Concentrating solutions for the Hénon equation in $\mathbb{R}^2$ *

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Abstract

We consider the boundary value problem $\Delta u + |x|^{2\alpha} u^p = 0$, $\alpha > 0$, in the unit ball B with homogeneous Dirichlet boundary condition and p a large exponent. We find a condition which ensures the existence of a positive solution u_p concentrating outside the origin at k symmetric points as p goes to $+\infty$. The same techniques lead also to a more general result on general domains. In particular, we have that concentration at the origin is always possible, provided $\alpha \notin \mathbb{N}$.

Keywords: Hénon equation, Large exponent, Concentrating solutions, Green's function, Finite dimensional reduction

AMS subject classification: 35J60, 35B33, 35J25, 35J20, 35B40

1 Introduction and statement of main results

In this paper, we consider the following so-called Hénon equation ([17])

$$\begin{cases} \Delta u + |x|^{2\alpha} u^p = 0 & \text{in } \Omega, \\ u > 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

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where $\alpha > 0$, Ω is a bounded domain in $\mathbb{R}^N$ $(N \geq 2)$ containing the origin, and $p > 1$.

Problem (1.1) has attracted a lot of studies in recent years. In [19], Ni showed the existence of a radially symmetric solution when $p < \frac{N+2+2\alpha}{N-2-2\alpha}$ for $N \geq 3$ and $\Omega = B_1(0)$. When $\Omega = B_1(0) \subset \mathbb{R}^2$, numerical computations by Chen, Ni and Zhou [9] suggest that for some parameters (α, p), the ground state solutions are nonradial. This was partially confirmed recently by Smets, Su and William in [24], in which it was proved that for each $2 < p+1 < 2^*(= \frac{2N}{N-2}$ if $N \geq 3; = +\infty$ if $N = 2)$, there exists α^* such that for $\alpha > \alpha^*$ the ground states are nonradial. They also showed that for fixed α, the ground state solution must be radial if p is close to 1. When $N \geq 2$, the asymptotic behavior of (radial or nonradial) ground state solutions as $\alpha \to +\infty$ is studied by Byeon and Wang in [3, 4], in which they proved that the ground state solution develop boundary concentrations. In another direction, when $N \geq 3$, α is fixed, $\Omega = B_1$, and $p + 1 \to \frac{2N}{N-2}$, Cao and Peng [5] showed that the ground state solution develops a boundary bubble (hence must be nonradial). In [6] and [20], multiple boundary concentrations have been constructed when $N \geq 3, \Omega = B_1$ and $p \to \frac{N+2}{N-2}$.

In this paper, we consider the problem (1.1) when $N = 2$ and p is large, i.e., the following boundary value problem:

$$\begin{cases} \Delta u + |x|^{2\alpha} u^p = 0 & \text{in } B, \\ u > 0 & \text{in } B, \\ u = 0 & \text{on } \partial B, \end{cases} \tag{1.2}$$

where $\alpha > 0$, $B = B(0,1)$ is the unit ball in $\mathbb{R}^2$ and p is a large exponent. Unlike [5], as $p \to +\infty$, there are no boundary concentration solutions. The proof of this fact follows from the same proof of Proposition 5 of [18]. One of the main results of this paper is to show the presence of solutions concentrating at the origin or outside the origin as long as $\alpha \notin I\!N$ and Ω contains the origin.

Let $K_\alpha = \max\{k \in I\!N : k < \alpha + 1\}$. Concerning concentration outside the origin, the main result we obtain for (1.2) is the following:

Theorem 1.1 *There exists $p_0 > 0$ large such that for any $1 \leq k \leq K_\alpha$ and $p \geq p_0$ problem (1.2) has a solution u_p which concentrates at k (symmetric) different points of $B \setminus \{0\}$, i.e. as p goes to $+\infty$*

$$p|x|^{2\alpha} u_p^{p+1} \rightharpoonup 8\pi e \sum_{i=1}^{k} \delta_{\xi_i} \text{ weakly in the sense of measure in } \overline{B}$$

for some $\xi = (\xi_1, \ldots, \xi_k)$. More precisely, for any $\delta > 0$ as $p \to +\infty$:

$$\max_{B \setminus \cup_{i=1}^k B(\xi_i, \delta)} u_p \to 0\,, \qquad \sup_{B(\xi_i,\delta)} u_p \to \sqrt{e}.$$

Theorem 1.1 is based on a constructive method which works also for a more general problem:

$$\begin{cases} \Delta u + a(x)u^p = 0 & \text{in } \Omega, \\ u > 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.3}$$

where Ω is a smooth, bounded domain in $\mathbb{R}^2$, p is a large exponent and $a(x) \geq 0$ is a potential (eventually vanishing somewhere in Ω).
Set

$$Z := \{q \in \Omega : \ a(x) = |x-q|^{2\alpha_q} a_q(x), \ a_q(q) > 0\}.$$

Let $G(x,y)$ be the Green's function, i.e. the solution of the problem

$$\begin{cases} -\Delta_x G(x,y) = \delta_y(x) & x \in \Omega, \\ G(x,y) = 0 & x \in \partial\Omega, \end{cases}$$

and let $H(x,y)$ be the regular part defined as

$$H(x,y) = G(x,y) - \frac{1}{2\pi} \log \frac{1}{|x-y|}.$$

Let $q_1, \ldots, q_m \in Z$ be different points so that $\alpha_i := \alpha_{q_i} \notin I\!N$ for any $i = 1, \ldots, m$. In order to find a solution concentrating at $q_1, \ldots, q_m$ and at $\xi_1, \ldots, \xi_k \in \Omega \setminus Z$, the location of the concentration points $\xi_1, \ldots, \xi_k$ should be a critical point of the following function:

$$\begin{aligned} \Phi(\xi_1, \ldots, \xi_k) &= \sum_{i=1}^{k} H(\xi_i, \xi_i) + \sum_{\substack{i,j=1 \\ i \neq j}}^{k} G(\xi_i, \xi_j) + \frac{1}{4\pi} \sum_{i=1}^{k} \log a(\xi_i) \\ &\quad + 2 \sum_{i=1}^{k} \sum_{j=1}^{m} (1+\alpha_j) G(\xi_i, q_j), \end{aligned}$$

where

$$(\xi_1, \ldots, \xi_k) \in \mathcal{M} := \left\{ (\xi_1, \ldots, \xi_k) \in (\Omega \setminus Z)^k \ : \ \xi_i \neq \xi_j \text{ for } i \neq j \right\}.$$

The role of the function Φ in concentration phenomena was already shown for (1.3) with $a(x) = 1$ in [15] (see also [2, 10, 14] in the context of the mean field equation). Considering changing sign solutions of (1.2) (u^p replaced by $|u|^{p-1}u$ in the equation), we can allow also negative concentration phenomena and the function responsible to locate the concentration points is "essentially" Φ as already shown in [16] for $a(x) = 1$. To understand the role of Φ in presence of some concentration point in Z, we refer to [12, 13] where in the context of the mean field equation blowing up solutions are constructed.

The result we have is the following:

Theorem 1.2 *Let $q_1,\dots,q_m \in Z$ be different points so that $\alpha_i=\alpha(q_i)\notin I\!N$ for any $i=1,\dots,m$. Let $k\ge 1$ and assume that $(\xi_1^*,\dots,\xi_k^*)\in\mathcal{M}$ is a C^0-stable critical point of Φ (according to Definition 3.1). Then, there exists $p_0>0$ such that for any $p\ge p_0$ problem (1.3) has a solution u_p which concentrates at $m+k$ different points of Ω, i.e. as p goes to $+\infty$*

$$pa(x)u_p^{p+1} \rightharpoonup 8\pi e\sum_{i=1}^{m}(\alpha_i+1)\delta_{q_i}+8\pi e\sum_{i=1}^{k}\delta_{\xi_i}$$

weakly in the sense of measure in $\overline{\Omega}$, for some $\xi\in\mathcal{M}$ such that $\Phi(\xi_1,\dots,\xi_k)=\Phi(\xi_1^*,\dots,\xi_k^*)$. More precisely, for any $\delta>0$ as p goes to $+\infty$:

$$u_p\to 0 \quad \text{uniformly in } \Omega\setminus(\cup_{i=1}^m B(q_i,\delta))\cup\left(\cup_{i=1}^k B(\xi_i,\delta)\right)$$

and

$$\sup_{x\in B(q_i,\delta)} u_p(x)\to\sqrt{e}\,,\qquad \sup_{x\in B(\xi_j,\delta)} u_p(x)\to\sqrt{e}$$

for any $i=1,\dots,m$ and $j=1,\dots,k$.

Remark 1.1 Let us remark that Theorem 1.2 implies the existence of solutions for (1.2) concentrating at the origin, provided $\alpha\notin I\!N$. Moreover, by means of a Pohozaev identity, it is easy to show that, in the class we are considering (according to a specific ansatz we will describe below), it is not possible to construct solutions for (1.2) concentrating at the origin and some other points.

As in the mean field equation, it is possible to identify a limit profile problem of Liouville-type (for $a(x)=1$ see the aymptotic analysis in [1, 11, 22, 23]):

$$\begin{cases}\Delta u+|x|^{2\alpha}e^u=0 \text{ in } \mathbb{R}^2,\\ \int_{\mathbb{R}^2}|x|^{2\alpha}e^u<\infty\end{cases}\tag{1.4}$$

with $\alpha\ge 0$. Problem (1.4) possesses exactly a three-parameters family of solutions:

$$U_{\delta,\xi}(x)=\log\frac{8\delta^2}{(\delta^2+|x-\xi|^2)^2},\ \delta>0,\ \xi\in\mathbb{R}^2\tag{1.5}$$

if $\alpha=0$ (see [8]), and a one-parameter family of solutions:

$$U_\delta(x)=\log\frac{8(\alpha+1)^2\delta^2}{(\delta^2+|x|^{2(\alpha+1)})^2},\ \delta>0\tag{1.6}$$

if $\alpha\notin I\!N$ (see [21]).

We will build solutions for problem (1.3) that, up to a suitable normalization, look like a sum of concentrated solutions for the limit profile problem (1.4) centered at several points $q_1,\dots,q_m,\xi_1,\dots,\xi_k$ as $p\to\infty$. We are going to use some arguments and ideas introduced in [15, 16].

The paper is organized as follows. In Section 2 we describe exactly the ansatz for the solution we are looking for and we rewrite the problem in term of a linear operator L (for which a solvability theory is performed in Appendix C). In Section 3 we solve an auxiliary non linear problem and, by reducing (1.3) to solve a finite system $c_{ij} = 0$, we will give in Section 4 the proof of Theorem 1.2. In Section 5, we provide in a radial setting the proof of Theorem 1.1.

2 Approximating solutions

Let us consider the problem

$$\begin{cases} -\Delta u = a(x) g_p(u) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{2.1}$$

Here $g_p(s) = (s^+)^p$. Let $q_1, \ldots, q_m \in Z$ and set $\alpha_i = \alpha_{q_i}$, $a_i(x) = a_{q_i}(x)$, for any $i = 1, \ldots, m$. Assume that $\alpha_i \notin I\!N$ and $|q_i - q_j| \geq 2\varepsilon$ for any $i \neq j$, for some small $\varepsilon > 0$. Take a $k-$tuple $\xi = (\xi_1, \ldots, \xi_k) \in \mathcal{O}_\varepsilon$, where

$$\mathcal{O}_\varepsilon = \left\{ \xi \in \Omega^k : \ \text{dist}\,(\xi_i, \partial(\Omega \setminus Z)) \geq 2\varepsilon,\ |\xi_i - \xi_j| \geq 2\varepsilon,\ i \neq j \right\}.$$

Define $q_i = \xi_{i-m}$, $\alpha_i = 0$ and $a_i(x) = a(x)$ for any $i = m+1, \ldots, m+k$.

Let $i = 1, \ldots, m+k$. Let us set $U^i(y) := \log \frac{8(\alpha_i+1)^2}{(1+|y|^{2(\alpha_i+1)})^2}$. Let f^{0i}, f^{1i} be defined in (A.1), (A.2) and V^i, W^i be the solutions of (A.1), (A.2) with $\alpha = \alpha_i$, for any $i = 1, \ldots, m+k$. Define

$$U_{\delta_i, q_i}(x) = U^i\left(\delta_i^{-\frac{1}{\alpha+1}}(x - q_i)\right) - 2\log\delta_i = \log \frac{8(\alpha_i+1)^2\delta_i^2}{(\delta_i^2 + |x - q_i|^{2(\alpha_i+1)})^2}$$

and

$$V_{\delta_i, q_i}(x) = V^i\left(\delta_i^{-\frac{1}{\alpha+1}}(x - q_i)\right), \quad W_{\delta_i, q_i}(x) = W^i\left(\delta_i^{-\frac{1}{\alpha+1}}(x - q_i)\right).$$

Set:

$$U_\xi(x) := \sum_{i=1}^{m+k} \frac{1}{\gamma \mu_i^{\frac{2}{p-1}} a_i(q_i)^{\frac{1}{p-1}}} P\left(U_{\delta_i, q_i} + \frac{1}{p} V_{\delta_i, q_i} + \frac{1}{p^2} W_{\delta_i, q_i}\right),$$

where

$$\gamma := p^{\frac{p}{p-1}} e^{-\frac{p}{2(p-1)}}$$

and the concentration parameters satisfy

$$\delta_i = \mu_i e^{-\frac{p}{4}} \tag{2.2}$$

(with μ_i to be chosen below).

