INFINITELY MANY TURNING POINTS FOR AN ELLIPTIC PROBLEM WITH A SINGULAR NONLINEARITY

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Abstract. We consider the following problem

$$-\Delta u = \frac{\lambda |x|^\alpha}{(1-u)^p} \text{ in } B, \ \ u = 0 \text{ on } \partial B, \ \ 0 < u < 1 \text{ in } B$$

where $\alpha \geq 0, p \geq 1$ and B is the unit ball in $\mathbb{R}^N$ $(N \geq 2)$. We show that there exists $\lambda_* > 0$ such that for $\lambda < \lambda_*$, the minimizer is the only positive radial solution. Furthermore, if $2 \leq N < 2 + \frac{2(2+\alpha)}{p+1}(p + \sqrt{p^2+p})$, the branch of positive radial solutions must undergo infinitely many turning points as the maximums of the radial solutions on the branch go to 1. This solves Conjecture B in [10]. The key ingredient is the use of monotonicity formula.

1. Introduction

We consider the structure of positive radial solutions of the problem

$$-\Delta u = \frac{\lambda |x|^\alpha}{(1-u)^2} \text{ in } B, \ \ 0 < u < 1 \text{ in } B, \ \ u = 0 \text{ on } \partial B \qquad (S_\lambda)$$

where $\lambda > 0$, $B \subset \mathbb{R}^N$ is the unit ball.

(S_λ) models a simple electrostatic Micro-Electromechanical System (MEMS) device consisting of a thin dielectric elastic membrane with boundary supported at 0 below a rigid plate located at $+1$. When a voltage-represented here by λ-is applied, the membrane deflects towards the ceiling plate and a snap-through may occur when it exceeds a certain critical value λ^* (pull-in voltage). This creates a so-called "pull-in instability" which greatly affects the design of many devices (see [7], [16], [9], [21], [22] for a detailed discussion on MEMS devices). Note that only two-dimensional domains are of real physical relevance.

In recent papers [10]-[12] and [8], the authors studied the problem

$$(P_\lambda) \qquad \begin{cases} -\Delta u = \frac{\lambda g(x)}{(1-u)^2} \text{ in } \Omega \\ 0 < u < 1 \text{ in } \Omega \\ u = 0 \text{ on } \partial\Omega \end{cases}$$

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where $\Omega \subset \mathbb{R}^N$ is a bounded smooth domain and $g \in C(\overline{\Omega})$ is a nonnegative function. They gave a detailed study on the minimal solutions of the problem (P_λ) with different forms of $g(x)$. The following theorem was obtained.

Theorem 1.1. *(Theorem 1.1-1.3 in* [10]*): Suppose $g \in C(\overline{\Omega})$ is a nonnegative function on Ω. Then, there exists a finite $\lambda^* > 0$ such that*

1. If $0 \le \lambda < \lambda^*$, there exists a unique minimal solution $\underline{u}_\lambda$ of (P_λ) such that $\mu_{1,\lambda}(\underline{u}_\lambda) > 0$. Moreover, $\underline{u}_\lambda \to 0$ as $\lambda \to 0$.

2. If $\lambda > \lambda^*$, there is no solution for (P_λ).

3. If $1 \le N \le 7$, then-by means of energy estimates-one has

$$\sup_{\lambda \in (0,\lambda^*)} \|\underline{u}_\lambda\|_\infty < 1$$

and consequently $u^* = \lim_{\lambda \uparrow \lambda^*} \underline{u}_\lambda$ is a solution of (P_{λ^*}) such that

$$\mu_{1,\lambda^*}(u^*) = 0.$$

4. If $g(x) = |x|^\alpha$ and Ω is the unit ball, then $u^*(x) = 1 - |x|^{\frac{2+\alpha}{3}}$ and $\lambda^* = \frac{(2+\alpha)(3N+\alpha-4)}{9}$, provided $N \ge 8$ and $0 \le \alpha \le \alpha_N = \frac{3N-14-4\sqrt{6}}{4+2\sqrt{6}}$.

Issues of uniqueness, multiplicity and other qualitative properties of the solutions for (P_λ) are still far from being well understood, even in the radially symmetric case. In their paper [10], Ghoussoub and Guo present some numerical evidence for various conjectures relating the case $g(x) = |x|^\alpha$. (See Figure 4 of [10].) In particular, they conjectured if

$$(*) \quad 2 \le N \le 7, \alpha > 0 \text{ and } N \ge 8, \alpha > \frac{3N - 14 - 4\sqrt{6}}{4 + 2\sqrt{6}},$$

then there exists an infinite number of branches of solutions. The purpose of this paper is to prove this conjecture, at least in the radially symmetric case.

More precisely, we shall consider a more general form of (S_λ):

$$-\Delta u = \frac{\lambda |x|^\alpha}{(1-u)^p} \text{ in } B, \quad 0 < u < 1, u = u(r) \text{ in } B, \quad u = 0 \text{ on } \partial B \qquad (S_{\lambda,p})$$

where $p \ge 1$. Note that when $p = 1$, $(S_{\lambda,p})$ arises in the study of singular minimal hypersurfaces with symmetry. See [19], [23] and the references therein. For general $p > 0$, $(S_{\lambda,p})$ also arises in relation to chemical catalyst kinetics (see [2] and [7]).

By a minimal solution $\underline{u}_\lambda$ of the equation $(S_{\lambda,p})$, we mean that $\underline{u}_\lambda \in C^2(B)$ satisfies $\underline{u}_\lambda \le u$ in B for any solution u of $(S_{\lambda,p})$. Throughout this paper, unless otherwise specified, solutions for $(S_{\lambda,p})$ are considered to be in the classical sense. Now for any solution u of $(S_{\lambda,p})$, one can introduce the linearized operator at u defined by:

$$L_{u,\lambda} = -G - \frac{p\lambda r^\alpha}{(1-u)^{p+1}} I,$$

where $G(h) = r^{1-N}(r^{N-1}h'(r))'$ with $h'(0) = 0$, $h(1) = 0$ and its corresponding eigenvalues $\{\mu_{k,\lambda} : \ k = 1, 2, \ldots\}$. Note that the first eigenvalue is simple and is given by:

$$\begin{aligned} \mu_{1,\lambda}(u) \ &= \ \inf\Big\{ \int_0^1 \Big[r^{N-1}|\phi'(r)|^2 - \frac{p\lambda r^{N-1+\alpha}}{(1-u)^{p+1}}\phi^2(r)\Big] dr : \\ &\qquad \phi = \phi(r), \phi(1) = 0, \ \int_0^1 r^{N-1}|\phi(r)|^2 dr = 1 \Big\} \end{aligned}$$

with the infimum being attained at a first eigenfunction ϕ_1, while the second eigenvalue is given by the formula:

$$\begin{aligned} \mu_{2,\lambda}(u) \ &= \ \inf\Big\{ \int_0^1 \Big[r^{N-1}|\phi'(r)|^2 - \frac{p\lambda r^{N-1+\alpha}}{(1-u)^{p+1}}\phi^2(r)\Big] dr : \\ &\qquad \phi = \phi(r), \phi(1) = 0, \ \int_0^1 r^{N-1}|\phi(r)|^2 dr = 1, \ \int_0^1 r^{N-1}\phi(r)\phi_1(r) dr = 0 \Big\}. \end{aligned}$$

This construction can then be iterated to obtain the k-th eigenvalue $\mu_{k,\lambda}(u)$ with the convention that eigenvalues are repeated according to their multiplicities.

