4$, we have*

$$
I(W)=k\Big(A_0-A_1\Lambda\gamma\varepsilon-A_2\Lambda^{N-2}\varepsilon+o(\varepsilon)\Big),
$$

where A_i, $i=0,1,2$, is some positive constant, and γ is the mean curvature of $\partial\Omega$ along Γ.

Proof. By using the symmetry, (A.23) and (A.29), we have

$$
\begin{aligned}
&\frac{1}{2}\int_{\Omega_\varepsilon}\big(|DW|^2+\mu\varepsilon^2W^2\big)\\
=&k\Big(\frac{1}{2}\int_{\Omega_\varepsilon}\big(|DW_{\Lambda,x_1}|^2+\mu\varepsilon^2W^2_{\Lambda,x_1}\big)+\sum_{j=2}^{k}\int_{\Omega_\varepsilon}\big(DW_{\Lambda,x_1}DW_{\Lambda,x_j}+\mu\varepsilon^2W_{\Lambda,x_1}W_{\Lambda,x_j}\big)\Big)\\
=&k\frac{1}{2}\bigg(\bar{A}_0+(\bar{A}_3-\bar{A}_1)\gamma\Lambda\varepsilon+o(\varepsilon)\\
&\quad+\sum_{j=2}^{k}\Big(\frac{B_1\Lambda^{N-2}}{|x_1-x_j|^{N-2}}+O\big(\frac{\varepsilon}{|x_1-x_j|^{N-3}}+\frac{1}{|x_1-x_j|^{N-2+\sigma}}\big)\Big)\bigg).
\end{aligned}
\tag{A.30}
$$

Let

$$\Omega_j=\big\{y=(y',y'')\in\Omega_\varepsilon:\langle\frac{y'}{|y'|},\frac{x_j}{|x_j|}\rangle\ge\cos\frac{\pi}{k}\big\}.$$

We have

$$
\begin{aligned}
&\frac{\alpha_N}{2^*}\int_{\Omega_\varepsilon}W^{2^*}=\frac{\alpha_Nk}{2^*}\int_{\Omega_1}W^{2^*}\\
=&\frac{\alpha_Nk}{2^*}\bigg(\int_{\Omega_1}W^{2^*}_{\Lambda,x_1}+2^*\int_{\Omega_1}\sum_{i=2}^{k}W^{2^*-1}_{\Lambda,x_1}W_{\Lambda,x_i}+O\Big(\int_{\Omega_1}W^{2^*-2}_{\Lambda,x_1}(\sum_{i=2}^{k}W_{\Lambda,x_i})^2\Big)\bigg).
\end{aligned}
$$

It is easy to check

$$
\begin{aligned}
&\frac{1}{2^*}\alpha_N\int_{\Omega_1}W^{2^*}_{\Lambda,x_1}=\frac{1}{2^*}\alpha_N\int_{\Omega_\varepsilon}W^{2^*}_{\Lambda,x_1}+O\big(\varepsilon^Nk^N\ln\frac{1}{\varepsilon}\big)\\
=&\frac{1}{2^*}\bar{A}_0+\big(\bar{A}_3-\frac{1}{2^*}\bar{A}_1\big)\gamma\Lambda\varepsilon+O(\varepsilon^{1+\sigma}),
\end{aligned}
$$

and

$$\alpha_N\int_{\Omega_1}W^{2^*-1}_{\Lambda,x_1}W_{\Lambda,x_i}=\frac{B_1\Lambda^{N-2}}{|x_i-x_j|^{N-2}}+O\Big(\frac{1}{|x_i-x_j|^{N-2+\sigma}}\Big).$$

Thus, we obtain

$$
\begin{aligned}
\frac{\alpha_N}{2^*}\int_{\Omega_\varepsilon}W^{2^*}=&k\Big(\frac{1}{2^*}\bar{A}_0+\big(\bar{A}_3-\frac{1}{2^*}\bar{A}_1\big)\gamma\Lambda\varepsilon+\sum_{j=2}^{k}\frac{B_1\Lambda^{N-2}}{|x_i-x_j|^{N-2}}\\
&+O\Big(\varepsilon^{1+\sigma}+\big(\ln\frac{1}{\varepsilon}\big)^{2^*}\int_{\Omega_1}U^{2^*-2}_{\frac{1}{\Lambda},x_1}(\sum_{i=2}^{k}U_{\frac{1}{\Lambda},x_i})^2\Big)\Big).
\end{aligned}
\tag{A.31}
$$

Here, we have used

$$|W_{\Lambda,x_j}| \le C|\ln\varepsilon|U_{\frac{1}{\Lambda},x_j},$$

which can be obtained directly from Lemma A.1.