By Lemmata B.1-B.2 we have that for $|x-q_i|\le\varepsilon$:

$$\begin{aligned} U_\xi(x) &= \frac{1}{\gamma\mu_i^{\frac{2}{p-1}}a_i(q_i)^{\frac{1}{p-1}}}\Bigg(p+U^i(y)-\log 8(\alpha_i+1)^2\mu_i^4+\frac{1}{p}V^i(y)+\frac{1}{p^2}W^i(y)\\ &+ 8\pi H(x,q_i)\left(\alpha_i+1-\frac{C_0(\alpha_i)}{4p}-\frac{C_1(\alpha_i)}{4p^2}\right)+\frac{1}{\alpha_i+1}\frac{\log\delta_i}{p}\left(C_0(\alpha_i)+\frac{C_1(\alpha_i)}{p}\right)\\ &+ 8\pi\sum_{j\neq i}\left(\frac{\mu_i^2a_i(q_i)}{\mu_j^2a_j(q_j)}\right)^{\frac{1}{p-1}}G(x,q_j)\left(\alpha_j+1-\frac{C_0(\alpha_j)}{4p}-\frac{C_1(\alpha_j)}{4p^2}\right)+O(e^{-\frac{p}{4}})\Bigg), \end{aligned}$$

where $y=\delta_i^{-\frac{1}{\alpha_i+1}}(x-q_i)$.

Let us choose $\{\mu_i\}$ as the solution of the following system:

$$\begin{aligned} &\log(8(\alpha_i+1)^2\mu_i^4)=8\pi H(q_i,q_i)\left(\alpha_i+1-\frac{C_0(\alpha_i)}{4p}-\frac{C_1(\alpha_i)}{4p^2}\right)\\ &+\frac{\log\delta_i}{p(\alpha_i+1)}\left(C_0(\alpha_i)+\frac{C_1(\alpha_i)}{p}\right)+\\ &8\pi\sum_{j\neq i}\left(\frac{\mu_i^2a_i(q_i)}{\mu_j^2a_j(q_j)}\right)^{\frac{1}{p-1}}G(q_i,q_j)\left(\alpha_j+1-\frac{C_0(\alpha_j)}{4p}-\frac{C_1(\alpha_j)}{4p^2}\right), \end{aligned} \tag{2.3}$$

in order to get that

$$\begin{aligned} U_\xi(x) &= \frac{1}{\gamma\mu_i^{\frac{2}{p-1}}a_i(q_i)^{\frac{1}{p-1}}}\left(p+U^i(y)+\frac{1}{p}V^i(y)+\frac{1}{p^2}W^i(y)\right)\\ &+O\left(\frac{e^{-\frac{p}{4(\alpha_i+1)}|y|}}{\gamma}+\frac{e^{-\frac{p}{4}}}{\gamma}\right) \end{aligned} \tag{2.4}$$

for $|y|\le\varepsilon\delta_i^{-\frac{1}{\alpha_i+1}}$.

For p large, μ_i bifurcates from the solution of (2.3) with $p=+\infty$:

$$\mu_i=e^{-\frac{3}{4}}e^{2\pi(\alpha_i+1)H(q_i,q_i)+2\pi\sum_{j\neq i}(\alpha_j+1)G(q_j,q_i)}(1+O(\frac{1}{p})) \tag{2.5}$$

in view of the value of $C_0(\alpha)$ (see (A.5)).

Remark 2.1 Let us remark that U_ξ is a positive function. Since

$$p+U^i+\frac{1}{p}V^i+\frac{1}{p^2}W^i\ge\log\frac{2(\alpha_i+1)^2\mu_i^4}{\varepsilon^{4(\alpha_i+1)}}-C$$

in $|y|\le\varepsilon\delta_i^{-\frac{1}{\alpha_i+1}}$, by (2.4) we get that U_ξ is positive in $B(q_i,\varepsilon)$ for any $i=1,\dots,m+k$ for ε sufficiently small. Moreover, by elliptic regularity theory Lemmata B.1-B.2 imply that for any $i=1,\dots,m+k$:

$$P\left(U_{\delta_i,q_i}+\frac{1}{p}V_{\delta_i,q_i}+\frac{1}{p^2}W_{\delta_i,q_i}\right)\to 8\pi(\alpha_i+1)G(\cdot,q_i)$$

in C^1-norm on $|x-q_i|\geq\varepsilon$. Hence, since $\frac{\partial G}{\partial n}(\cdot,q_i)<0$ on $\partial\Omega$, U_ξ is a positive function in Ω.

We will look for solutions u of problem (2.1) in the form $u=U_\xi+\phi$, where ϕ will represent an higher order term in the expansion of u. In terms of ϕ, problem (2.1) becomes

$$\begin{cases} L(\phi)=-[R+N(\phi)] & \text{in } \Omega,\\ \phi=0 & \text{on } \partial\Omega,\end{cases}$$

where

$$L(\phi):=\Delta\phi+\ a(x)g_p'(U_\xi)\phi, \tag{2.6}$$
$$R:=\Delta U_\xi+a(x)g_p(U_\xi), \tag{2.7}$$
$$N(\phi)=a(x)[g_p(U_\xi+\phi)-g_p(U_\xi)-g_p'(U_\xi)\phi]. \tag{2.8}$$

For any $h\in L^\infty(\Omega)$, define

$$\|h\|_*=\sup_{x\in\Omega}\Big|\left(\sum_{i=1}^{m+k}\frac{\delta_i|x-q_i|^{2\alpha_i}}{(\delta_i^2+|x-q_i|^{2(\alpha_i+1)})^{\frac{3}{2}}}\right)^{-1}h(x)\Big|. \tag{2.9}$$

We conclude this section by proving an estimate on R in $\|\cdot\|_*$.

Proposition 2.1 *There exist $C>0$ and $p_0>0$ such that for any $\xi\in\mathcal{O}_\varepsilon$ and $p\geq p_0$*

$$\|\Delta U_\xi+a(x)U_\xi^p\|_*\leq\frac{C}{p^4}. \tag{2.10}$$

Proof. Observe that by equations (A.1)-(A.2):

$$\begin{aligned}\Delta U_\xi(x) &= \sum_{i=1}^{m+k}\frac{\delta_i^{-\frac{2}{\alpha_i+1}}}{\gamma\mu_i^{\frac{2}{p-1}}a_i(q_i)^{\frac{1}{p-1}}}\left(-|y_i|^{2\alpha_i}e^{U^i(y_i)}+\frac{1}{p}\Delta V^i(y_i)+\frac{1}{p^2}\Delta W^i(y_i)\right)\\ &= \sum_{i=1}^{m+k}\frac{\delta_i^{-\frac{2}{\alpha_i+1}}}{\gamma\mu_i^{\frac{2}{p-1}}a_i(q_i)^{\frac{1}{p-1}}}|y_i|^{2\alpha_i}\left(-e^{U^i(y_i)}+\frac{1}{p}f^{0i}(y_i)+\frac{1}{p^2}f^{1i}(y_i)\right.\\ &\qquad \left.-\frac{1}{p}e^{U^i(y_i)}V^i(y_i)-\frac{1}{p^2}e^{U^i(y_i)}W^i(y_i)\right),\end{aligned} \tag{2.11}$$

where $y_i=\delta_i^{-\frac{1}{\alpha_i+1}}(x-q_i)$. If $|x-q_i|\geq\varepsilon$ for any $i=1,\ldots,m+k$, by (B.2), (B.4) formula (2.11) gives that:

$$\begin{aligned}&\Big|\left(\sum_{j=1}^{m+k}\frac{\delta_j|x-q_j|^{2\alpha_j}}{(\delta_j^2+|x-q_j|^{2(\alpha_j+1)})^{\frac{3}{2}}}\right)^{-1}\left(\Delta U_\xi+a(x)U_\xi^p\right)(x)\Big| \\ &\leq Ce^{\frac{p}{4}}\left((\frac{C}{p})^p+pe^{-\frac{p}{2}}\right)=O(pe^{-\frac{p}{4}}).\end{aligned} \tag{2.12}$$

While, if $|x-q_i|\le\varepsilon$ for some $i=1,\dots,m+k$,

$$|\Delta U_\xi + a(x)U_\xi^p| = \Big|\frac{\delta_i^{-\frac{2}{\alpha_i+1}}}{\gamma\mu_i^{\frac{2}{p-1}}a_i(q_i)^{\frac{1}{p-1}}}|y|^{2\alpha_i}\left(-e^{U^i}+\frac{1}{p}f^{0i}+\frac{1}{p^2}f^{1i}-\frac{1}{p}e^{U^i}V^i\right.$$
$$\left.-\frac{1}{p^2}e^{U^i}W^i\right)+\delta_i^{\frac{2\alpha_i}{\alpha_i+1}}|y|^{2\alpha_i}a_i(\delta_i^{\frac{1}{\alpha_i+1}}y+q_i)U_\xi^p(\delta_i^{\frac{1}{\alpha_i+1}}y+q_i)+O(pe^{-\frac{p}{2}})\Big|$$

where $y=\delta_i^{-\frac{1}{\alpha_i+1}}(x-q_i)$. By (2.4) we deduce that, for $x=\delta_i^{\frac{1}{\alpha_i+1}}y+q_i$:

$$\begin{aligned} U_\xi^p(x) &= \left(\frac{p}{\gamma\mu_i^{\frac{2}{p-1}}a_i(q_i)^{\frac{1}{p-1}}}\right)^p\left(1+\frac{1}{p}U^i(y)+\frac{1}{p^2}V^i(y)+\frac{1}{p^3}W^i(y)\right. \\ &+ \left. O\left(\frac{e^{-\frac{p}{4(\alpha_i+1)}}}{p}|y|+\frac{e^{-\frac{p}{4}}}{p}\right)\right)^p. \end{aligned} \tag{2.13}$$

By Taylor expansions of exponential and logarithmic function, we have that, for $|y|\le Ce^{\frac{p}{8(\alpha+1)}}$,

$$\begin{aligned} \left(1+\frac{a}{p}+\frac{b}{p^2}+\frac{c}{p^3}\right)^p &= e^a\left[1+\frac{1}{p}(b-\frac{a^2}{2})+\frac{1}{p^2}\left(c-ab+\frac{a^3}{3}\right.\right. \\ &\quad \left.\left.+\frac{b^2}{2}+\frac{a^4}{8}-\frac{a^2b}{2}\right)+O\left(\frac{\log^6(|y|+2)}{p^3}\right)\right] \end{aligned} \tag{2.14}$$

provided $-5(\alpha+1)\log(|y|+2)\le a(y)\le C$ and $|b(y)|+|c(y)|\le C\log(|y|+2)$. Since $(\frac{p}{\gamma\mu_i^{\frac{2}{p-1}}})^p=\frac{1}{\gamma\delta_i^2\mu_i^{\frac{2}{p-1}}}$, by (2.14) we get that for $|x-q_i|\le\varepsilon\delta_i^{\frac{1}{2(\alpha_i+1)}}$

$$\begin{aligned} U_\xi^p(x) &= \frac{1}{\gamma\delta_i^2\mu_i^{\frac{2}{p-1}}a_i(q_i)^{\frac{p}{p-1}}}e^{U^i(y)}\left[1+\frac{1}{p}\left(V^i-\frac{1}{2}(U^i)^2\right)(y)\right. \\ &\quad+\frac{1}{p^2}\left(W^i-U^iV^i+\frac{1}{3}(U^i)^3+\frac{1}{2}(V^i)^2+\frac{1}{8}(U^i)^4-\frac{1}{2}V^i(U^i)^2\right)(y) \\ &\quad\left.+O\left(\frac{\log^6(|y|+2)}{p^3}+e^{-\frac{p}{4(\alpha_i+1)}}|y|+e^{-\frac{p}{4}}\right)\right], \end{aligned}$$

where $y=\delta_i^{-\frac{1}{\alpha_i+1}}(x-q_i)$. Since

$$\delta_i^{\frac{2\alpha_i}{\alpha_i+1}}|y|^{2\alpha_i}\frac{a_i(\delta_i^{\frac{1}{\alpha_i+1}}y+q_i)}{a_i(q_i)}U_\xi^p(\delta_i^{\frac{1}{\alpha_i+1}}y+q_i)=O(p^2\delta_i^{-\frac{1}{\alpha_i+1}}|y|^{2\alpha_i+1}e^{U^i(y)}),$$

we get that in this region

$$|\Delta U_\xi+a(x)U_\xi^p| = \Big|\frac{\delta_i^{-\frac{2}{\alpha_i+1}}}{\gamma\mu_i^{\frac{2}{p-1}}a_i(q_i)^{\frac{1}{p-1}}}|y|^{2\alpha_i}\left(-e^{U^i}+\frac{1}{p}f^{0i}+\frac{1}{p^2}f^{1i}\right. \tag{2.15}$$

$$-\frac{1}{p}e^{U^i}V^i-\frac{1}{p^2}e^{U^i}W^i\Big)+\delta_i^{\frac{2\alpha_i}{\alpha_i+1}}|y|^{2\alpha_i}a_i(q_i)U_\xi^p(\delta_i^{\frac{1}{\alpha_i+1}}y+q_i)$$
$$+O(p^2\delta_i^{-\frac{1}{\alpha_i+1}}|y|^{2\alpha_i+1}e^{U^i(y)}+pe^{-\frac{p}{2}})\Big|$$
$$=\frac{1}{\delta_i^{\frac{2}{\alpha_i+1}}}|y|^{2\alpha_i}e^{U^i(y)}O(\frac{1}{p^4}\log^6(|y|+2)+p^2\delta_i^{\frac{1}{\alpha_i+1}}|y|)+O(pe^{-\frac{p}{2}}).$$