In this paper, we shall give the exact shape of the branch of positive radial solutions of $(S_{\lambda,p})$. We show that there exists $\lambda_* > 0$ such that for $\lambda < \lambda_*$, the minimizer is the only radial solution. Moreover, the branch of positive radial solutions of $(S_{\lambda,p})$ must undergo infinitely many turning points (the points where the branch changes direction, i.e. the points where the branch locally "bends back"). This agrees with the numerical evidence on the radial solutions of the problem on B (see Figure 4 of [10]). Note that in our case here, the maximum of the solution of $(S_{\lambda,p})$ may close to 1 and this makes the problem more difficult to deal with.

Let D denote the component of $\{(u(r), \lambda) \in C([0,1]) \times \mathbf{R}^+ : \ -r^{1-N}(r^{N-1}u'(r))' = \frac{\lambda r^\alpha}{(1-u)^p}, \ 0 < u < 1 \text{ in } (0,1), \ u(1) = 0\}$. We see from Theorem 1.1 that D contains $(0,0)$ in its closure, since the minimal solutions $\underline{u}_\lambda$ of $(S_{\lambda,p})$ are radially symmetric (see [10]). As mentioned in [10], the main conclusions of Theorem 1.1 are still true for $p \geq 1$. Note that we can talk about the component since it is a simple curve near the end point. It is convenient to add $(0,0)$ to D.

Our main result of this paper is the following theorem.

Theorem 1.2. *(1)* $\lambda_* = \inf\{\lambda > 0 : \ (u, \lambda) \in D$ *for some non-minimal* $u\} > 0$.

(2) If N lies in the range

$$2 \leq N < 2 + \frac{2(2+\alpha)}{p+1}(p + \sqrt{p^2+p}), \tag{1.1}$$

then D has infinitely many turning points.

Remark 1.3. It is easy to see that when $p = 2$, condition (1.1) is equivalent to condition (*). So we have given a rigorous proof of the Conjecture B of [10], in the radially symmetric case.

There already exist in the literature many interesting results concerning the properties of the branch of solutions for Dirichlet boundary value problems of the form $-\Delta u = \lambda h(u)$ where h is a regular nonlinearity (for example of the form e^u or $(1+u)^p$ for $p > 1$). See, for example, [3], [4], [17], [18] and the references therein. The singular situation was considered in a very general context in [20].

The key ingredient of our proof is the critical use of the monotonicity formula derived in Section 2.

The organization of the paper is as follows: in Section 2, we derive the key estimate–monotonicity formula. In Section 3, we use the monotonicity formula to prove the uniqueness of minimizers for small λ. Finally, we adopt Dancer's idea in [6] to prove (2) of Theorem 1.2 in Section 4.

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2. A monotonicity inequality

In this section we will obtain a monotonicity inequality similar to that in [14] for nonnegative finite energy stationary solutions $u \in H^1(\Omega)$ and $|x|^\alpha u^{-p} \in L^1_{loc}(\Omega)$ of the equation

$$\Delta u = \lambda |x|^\alpha u^{-p} \text{ in } \Omega \tag{2.1}$$

where $p \geq 1$ and Ω is a bounded smooth domain in $\mathbb{R}^N$ $(N \geq 2)$ and $0 \in \Omega$. From now on, we will assume that $p > 1$. The case of $p = 1$ can be modified by changing $\frac{1}{1-p}u^{1-p}$ to $\log u$.

We call u a *nonnegative finite energy solution* to (2.1) in Ω if $u \geq 0$ in Ω and $u \in H^1(\Omega)$ with $\int_\Omega u^{1-p} dx < \infty$. We also say such a finite energy solution u is *stationary* if, in addition, it satisfies

$$(2.2) \quad \int_\Omega \Big[\frac{\partial u}{\partial x_i}\frac{\partial u}{\partial x_j}\frac{\partial \phi^j}{\partial x_i} - \frac{1}{2}|\nabla u|^2 \frac{\partial \phi^i}{\partial x_i} - \frac{\lambda}{1-p} u^{1-p}\frac{\partial |x|^\alpha}{\partial x_i}\phi^i - \frac{\lambda}{1-p}u^{1-p}|x|^\alpha \frac{\partial \phi^i}{\partial x_i}\Big] dx = 0$$

for all regular vector field ϕ with compact support in Ω (summation over i and j is understood.) The identity (2.2) can be obtained by multiplying (2.1) by $\phi \cdot \nabla u$ and integrating it by parts in Ω (if it can be integrated by parts). Note that if $u > 0$ in Ω and $u \in C^2(\Omega)$, then u is stationary.

Theorem 2.1. *For any $r_0 > 0$ with $B(0, r_0) \subset \Omega$, if u is a nonnegative finite energy stationary solution of (2.1), then*

$$(2.3) \quad \mathcal{E}_u(r) \equiv -\frac{(p+1)\lambda}{2(p-1)} r^{-\mu} \int_{B(0,r)} |x|^\alpha u^{1-p} dx + \frac{1}{4}\frac{d}{dr}\Big[r^{-\mu}\int_{\partial B(0,r)} u^2 dS\Big] - \frac{1}{4} r^{-\mu-1}\int_{\partial B(0,r)} u^2 dS$$

is an increasing function of r for $r \in (0, r_0)$, where $\mu = N - 2 + \frac{4+2\alpha}{p+1}$.