Note that for $y \in \Omega_1$, $|y-x_i| \ge \frac{1}{2}|x_i - x_1|$. Thus

$$\begin{aligned}\sum_{i=2}^{k} U_{\frac{1}{\Lambda},x_i} &\le C\sum_{i=2}^{k}\frac{1}{(1+|y-x_1|)^{\frac{N-3}{2}}}\frac{1}{|x_1-x_i|^{\frac{N-1}{2}}}\\ &\le \frac{1}{(1+|y-x_1|)^{\frac{N-3}{2}}}\sum_{i=2}^{k}\frac{1}{|x_1-x_i|^{\frac{N-1}{2}}}.\end{aligned}$$

As a result,

$$\int_{\Omega_\varepsilon} U_{\frac{1}{\Lambda},x_1}^{2^*-2}\Big(\sum_{i=2}^{k} U_{\frac{1}{\Lambda}.x_i}\Big)^2 = O\big(\varepsilon^{N-1}k^{N-1}\big),$$

which, together with (A.31), gives

$$\frac{\alpha_N}{2^*}\int_{\Omega_\varepsilon} W^{2^*} = k\Big(\frac{1}{2^*}\bar{A}_0 + \big(\bar{A}_3 - \frac{1}{2^*}\bar{A}_1\big)\gamma\Lambda\varepsilon + \sum_{j=2}^{k}\frac{B_1\Lambda^{N-2}}{|x_i-x_j|^{N-2}} + O\big(\varepsilon^{1+\sigma}\big)\Big). \tag{A.32}$$

Combining (A.30) and (A.32), we are led to

$$I(W) = k\Big(A_0 - A_1\gamma\Lambda\varepsilon - \frac{1}{2}\sum_{i=2}^{k}\frac{B_1\Lambda^{N-2}}{|x_1-x_i|^{N-2}} + O\big(\varepsilon^{1+\sigma}\big)\Big), \tag{A.33}$$

where A_0 and A_1 are some positive constants.

Since

$$|x_j - x_1| = 2|x_1|\sin\frac{2(j-1)\pi}{k}, \quad j = 2,\dots,k,$$

we have

$$\sum_{j=2}^{k}\frac{1}{|x_j-x_1|^{N-2}}=\frac{1}{(2|x_1|)^{N-2}}\sum_{j=2}^{k}\frac{1}{(\sin\frac{(j-1)\pi}{k})^{N-2}}$$

$$=\begin{cases}\frac{2}{(2|x_1|)^{N-2}}\sum_{j=2}^{\frac{k}{2}}\frac{1}{(\sin\frac{(j-1)\pi}{k})^{N-2}}+\frac{1}{(2|x_1|)^{N-2}}, & \text{if } k \text{ is even;}\\ \frac{2}{(2|x_1|)^{N-2}}\sum_{j=2}^{[\frac{k}{2}]}\frac{1}{(\sin\frac{(j-1)\pi}{k})^{N-2}}, & \text{if } k \text{ is old.}\end{cases}$$

But

$$0<c'\le\frac{\sin\frac{(j-1)\pi}{k}}{\frac{(j-1)\pi}{k}}\le c'',\quad j=2,\cdots,[\frac{k}{2}].$$

So, there is a constant $B_4>0$, such that

$$\sum_{j=2}^{k}\frac{1}{|x_j-x_1|^{N-2}}=B_4(\varepsilon k)^{N-2}+O\Big(\varepsilon^{N-2}k\Big).$$

Using $\varepsilon=k^{-\frac{N-2}{N-3}}$, we obtain

$$I(W)=k\big(A_0-A_1\gamma\Lambda\varepsilon-A_2\Lambda^{N-2}\varepsilon+o(\varepsilon)\big),$$

where A_0, A_1 and A_2 are some positive constants.