Hence, in this region we obtain that

$$\left|\left(\sum_{j=1}^{m+k}\frac{\delta_j|x-q_j|^{2\alpha_j}}{(\delta_j^2+|x-q_j|^{2(\alpha_j+1)})^{\frac{3}{2}}}\right)^{-1}\left(\Delta U_\xi+a(x)U_\xi^p\right)(x)\right| \tag{2.16}$$
$$\leq C\delta_i^{\frac{2}{\alpha_i+1}}\frac{(1+|y|^{2(\alpha_i+1)})^{\frac{3}{2}}}{|y|^{2\alpha_i}}\frac{1}{\delta_i^{\frac{2}{\alpha_i+1}}}|y|^{2\alpha_i}e^{U^i(y)}(\frac{1}{p^4}\log^6(|y|+2)$$
$$+p^2\delta_i^{\frac{1}{\alpha_i+1}}|y|)+Cpe^{-\frac{p}{4}}\leq\frac{C}{p^4},$$

where $y=\delta_i^{-\frac{1}{\alpha_i+1}}(x-q_i)$. Let us remark that, if $m+k=1$ the weighted $\|\cdot\|_*$-norm has a singular weight at q_1. However, the expression for $\Delta U_\xi+a(x)U_\xi^p$ in (2.15) reduces to take the form:

$$\Delta U_\xi+a(x)U_\xi^p=\frac{1}{\delta_1^{\frac{2}{\alpha_1+1}}}|y|^{2\alpha_1}e^{U^1(y)}O(\frac{1}{p^4}\log^6(|y|+2)+p^2\delta_1^{\frac{1}{\alpha_1+1}}|y|)$$

since the term $O(pe^{-\frac{p}{2}})$ comes out from the interaction with all the other concentration points. Hence, the estimate (2.16) does not present any problem.

On the other hand, if $\varepsilon\delta_i^{\frac{1}{2(\alpha_i+1)}}\leq|x-q_i|\leq\varepsilon$ we have that by (2.11):

$$|\Delta U_\xi|=O\left(pe^{-\frac{p}{2}}+p\delta_i^{-\frac{2}{\alpha_i+1}}|y|^{2\alpha_i}e^{U^i(y)}\right),$$

and by (2.13)

$$a(x)U_\xi^p(x)=O\left(\frac{1}{\gamma}\delta_i^{-\frac{2}{\alpha_i+1}}|y|^{2\alpha_i}e^{U^i(y)}\right),$$

since $(1+\frac{s}{p})^p\leq e^s$, where $y=\delta_i^{-\frac{1}{\alpha_i+1}}(x-q_i)$. Thus, in this region

$$\left|\left(\sum_{j=1}^{m+k}\frac{\delta_j|x-q_j|^{2\alpha_j}}{(\delta_j^2+|x-q_j|^{2(\alpha_j+1)})^{\frac{3}{2}}}\right)^{-1}\left(\Delta U_\xi+a(x)U_\xi^p\right)(x)\right| \tag{2.17}$$
$$\leq Cpe^{-\frac{p}{4}}+\frac{Cp}{(1+|y|^{2(\alpha_i+1)})^{\frac{1}{2}}}\leq Cpe^{-\frac{p}{8}},\ y=\delta_i^{-\frac{1}{\alpha_i+1}}(x-q_i).$$

By (2.12), (2.16) and (2.17) we obtain the desired result. ∎

3 The finite dimensional reduction

First of all, we will solve the following linear problem: given $h \in C(\bar{\Omega})$, we consider the linear problem of finding a function $\phi \in W^{2,2}(\Omega)$ such that

$$\begin{cases} L(\phi) \;=\; h + \sum_{j=1}^{2} \sum_{i=m+1}^{m+k} c_{ij} e^{U_{\delta_i,q_i}} Z_{ij} & \text{in } \Omega, \\ \phi = 0 & \text{on } \partial\Omega, \\ \int_\Omega e^{U_{\delta_i,q_i}} Z_{ij} \phi = 0 \quad j = 1,2,\; i = m+1, \dots, m+k, \end{cases} \tag{3.1}$$

for some coefficients c_{ij}, $j = 1, 2$ and $i = m+1, \dots, m+k$. Here and in the sequel, we denote for any $i = 1, \dots, m+k$:

$$Z_{i0}(x) := \frac{|x-q_i|^{2(\alpha_i+1)} - \delta_i^2}{|x-q_i|^{2(\alpha_i+1)} + \delta_i^2}$$

and for any $j = 1, 2$, $i = m+1, \dots, m+k$:

$$Z_{ij}(x) := \frac{4\delta_i (x-\xi_i)_j}{\delta_i^2 + |x-\xi_i|^2}.$$

Following the approach in [15, 16] for $a(x) = 1$ (see also [10, 14]), in Appendix C we prove:

Proposition 3.1 *There exist $p_0 > 0$ and $C > 0$ such that, for $h \in C(\bar{\Omega})$ there is a unique solution to problem (3.1), for any $p > p_0$ and $\xi \in \mathcal{O}_\varepsilon$, which satisfies*

$$\|\phi\|_\infty \le C p \|h\|_*. \tag{3.2}$$

Moreover

$$\sum_{j=1}^{2} \sum_{i=m+1}^{m+k} |c_{ij}| \le C \left(\frac{1}{p} \|\phi\|_\infty + \|h\|_* \right) \tag{3.3}$$

and

$$\|\phi\| \le C \left(\|\phi\|_\infty + \|h\|_* \right). \tag{3.4}$$

Let us now introduce the following nonlinear auxiliary problem:

$$\begin{cases} \Delta(U_\xi + \phi) + a(x) g_p \left(U_\xi + \phi \right) = \sum_{j=1}^{2} \sum_{i=m+1}^{m+k} c_{ij} e^{U_{\delta_i,q_i}} Z_{ij} & \text{in } \Omega, \\ \phi = 0 & \text{on } \partial\Omega, \\ \int\limits_\Omega e^{U_{\delta_i,q_i}} Z_{ij} \phi = 0 \quad j = 1,2,\; i = m+1, \dots, m+k, \end{cases} \tag{3.5}$$

for some coefficients c_{ij}. The following result holds:

Proposition 3.2 *Let $\varepsilon > 0$ be fixed. There exist $c > 0$ and $p_0 > 0$ such that for any $p > p_0$ and $\xi \in \mathcal{O}_\varepsilon$ problem (3.5) has a unique solution $\phi_p(\xi)$ which satisfies $\|\phi_p(\xi)\|_\infty \le \frac{c}{p^3}$. Furthermore, the function $\xi \to \phi_p(\xi)$ is a C^1 function in $L^\infty(\Omega)$ and in $H_0^1(\Omega)$.*

Proof. Using (2.6)-(2.8) we can rewrite problem (3.5) in the following way

$$L(\phi) = -\left(R + N(\phi)\right) + \sum_{i,j} c_{ij} e^{U_{\delta_i, q_i}} Z_{ij}.$$

Let us denote by C_* the function space $C(\bar{\Omega})$ endowed with the norm $\|\cdot\|_*$. Proposition 3.1 ensures that the unique solution $\phi = T(h)$ of (3.1) defines a continuous linear map from the Banach space C_* into $C(\bar{\Omega})$, with a norm bounded by a multiple of p. Then, problem (3.5) becomes

$$\phi = \mathcal{A}(\phi) := -T\big[R + N(\phi)\big].$$

Let $\mathcal{B}_r := \left\{ \phi \in \mathrm{C}(\overline{\Omega}) : \ \phi = 0 \text{ on } \partial\Omega, \ \|\phi\|_\infty \le \frac{r}{p^3} \right\}$, for some $r > 0$. Arguing as in [15], using Remark C.1 we can prove that the following estimates hold for any $\phi, \phi_1, \phi_2 \in \mathcal{B}_r$

$$\|N(\phi)\|_* \le cp\|\phi\|_\infty^2, \ \|N(\phi_1) - N(\phi_2)\|_* \le cp \max_{i=1,2} \|\phi_i\|_\infty \|\phi_1 - \phi_2\|_\infty. \quad (3.6)$$

By (3.6), Proposition 2.1 and Proposition 3.1, we easily deduce that $\mathcal{A}$ is a contraction mapping of $\mathcal{B}_r$ for a suitable $r > 0$. Finally, a unique fixed point of $\mathcal{A}$ exists in $\mathcal{B}_r$. The regularity of the map $\xi \to \phi_p(\xi)$ follows using standard arguments as in [15]. ■

After problem (3.5) has been solved, we find a solution to problem (2.1), if we are able to find a point $\xi = (\xi_1, \ldots, \xi_k)$ such that coefficients $c_{ij}(\xi)$ in (3.5) satisfy

$$c_{ij}(\xi) = 0 \text{ for } i = m+1, \ldots, m+k, \ j = 1, 2.$$

Let us introduce the energy functional $J_p : \mathrm{H}_0^1(\Omega) \to \mathbb{R}$ given by

$$J_p(u) := \frac{1}{2} \int_\Omega |\nabla u|^2 dx - \frac{1}{p+1} \int_\Omega a(x)(u^+)^{p+1} dx,$$

whose critical points are solutions to (2.1). We also introduce the finite dimensional restriction $\widetilde{J}_p : \mathcal{M} \to \mathbb{R}$ given by

$$\widetilde{J}_p(\xi) := J_p\big(U_\xi + \phi_p(\xi)\big). \quad (3.7)$$

The following result can be proved using standard arguments as in [15, 16]:

Lemma 3.1 *For all p sufficiently large, if $\xi \in \mathcal{M}$ is a critical point of $\widetilde{J}_p$, then $U_\xi + \phi_p(\xi)$ is a critical point of J_p, namely a solution to problem (2.1).*

Next, we need to write the expansion of $\widetilde{J}_p$ as p goes to $+\infty$,

Lemma 3.2 *It holds*

$$\widetilde{J}_p(\xi) = \frac{c_1}{p} + \frac{c_2}{p^2} - \frac{c_3}{p^2}\Phi(\xi) + R_p(\xi),$$

where $R_p = O(\frac{\log^2 p}{p^3})$ *uniformly with respect to* ξ *in compact sets of* $\mathcal{M}$*. Here* c_1*,* c_2 *and* $c_3 \neq 0$ *are constants (depending only on* $q_1, \dots, q_m$*) and the function* $\Phi : \mathcal{M} \to \mathbb{R}$ *is defined by*

$$\Phi(\xi_1,\dots,\xi_k) = \sum_{i=1}^{k} H(\xi_i,\xi_i) + \sum_{\substack{i,j=1\\ i\neq j}}^{k} G(\xi_i,\xi_j) + \frac{1}{4\pi}\sum_{i=1}^{k}\log a(\xi_i) + 2\sum_{i=1}^{k}\sum_{j=1}^{m}(\alpha_j+1)G(\xi_i,q_j).$$

Proof. Multiplying equation in (3.5) by $U_\xi + \phi_p(\xi)$ and integrating by parts, we get that

$$\widetilde{J}_p(\xi) = \left(\frac{1}{2} - \frac{1}{p+1}\right)\int_\Omega |\nabla(U_\xi + \phi_p(\xi))|^2 - \frac{1}{p+1}\sum_{i,j} c_{ij}(\xi)\int_\Omega e^{U_{\delta_i,q_i}} Z_{ij} U_\xi.$$