Proof. Fix $r_0 > 0$ such that $B(0, r_0) \subset \Omega$. Let r, $m > 0$ be such that $r + m < r_0$. Set $\phi(x) = \xi(|x|)x$, where

$$\xi(|x|) \equiv \begin{cases} 1 & \text{for } |x| \leq r, \\ 1 + \frac{r-|x|}{m} & \text{for } r \leq |x| \leq r+m, \\ 0 & \text{for } |x| \geq r+m. \end{cases}$$

We derive from (2.2), letting $m \to 0^+$, that the following identity holds for $0 < r < r_0$:

$$\frac{N\lambda}{p-1}\int_{B(0,r)} |x|^\alpha u^{1-p} dx - \frac{N-2}{2}\int_{B(0,r)} |\nabla u|^2 dx + \frac{r}{2}\int_{\partial B(0,r)} |\nabla u|^2 dS$$

$$(2.4) \quad + \frac{\lambda}{p-1}\int_{B(0,r)} (x\cdot\nabla|x|^\alpha) u^{1-p} dx - \frac{\lambda r}{p-1}\int_{\partial B(0,r)} |x|^\alpha u^{1-p} dS = r\int_{\partial B(0,r)} (u_r)^2 dS.$$

Since

$$x \cdot \nabla |x|^\alpha = \alpha |x|^\alpha$$

we see from (2.4) that

$$\frac{(N+\alpha)\lambda}{p-1}\int_{B(0,r)} |x|^\alpha u^{1-p} dx - \frac{N-2}{2}\int_{B(0,r)} |\nabla u|^2 dx + \frac{r}{2}\int_{\partial B(0,r)} |\nabla u|^2 dS$$

$$(2.5) \quad -\frac{\lambda r}{p-1}\int_{\partial B(0,r)} |x|^\alpha u^{1-p} dS = r\int_{\partial B(0,r)} (u_r)^2 dS.$$

On the other hand, multiplying (2.1) by u and integrating over $B(0,r)$ we find, for almost every $0<r<r_0$,

$$\int_{B(0,r)} |\nabla u|^2 dx = \int_{\partial B(0,r)} u u_r dS - \lambda \int_{B(0,r)} |x|^\alpha u^{1-p} dx. \tag{2.6}$$

Taking the derivative of (2.6) with respect to r, we obtain

$$\int_{\partial B(0,r)} |\nabla u|^2 dS = \frac{d}{dr}\int_{\partial B(0,r)} u u_r dS - \lambda \int_{\partial B(0,r)} |x|^\alpha u^{1-p} dS. \tag{2.7}$$

Substituting $\int_{B(0,r)} |\nabla u|^2 dx$ of (2.6) and $\int_{\partial B(0,r)} |\nabla u|^2 dS$ of (2.7) into (2.5), we finally obtain

$$\begin{aligned}\Big(\frac{N+\alpha}{p-1}+\frac{N-2}{2}\Big)\lambda\int_{B(0,r)} |x|^\alpha u^{1-p} dx - \Big(\frac{1}{2}+\frac{1}{p-1}\Big)\lambda r\int_{\partial B(0,r)} |x|^\alpha u^{1-p} dS \\ +\frac{r}{2}\frac{d}{dr}\int_{\partial B(0,r)} u u_r dS - \frac{N-2}{2}\int_{\partial B(0,r)} u u_r dS = r\int_{\partial B(0,r)} (u_r)^2 dS.\end{aligned} \tag{2.8}$$

Rewriting (2.8), we have

$$\begin{aligned}-\frac{(p+1)\lambda}{2(p-1)}\frac{d}{dr}\Big[r^{-\mu}\int_{B(0,r)} |x|^\alpha u^{1-p} dx\Big] + \frac{1}{2}r^{-\mu}\frac{d}{dr}\Big[\int_{\partial B(0,r)} u u_r dS\Big] \\ = r^{-\mu}\int_{\partial B(0,r)} \Big[(u_r)^2 + \frac{N-2}{2} r^{-1} u u_r\Big] dS,\end{aligned} \tag{2.9}$$

where $\mu = N-2+\frac{4+2\alpha}{p+1}$. Using the identity

$$\frac{d}{dr}\Big[\int_{\partial B(0,r)} u^2 dS\Big] = 2\int_{\partial B(0,r)} u u_r dS + (N-1)\int_{\partial B(0,r)} r^{-1} u^2 dS$$

we have that

$$\begin{aligned}\frac{1}{2}\frac{d^2}{dr^2}\Big[r^{-\mu}\int_{\partial B(0,r)} u^2 dS\Big] - \frac{d}{dr}\Big[r^{-\mu}\int_{\partial B(0,r)} u u_r dS\Big] \\ = (N-\mu-1) r^{-\mu}\int_{\partial B(0,r)} \Big[\frac{(N-2-\mu)}{2} r^{-2} u^2 + r^{-1} u u_r\Big] dS.\end{aligned} \tag{2.10}$$

Note that

$$r^{-\mu}\frac{d}{dr}\Big[\int_{\partial B(0,r)} u u_r dS\Big] = \frac{d}{dr}\Big[r^{-\mu}\int_{\partial B(0,r)} u u_r dS\Big] + \mu r^{-\mu-1}\int_{\partial B(0,r)} u u_r dS. \tag{2.11}$$

Substituting (2.11) and (2.10) into (2.9), we obtain that

$$\begin{aligned}-\frac{(p+1)\lambda}{2(p-1)}\frac{d}{dr}\Big[r^{-\mu}\int_{B(0,r)} |x|^\alpha u^{1-p} dx\Big] + \frac{1}{4}\frac{d^2}{dr^2}\Big[r^{-\mu}\int_{\partial B(0,r)} u^2 dS\Big] \\ = r^{-\mu}\int_{\partial B(0,r)} \Big[(u_r)^2 + \frac{2N-2\mu-3}{2} r^{-1} u u_r + \frac{1}{4}(N-\mu-1)(N-\mu-2) r^{-2} u^2\Big] dS\end{aligned}$$

which yields that

$$
\begin{aligned}
&-\frac{(p+1)\lambda}{2(p-1)}\frac{d}{dr}\Big[r^{-\mu}\int_{B(0,r)}|x|^\alpha u^{1-p}dx\Big]\\
&\quad+\frac{1}{4}\frac{d^2}{dr^2}\Big[r^{-\mu}\int_{\partial B(0,r)}u^2dS\Big]-\frac{1}{4}\frac{d}{dr}\Big[r^{-\mu-1}\int_{\partial B(0,r)}u^2dS\Big]\\
&= r^{-\mu}\int_{\partial B(0,r)}\Big(u_r+\frac{N-\mu-2}{2}r^{-1}u\Big)^2dS\geq 0. \qquad (2.12)
\end{aligned}
$$

We conclude from (2.12) that

$$
\begin{aligned}
\mathcal{E}_u(r) \;\equiv\; & -\frac{(p+1)\lambda}{2(p-1)}r^{-\mu}\int_{B(0,r)}|x|^\alpha u^{1-p}dx+\frac{1}{4}\frac{d}{dr}\Big[r^{-\mu}\int_{\partial B(0,r)}u^2dS\Big]\\
&-\frac{1}{4}r^{-\mu-1}\int_{\partial B(0,r)}u^2dS
\end{aligned}
$$

is an increasing function of r for $r\in(0,r_0)$. This completes the proof. $\square$

3. Uniqueness of solutions for small λ

In the following we focus on the uniqueness of solutions of $(S_{\lambda,p})$ when λ is small enough, thereby proving (1) of Theorem 1.2. It is known from Theorem 1.1 that there exists a unique minimal solution $\underline{u}_\lambda$ of $(S_{\lambda,p})$ for $0\leq\lambda<\lambda^*$. By arguments similar to those in the proof of Theorem 5.5 of [10], we can show that for every $M>0$ there exists $0<\lambda_1^*(M)<\lambda^*$ such that for $\lambda\in(0,\lambda_1^*(M))$, $(S_{\lambda,p})$ has a unique solution u_λ satisfying

$$\Big\|\frac{1}{(1-u_\lambda)^{p+1}}\Big\|_{L^{1+\epsilon}(\Omega)}\leq M,\quad \text{for } N=2$$

and

$$\Big\|\frac{1}{(1-u_\lambda)^{p+1}}\Big\|_{L^{N/2}(\Omega)}\leq M,\quad \text{for } N\geq 3$$

where $0<\epsilon<1$ is a small number.