□

For the case $N=3$, we have

Proposition A.4. *For $N=3$, we have*

$$I(W)=k\Big(D_1-D_2\gamma\varepsilon\Lambda\ln\frac{1}{\Lambda\varepsilon}-D_3\varepsilon\Lambda\beta_k k\ln k+O(\varepsilon)\Big),$$

where D_i, $i=1,2,3$, is some positive constant, and $\beta_k\to1$ as $k\to+\infty$.

Proof. Similar to the proof of Proposition A.3, we find

$$I(W)=k\Big(D_1-D_2\gamma\varepsilon\Lambda\ln\frac{1}{\Lambda\varepsilon}-\sum_{j=2}^{k}\frac{\bar D\Lambda}{|x_j-x_1|}+O(\varepsilon)\Big),$$

where D_1, D_2 and $\bar D$ are some positive constant. Noting that $|x_j-x_1|=\frac{2}{\varepsilon}\sin\frac{2(j-1)\pi}{k}$, and

$$\sum_{j=2}^{k}\frac{1}{j}=(c_0+o(1))\ln k,$$

we obtain

$$\sum_{j=2}^{k} \frac{\bar{D}\Lambda}{|x_j - x_1|} = D_3 \varepsilon \Lambda \beta_k k \ln k,$$

where $\beta_k \to 1$ as $k \to \infty$. Thus, the result follows.

□

Appendix B. Basic Estimates

Firstly, we prove that $W \leq C$, where $C > 0$ is a constant, independent of k. We have a more general result.

Lemma B.1. *For any $\alpha > 0$,*

$$\sum_{j=1}^{k} \frac{1}{(1 + |y - x_j|)^\alpha} \leq C\Big(1 + \sum_{j=2}^{k} \frac{1}{|x_1 - x_j|^\alpha}\Big),$$

where $C > 0$ is a constant, independent of k.

Proof. Define

$$\Omega_j = \Big\{ y = (y', y'') \in \mathbb{R}^2 \times \mathbb{R}^{N-2} : \Big\langle \frac{y'}{|y'|}, \frac{x_j}{|x_j|} \Big\rangle \geq \cos \frac{\pi}{k} \Big\}.$$

Without loss of generality, we assume $y \in \Omega_1$. Then,

$$|y - x_j| \geq |y - x_1|, \quad \forall\, y \in \Omega_1.$$

If $|y - x_1| \leq \frac{1}{2}|x_1 - x_j|$,

$$|y - x_j| \geq |x_j - x_1| - |y - x_1| \geq \frac{1}{2}|x_1 - x_j|.$$

But if $|y - x_1| \geq \frac{1}{2}|x_1 - x_j|$,

$$|y - x_j| \geq |y - x_1| \geq \frac{1}{2}|x_1 - x_j|, \quad \forall\, y \in \Omega_1.$$

Thus,

$$|y - x_j| \geq \frac{1}{2}|x_1 - x_j|, \qquad \forall\, y \in \Omega_1,\ j = 2, \cdots, k.$$

Hence,

$$\sum_{j=1}^{k}\frac{1}{(1+|y-x_j|)^\alpha}\le C+\sum_{j=2}^{k}\frac{1}{(1+|y-x_j|)^\alpha}$$
$$\le C\Big(1+\sum_{j=2}^{k}\frac{1}{|x_1-x_j|^\alpha}\Big).$$

□

For each fixed i and j, $i\neq j$, consider the following function

$$g_{ij}(y)=\frac{1}{(1+|y-x_j|)^\alpha}\frac{1}{(1+|y-x_i|)^\beta}, \tag{B.1}$$

where $\alpha\ge 1$ and $\beta\ge 1$ are two constants. The following two lemmas can be found in Appendix B in [64].