Let us expand the leading term $\int_\Omega |\nabla U_\xi|^2$: in view of (2.4) we have that

$$\begin{aligned}
&\int_\Omega |\nabla U_\xi|^2 = -\int_\Omega \Delta U_\xi(x) U_\xi(x)dx\\
&= \sum_{i=1}^{m+k} \frac{1}{\gamma \mu_i^{\frac{2}{p-1}} a_i(q_i)^{\frac{1}{p-1}}} \int_{B(q_i,\varepsilon)} \left(|x-q_i|^{2\alpha_i} e^{U_{\delta_i,q_i}} - \frac{1}{p}\Delta V_{\delta_i,q_i} - \frac{1}{p^2}\Delta W_{\delta_i,q_i}\right) U_\xi(x)dx\\
&+O\left(e^{-\frac{p}{2}}\right) \qquad (\text{setting } x-q_i = \delta_i^{\frac{1}{\alpha_i+1}} y)\\
&= \sum_{i=1}^{m+k} \frac{1}{\gamma^2 \mu_i^{\frac{4}{p-1}} a_i(q_i)^{\frac{2}{p-1}}} \int_{B(0,\varepsilon\delta_i^{-\frac{1}{\alpha_i+1}})} \left(|y|^{2\alpha_i} e^{U^i} - \frac{\delta_i^{\frac{2}{\alpha_i+1}}}{p}\Delta V_{\delta_i,q_i} - \frac{\delta_i^{\frac{2}{\alpha_i+1}}}{p^2}\Delta W_{\delta_i,q_i}\right) \times\\
&\times \left(p + U^i(y) + \frac{1}{p}V^i(y) + \frac{1}{p^2}W^i(y)\right) dy + O\left(\frac{1}{p^3}\right)\\
&= \sum_{i=1}^{m+k} \frac{1}{\gamma^2 \mu_i^{\frac{4}{p-1}} a_i(q_i)^{\frac{2}{p-1}}} \int_{B(0,\varepsilon\delta_i^{-\frac{1}{\alpha_i+1}})} |y|^{2\alpha_i}\left(e^{U^i} - \frac{1}{p}f^{0i} + \frac{1}{p}e^{U^i}V^i - \frac{1}{p^2}f^{1i} + \frac{1}{p^2}e^{U^i}W^i\right) \times\\
&\times \left(p + U^i(y) + \frac{1}{p}V^i(y) + \frac{1}{p^2}W^i(y)\right) dy + O\left(\frac{1}{p^3}\right)\\
&= \sum_{i=1}^{m+k} \frac{1}{\gamma^2 \mu_i^{\frac{4}{p-1}} a_i(q_i)^{\frac{2}{p-1}}} \left(p\int_{\mathbb{R}^2} |y|^{2\alpha_i} e^{U^i}dy + \int_{\mathbb{R}^2} |y|^{2\alpha_i} U^i e^{U^i}dy - \int_{\mathbb{R}^2} |y|^{2\alpha_i} f^{0i}dy\right.\\
&\left. + \int_{\mathbb{R}^2} |y|^{2\alpha_i} V^i e^{U^i} + O(\frac{1}{p})\right) =\\
&= \sum_{i=1}^{m+k}\left[\frac{e}{p}\left(1 - 2\frac{\log p}{p} + \frac{1}{p} - \frac{2}{p}\log a_i(q_i)\right)A_i + \frac{e}{p^2}B_i - \frac{4e}{p^2}A_i\log\mu_i\right] + O\left(\frac{\log^2 p}{p^3}\right),
\end{aligned}$$

where

$$\begin{aligned}
A_i &:= \textstyle\int_{\mathbb{R}^2} |y|^{2\alpha_i} e^{U^i}dy = 8\pi(\alpha_i+1)\\
B_i &:= \textstyle\int_{\mathbb{R}^2} |y|^{2\alpha_i} e^{U^i}\left(U^i - \frac{1}{2}(U^i)^2 + V^i\right)dy,
\end{aligned}$$

because

$$\begin{aligned}
\mu_i^{-\frac{4}{p-1}} &= 1 - \tfrac{4}{p}\log\mu_i + O(\tfrac{1}{p^2}),\\
a_i(q_i)^{-\frac{2}{p-1}} &= 1 - \tfrac{2}{p}\log a_i(q_i) + O(\tfrac{1}{p^2}),\\
\tfrac{1}{\gamma^2} &= \tfrac{e}{p^2}\left(1 - 2\tfrac{\log p}{p} + \tfrac{1}{p} + O(\tfrac{\log^2 p}{p^2})\right).
\end{aligned}$$

Recalling the expansion of μ_i in (2.5), we get that

$$\begin{aligned}
&\int_\Omega |\nabla U_\xi|^2 = \frac{8\pi e}{p}\left(1 - 2\frac{\log p}{p} + \frac{4}{p}\right)\sum_{i=1}^{m+k}(\alpha_i+1) + \frac{e}{p^2}\sum_{i=1}^{m+k} B_i\\
&-\frac{16\pi e}{p^2}\sum_{i=1}^{m+k}(\alpha_i+1)\left(\log a_i(q_i) + 4\pi(\alpha_i+1)H(q_i,q_i) + 4\pi\sum_{j\neq i}(\alpha_j+1)G(q_j,q_i)\right)\\
&+O\left(\frac{\log^2 p}{p^3}\right) = \frac{8\pi e}{p}\left(1 - 2\frac{\log p}{p} + \frac{4}{p}\right)\sum_{i=1}^{m+k}(\alpha_i+1) + \frac{e}{p^2}\sum_{i=1}^{m+k} B_i\\
&-\frac{16\pi e}{p^2}\sum_{i=1}^{m}(\alpha_i+1)\left(\log a_i(q_i) + 4\pi(\alpha_i+1)H(q_i,q_i) + 4\pi\sum_{\substack{j=1\\ j\neq i}}^{m}(\alpha_j+1)G(q_i,q_j)\right)\\
&-\frac{64\pi^2 e}{p^2}\Phi(\xi_1,\dots,\xi_k) + O\left(\frac{\log^2 p}{p^3}\right)
\end{aligned}$$

uniformly for ξ in a compact set of $\mathcal{M}$. In particular,

$$\int_\Omega |\nabla U_\xi|^2 = O\left(\frac{1}{p}\right). \tag{3.8}$$

Now, using Proposition 3.2 and estimates (2.10), (3.6), by (3.3)-(3.4) we deduce that

$$|c_{ij}(\xi)| = O\left(\frac{1}{p}||\phi_p(\xi)||_\infty + \|N(\phi_p(\xi))\|_* + \|R\|_*\right) = O\left(\frac{1}{p^4}\right)$$

and

$$\|\phi_p(\xi)\| = O\left(\|\phi_p(\xi)\|_\infty + \|N(\phi_p(\xi))\|_* + \|R\|_*\right) = O\left(\frac{1}{p^3}\right).$$

Therefore, by (3.8) we have that

$$\widetilde{J}_p(\xi) = \left(\frac{1}{2} - \frac{1}{p+1}\right)\int_\Omega |\nabla U_\xi|^2 + O\left(\frac{1}{p^3}\right)$$

and our claim follows with suitable constant c_1, c_2 and $c_3 = 32\pi^2 e \neq 0$.

■

We introduce the following definition.

Definition 3.1 *We say that ξ is a C^0-stable critical point of $\Phi : \mathcal{M} \to \mathbb{R}$ if for any sequence of functions $\Phi_n : \mathcal{M} \to \mathbb{R}$ such that $\Phi_n \to \Phi$ uniformly on compact sets of $\mathcal{M}$, we have that Φ_n has a critical point ξ_n such that $\Phi_n(\xi_n) \to \Phi(\xi)$.*

In particular, if ξ is a strict local minimum/maximum point of Φ, then ξ is a C^0-stable critical point.

Proof of Theorem 1.2. According to Lemma 3.1, we have a solution to problem (1.3) if we find a critical point ξ_p of $\widetilde{J}_p$. This is equivalent to find a critical point of the function $\Phi_p : \mathcal{M} \to \mathbb{R}$ defined by $\Phi_p(\xi) := \left(pc_1 + c_2 - p^2 \widetilde{J}_p(\xi)\right)/c_3$ (see Lemma 3.2). On the other hand, $\Phi_p \to \Phi$ uniformly on compact sets of $\mathcal{M}$ as p goes to $+\infty$, because of Lemma 3.2. Now, by Definition 3.1 we deduce that, if p is large enough, there exists a critical point $\xi^p \in \mathcal{M}$ of Φ_p such that $\Phi_p(\xi^p) \to \Phi(\xi^*)$. Moreover, up to a subsequence, we have that $\xi^p \to \xi$ as p goes to $+\infty$, with $\Phi(\xi) = \Phi(\xi^*)$. The function $u_p = U_{\xi^p} + \phi_{\xi^p}$ is therefore a solution to (1.3) with the qualitative properties predicted by the theorem, as it can be easily shown. The proof of the positivity of u_p follows the lines of Remark 2.1. ■

4 Proof of Theorem 1.1

Let $\Omega = \{x \in \mathbb{R}^2 \ : \ |x| < 1\}$ be the unit ball and let $a(x) = |x|^{2\alpha}$ for some $\alpha > 0$. Let $k \geq 1$ be a fixed integer and set

$$\xi_i^* := \left(\cos \frac{2\pi}{k}(i-1), \sin \frac{2\pi}{k}(i-1)\right) \text{ for any } i = 1, \dots, k.$$

We will look for a solution to problem (2.1) as $u_p = U_\rho + \phi_p(\rho)$, where

$$U_\rho := \sum_{i=1}^{k} \frac{1}{\gamma \mu_i^{\frac{2}{p-1}} \rho^{\frac{2\alpha}{p-1}}} \left(PU_{\delta_i,\xi_i} + \frac{1}{p} PV_{\delta_i,\xi_i} + \frac{1}{p^2} PW_{\delta_i,\xi_i} \right)$$

and the concentration parameters δ_i are given in (2.2), μ_i are defined in (2.3) and the concentration points ξ are given, for any $i = 1, \dots, k$, by

$$\xi_i := \xi_i(\rho) = \rho \xi_i^* = \left(\rho \cos \frac{2\pi}{k}(i-1), \rho \sin \frac{2\pi}{k}(i-1)\right) \ , \qquad \rho \in (0,1).$$

The rest term $\phi_p(\rho)$ can be found symmetric with respect to the variable x_2 and each line $\{t\xi_i^* \ : \ t \in \mathbb{R}\}$, for any $i = 1, \dots, k$.

Using results obtained in the previous Sections and taking into account the symmetry of the domain and the function a, we reduce the problem of finding solutions to (2.1) to that of finding critical points of the function $\widetilde{J}_p : (0,1) \to \mathbb{R}$ defined as in (3.7) by $\widetilde{J}_p(\rho) := J_p(U_\rho + \phi_p(\rho))$. Using Lemma 3.2, it is not difficult to check that

$$\widetilde{J}_p(\rho) = \frac{c_1}{p} + \frac{c_2}{p^2} - \frac{c_3}{p^2}\Phi(\rho) + R_p(\rho),$$

where $R_p(\rho) = O\left(\frac{\log^2 p}{p^3}\right)$ uniformly for ρ in compact sets of $(0,1)$. Moreover, c_1, c_2 and $c_3 \neq 0$ are constants and

$$\Phi(\rho) := H(\rho\xi_1^*, \rho\xi_1^*) + \sum_{i=2}^{k} G(\rho\xi_1^*, \rho\xi_i^*) + \frac{\alpha}{2\pi}\log\rho,\ \ \rho \in (0,1).$$

In this case, we have

$$G(x,y) = \frac{1}{2\pi}\log\frac{1}{|x-y|} - \frac{1}{2\pi}\log\frac{1}{\sqrt{|x|^2|y|^2+1-2(x,y)}},\ H(x,x) = -\frac{1}{2\pi}\log\frac{1}{1-|x|^2},$$

and so, function Φ reduces to

$$\Phi(\rho) = \frac{1}{2\pi}\log(1-\rho^2) + \frac{\alpha-(k-1)}{2\pi}\log\rho + \frac{1}{2\pi}\sum_{i=2}^{k}\log\frac{\sqrt{\rho^4+1-2\rho^2(\xi_1^*,\xi_i^*)}}{|\xi_1^*-\xi_i^*|}.$$

Now, there exists $\rho_0 \in (0,1)$ such that $\Phi(\rho_0) = \max\limits_{\rho\in(0,1)}\Phi(\rho)$, provided $\alpha-k+1 > 0$, since $\lim\limits_{\rho\to 1^-}\Phi(\rho) = \lim\limits_{\rho\to 0^+}\Phi(\rho) = -\infty$. Then, ρ_0 is a C^0-stable critical point of Φ and so, function $\widetilde{J}_p$ has for p large enough a critical point ρ_p. That proves our claim for any $k \leq K_\alpha$.