In this section, we shall show that there exists $0<\lambda_*<\lambda^*$ such that for $\lambda\in(0,\lambda_*)$, $(S_{\lambda,p})$ has only the minimal solution $\underline{u}_\lambda$.

Theorem 3.1. *There exists $0<\lambda_*<\lambda^*$ such that for $\lambda\in(0,\lambda_*)$, $(S_{\lambda,p})$ has a unique radial solution, i.e. the minimal solution $\underline{u}_\lambda$.*

We prove this theorem by a contradiction argument. On the contrary, we see that there are sequences $\{\lambda_i\}$ and $\{u_i\}\equiv\{u_{\lambda_i}\}$ with $\lambda_i\to 0$ as $i\to\infty$ such that u_i is a non-minimal solution of $(S_{\lambda_i,p})$. A solution $u(r)$ is said to be a non-minimal solution of $(S_{\lambda,p})$, if $0 \le u < 1$ in B and there exists another solution $0 \le v < 1$ of $(S_{\lambda,p})$ and a point $r \in B$ such that $u(r) > v(r)$.

We consider two cases here:

(i) there is a $0 < \rho < 1$ such that $\|u_i\|_\infty \le 1 - \rho$ for all i (we can choose subsequences if necessary).

(ii) $\|u_i\|_\infty \to 1$ as $i \to \infty$ (we can choose subsequences if necessary).

By arguments similar to those in the proof of Theorem 5.5 of [10] we easily see that the first case does not occur since for this case $(S_{\lambda_i,p})$ has only the minimal solution, but u_i is a non-minimal solution by our assumption. Note that there exists $M > 0$ such that

$$\|\frac{1}{(1-u_i)^{p+1}}\|_{L^{1+\epsilon}(B)} \le M \ \text{ for } N = 2$$

and

$$\|\frac{1}{(1-u_i)^{p+1}}\|_{L^{N/2}(B)} \le M \ \text{ for } N \ge 3.$$

We only need to consider the second case. Defining $z_i(r) = 1 - u_i(r)$, we see that $\min_B z_i \to 0$ as $i \to \infty$. Note that z_i satisfies the problem

$$z_i'' + \frac{N-1}{r} z_i' = \frac{\lambda_i r^\alpha}{z_i^p} \ \text{ in } [0,1], \quad z_i'(0) = 0, \quad z_i(1) = 1. \tag{3.1}$$

Lemma 3.2. *For each i, $z_i'(r) \ge 0$ for $r \in (0,1]$.*

Proof. We can write (3.1) in the form

$$(r^{N-1} z_i')' = \frac{\lambda_i r^{N-1+\alpha}}{z_i^p} \ \text{ for } r \in (0,1). \tag{3.2}$$

Then integrating (3.2) from 0 to r, we see from (3.2) that

$$r^{N-1} z_i'(r) \ge 0 \ \text{ for } r \in (0,1].$$

This completes the proof. $\square$

We define $\epsilon_i = \min_B z_i$ $(:= z_i(0))$ and

$$U_i(y) = \frac{z_i(\epsilon_i^{(1+p)/(2+\alpha)} y)}{\epsilon_i}, \ y \in B_i := \{y : \ \epsilon_i^{(1+p)/(2+\alpha)} y \in (0,1]\}. \tag{3.3}$$

Then

$$U_i'' + \frac{N-1}{y} U_i' = \frac{\lambda_i y^\alpha}{U_i^p} \ \text{ in } (0, \epsilon_i^{-(1+p)/(2+\alpha)}), \quad U_i'(0) = 0, \quad U_i(0) = 1.$$

We shall show that such U_i does not exist. In the following, $O(1)$ means $|O(1)| \le C$ for some $C > 0$.

Lemma 3.3. *For any $\beta \in (0,1)$, there is $\kappa := \kappa(\beta) \in (0,1)$ independent of i such that for i sufficiently large,*

$$z_i(r) \geq \kappa, \quad z_i'(r) = O(1) \ \text{for } r \in [\beta, 1] \tag{3.4}$$

Proof. By the well-known Pohozaev identity,

$$\frac{d}{dr}\Big\{r^N\Big[\frac{1}{2}z_i'^2(r) + \frac{\lambda_i}{p-1}r^\alpha z_i^{1-p}\Big]\Big\}$$

$$= r^{N-1}\Big[(1-\frac{N}{2})z_i'^2(r) + \frac{N+\alpha}{p-1}\lambda_i r^\alpha z_i^{1-p}\Big] \tag{3.5}$$

integrating (3.5) from 0 to 1, we see that there is $C > 0$ independent of i such that

$$z_i'^2(1) \leq C\lambda_i \int_0^1 r^{N-1+\alpha} z_i^{1-p} dr.$$

On the other hand, it follows from the equation of z_i that

$$z_i'(1) = \lambda_i \int_0^1 \frac{r^{N-1+\alpha}}{z_i^p(r)} dr.$$

Thus,

$$\Big(\lambda_i \int_0^1 \frac{r^{N-1+\alpha}}{z_i^p(r)}dr\Big)^2 \leq C\lambda_i \int_0^1 \frac{r^{N-1+\alpha}}{z_i^{p-1}(r)}dr \leq C\lambda_i\Big(\int_0^1 \frac{r^{N-1+\alpha}}{z_i^p(r)}dr\Big)^{(p-1)/p}\Big(\int_0^1 r^{N-1+\alpha}dr\Big)^{1/p}.$$

This implies

$$\int_0^1 r^{N-1+\alpha} z_i^{-p}(r)dr \leq C\lambda_i^{-p/(p+1)}. \tag{3.6}$$

Since $z_i'(r) \geq 0$, we see that

$$\lambda_i r^\alpha z_i^{-p}(r) \leq C\lambda_i \int_{\beta/2}^r \xi^{N-1+\alpha} z_i^{-p}(\xi)d\xi \ \text{ for } r \in [\beta, 1]. \tag{3.7}$$

Thus, it follows from (3.6) and (3.7) that

$$\lambda_i r^\alpha z_i^{-p}(r) \leq C\lambda_i \int_0^1 r^{N-1+\alpha} z_i^{-p}(r)dr \leq C\lambda_i^{1/(p+1)} \ \text{ for } r \in [\beta, 1].$$