Lemma B.2. *For any constant $0\le\sigma\le\min(\alpha,\beta)$, there is a constant $C>0$, such that*

$$g_{ij}(y)\le\frac{C}{|x_i-x_j|^\sigma}\Big(\frac{1}{(1+|y-x_i|)^{\alpha+\beta-\sigma}}+\frac{1}{(1+|y-x_j|)^{\alpha+\beta-\sigma}}\Big).$$

Lemma B.3. *For any constant $0<\sigma<N-2$, there is a constant $C>0$, such that*

$$\int_{\mathbb{R}^N}\frac{1}{|y-z|^{N-2}}\frac{1}{(1+|z|)^{2+\sigma}}\,dz\le\frac{C}{(1+|y|)^\sigma}.$$

Let us recall that

$$\varepsilon=k^{-\frac{N-2}{N-3}}\quad\text{if}\quad N\ge 4,\quad \varepsilon=e^{\frac{D_3}{D_2\gamma}\beta_k k\ln k},\quad\text{if}\quad N=3.$$

Lemma B.4. *Suppose that $\tau=\frac{N-3}{N-2}$. Then there is a small $\theta>0$, such that*

$$\int_{\mathbb{R}^N}\frac{1}{|y-z|^{N-2}}W^{\frac{4}{N-2}}(z)\sum_{j=1}^{k}\frac{1}{(1+|z-x_j|)^{\frac{N-2}{2}+\tau}}\,dz$$
$$\le C\sum_{j=1}^{k}\frac{1}{(1+|y-x_j|)^{\frac{N-2}{2}+\tau+\theta}}+o(1)\sum_{j=1}^{k}\frac{1}{(1+|y-x_j|)^{\frac{N-2}{2}+\tau}},$$

where $o(1)\to 0$ as $k\to+\infty$.

Proof. Firstly, we consider $N \geq 6$. Then $\frac{4}{N-2} \leq 1$. Thus

$$W^{\frac{4}{N-2}}(z) \leq \sum_{i=1}^{k} \frac{1}{(1+|z-x_i|)^{4(1-\beta)}}.$$

where $\beta > 0$ can be chosen as any small fixed constant. So, we obtain

$$\begin{aligned}
&\int_{\mathbb{R}^N} \frac{1}{|y-z|^{N-2}} W^{\frac{4}{N-2}}(z) \sum_{j=1}^{k} \frac{1}{(1+|z-x_j|)^{\frac{N-2}{2}+\tau}}\,dz \\
\leq &\sum_{j=1}^{k} \int_{\mathbb{R}^N} \frac{1}{|y-z|^{N-2}} \frac{1}{(1+|z-x_j|)^{4(1-\beta)+\frac{N-2}{2}+\tau}}\,dz \\
&+ \sum_{j=1}^{k} \sum_{i\neq j} \int_{\mathbb{R}^N} \frac{1}{|y-z|^{N-2}} \frac{1}{(1+|z-x_i|)^{4(1-\beta)}} \frac{1}{(1+|z-x_j|)^{\frac{N-2}{2}+\tau}}\,dz.
\end{aligned}$$

By Lemma B.3, if $\theta > 0$ is so small that $\frac{N-2}{2} + \tau + \theta < N-2$, then

$$\begin{aligned}
&\int_{\mathbb{R}^N} \frac{1}{|y-z|^{N-2}} \frac{1}{(1+|z-x_j|)^{4(1-\beta)+\frac{N-2}{2}+\tau}}\,dz \\
\leq &\int_{\mathbb{R}^N} \frac{1}{|y-z|^{N-2}} \frac{1}{(1+|z-x_j|)^{2+\frac{N-2}{2}+\tau+\theta}}\,dz \leq \frac{C}{(1+|y-x_j|)^{\frac{N-2}{2}+\tau+\theta}}.
\end{aligned}$$

On the other hand, it follows from Lemmas B.2 and B.3 that for $i \neq j$,

$$\begin{aligned}
&\int_{\mathbb{R}^N} \frac{1}{|y-z|^{N-2}} \frac{1}{(1+|z-x_i|)^{4(1-\beta)}} \frac{1}{(1+|z-x_j|)^{\frac{N-2}{2}+\tau}}\,dz \\
\leq &\frac{C}{|x_i-x_j|^2} \int_{\mathbb{R}^N} \frac{1}{|y-z|^{N-2}} \Big(\frac{1}{(1+|z-x_i|)^{2+\frac{N-2}{2}+\tau}} + \frac{1}{(1+|z-x_j|)^{2+\frac{N-2}{2}+\tau}} \Big)\,dz \\
\leq &\frac{C}{|x_i-x_j|^{2-4\beta}} \Big(\frac{1}{(1+|y-x_i|)^{\frac{N-2}{2}+\tau}} + \frac{1}{(1+|y-x_j|)^{\frac{N-2}{2}+\tau}} \Big).
\end{aligned}$$