A Appendix

Let us recall the following basic result stated by Chae and Imanuvilov in [7]: for any $f(t) \in C^1[0,+\infty)$ there exists a smooth radial solution

$$w(r) = \frac{r^{2(\alpha+1)}-1}{r^{2(\alpha+1)}+1}\left(\int_0^r \frac{\phi_f(s)-\phi_f(1)}{(s-1)^2}ds + \phi_f(1)\frac{r}{1-r}\right)$$

for the equation

$$\Delta w + \frac{8(\alpha+1)^2|y|^{2\alpha}}{(1+|y|^{2(\alpha+1)})^2}w = |y|^{2\alpha}f(|y|),$$

where $\phi_f(s) = (\frac{s^{2(\alpha+1)}+1}{s^{2(\alpha+1)}-1})^2\frac{(s-1)^2}{s}\int_0^s t^{2\alpha+1}\frac{t^{2(\alpha+1)}-1}{t^{2(\alpha+1)}+1}f(t)dt$ for $s \neq 1$ and $\phi_f(1) = \lim_{s\to 1}\phi_f(s)$.

Assume that $\int_0^\infty t^{2\alpha+1}|\log t||f|(t)dt < +\infty$. It is a straightforward computation to show that

$$w(r) = C_f\log r + D_f + O(\int_r^{+\infty} t^{2\alpha+1}|\log t||f|(t)dt + \frac{1}{r^{2(\alpha+1)}}) \text{ as } r \to +\infty,$$

where $C_f = \int_0^{+\infty} t^{2\alpha+1}\frac{t^{2(\alpha+1)}-1}{t^{2(\alpha+1)}+1}f(t)dt$. A similar computation can be performed also for $\partial_r w(r)$. Therefore, up to replacing $w(r)$ with $w(r) - D_f\frac{r^{2(\alpha+1)}-1}{r^{2(\alpha+1)}+1}$, we have shown

Lemma A.1 *Let $f \in C^1[0,+\infty)$ such that $\int_0^{+\infty} t^{2\alpha+1}|\log t||f|(t)dt < +\infty$. There exists a C^2 radial solution $w(r)$ of equation*

$$\Delta w + \frac{8(\alpha+1)^2|y|^{2\alpha}}{(1+|y|^{2(\alpha+1)})^2} w = |y|^{2\alpha} f(|y|) \ \mathit{in}\ \mathbb{R}^2$$

such that as $r \to +\infty$

$$w(r) = C_f \log r + O(\int_r^{+\infty} t^{2\alpha+1}|\log t||f|(t)dt + \frac{1}{r^{2(\alpha+1)}})$$

and

$$\partial_r w(r) = \frac{C_f}{r} + O(\frac{1}{r}\int_r^{+\infty} t^{2\alpha+1}|f|(t)dt + \frac{|\log r|}{r^{2\alpha+3}}),$$

where $C_f = \left(\int_0^{+\infty} t^{2\alpha+1}\frac{t^{2(\alpha+1)}-1}{t^{2(\alpha+1)}+1} f(t)dt\right)$.

Now, let $U(y) = \log \frac{8(\alpha+1)^2}{(1+|y|^{2(\alpha+1)})^2}$. Let V, W be radial solutions of

$$\Delta V + |y|^{2\alpha} e^U V = |y|^{2\alpha} f^0 \quad \text{in } \mathbb{R}^2,\ f^0(y) := \frac{1}{2} e^{U(y)} U^2(y), \tag{A.1}$$

and

$$\Delta W + |y|^{2\alpha} e^U W = |y|^{2\alpha} f^1 \quad \text{in } \mathbb{R}^2, \tag{A.2}$$
$$f^1(y) := e^{U(y)}\left(VU - \frac{1}{2}V^2 - \frac{1}{3}U^3 - \frac{1}{8}U^4 + \frac{1}{2}VU^2\right)(y)$$

such that as $|y| \to +\infty$:

$$V(y) = C_0(\alpha)\log|y| + O(\frac{1}{|y|^{\alpha+1}}), \tag{A.3}$$
$$W(y) = C_1(\alpha)\log|y| + O(\frac{1}{|y|^{\alpha+1}}),$$

where $C_i(\alpha) = \int_0^{+\infty} t^{2\alpha+1}\frac{t^{2(\alpha+1)}-1}{t^{2(\alpha+1)}+1} f^i(t)dt$, $i = 1, 2$.

Let us remark that it is possible to construct W since by (A.3) V has logarithmic growth at infinity. Since later we will need the exact expression of V, we have that

$$\begin{aligned} V(y) \ &= \ \frac{1}{2}U^2(y) + 6\log(|y|^{2(\alpha+1)}+1) + \frac{2\log 8(\alpha+1)^2 - 10}{|y|^{2(\alpha+1)}+1} \\ &+ \frac{|y|^{2(\alpha+1)}-1}{|y|^{2(\alpha+1)}+1}\left(2\log^2(|y|^{2(\alpha+1)}+1) - \frac{1}{2}\log^2 8(\alpha+1)^2\right. \\ &\left. +4\int_{|y|^{2(\alpha+1)}}^{+\infty} \frac{ds}{s+1}\log\frac{s+1}{s} - 8(\alpha+1)\log|y|\log(|y|^{2(\alpha+1)}+1)\right), \end{aligned} \tag{A.4}$$

as we can see by direct inspection. Moreover, it is easy to compute the value of $C_0(\alpha)$:

$$C_0(\alpha) = 12(\alpha+1) - 4(\alpha+1)\log 8(\alpha+1)^2 \tag{A.5}$$

B Appendix

Let $\alpha \geq 0$. Let $U_{\delta,\xi}$ be the function defined as

$$U_{\delta,\xi}(x) = \log \frac{8(\alpha+1)^2\delta^2}{(\delta^2 + |x-\xi|^{2(\alpha+1)})^2},\ \delta > 0,\ \xi \in \mathbb{R}^2,$$

a solution of $-\Delta U_{\delta,\xi} = |x-\xi|^{2\alpha} e^{U_{\delta,\xi}}$ in $\mathbb{R}^2$ (see (1.5)-(1.6)). Let $P : H^1(\Omega) \to H_0^1(\Omega)$ be the projection operator onto $H_0^1(\Omega)$. The following expansions hold:

Lemma B.1 *We have as $\delta \to 0$:*

$$PU_{\delta,\xi}(x) = U_{\delta,\xi}(x) - \log 8(\alpha+1)^2\delta^2 + 8\pi(\alpha+1)H(x,\xi) + O\left(\delta^2\right) \tag{B.1}$$

in $C(\bar{\Omega})$ and

$$PU_{\delta,\xi}(x) = 8\pi(\alpha+1)G(x,\xi) + O\left(\delta^2\right) \tag{B.2}$$

in $C_{loc}(\bar{\Omega} \setminus \{\xi\})$, uniformly for ξ away from $\partial\Omega$.

Proof. Since $PU_{\delta,\xi}(x) - U_{\delta,\xi}(x) + \log 8(\alpha+1)^2\delta^2 = -4(\alpha+1)\log\frac{1}{|x-\xi|} + O(\delta^2)$ as $\delta \to 0$ uniformly for $x \in \partial\Omega$ and ξ away from $\partial\Omega$, by harmonicity and the maximum principle (B.1) readily follows.
On the other hand, away from ξ, we have $U_{\delta,\xi}(x) - \log 8(\alpha+1)^2\delta^2 = 4(\alpha+1)\log\frac{1}{|x-\xi|} + O(\delta^2)$. This fact, together with (B.1) gives (B.2). ∎

Let V, W be the radial solutions of (A.1), (A.2) respectively, which satisfy (A.3):

$$V(y) = C_0(\alpha)\log|y| + O\left(\frac{1}{|y|^{\alpha+1}}\right), W(y) = C_1(\alpha)\log|y| + O\left(\frac{1}{|y|^{\alpha+1}}\right) \text{ as } |y| \to +\infty,$$

for some constants $C_0(\alpha)$, $C_1(\alpha)$. For any $\delta > 0$ and $\xi \in \mathbb{R}^2$, we define

$$V_{\delta,\xi}(x) := V\left(\delta^{-\frac{1}{\alpha+1}}(x-\xi)\right),\ W_{\delta,\xi}(x) := W\left(\delta^{-\frac{1}{\alpha+1}}(x-\xi)\right)$$

for $x \in \Omega$. Then, $V_{\delta,\xi}$ and $W_{\delta,\xi}$ satisfy

$$\Delta V_{\delta,\xi} + |x-\xi|^{2\alpha} e^{U_{\delta,\xi}} V_{\delta,\xi} = |x-\xi|^{2\alpha} f^0_{\delta,\xi} \quad \text{in } \mathbb{R}^2,$$

and

$$\Delta W_{\delta,\xi} + |x-\xi|^{2\alpha} e^{U_{\delta,\xi}} W_{\delta,\xi} = |x-\xi|^{2\alpha} f^1_{\delta,\xi} \quad \text{in } \mathbb{R}^2,$$

where

$$f^j_{\delta,\xi}(x) := \frac{1}{\delta^2} f^j\left(\frac{x-\xi}{\delta^{\frac{1}{\alpha+1}}}\right),\ j = 0, 1.$$

By (A.3) we deduce the following expansions:

Lemma B.2 *We have as $\delta \to 0$:*

$$\begin{aligned} PV_{\delta,\xi}(x) &= V_{\delta,\xi}(x) - 2\pi C_0(\alpha)H(x,\xi) + \tfrac{C_0(\alpha)}{\alpha+1}\log\delta + O(\delta) \\ PW_{\delta,\xi}(x) &= W_{\delta,\xi}(x) - 2\pi C_1(\alpha)H(x,\xi) + \tfrac{C_1(\alpha)}{\alpha+1}\log\delta + O(\delta) \end{aligned} \tag{B.3}$$

in $C(\bar\Omega)$, and

$$\begin{aligned} PV_{\delta,\xi}(x) &= -2\pi C_0(\alpha)G(x,\xi) + O(\delta) \\ PW_{\delta,\xi}(x) &= -2\pi C_1(\alpha)G(x,\xi) + O(\delta) \end{aligned} \tag{B.4}$$

in $C_{loc}(\bar\Omega\setminus\{\xi\})$, uniformly for ξ away from $\partial\Omega$. In particular, the following global estimate holds: for any $\varepsilon>0$ there exists $c>0$ such that for any δ small and $\xi\in\Omega$ with dist $(\xi,\partial\Omega)\ge\varepsilon$ *we have that*

$$\|PV_{\delta,\xi}\|_\infty + \|PW_{\delta,\xi}\|_\infty \le c|\log\delta|.$$

Proof. The proof follows from the same argument used to prove Lemma B.1 and from estimates (A.3). ∎

C Appendix

In this Section, we prove invertibility of the operator L and we give a bound (uniformly on $\xi\in\mathcal{O}_\varepsilon$) on its inverse norm by using L^∞-norms introduced in (2.9). Let us recall that $L(\phi)=\Delta\phi + a(x)W_\xi\phi$, where $W_\xi(x)=pU_\xi^{p-1}(x)$.

As in Proposition 2.1, we have for the potential $a(x)W_\xi(x)$ the following expansions. By (2.13), if $|x-q_i|\le\varepsilon$ for some $i=1,\dots,m+k$ we have that:

$$\begin{aligned} a(x)W_\xi(x) &= p\Big(\frac{p}{\gamma\mu_i^{\frac{2}{p-1}}a_i(q_i)^{\frac{1}{p-1}}}\Big)^{p-1}a(x)\Bigg(1+\frac1p U^i(y)+\frac{1}{p^2}V^i(y) \\ &+\frac{1}{p^3}W^i(y) + O\Big(\frac{e^{-\frac{p}{4(\alpha_i+1)}}}{p}|y| + \frac{e^{-\frac p4}}{p}\Big)\Bigg)^{p-1} \\ &= \delta_i^{-\frac{2}{\alpha_i+1}}|y|^{2\alpha_i}\Big(1+O(\delta_i^{\frac{1}{\alpha_i+1}}|y|)\Big)\Bigg(1+\frac1p U^i(y)+\frac{1}{p^2}V^i(y)+\frac{1}{p^3}W^i(y) \\ &+O\Big(\frac{e^{-\frac{p}{4(\alpha_i+1)}}}{p}|y| + \frac{e^{-\frac p4}}{p}\Big)\Bigg)^{p-1}, \end{aligned}$$

where again we use the notation $y=\delta_i^{-\frac{1}{\alpha_i+1}}(x-q_i)$. In this region, we have that

$$a(x)W_\xi(x)\le C\delta_i^{-\frac{2}{\alpha_i+1}}|y|^{2\alpha_i}e^{U^i(y)} = O\left(|x-q_i|^{2\alpha_i}e^{U_{\delta_i,q_i}(x)}\right).$$

Furthermore, by Taylor expansions of exponential and logarithmic functions as in (2.14), we obtain that, if $|x-q_i|\le\varepsilon\delta_i^{\frac{1}{2(\alpha_i+1)}}$ (and $|y|\le\varepsilon\delta_i^{-\frac{1}{2(\alpha_i+1)}}$),

$$a(x)W_\xi(x) \;=\; \delta_i^{-\frac{2}{\alpha_i+1}}|y|^{2\alpha_i}\Big(1+O(\delta_i^{\frac{1}{\alpha_i+1}}|y|)\Big)\Big(1+\frac1p U^i(y)+\frac{1}{p^2}V^i(y)$$

$$
\begin{aligned}
&+ \quad \frac{1}{p^3}W^i(y) + O(\frac{e^{-\frac{p}{4(\alpha_i+1)}}}{p}|y| + \frac{e^{-\frac{p}{4}}}{p})\Bigg)^{p-1} \\
&= \quad \delta_i^{-\frac{2}{\alpha_i+1}}|y|^{2\alpha_i}e^{U^i(y)}\left[1 + \frac{1}{p}\left(V^i - U^i - \frac{1}{2}(U^i)^2\right) + O(\frac{\log^4(|y|+2)}{p^2})\right].
\end{aligned}
$$

If $|x - q_i| \geq \varepsilon$ for any $i = 1, \ldots, m+k$:

$$
a(x)W_\xi(x) = O(p(\frac{C}{p})^{p-1}).
$$

Summing up, we have that

Lemma C.1 *There exist $D_0 > 0$ and $p_0 > 0$ such that*

$$
a(x)W_\xi(x) \leq D_0 \sum_{i=1}^{m+k} |x - q_i|^{2\alpha_i} e^{U_{\delta_i, q_i}(x)}
$$

for any $\xi \in \mathcal{O}_\varepsilon$ and $p \geq p_0$. Furthermore,

$$
a(x)W_\xi(x) = \delta_i^{-\frac{2}{\alpha_i+1}}|y|^{2\alpha_i}e^{U^i(y)}\left[1 + \frac{1}{p}\left(V^i - U^i - \frac{1}{2}(U^i)^2\right) + O(\frac{\log^4(|y|+2)}{p^2})\right]
$$

for any $|x - q_i| \leq \varepsilon\delta_i^{\frac{1}{2(\alpha_i+1)}}$, where $y = \delta_i^{-\frac{1}{\alpha_i+1}}(x - q_i)$.