Let $k_i(r)$ be the solution of the problem

$$k_i'' + \frac{N-1}{r}k_i' = C\lambda_i^{1/(p+1)} \ \text{ in } (\beta, 1), \quad k_i(1) = 1, \ \ k_i(\beta) = 0.$$

The maximum principle implies that

$$z_i(r) \geq k_i(r) \ \text{ for } r \in [\beta, 1].$$

Since $k_i = k_0 + C\lambda_i^{1/(p+1)}k_1$, where

$$k_0'' + \frac{N-1}{r}k_0' = 0 \ \text{ in } (\beta, 1), \quad k_0(1) = 1, \ \ k_0(\beta) = 0$$

and

$$k_1'' + \frac{N-1}{r}k_1' = 1 \text{ in } (\beta, 1), \quad k_1(1) = 0, \ \ k_1(\beta) = 0$$

we see that

$$z_i \geq k_0 + C\lambda_i^{1/(p+1)}k_1.$$

Note that the maximum principle implies that $k_0 > 0$ in $(\beta, 1)$ and $|k_1(r)| \leq C$ for $r \in (\beta, 1)$. We see that

$$z_i \geq C \ \text{ for } r \in (\tau\beta, 1] \text{ and } i \text{ sufficiently large}$$

where $\tau > 1$ is close to 1. The arbitrary of β implies the first inequality of (3.4) holds. The second identity in (3.4) can be obtained from the regularity of Δ or direct calculations. This completes the proof. □

Proof of Theorem 3.1

Let U_i be defined at (3.3). We see that $U_i(0) = 1$, $U_i'(0) = 0$ and

$$(y^{N-1}U_i')' = \lambda_i \frac{y^{N-1+\alpha}}{U_i^p}$$

which implies that $U_i' \geq 0$ and

$$y^{N-1}U_i'(y) = \lambda_i \int_0^y \frac{t^{N-1+\alpha}}{U_i^p(t)} \geq \frac{\lambda_i}{(N+\alpha)U_i^p(y)} y^{N+\alpha}$$

which implies that

$$U_i(y) \geq C\lambda_i y^{(2+\alpha)/(p+1)} \ \text{ for all } y \geq 0. \tag{3.8}$$

By the Emden-Fowler transformation:

$$v_i(s) = y^{-(2+\alpha)/(p+1)}U_i(y), \quad y = e^s$$

we see that v_i satisfies the equation

$$v_i''(s) + \Big(N - 2 + \frac{4+2\alpha}{p+1}\Big)v_i'(s) + \frac{2+\alpha}{p+1}\Big(N - 2 + \frac{2+\alpha}{p+1}\Big)v_i = \lambda_i v_i^{-p} \tag{3.9}$$

and it follows from (3.8) that $v_i(s) \geq C\lambda_i$. Define $w_i(s) = v_i^2(s)$. In the following, we omit the subscript i from v_i and w_i for convenience.

Let $R_i = \ln \epsilon_i^{-(1+p)/(2+\alpha)}$. We see that $R_i \to \infty$ as $i \to \infty$. Moreover, we see from $U_i'(y) \geq 0$ that

$$v'(s) + \frac{2+\alpha}{p+1}v(s) \geq 0 \ \text{ for } s \in (-\infty, R_i] \tag{3.10}$$

Moreover,

$$v(R_i) = 1, \ \ v'(R_i) = O(1), \quad C^{-1} \leq v(0) \leq C, \ \ v'(0) = O(1), \tag{3.11}$$

and for any $0 < \tilde{\beta} < 1/2$

$$C^{-1} \le v(s) \le C, \quad v'(s) = O(1) \text{ for } s \in [0, \tilde{\beta}] \tag{3.12}$$

$$C^{-1} \le v(s) \le C, \quad v'(s) = O(1) \text{ for } s \in [R_i + \ln \tilde{\beta}, R_i]. \tag{3.13}$$

The above identities can be obtained from simple calculations, the fact that $U_i(y) \to 1$ in $C^1_{loc}(0, \infty)$ as $i \to \infty$ and Lemma 3.3. From (3.9) we see that

$$v'''(s) + \Big(N-2+\frac{4+2\alpha}{p+1}\Big)v''(s) + \frac{2+\alpha}{p+1}\Big(N-2+\frac{2+\alpha}{p+1}\Big)v'(s) = -p\lambda_i v^{-(p+1)}(s)v'(s)$$

and using (3.10), we obtain

$$v'''(s) + \Big(N-2+\frac{4+2\alpha}{p+1}\Big)v''(s) + \frac{2+\alpha}{p+1}\Big(N-2+\frac{2+\alpha}{p+1}\Big)v'(s) \le \frac{(2+\alpha)p}{p+1}\lambda_i v^{-p}(s)$$

$$= \frac{(2+\alpha)p}{p+1}\Big[v''(s) + \Big(N-2+\frac{4+2\alpha}{p+1}\Big)v'(s) + \frac{2+\alpha}{p+1}\Big(N-2+\frac{2+\alpha}{p+1}\Big)v(s)\Big].$$

Setting $q(s) = v'(s) - \frac{2+\alpha}{p+1}v(s)$, we see that

$$q''(s) + \Big(N-2+\frac{4+2\alpha}{p+1}\Big)q'(s) + \frac{2+\alpha}{p+1}\Big(N-2+\frac{2+\alpha}{p+1}\Big)q(s) \le 0$$

i.e.,

$$\Big[e^{\frac{2+\alpha}{p+1}s}\Big(q'(s) + \Big(N-2+\frac{2+\alpha}{p+1}\Big)q(s)\Big)\Big]' \le 0 \text{ for } s \in (-\infty, R_i]. \tag{3.14}$$

Note that

$$e^{\frac{2+\alpha}{p+1}s}v(s) \to 1 \text{ as } s \to -\infty$$

and

$$e^{\frac{2+\alpha}{p+1}s}\Big(v'(s) + \frac{2+\alpha}{p+1}v(s)\Big) \to 0 \text{ as } s \to -\infty.$$

We see that

$$e^{\frac{2+\alpha}{p+1}s}q(s) \to -\frac{2(2+\alpha)}{p+1} \quad \text{as } s \to -\infty. \tag{3.15}$$

This implies that

$$(e^{\frac{2+\alpha}{p+1}s}q(s))' \to 0 \text{ as } s \to -\infty. \tag{3.16}$$

(3.15) and (3.16) imply that

$$e^{\frac{2+\alpha}{p+1}s}\Big(q'(s) + \Big(N-2+\frac{2+\alpha}{p+1}\Big)q(s)\Big)$$
$$= \Big(e^{\frac{2+\alpha}{p+1}s}q(s)\Big)' + (N-2)e^{\frac{2+\alpha}{p+1}s}q(s)$$
$$\to -\frac{2(N-2)(2+\alpha)}{p+1} \text{ as } s \to -\infty.$$