Noting that

$$\sum_{j\neq i} \frac{1}{|x_i-x_j|^{2-4\beta}} \leq C(\varepsilon k)^{2-4\beta} \sum_{j=1}^{k} \frac{1}{j^{2-4\beta}} \leq C(\varepsilon k)^{2-4\beta} = o(1),$$

we obtain

$$
\begin{aligned}
&\sum_{j=1}^{k}\sum_{i\neq j}\int_{\mathbb{R}^N}\frac{1}{|y-z|^{N-2}}\frac{1}{(1+|z-x_i|)^{4(1-\beta)}}\frac{1}{(1+|z-x_j|)^{\frac{N-2}{2}+\tau}}\,dz\\
=&o(1)\sum_{j=1}^{k}\frac{1}{(1+|y-x_j|)^{\frac{N-2}{2}+\tau}}.
\end{aligned}
$$

Suppose now that $N=5$. Recall that $\varepsilon=k^{-\frac{3}{2}}$ and

$$
\Omega_j=\big\{y=(y',y'')\in\Omega_\varepsilon:\big\langle\frac{y'}{|y'|},\frac{x_j}{|x_j|}\big\rangle\geq\cos\frac{\pi}{k}\big\}.
$$

For $z\in\Omega_1$, we have $|z-x_j|\geq|z-x_1|$. Using Lemma B.2, we obtain

$$
\begin{aligned}
&\sum_{j=2}^{k}\frac{1}{(1+|z-x_j|)^{3(1-\beta)}}\leq\frac{1}{(1+|z-x_1|)^{\frac{3}{2}}}\sum_{j=2}^{k}\frac{1}{(1+|z-x_j|)^{\frac{3}{2}-3\beta}}\\
\leq&\frac{C}{(1+|z-x_1|)^{\frac{7}{3}-3\beta}}\sum_{j=2}^{k}\frac{1}{|x_j-x_1|^{\frac{2}{3}}}\leq\frac{C}{(1+|z-x_1|)^{\frac{7}{3}-3\beta}}
\end{aligned}
$$

since

$$
\sum_{j=2}^{k}\frac{1}{|x_j-x_1|^{\frac{2}{3}}}\leq C(\varepsilon k)^{\frac{2}{3}}\sum_{j=2}^{k}\frac{1}{j^{\frac{1}{3}}}=O(\varepsilon^{\frac{2}{3}}k)=O(1).
$$

Thus,

$$
W^{\frac{4}{3}}(z)\leq\Big(\frac{C}{1+|z-x_1|)^{3(1-\beta)}}+\frac{C}{(1+|z-x_1|)^{\frac{7}{3}-3\beta}}\Big)^{\frac{4}{3}}\leq\frac{C}{(1+|z-x_1|)^{\frac{28}{9}-4\beta}}.
$$

As a result, for $z\in\Omega_1$, using Lemma B.2 again, we find that for $\theta>0$ small,

$$
\begin{aligned}
&W^{\frac{4}{3}}(z)\sum_{j=1}^{k}\frac{1}{(1+|z-x_j|)^{\frac{3}{2}+\tau}}\\
\leq&\frac{C}{(1+|z-x_1|)^{\frac{28}{9}+\frac{3}{2}+\tau-4\beta}}+\frac{C}{(1+|z-x_1|)^{2+\frac{3}{2}+\tau+\theta}}\sum_{j=2}^{k}\frac{1}{|x_j-x_1|^{\frac{10}{9}-\theta-4\beta}}\\
\leq&\frac{C}{(1+|z-x_1|)^{2+\frac{3}{2}+\tau+\theta}}.
\end{aligned}
$$