Remark C.1 As for W_ξ, let us point out that, if $|x - q_i| \leq \varepsilon$ for some $i = 1, \ldots, m+k$, there holds

$$
pa(x)(U_\xi + O(\frac{1}{p^3}))^{p-2} \leq Cp(\frac{p}{\gamma})^{p-2}|x - q_i|^{2\alpha_i}e^{U^i(y)} = O\left(|x - q_i|^{2\alpha_i}e^{U_{\delta_i, q_i}(x)}\right)
$$

where $y = \delta_i^{-\frac{1}{\alpha_i+1}}(x - q_i)$. Since this estimate is true if $|x - q_i| \geq \varepsilon$ for any $i = 1, \ldots, m+k$, we have that

$$
pa(x)(U_\xi + O(\frac{1}{p^3}))^{p-2} \leq C \sum_{i=1}^{m+k} |x - q_i|^{2\alpha_i} e^{U_{\delta_i, q_i}(x)}.
$$

In an heuristic way, the operator L is close to $\tilde{L}$ defined by

$$
\tilde{L}(\phi) = \Delta\phi + \left(\sum_{i=1}^{m+k} |x - q_i|^{2\alpha_i} e^{U_{\delta_i, q_i}}\right)\phi.
$$

The operator $\tilde{L}$ is "essentially" a superposition of linear operators which, after a dilation and translation, approach, as $p \to \infty$, the linear operator in $\mathbb{R}^2$:

$$
\phi \ \to \ \Delta\phi + \frac{8(\alpha_i+1)^2|y|^{2\alpha_i}}{(1 + |y|^{2(\alpha_i+1)})^2}\phi, \ i = 1, \ldots, m+k,
$$

namely equation $\Delta v+|y|^{2\alpha_i}e^v=0$ linearized around the radial solution $\log\frac{8(\alpha_i+1)^2}{(1+|y|^{2(\alpha_i+1)})^2}$.

Set $z_0^i(y)=\frac{|y|^{2(\alpha_i+1)}-1}{|y|^{2(\alpha_i+1)}+1}$ for any $i=1,\dots,m+k$ and $z_j(y)=\frac{4y_j}{1+|y|^2}$, $j=1,2$. The first ingredient to develop the desired solvability theory for L is the well known fact that any bounded solution of $L(\phi)=0$ in $\mathbb{R}^2$ is precisely:
- for $i=1,\dots,m$ proportional to z_0^i;
- for $i=m+1,\dots,m+k$ a linear combination of z_0^i and z_j, $j=1,2$.
The second ingredient is a detailed analysis of $L-\tilde{L}$. Let us rewrite problem (3.1). Given $h\in C(\bar{\Omega})$, we consider the linear problem of finding a function $\phi\in W^{2,2}(\Omega)$ such that

$$L(\phi)\ =\ h+\sum_{j=1}^{2}\sum_{i=m+1}^{m+k}c_{ij}e^{U_{\delta_i,q_i}}Z_{ij}\ \text{in}\ \Omega, \tag{C.1}$$

$$\phi=0\ \text{on}\ \partial\Omega, \tag{C.2}$$

$$\int_\Omega e^{U_{\delta_i,q_i}}Z_{ij}\phi=0\quad j=1,2,\ i=m+1,\dots,m+k, \tag{C.3}$$

for some coefficients c_{ij}, $j=1,2$ and $i=m+1,\dots,m+k$. Here and in the sequel, we denote for any $i=1,\dots,m+k$:

$$Z_{i0}(x):=z_0^i\left(\delta_i^{-\frac{1}{\alpha_i+1}}(x-q_i)\right)=\frac{|x-q_i|^{2(\alpha_i+1)}-\delta_i^2}{|x-q_i|^{2(\alpha_i+1)}+\delta_i^2}$$

and for any $j=1,2$, $i=m+1,\dots,m+k$:

$$Z_{ij}(x):=z_j\left(\delta_i^{-1}(x-\xi_i)\right)=\frac{4\delta_i(x-\xi_i)_j}{\delta_i^2+|x-\xi_i|^2}.$$

Following some ideas in [15] for $a(x)=1$, we give the proof of Proposition 3.1. The proof consists of six steps.

1^{st} **Step**. The operator L satisfies the maximum principle in $\tilde{\Omega}:=\Omega\setminus\cup_{i=1}^{m+k}B(q_i,R\delta_i^{\frac{1}{\alpha_i+1}})$ for R large, independent on p. Namely,

$$\text{if } L(\psi)\le 0 \text{ in } \tilde{\Omega} \text{ and } \psi\ge 0 \text{ on } \partial\tilde{\Omega}, \text{ then } \psi\ge 0 \text{ in } \tilde{\Omega}.$$

In order to prove this fact, we show the existence of a positive function Z in $\tilde{\Omega}$ satisfying $L(Z)<0$. We define Z to be

$$Z(x)=\sum_{i=1}^{m+k}z_0^i\left(a^{\frac{1}{\alpha_i+1}}\delta_i^{-\frac{1}{\alpha_i+1}}(x-q_i)\right)\ ,\ a>0.$$

First, observe that for $x\in\tilde{\Omega}$, if $R>\frac{1}{a^{\frac{1}{\alpha_i+1}}}$ for any $i=1,\dots,m+k$, then $Z(x)>0$. On the other hand, we have:

$$a(x)W_\xi(x)\le D_0\left(\sum_{i=1}^{m+k}|x-q_i|^{2\alpha_i}e^{U_{\delta_i,q_i}(x)}\right)\le D_0\sum_{i=1}^{m+k}\frac{8(\alpha_i+1)^2\delta_i^2}{|x-q_i|^{2\alpha_i+4}},$$

where D_0 is the constant in Lemma C.1. Further, by definition of z_0^i, we have that for any $x \in \tilde{\Omega}$:

$$\begin{aligned} -\Delta Z(x) &= \sum_{i=1}^{m+k} a^2|x-q_i|^{2\alpha_i} \frac{8(\alpha_i+1)^2\delta_i^2(a^2|x-q_i|^{2(\alpha_i+1)}-\delta_i^2)}{(a^2|x-q_i|^{2(\alpha_i+1)}+\delta_i^2)^3} \\ &\geq \frac{1}{3}\sum_{i=1}^{m+k} \frac{8a^2(\alpha_i+1)^2\delta_i^2|x-q_i|^{2\alpha_i}}{(a^2|x-q_i|^{2(\alpha_i+1)}+\delta_i^2)^2} \geq \frac{4}{27}\sum_{i=1}^{m+k} \frac{8(\alpha_i+1)^2\delta_i^2}{a^2|x-q_i|^{2\alpha_i+4}} \end{aligned}$$

provided $R > (\frac{\sqrt{2}}{a})^{\frac{1}{\alpha_i+1}}$ for any $i = 1, \ldots, m+k$. Hence,

$$LZ(x) \leq \left(-\frac{4}{27a^2} + D_0(m+k)\right)\sum_{i=1}^{m+k} \frac{8(\alpha_i+1)^2\delta_i^2}{|x-q_i|^{2\alpha_i+4}} < 0$$

since $Z(x) \leq m+k$, provided that a is chosen sufficiently small (independent of p). The function $Z(x)$ is what we are looking for.

$2^{\mathbf{nd}}$ **Step**. Let R be as before. Let us define the "inner norm" of ϕ in the following way

$$\|\phi\|_i = \sup_{x \in \cup_{i=1}^{m+k} B(q_i, R\delta_i^{\frac{1}{\alpha_i+1}})} |\phi|(x).$$

We claim that there is a constant $C > 0$ such that, if $L(\phi) = h$ in Ω, $\phi = 0$ on $\partial\Omega$, then

$$\|\phi\|_\infty \leq C[\|\phi\|_i + \|h\|_*],$$

for any $h \in C^{0,\alpha}(\bar{\Omega})$. We will establish this estimate with the use of suitable barriers. Let $M = 2 \operatorname{diam} \Omega$. Consider the solution $\psi_i(x)$ of the problem:

$$\begin{cases} -\Delta\psi_i = \frac{2\delta_i}{|x-q_i|^{\alpha_i+3}} & \text{in } R\delta_i^{\frac{1}{\alpha_i+1}} < |x-q_i| < M \\ \psi_i(x) = 0 & \text{on } |x-q_i| = R\delta_i^{\frac{1}{\alpha_i+1}} \text{ and } |x-q_i| = M. \end{cases}$$

Namely, the function $\psi_i(x)$ is the positive function defined by:

$$\psi_i(x) = (\alpha_i+1)^{-2}\left(-\frac{2\delta_i}{|x-q_i|^{\alpha_i+1}} + A_i + B_i \log|x-q_i|\right),$$

where

$$B_i = 2\left(\frac{\delta_i}{M^{\alpha_i+1}} - \frac{1}{R^{\alpha_i+1}}\right)\frac{1}{\log\left(\frac{M}{R\delta_i^{\frac{1}{\alpha_i+1}}}\right)} < 0$$

and

$$A_i = \frac{2\delta_i}{M^{\alpha_i+1}} - B_i \log M.$$

Hence, the function $\psi_i(x)$ is uniformly bounded from above by a constant independent of p, since we have that, for $R\delta_i^{\frac{1}{\alpha_i+1}} \leq |x-q_i| \leq M$,

$$\psi_i(x) \leq (\alpha_i+1)^{-2}\left(A_i + B_i \log(R\delta_i^{\frac{1}{\alpha_i+1}})\right)$$
$$= (\alpha_i+1)^{-2}\left(\frac{2\delta_i}{M^{\alpha_i+1}} - B_i \log\frac{M}{R\delta_i^{\frac{1}{\alpha_i+1}}}\right) = \frac{2}{R^{\alpha_i+1}}(\alpha_i+1)^{-2} \leq \frac{2}{R}.$$