Then (3.14) implies that

(3.17)
$$e^{\frac{2+\alpha}{p+1}s}\Big(q'(s)+\Big(N-2+\frac{2+\alpha}{p+1}\Big)q(s)\Big)\le -\frac{2(N-2)(2+\alpha)}{p+1}<0 \quad \text{for } s\in(-\infty,R_i]$$

On the other hand, we see from (3.17) that

$$\begin{aligned} 0 &\ge q'(s)+\Big(N-2+\frac{2+\alpha}{p+1}\Big)q(s)\\ &= v''(s)+(N-2)v'(s)-\frac{(2+\alpha)}{p+1}\Big(N-2+\frac{2+\alpha}{p+1}\Big)v(s)\\ &= \lambda_i v^{-p}(s)-\frac{2(2+\alpha)}{p+1}v'(s)-\frac{2(2+\alpha)}{p+1}\Big(N-2+\frac{2+\alpha}{p+1}\Big)v(s). \end{aligned}$$

Then

$$\lambda_i v^{-p}(s)\le C[|v'(s)|+v(s)] \ \text{ for } s\in(-\infty,R_i]. \tag{3.18}$$

On the other hand, if we define

$$J(s)=\frac{1}{2}v'^2(s)+\frac{2+\alpha}{2(p+1)}\Big(N-2+\frac{2+\alpha}{p+1}\Big)v^2(s)+\frac{\lambda_i}{p-1}v^{1-p}(s),$$

we see that

$$J'(s)=-\Big(N-2+\frac{4+2\alpha}{p+1}\Big)(v')^2(s)\le 0.$$

This implies that

$$J(R_i)\le J(s)\le J(0) \ \text{ for } s\in[0,R_i] \tag{3.19}$$

We see from (3.12) and (3.19) that

$$|v'(s)|\le C,\quad v(s)\le C \quad \text{for } s\in[0,R_i]. \tag{3.20}$$

This and (3.18) imply that

$$\lambda_i v^{-p}(s)\le C \ \text{ for } s\in[0,R_i]. \tag{3.21}$$

This also implies that

$$\lambda_i v^{1-p}(s)\le C\lambda_i^{1/p} \ \text{ for } s\in[0,R_i]. \tag{3.22}$$

Now we use the monotonicity formula in Theorem 2.1: since $U_i\in C^2(B_i)$, we easily see that U_i is stationary, where $B_i=\{z:\epsilon_i^{(p+1)/(2+\alpha)}z\in B\}$. By Theorem 2.1, the function

$$\mathcal{E}_{U_i}(r)=-\frac{(p+1)\lambda}{2(p-1)}r^{-\mu}\int_{B(0,r)}|z|^\alpha U_i^{1-p}dz+\frac{1}{4}\frac{d}{dr}\Big[r^{-\mu}\int_{\partial B(0,r)}U_i^2dS\Big]-\frac{1}{4}r^{-\mu-1}\int_{\partial B(0,r)}U_i^2dS$$

is a nondecreasing function of $r \in (0, \epsilon_i^{-(p+1)/(2+\alpha)})$. Moreover, a simple calculation implies that under the changes:

$$v_i(s) = |z|^{-(2+\alpha)/(p+1)} U_i, \quad |z| = e^s$$

the function $\mathcal{E}_{U_i}(r)$ is just a positive multiple of

$$\mathcal{E}_{v_i}(s) = w'(s) - \frac{2(p+1)}{p-1} h_i(s)$$

where $h_i(s) = \lambda_i \int_{-\infty}^{s} e^{\frac{(N-2)(p+1)+4+2\alpha}{p+1}(\tau - s)} v^{1-p}(\tau) d\tau$. Note that U_i is a radial function. Hence $\mathcal{E}_{v_i}(s)$ is a nondecreasing function of s for $s \in [0, R_i)$. We see from (3.22) that

$$h_i(s) \le C\lambda_i^{1/p} \ \text{ for } s \in [0, R_i]. \tag{3.23}$$

Now we claim that

$$w'(R_i) \le C\tilde{\beta}^{\frac{2+\alpha}{p+1}} + o(1) - 2\Big(N - 2 + \frac{2+\alpha}{p+1}\Big). \tag{3.24}$$

Note $v(R_i) = 1$ implies $w(R_i) = 1$. Indeed, it follows from the equation of v that

$$\begin{aligned}
&v'(R_i) + \Big(N - 2 + \frac{2+\alpha}{p+1}\Big) v(R_i) \\
&= \lambda_i \int_{-\infty}^{R_i} e^{\frac{2+\alpha}{p+1}(\tau - R_i)} v^{-p}(\tau) d\tau \\
&\le C\lambda_i \int_{R_i + \ln\tilde{\beta}}^{R_i} e^{\frac{2+\alpha}{p+1}(\tau - R_i)} d\tau + \lambda_i \int_{-\infty}^{R_i + \ln\tilde{\beta}} e^{\frac{2+\alpha}{p+1}(\tau - R_i)} v^{-p} d\tau \\
&\le C\lambda_i (1 - e^{\frac{2+\alpha}{p+1}\ln\tilde{\beta}}) + e^{-\frac{2+\alpha}{p+1}R_i} \lambda_i \int_{-\infty}^{0} e^{\frac{2+\alpha}{p+1}\tau} v^{-p} d\tau + \lambda_i \int_{0}^{R_i + \ln\tilde{\beta}} e^{\frac{2+\alpha}{p+1}(\tau - R_i)} v^{-p} d\tau \\
&\le C\tilde{\beta}^{\frac{2+\alpha}{p+1}} + o(1) + Ce^{-\frac{2+\alpha}{p+1}R_i} \quad \text{for } i \text{ sufficiently large}
\end{aligned}$$

Here we have used (3.13) and (3.21) and the fact that $\mathcal{E}_{v_i}(0)$ is bounded. Thus, we see that

$$\begin{aligned}
w'(R_i) &= 2v'(R_i) \\
&= 2\Big[v'(R_i) + \Big(N - 2 + \frac{2+\alpha}{p+1}\Big)\Big] - 2\Big(N - 2 + \frac{2+\alpha}{p+1}\Big) \\
&\le C\tilde{\beta}^{\frac{2+\alpha}{p+1}} + o(1) - 2\Big(N - 2 + \frac{2+\alpha}{p+1}\Big).
\end{aligned}$$

This is our claim. By choosing $\tilde{\beta}$ sufficiently small, we see from (3.24) that

$$w'(R_i) \le -C$$

and

$$\mathcal{E}_{v_i}(R_i) \le -C.$$

The monotonicity of $\mathcal{E}_{v_i}(s)$ of s implies

$$w'(s) - \frac{2(p+1)}{p-1} h_i(s) \le -C \quad \text{for } s \in [0, R_i]$$

Then

$$h_i(s) \ge C(1 + w'(s)) \quad \text{for } s \in [0, R_i]. \tag{3.25}$$

Integrating (3.25) from 0 to R_i and using (3.11) and (3.23), we see that

$$\lambda_i^{1/p} R_i \ge C R_i + C$$

and

$$\lambda_i^{1/p} \ge C > 0.$$

This is a contradiction. This completes the proof of Theorem 3.1. $\square$

4. Proof of Theorem 1.2

In this section we complete the proof of Theorem 1.2. The main ideas of the proof are similar to those in [6]. To prove (2) of Theorem 1.2, we first show the following lemma.