So, we obtain

$$
\begin{aligned}
&\int_{\Omega_1} \frac{1}{|y-z|^3} W^{\frac{4}{3}}(z) \sum_{j=1}^{k} \frac{1}{(1+|z-x_j|)^{\frac{3}{2}+\tau}}\, dz \\
\le &\int_{\Omega_1} \frac{1}{|y-z|^3} \frac{C}{(1+|z-x_1|)^{2+\frac{3}{2}+\tau+\theta}}\, dz \le \frac{C}{(1+|y-x_1|)^{\frac{3}{2}+\tau+\theta}},
\end{aligned}
$$

which gives

$$
\begin{aligned}
&\int_{\Omega_\varepsilon} \frac{1}{|y-z|^3} W^{\frac{4}{3}}(z) \sum_{j=1}^{k} \frac{1}{(1+|z-x_j|)^{\frac{3}{2}+\tau}}\, dz \\
= &\sum_{i=1}^{k} \int_{\Omega_i} \frac{1}{|y-z|^3} W^{\frac{4}{3}}(z) \sum_{j=1}^{k} \frac{1}{(1+|z-x_j|)^{\frac{3}{2}+\tau}}\, dz \\
\le &\sum_{i=1}^{k} \frac{C}{(1+|y-x_i|)^{\frac{3}{2}+\tau+\theta}}.
\end{aligned}
$$

Suppose that $N=4$. In this case, $\varepsilon = k^{-2}$. We have that for $z \in \Omega_1$,

$$
\begin{aligned}
&\sum_{j=2}^{k} \frac{1}{(1+|z-x_j|)^{2(1-\beta)}} \le \frac{C}{(1+|z-x_1|)^{\frac{2}{3}-2\beta}} \sum_{j=2}^{k} \frac{1}{|x_j-x_1|^{\frac{1}{2}}} \\
\le &\frac{C\varepsilon^{\frac{1}{2}} k}{(1+|z-x_1|)^{\frac{3}{2}-2\beta}} \le \frac{C}{(1+|z-x_1|)^{\frac{3}{2}-2\beta}}
\end{aligned}
$$

and thus,

$$
\begin{aligned}
&W^2(z) \sum_{j=1}^{k} \frac{1}{(1+|z-x_j|)^{1+\tau}} \le \frac{C}{(1+|z-x_1|)^{3-4\beta}} \sum_{j=1}^{k} \frac{1}{(1+|z-x_j|)^{1+\tau}} \\
\le &\frac{C}{(1+|z-x_1|)^{4+\tau-4\beta}} + \frac{C}{(1+|z-x_1|)^{2+1+\tau+\frac{1}{2}-4\beta}} \sum_{j=1}^{k} \frac{1}{|x_1-x_j|^{\frac{1}{2}}} \\
\le &\frac{C}{(1+|z-x_1|)^{2+1+\tau+\frac{1}{2}-4\beta}},
\end{aligned}
$$

which gives

$$\begin{aligned}&\int_{\Omega_\varepsilon}\frac{1}{|y-z|^2}W^2(z)\sum_{j=1}^k\frac{1}{(1+|z-x_j|)^{1+\tau}}\,dz\\&=\sum_{i=1}^k\int_{\Omega_i}\frac{1}{|y-z|^2}W^2(z)\sum_{j=1}^k\frac{1}{(1+|z-x_j|)^{1+\tau}}\,dz\\&\le\sum_{i=1}^k\frac{C}{(1+|y-x_i|)^{\frac{1}{2}+1+\tau-4\beta}}.\end{aligned}$$

For $N=3, z\in\Omega_1$, since $k^n\lambda^{-\alpha}=o(1)$ for any $n>0$ and $\alpha>0$ as $k\to+\infty$, we have for $\alpha>\beta>0$,

$$\sum_{j=2}^k\frac{1}{(1+|z-x_j|)^{1-\beta}}\le C\frac{1}{(1+|z-x_1|)^{1-\alpha}}$$

and

$$W^4(z)\sum_{j=1}^k\frac{1}{(1+|z-x_j|)^{\frac{1}{2}+\tau}}\le\frac{C}{(1+|z-x_1|)^{2+\frac{1}{2}+\tau+2-5\alpha}}$$

which gives

$$\begin{aligned}&\int_{\Omega_\varepsilon}\frac{1}{|y-z|}W^4(z)\sum_{j=1}^k\frac{1}{(1+|z-x_j|)^{\frac{1}{2}+\tau}}\,dz\\&=\sum_{i=1}^k\int_{\Omega_i}\frac{1}{|y-z|}W^4(z)\sum_{j=1}^k\frac{1}{(1+|z-x_j|)^{\frac{1}{2}+\tau}}\,dz\\&\le\sum_{i=1}^k\frac{C}{(1+|y-x_i|)^{\frac{1}{2}+\tau-5\alpha+2}}.\end{aligned}$$

□

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