Define now the function

$$\tilde{\phi}(x) = 3\|\phi\|_i Z(x) + \|h\|_* \sum_{i=1}^{m+k} \psi_i(x),$$

where Z was defined in the previous Step. First of all, observe that by the definition of Z

$$\tilde{\phi}(x) \geq 3\|\phi\|_i Z(x) \geq \|\phi\|_i \geq |\phi|(x) \text{ for } |x-q_i| = R\delta_i^{\frac{1}{\alpha_i+1}},\ i=1,\dots,m+k,$$

and, by the positivity of $Z(x)$ and $\psi_i(x)$,

$$\tilde{\phi}(x) \geq 0 = |\phi|(x) \text{ for } x \in \partial\Omega.$$

Since by definition of $\|\cdot\|_*$ we have that

$$\left(\sum_{i=1}^{m+k} \frac{\delta_i |x-q_i|^{2\alpha_i}}{(\delta_i^2 + |x-q_i|^{2(\alpha_i+1)})^{\frac{3}{2}}}\right)\|h\|_* \geq |h(x)|, \tag{C.4}$$

finally we obtain that

$$\begin{aligned} L\tilde{\phi} &\leq \|h\|_* \sum_{i=1}^{m+k} L\psi_i(x) = \|h\|_* \sum_{i=1}^{m+k}\left(-\frac{2\delta_i}{|x-q_i|^{\alpha_i+3}} + a(x)W(x)\psi_i(x)\right) \\ &\leq \|h\|_* \sum_{i=1}^{m+k} |x-q_i|^{2\alpha_i}\left(-\frac{2\delta_i}{|x-q_i|^{3(\alpha_i+3)}} + \frac{2(m+k)D_0}{R} e^{U_{\delta_i,q_i}(x)}\right) \\ &\leq -\|h\|_*\left(\sum_{i=1}^{m+k} \frac{\delta_i |x-q_i|^{2\alpha_i}}{(\delta_i^2 + |x-q_i|^{2(\alpha_i+1)})^{\frac{3}{2}}}\right) \leq -|h(x)| \leq -|L\phi|(x) \end{aligned}$$

provided $R \geq 16(m+k)D_0(\alpha_i+1)^2$ for any $i=1,\dots,m+k$ and p large enough. Hence, by the maximum principle in Step 1 we obtain that

$$|\phi|(x) \leq \tilde{\phi}(x) \text{ for } x \in \tilde{\Omega},$$

and therefore, since $Z(x) \leq m+k$ and $\psi_i(x) \leq \frac{2}{R}$,

$$\|\phi\|_\infty \leq C[\|\phi\|_i + \|h\|_*].$$

$3^{\mathbf{rd}}$ **Step**. We prove uniform a priori estimates for solutions ϕ of problem $L\phi = h$ in Ω, $\phi = 0$ on $\partial\Omega$, when $h \in C^{0,\alpha}(\bar\Omega)$ and ϕ satisfies (C.3) and in addition the orthogonality conditions:

$$\int_\Omega |x-q_i|^{2\alpha_i} e^{U_{de_i,q_i}} Z_{i0}\phi = 0, \qquad \text{for } i = 1,\dots,m+k. \tag{C.5}$$

Namely, we prove that there exists a positive constant C such that for any $\xi \in \mathcal{O}_\varepsilon$ and $h \in C^{0,\alpha}(\bar\Omega)$

$$\|\phi\|_\infty \le C\|h\|_*,$$

for p sufficiently large. By contradiction, assume the existence of sequences $p_n \to \infty$, points $\xi^n \in \mathcal{O}_\varepsilon$, functions h_n and associated solutions ϕ_n such that $\|h_n\|_* \to 0$ and $\|\phi_n\|_\infty = 1$.
Since $\|\phi_n\|_\infty = 1$, Step 2 shows that $\liminf_{n\to+\infty} \|\phi_n\|_i > 0$. Let us set $\hat\phi_i^n(y) = \phi_n((\delta_i^n)^{\frac{1}{\alpha_i+1}} y + q_i^n)$ for $i = 1,\dots,m+k$, where we set $q_i^n = q_i$ for $i = 1,\dots,m$ and $q_i^n = \xi_{i-m}^n$ for $i = m+1,\dots,m+k$. By Lemma C.1 and (C.4), elliptic estimates readily imply that $\hat\phi_i^n$ converges uniformly over compact sets to a bounded solution $\hat\phi_i^\infty$ of the equation in $\mathbb{R}^2$:

$$\Delta\phi + \frac{8(\alpha_i+1)^2|y|^{2\alpha_i}}{(1+|y|^{2(\alpha_i+1)})^2}\phi = 0.$$

This implies that $\hat\phi_i^\infty$ is proportional to z_0^i if $i = 1,\dots,m$ and is a linear combination of the functions z_0^i and z_j, $j = 1, 2$, if $i = m+1,\dots,m+k$. Since $\|\hat\phi_i^n\|_\infty \le 1$, by Lebesgue theorem the orthogonality conditions (C.3) and (C.5) on ϕ_n pass to the limit and give rise to:
$\int_{\mathbb{R}^2} \frac{8(\alpha_i+1)^2|y|^{2\alpha_i}}{(1+|y|^{2(\alpha_i+1)})^2} z_0^i(y)\hat\phi_i^\infty = 0$ for any $i = 1,\dots,m+k$;
$\int_{\mathbb{R}^2} \frac{8}{(1+|y|^2)^2} z_j(y)\hat\phi_i^\infty = 0$ for any $j = 1,2$ and $i = m+1,\dots,m+k$.
Hence, $\hat\phi_i^\infty \equiv 0$ for any $i = 1,\dots,m+k$ contradicting $\liminf_{n\to+\infty}\|\phi_n\|_i > 0$.

$4^{\mathbf{th}}$ **Step**. We prove that there exists a positive constant $C > 0$ such that any solution ϕ of equation $L\phi = h$ in Ω, $\phi = 0$ on $\partial\Omega$, satisfies

$$\|\phi\|_\infty \le Cp\|h\|_*,$$

when $h \in C^{0,\alpha}(\bar\Omega)$ and we assume on ϕ only the orthogonality conditions (C.3). Proceeding by contradiction as in Step 3, we can suppose further that

$$p_n\|h_n\|_* \to 0 \text{ as } n \to +\infty \tag{C.6}$$

but we loss in the limit the condition $\int_{\mathbb{R}^2} \frac{8(\alpha_i+1)^2|y|^{2\alpha_i}}{(1+|y|^{2(\alpha_i+1)})^2} z_0^i(y)\hat\phi_i^\infty = 0$. Hence, we have that

$$\hat\phi_i^n \to C_i \frac{|y|^{2(\alpha_i+1)}-1}{|y|^{2(\alpha_i+1)}+1} \text{ in } C^0_{\text{loc}}(\mathbb{R}^2) \tag{C.7}$$

for some constants C_i. To reach a contradiction, we have to show that $C_i = 0$ for any $i = 1, \dots, m+k$. We will obtain it from the stronger condition (C.6) on h_n.

To this end, we perform the following construction. By Lemma A.1, we find radial solutions w_i and t_i respectively of equations $\Delta w_i + |y|^{2\alpha_i} e^{U^i} w_i = |y|^{2\alpha_i} e^{U^i} z_0^i$ and $\Delta t_i + |y|^{2\alpha_i} e^{U^i} t_i = |y|^{2\alpha_i} e^{U^i}$ in $\mathbb{R}^2$, such that as $|y| \to +\infty$

$$w_i(y) = \frac{4}{3}(\alpha_i + 1)\log|y| + O(\frac{1}{|y|^{\alpha_i+1}}) \ , \quad t_i(y) = O(\frac{1}{|y|^{\alpha_i+1}}),$$

since $\int_0^{+\infty} t^{2\alpha_i+1} \frac{(t^{2(\alpha_i+1)}-1)^2}{(t^{2(\alpha_i+1)}+1)^4} dt = \frac{1}{6(\alpha_i+1)}$ and $\int_0^{+\infty} t^{2\alpha_i+1} \frac{t^{2(\alpha_i+1)}-1}{(t^{2(\alpha_i+1)}+1)^3} dt = 0$.

For simplicity, from now on we will omit the dependence on n. For $i = 1, \dots, m+k$, define now

$$\begin{aligned} u_i(x) &= w_i\left(\delta_i^{-\frac{1}{\alpha_i+1}}(x-q_i)\right) + \frac{4}{3}(\log\delta_i) Z_{i0}(x) \\ &\quad + \frac{8\pi}{3}(\alpha_i+1) H(q_i,q_i) t_i\left(\delta_i^{-\frac{1}{\alpha_i+1}}(x-q_i)\right) \end{aligned}$$

and denote by Pu_i the projection of u_i onto $H_0^1(\Omega)$. Since $u_i - Pu_i - \frac{4}{3}(\alpha_i + 1)\log|\cdot - q_i| = O(\delta_i)$ on $\partial\Omega$ (together with boundary derivatives), by harmonicity we get

$$\begin{aligned} &Pu_i = u_i - \tfrac{8\pi}{3}(\alpha_i+1)H(\cdot,q_i) + O(e^{-\frac{p}{4}}) \text{ in } C^1(\bar\Omega), \\ &Pu_i = -\tfrac{8\pi}{3}(\alpha_i+1)G(\cdot,q_i) + O(e^{-\frac{p}{4}}) \text{ in } C^1_{\text{loc}}(\bar\Omega \setminus \{q_i\}). \end{aligned} \tag{C.8}$$

The function Pu_i solves

$$\begin{aligned} &\Delta Pu_i + a(x)W_\xi(x) Pu_i = |x-q_i|^{2\alpha_i} e^{U_{\delta_i,q_i}} Z_{i0} \\ &+ \left(a(x)W_\xi(x) - |x-q_i|^{2\alpha_i} e^{U_{\delta_i,q_i}}\right) Pu_i + R_i, \end{aligned} \tag{C.9}$$

where

$$R_i(x) = \left(Pu_i - u_i + \frac{8\pi}{3}(\alpha_i+1)H(q_i,q_i)\right)|x-q_i|^{2\alpha_i} e^{U_{\delta_i,q_i}}.$$

Multiply (C.9) by ϕ and integrate by parts to obtain:

$$\begin{aligned} &\int_\Omega |x-q_i|^{2\alpha_i} e^{U_{\delta_i,q_i}} Z_{i0}\phi + \int_\Omega \left(a(x)W_\xi(x) - |x-q_i|^{2\alpha_i} e^{U_{\delta_i,q_i}}\right) Pu_i\phi \\ &= \int_\Omega Pu_i h - \int_\Omega R_i \phi. \end{aligned} \tag{C.10}$$

First of all, by Lebesgue theorem and (C.7) we get that

$$\int_\Omega |x-q_i|^{2\alpha_i} e^{U_{\delta_i,q_i}} Z_{i0}\phi \to C_i \int_{\mathbb{R}^2} |y|^{2\alpha_i} e^{U^i} (z_0^i)^2 = \frac{8\pi}{3}(\alpha_i+1)C_i. \tag{C.11}$$

The more delicate term is $\int_\Omega \left(a(x)W_\xi(x) - |x-q_i|^{2\alpha_i}e^{U_{\delta_i,q_i}}\right)Pu_i\phi$. By Lemma C.1 and (C.8) we have that

$$
\begin{aligned}
&\int_\Omega \left(a(x)W_\xi(x) - |x-q_i|^{2\alpha_i}e^{U_{\delta_i,q_i}}\right)Pu_i\phi \\
&= \int_{B(q_i,\varepsilon\delta_i^{\frac{1}{2(\alpha_i+1)}})} \left(a(x)W_\xi(x) - |x-q_i|^{2\alpha_i}e^{U_{\delta_i,q_i}}\right)Pu_i\phi \\
&-\frac{8\pi}{3}(\alpha_i+1)\sum_{j\neq i} G(q_j,q_i)\int_{B(q_j,\varepsilon\delta_j^{\frac{1}{2(\alpha_j+1)}})} a(x)W_\xi(x)\phi + O(\frac{1}{p}) \\
&= \frac{4}{3}\frac{\log\delta_i}{p}\int_{B(0,\varepsilon\delta_i^{-\frac{1}{2(\alpha_i+1)}})} |y|^{2\alpha_i}e^{U^i}\left(V^i - U^i - \frac{1}{2}(U^i)^2\right)z_0^i(y)\hat{\phi}_i \\
&-\frac{8\pi}{3}(\alpha_i+1)\sum_{j\neq i} G(q_j,q_i)\int_{B(0,\varepsilon\delta_j^{-\frac{1}{2(\alpha_j+1)}})} |y|^{2\alpha_j}e^{U^j}\hat{\phi}_j + O(\frac{1}{p}) \\
&= -\frac{C_i}{3}\int_{\mathbb{R}^2}|y|^{2\alpha_i}e^{U^i}(z_0^i)^2\left(V^i - U^i - \frac{1}{2}(U^i)^2\right)(y) + o(1)
\end{aligned}
$$

since Lebesgue theorem and (C.7) imply:

$$
\int_{B(0,\varepsilon\delta_i^{-\frac{1}{2(\alpha_i+1)}})} |y|^{2\alpha_i}e^{U^i}\left(V^i - U^i - \frac{1}{2}(U^i)^2\right)z_0^i(y)\hat{\phi}_i \to
$$

$$
C_i\int_{\mathbb{R}^2}|y|^{2\alpha_i}e^{U^i}(z_0^i)^2\left(V^i - U^i - \frac{1}{2}(U^i)^2\right)
$$

and

$$
\int_{B(0,\varepsilon\delta_j^{-\frac{1}{2(\alpha_j+1)}})} |y|^{2\alpha_j}e^{U^j}\hat{\phi}_j \to C_j\int_{\mathbb{R}^2}|y|^{2\alpha_j}e^{U^j}z_0^j = 0.
$$

In a straightforward but tedious way, by (A.4) we can compute:

$$
\int_{\mathbb{R}^2}|y|^{2\alpha_i}e^{U^i}(z_0^i)^2\left(V^i - U^i - \frac{1}{2}(U^i)^2\right)(y) = -8\pi(\alpha_i+1),
$$

so that we obtain

$$
\int_\Omega \left(a(x)W_\xi(x) - |x-q_i|^{2\alpha_i}e^{U_{\delta_i,q_i}}\right)Pu_i\phi = \frac{8\pi}{3}(\alpha_i+1)C_i + o(1). \tag{C.12}
$$

As far as the R.H.S. in (C.10), we have that by (C.8)

$$
\begin{aligned}
|\int_\Omega Pu_ih| &= O\left(\|h\|_*\int_\Omega(\sum_{j=1}^{m+k}\frac{\delta_j|x-q_j|^{2\alpha_j}}{(\delta_j^2+|x-q_j|^{2(\alpha_j+1)})^{\frac{3}{2}}})|u_i|\right) \\
&+O(\|h\|_*) = O(p\|h\|_*)
\end{aligned} \tag{C.13}
$$

since $|u_i| = O(|\log\delta_i|) = O(p)$ in Ω and

$$
\int_{B(q_j,\varepsilon)}\frac{\delta_j|x-q_j|^{2\alpha_j}}{(\delta_j^2+|x-q_j|^{2(\alpha_j+1)})^{\frac{3}{2}}}|u_i| \le Cp\int_{\mathbb{R}^2}\frac{|y|^{2\alpha_j}}{(1+|y|^{2(\alpha_j+1)})^{\frac{3}{2}}} = O(p).
$$

Finally, by (C.8)

$$\int_\Omega R_i\phi = O\left(\int_\Omega |x-q_i|^{2\alpha_i}e^{U_{\delta_i,q_i}}(|x-q_i|+e^{-\frac{p}{4}})\right) = O(e^{-\frac{p}{4(\alpha_i+1)}}). \quad (C.14)$$

Hence, inserting (C.11)-(C.14) in (C.10) we obtain that

$$\frac{16\pi}{3}(\alpha_i+1)C_i = o(1)$$

for any $i = 1,\dots,m+k$. Necessarily, $C_i = 0$ and the claim is proved.