Lemma 4.1. *For any $\kappa \in (0,1)$, there is at most one $\tilde{\lambda} := \tilde{\lambda}(\kappa) \in (0, \lambda^*]$ with $(\tilde{\lambda}, u_{\tilde{\lambda}}) \in D$ and $u_{\tilde{\lambda}}(0) = \kappa$.*

Proof. Suppose there are λ_1, $\lambda_2 \in (0, \lambda^*]$ with $\lambda_1 \ne \lambda_2$ and $(\lambda_1, u_{\lambda_1})$, $(\lambda_2, u_{\lambda_2}) \in D$ such that $u_{\lambda_1}(0) = u_{\lambda_2}(0) = \kappa$. If we set $u_1 \equiv u_{\lambda_1}$, $u_2 \equiv u_{\lambda_2}$ and $z_j(r) = 1 - u_j(r)$ for $j = 1, 2$, then

$$z_j'' + \frac{N-1}{r} z_j' = \lambda_j r^\alpha z_j^{-p}, \quad z_j(0) = 1 - \kappa, \quad z_j'(0) = 0, \quad z_j(1) = 1. \tag{4.1}$$

Let $\tilde{z}_j(y) = \frac{z_j((1-\kappa)^{(1+p)/(2+\alpha)} \lambda_j^{-1/(2+\alpha)} y)}{1-\kappa}$. We see that $\tilde{z}_j$ $(j = 1, 2)$ satisfies

$$v''(y) + \frac{N-1}{y} v'(y) = y^\alpha v^{-p}(y), \quad v(0) = 1, \quad v'(0) = 0. \tag{4.2}$$

The standard ODE theory implies that (4.2) has a unique solution $v(y)$. Thus, $\tilde{z}_j(y) \equiv v(y)$ for $j = 1, 2$. On the other hand, since

$$\tilde{z}_1((1-\kappa)^{-(1+p)/(2+\alpha)} \lambda_1^{1/(2+\alpha)}) = \frac{1}{1-\kappa}, \quad \tilde{z}_2((1-\kappa)^{-(1+p)/(2+\alpha)} \lambda_2^{1/(2+\alpha)}) = \frac{1}{1-\kappa}$$

we see that

$$v((1-\kappa)^{-(1+p)/(2+\alpha)} \lambda_1^{1/(2+\alpha)}) = \frac{1}{1-\kappa}, \quad v((1-\kappa)^{-(1+p)/(2+\alpha)} \lambda_2^{1/(2+\alpha)}) = \frac{1}{1-\kappa}. \tag{4.3}$$

We easily see from (4.2) that $v'(y) > 0$. Then (4.3) implies that $\lambda_1 = \lambda_2$ and a contradiction. This completes the proof. $\square$

We first note that, by the implicit function theorem, for $\lambda \in (0, \lambda^*)$, the operator $I + \lambda A'(\underline{u}_\lambda)$ is invertible. Here $A(u) = G^{-1}(\frac{r^\alpha}{(1-u)^2})$ and $G(h) = r^{1-N}(r^{N-1}h'(r))'$ on $(0,1)$ with $h'(0) = 0$, $h(1) = 0$. Thus, $(\lambda, \underline{u}_\lambda)$ for $0 < \lambda < \lambda^*$ is a simple curve of D. We can argue as in section 2.1 of Buffoni, Dancer and Toland [1] and [3]-[4] in the space $C^1[0,1] \times \mathbb{R}$, to find a analytic curve $\lambda = \tilde{\lambda}(t)$, $u = \tilde{u}(t)$ for $t \geq 0$ such that $\|\tilde{u}(t)\|_\infty \to 1$ as $t \to \infty$, $(\tilde{u}(t), \tilde{\lambda}(t)) \in D$ for $t \geq 0$, $(\tilde{u}(0), \tilde{\lambda}(0)) = (0,0)$ and $I - \tilde{\lambda}(t)A'(\tilde{u}(t))$ is invertible except at isolated points. We see from Lemma 4.1 that the curve has no intersection. Let us denote this curve by T and parameterize it by $(\tilde{u}(t), \tilde{\lambda}(t))$ for $t \geq 0$. Let $\mu_{i,\tilde{\lambda}(t)}(\tilde{u}(t))$ be the ith eigenvalue counting multiplicity of

$$-G - \frac{pr^\alpha\tilde{\lambda}(t)}{(1-\tilde{u}(t))^{(p+1)}}I \tag{4.4}$$

on $(0,1)$ with the Dirichlet boundary condition. (The definition of $\mu_{i,\tilde{\lambda}(t)}(\tilde{u}(t))$ is given in Section 1.) By our comments above, $\mu_{i,\tilde{\lambda}(t)}(\tilde{u}(t))$ are continuous, piecewise analytic and have only isolated zeroes. We will show that $\mu_{i,\tilde{\lambda}(t)}(\tilde{u}(t)) < 0$ for large t. This means that for any $\zeta > 0$, (4.4) has at least ζ negative eigenvalues for t large. Hence we see that there is a sequence $\{t_i\}$ with $t_i \to \infty$ as $i \to \infty$ such that the number of negative eigenvalues of (4.4) (counting multiplicity) changes at t_i. (Recall that $\mu_{i,\tilde{\lambda}(0)}(\tilde{u}(0)) = \mu_i(-G) \to +\infty$ as $i \to \infty$). Each $(\tilde{u}(t_i), \tilde{\lambda}(t_i))$ must be a bifurcation point. Otherwise the solutions near $(\tilde{u}(t_i), \tilde{\lambda}(t_i))$ are a curve parametrized by λ, the critical groups of these solutions must be locally independent of λ by homotopy invariance of the critical groups (where critical groups are defined in Chang [5]). By the formula for the critical groups at a non-degenerate point (see [5], p.33), this implies that the number of negative eigenvalues of the linearization counting multiplicity must be constant in a deleted neighborhood of $(\tilde{u}(t_i), \tilde{\lambda}(t_i))$ which contradicts our choice of t_i. (There is a minor technical point here. We need to work in the space $H_0^1(B)$. We choose $\|\tilde{u}(t_i)\|_\infty < \tau < 1$ and then smoothly truncate the function $\frac{1}{(1-s)^p}$ such that it equals $\frac{1}{(1-\tau)^p}$ for $1 > s > \tau$ so the equation makes sense on $H_0^1(B)$. Note that the truncation will not affect the solutions close to $(\tilde{u}(t_i), \tilde{\lambda}(t_i))$ in $H_0^1(B) \times \mathbb{R}$.) We also see that each $(\tilde{u}(t_i), \tilde{\lambda}(t_i))$ is either a turning point, i.e. the point where T changes direction (the branch T locally "bends back") or a point of secondary bifurcation. Our Lemma 4.1 implies that it is not a secondary bifurcation point. Thus, it must be a turning point.

To prove our claim on $\mu_{i,\tilde{\lambda}(t)}(\tilde{u}(t))$ for large t, we need to consider positive solutions (u_i, λ_i) of $(S_{\lambda,p})$ such that $\lambda_i \to \kappa \in (0, \infty)$ as $i \to \infty$ and $\|u_i\|_\infty \to 1$ as $i \to \infty$. (Note that Theorem 3.1 implies that $\lambda_i \not\to 0$ as $i \to \infty$). Thus, we see that there is

t_i with $t_i \to \infty$ such that $\tilde{\lambda}(t_i) = \lambda_i$ and $\tilde{u}(t_i) = u_i$. We use a blowing up argument. If we define $\epsilon_i = 1 - \|u_i\|_\infty$ and

$$U_i(y) = \frac{1 - u_i(\epsilon_i^{(p+1)/(2+\alpha)} \lambda_i^{-1/(2+\alpha)} y)}{\epsilon_i}, \quad y \in B_i := \{y : \ \epsilon_i^{(p+1)/(2+\alpha)} \lambda_i^{-1/(2+\alpha)} y \in (0,1)\},$$

then $U_i(0) = \min_{B_i} U_i = 1$. A rather standard limiting argument shows that a subsequence of the U_i converges uniformly on compact set to a positive solution U of $y^{1-N}(y^{N-1}U')' = \frac{y^\alpha}{U^p}$ on $\mathbb{R}^N$ such that $U(0) = 1$, $U'(0) = 0$ and $U(y) \geq 1$. It follows easily from the equation that $U(y) \geq 1$ for all y, since $U'(y) \geq 0$ for $y > 0$.

Moreover, by a similar argument as in (3.8) and (3.20) of Theorem 3.1, we have

$$\frac{1}{C} y^{(2+\alpha)/(p+1)} \leq U(y) \leq C y^{(2+\alpha)/(p+1)} \tag{4.5}$$

and hence

$$\lim_{y \to \infty} y^{-(2+\alpha)/(p+1)} U(y) = \Big[\frac{2+\alpha}{p+1}\Big(N - 2 + \frac{2+\alpha}{p+1}\Big)\Big]^{-1/(p+1)}. \tag{4.6}$$

The proof of (4.6) is a little variant of the proof of Theorem 1.1 of [13]. See also Theorem 1.2 of [15].

We now claim that the solution q of

$$-h''(y) - \frac{N-1}{y} h'(y) = \frac{p y^\alpha}{U^{p+1}} h(y), \quad h(0) = 1 \tag{4.7}$$

has infinitely many positive zeroes provided $2 \leq N < 2 + \frac{2(2+\alpha)}{p+1}(p + \sqrt{p^2 + p})$. Note that

$$p y^\alpha U^{-(p+1)}(y) \sim \frac{p(2+\alpha)}{p+1}\Big(N - 2 + \frac{2+\alpha}{p+1}\Big) y^{-2} \quad \text{as } y \to \infty.$$

On the other hand, by explicitly solving the equation (it is an Euler equation), one finds that any non-trivial solution of

$$-k'' - \frac{N-1}{y} k' - (\mu/y^2) k = 0$$

has infinitely many (and unbounded) positive zeroes if $\mu > \frac{1}{4}(N-2)^2$. A simple calculation implies that

$$\frac{p(2+\alpha)}{p+1}\Big(N - 2 + \frac{2+\alpha}{p+1}\Big) > \frac{1}{4}(N-2)^2$$

provided $2 \leq N < 2 + \frac{2(2+\alpha)}{p+1}(p + \sqrt{p^2 + p})$. Thus, we can easily deduce that q has infinitely many positive zeroes. Our claim holds.

We now in the position to complete the proof of (2) of Theorem 1.2. If $M > 0$ and σ is small and negative, we see by continuous dependence that the solution $\tilde{q}$ of

$$-h''(y) - \frac{N-1}{y} h'(y) = \frac{p y^\alpha}{U^{p+1}} h(y) + \sigma h, \quad h(0) = 1 \tag{4.8}$$

has at least M positive zeroes. Note that the solution of (4.7) is unique. Let h_i be the function defined to be $\tilde{q}(y)$ for y between the ith and $(i+1)$th the zeroes of $\tilde{q}$ and to be zero otherwise. Then $h_i \in H^1(\mathbb{R}^N)$, h_i are orthogonal (in $L^2(\mathbb{R}^N)$ or $H^1(\mathbb{R}^N)$) and by multiplying (4.8) by h_i and integrating between these zeroes we see that

$$Q(h) = \int_{\mathbb{R}^N} \left[\frac{1}{2}|\nabla h|^2 - \frac{p|z|^\alpha}{2U^{p+1}} h^2\right]$$

is strictly negative at each h_i. Hence the span of h_i is an $(M-1)$-dimensional subspace of $C_0^\infty(\mathbb{R}^N)$ such that $Q(h) < \tilde{\mu} < 0$ if h is in the unit sphere of E, where E is the span of h_i in $H^1(\mathbb{R}^N)$. Since h_i has compact support it follows easily that there is an $(M-1)$-dimensional subspace of $H_0^1(B_i)$ such that

$$\int_{B_i} |\nabla h(z)|^2 - \frac{p|z|^\alpha(1-\|\tilde{u}(t_i)\|_\infty)^{p+1}}{(1-\tilde{u}(t_i)(\tau_i y))^{p+1}} h^2(z) < 0$$

where $\tau_i = (1-\|\tilde{u}(t_i)\|_\infty)^{(p+1)/(2+\alpha)}[\tilde{\lambda}(t_i)]^{-1/(2+\alpha)}$ for large t_i if h is in the unit sphere in E. (Note that B_i, which is B rescaled has the property that each function in E is supported in B_i for large i.)

Hence returning to the original scaling we see that there is an $(M-1)$-dimensional subspace E_i of $H_0^1(B)$ such that

$$\int_B |\nabla h|^2 - \frac{p\tilde{\lambda}(t)|x|^\alpha}{(1-(\tilde{u}(t))^{(p+1)}} h^2 < 0$$

for h is in the unit sphere of E_i and t large. By the variational characterization of eigenvalues, this implies that $\mu_{i,\tilde{\lambda}(t)}(\tilde{u}(t)) < 0$ for $1 \le i \le M-1$ if t is large. Since M is arbitrary, this proves our claim and completes the proof of Theorem 1.2.

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