$5^{\mathbf{th}}$ **Step.** We establish the validity of the a priori estimate:

$$\|\phi\|_\infty \le Cp\|h\|_* \tag{C.15}$$

for solutions of problem (C.1)-(C.3) and $h \in C^{0,\alpha}(\bar\Omega)$. The previous Step gives

$$\|\phi\|_\infty \le Cp\left(\|h\|_* + \sum_{j=1}^{2}\sum_{i=m+1}^{m+k}|c_{ij}|\right)$$

since

$$\|e^{U_{\delta_i,q_i}}Z_{ij}\|_* \le 2\|e^{U_{\delta_i,q_i}}\|_* \le 16\ ,\ \ \forall\, j=1,2,\ i=m+1,\dots,m+k.$$

Hence, arguing by contradiction of (C.15), we can proceed as in Step 3 and suppose further that

$$p_n\|h_n\|_* \to 0\ ,\ \ p_n\sum_{j=1}^{2}\sum_{i=m+1}^{m+k}|c^n_{ij}| \ge \delta > 0 \text{ as } n\to+\infty.$$

We omit the dependence on n. It suffices to estimate the values of the constants c_{ij}. For $j=1,2$ and $i=m+1,\dots,m+k$, multiply (C.1) by PZ_{ij} and, integrating by parts, get:

$$\begin{aligned}&\sum_{h=1}^{2}\sum_{l=m+1}^{m+k}c_{lh}\big(PZ_{lh},PZ_{ij}\big)_{\mathrm{H}^1_0} + \int_\Omega hPZ_{ij}\\ &= \int_\Omega a(x)W_\xi(x)\phi PZ_{ij} - \int_\Omega e^{U_{\delta_i,q_i}}Z_{ij}\phi,\end{aligned} \tag{C.16}$$

since $\Delta PZ_{ij} = \Delta Z_{ij} = -e^{U_{\delta_i,q_i}}Z_{ij}$.

We quote now some well known facts, see for example [14]. For $j=1,2$ and $i=m+1,\dots,m+k$ we have the following expansions:

$$\begin{aligned}PZ_{ij} &= Z_{ij} - 8\pi\delta_i\tfrac{\partial H}{\partial(q_i)_j}(\cdot,q_i) + O\left(\delta_i^3\right)\\ PZ_{i0} &= Z_{i0} - 1 + O\left(\delta_i^2\right)\end{aligned} \tag{C.17}$$

in $C^1(\bar\Omega)$ and

$$\begin{aligned} PZ_{ij} &= -8\pi\delta_i \tfrac{\partial G}{\partial(q_i)_j}(\cdot,q_i) + O\left(\delta_i^3\right) \\ PZ_{i0} &= O\left(\delta_i^2\right) \end{aligned} \tag{C.18}$$

in $C^1_{\rm loc}(\bar\Omega\setminus\{q_i\})$. By (C.17)-(C.18) we deduce the following "orthogonality" relations: for $j,h=1,2$ and $i,l=m+1,\dots,m+k$ with $i\neq l$,

$$\begin{aligned} \Big(PZ_{ij},PZ_{ih}\Big)_{H_0^1(\Omega)} &= \left(64\int\limits_{\mathbb{R}^2}\tfrac{|y|^2}{(1+|y|^2)^4}\right)\delta_{jh} + O(\delta_i^2) \\ \Big(PZ_{ij},PZ_{lh}\Big)_{H_0^1(\Omega)} &= O(\delta_i\delta_l) \end{aligned} \tag{C.19}$$

and

$$\begin{aligned} \Big(PZ_{i0},PZ_{ij}\Big)_{H_0^1(\Omega)} &= O(\delta_i^2) \\ \Big(PZ_{i0},PZ_{lh}\Big)_{H_0^1(\Omega)} &= O(\delta_i\delta_l) \end{aligned} \tag{C.20}$$

uniformly on $\xi\in\mathcal{O}_\varepsilon$, where δ_{jh} denotes the Kronecker's symbol.

Now, since

$$|\int_\Omega hPZ_{ij}| \le C'\int_\Omega |h| \le C\|h\|_*,$$

by (C.19) the L.H.S. of (C.16) is estimated as follows:

$$\text{L.H.S.} = Dc_{ij} + O\big(e^{-\frac{p}{2}}\sum_{h=1}^{2}\sum_{l=m+1}^{m+k}|c_{lh}|\big) + O(\|h\|_*), \tag{C.21}$$

where $D = 64\int\limits_{\mathbb{R}^2}\frac{|y|^2}{(1+|y|^2)^4}$. Moreover, by Lemma C.1 the R.H.S. of (C.16) takes the form:

$$\begin{aligned} \text{R.H.S.} &= \int\limits_{B(q_i,\varepsilon\sqrt{\delta_i})} a(x)W_\xi(x)\phi PZ_{ij} - \int_\Omega e^{U_{\delta_i,q_i}}\phi Z_{ij} + O(e^{-\frac{p}{4}}\|\phi\|_\infty) \qquad \text{(C.22)} \\ &= \int\limits_{B(q_i,\varepsilon\sqrt{\delta_i})} \left(a(x)W_\xi(x) - e^{U_{\delta_i,q_i}}\right)\phi PZ_{ij} + \int_\Omega e^{U_{\delta_i,q_i}}\phi\left(PZ_{ij}-Z_{ij}\right) + O(e^{-\frac{p}{4}}\|\phi\|_\infty) \\ &= \frac{1}{p}\int\limits_{B(0,\frac{\varepsilon}{\sqrt{\delta_i}})} \frac{32y_j}{(1+|y|^2)^3}\left(V^i - U^i - \frac{1}{2}(U^i)^2\right)\hat\phi_i + O(\frac{1}{p^2}\|\phi\|_\infty) \end{aligned}$$

in view of (C.17), where $\hat\phi_i(y)=\phi(\delta_i y+q_i)$. Inserting the estimates (C.21) and (C.22) into (C.16), we deduce that

$$Dc_{ij} + O\big(e^{-\frac{p}{2}}\sum_{h=1}^{2}\sum_{l=m+1}^{m+k}|c_{lh}|\big) = O(\|h\|_* + \frac{1}{p}\|\phi\|_\infty).$$

Hence, we obtain that

$$\sum_{h=1}^{2}\sum_{l=m+1}^{m+k}|c_{lh}|=O(\|h\|_*+\frac{1}{p}\|\phi\|_\infty). \tag{C.23}$$

Since $\sum_{h=1}^{2}\sum_{l=m+1}^{m+k}|c_{lh}|=o(1)$, as in Step 4 we have that

$$\hat{\phi}_i\to C_i\frac{|y|^2-1}{|y|^2+1}\ \text{in}\ C^0_{\text{loc}}(\mathbb{R}^2)$$

for some constant C_i, $i=m+1,\dots,m+k$. Hence, in (C.22) we have a better estimate since by Lebesgue theorem the term

$$\int\limits_{B(0,\frac{\varepsilon}{\sqrt{\delta_i}})}\frac{32y_j}{(1+|y|^2)^3}\left(V^i-U^i-\frac{1}{2}(U^i)^2\right)(y)\hat{\phi}_i(y)$$

converges to

$$C_i\int\limits_{\mathbb{R}^2}\frac{32y_j(|y|^2-1)}{(1+|y|^2)^4}\left(V^i-U^i-\frac{1}{2}(U^i)^2\right)(y)=0.$$

Therefore, we get that the R.H.S. in (C.16) satisfies: R.H.S. $=o(\frac{1}{p})$, and in turn, $\sum_{h=1}^{2}\sum_{l=m+1}^{m+k}|c_{lh}|=O(\|h\|_*)+o(\frac{1}{p})$. This contradicts

$$p\sum_{j=1}^{2}\sum_{i=m+1}^{m+k}|c_{ij}|\geq\delta>0,$$

and the claim is established.

6^{th} **Step**. We prove the solvability of (C.1)–(C.3). To this purpose, we consider the spaces:

$$K_\xi=\{\sum_{j=1}^{2}\sum_{i=m+1}^{m+k}c_{ij}PZ_{ij}:c_{ij}\in\mathbb{R}\ \text{for}\ j=1,2,\ i=m+1,\dots,m+k\}$$

and

$$K_\xi^\perp=\{\phi\in L^2(\Omega)\ :\int_\Omega e^{U_{\delta_i,q_i}}Z_{ij}\,\phi\ =\ 0\ \text{for}\ \ j=1,2,\ i=m+1,\dots,m+k\}.$$

Let $\Pi_\xi:\ L^2(\Omega)\to K_\xi$ defined as:

$$\Pi_\xi\phi=\sum_{j=1}^{2}\sum_{i=m+1}^{m+k}c_{ij}PZ_{ij},$$

where c_{ij} are uniquely determined (as it follows by (C.19)) by the system:

$$\int_\Omega e^{U_{\delta_l,q_l}} Z_{lh} \left(\phi - \sum_{j=1}^{2} \sum_{i=m+1}^{m+k} c_{ij} P Z_{ij} \right) = 0 \quad \text{for any } h = 1,2, l = m+1, \ldots, m+k.$$

Let $\Pi_\xi^\perp = \mathrm{Id} - \Pi_\xi : L^2(\Omega) \to K_\xi^\perp$. Problem (C.1)–(C.3), expressed in a weak form, is equivalent to find $\phi \in K_\xi^\perp \cap H_0^1(\Omega)$ such that

$$(\phi, \psi)_{\mathrm{H}_0^1(\Omega)} = \int_\Omega (a(x) W_\xi \phi - h)\ \psi\ dx, \qquad \text{for all } \psi \in K_\xi^\perp \cap H_0^1(\Omega).$$

With the aid of Riesz's representation theorem, this equation gets rewritten in $K_\xi^\perp \cap H_0^1(\Omega)$ in the operatorial form

$$(\mathrm{Id} - K)\phi = \tilde{h}, \tag{C.24}$$

where $\tilde{h} = \Pi_\xi^\perp \Delta^{-1} h$ and $K(\phi) = -\Pi_\xi^\perp \Delta^{-1} (a(x) W_\xi \phi)$ is a linear compact operator in $K_\xi^\perp \cap H_0^1(\Omega)$. The homogeneous equation $\phi = K(\phi)$ in $K_\xi^\perp \cap H_0^1(\Omega)$, which is equivalent to (C.1)-(C.3) with $h \equiv 0$, has only the trivial solution in view of the a priori estimate (C.15). Now, Fredholm's alternative guarantees unique solvability of (C.24) for any $\tilde{h} \in K_\xi^\perp$. Moreover, by elliptic regularity theory this solution is in $W^{2,2}(\Omega)$.
At $p > p_0$ fixed, by density of $C^{0,\alpha}(\bar{\Omega})$ in $(C(\bar{\Omega}), \|\cdot\|_\infty)$, we can approximate $h \in C(\bar{\Omega})$ by Hölderian functions and, by (C.15) and elliptic regularity theory, we can show that estimate $\|\phi\|_\infty \le C\|h\|_*$ holds for any $h \in C(\bar{\Omega})$. The proof is complete. ■

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