POSITIVE CLUSTERED LAYERED SOLUTIONS FOR THE GIERER-MEINHARDT SYSTEM

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ABSTRACT. We consider the stationary Gierer-Meinhardt system in a ball of $\mathbb{R}^N$:

$$\begin{cases} \varepsilon^2\Delta u - u + \frac{u^p}{v^q} = 0 & \text{in } \Omega, \\ \Delta v - v + \frac{u^m}{v^s} = 0 & \text{in } \Omega, \\ u, v > 0, \text{ and } \frac{\partial u}{\partial \nu} = \frac{\partial v}{\partial \nu} = 0 & \text{on } \partial\Omega \end{cases}$$

where $\Omega = B_R$ is a ball of $\mathbb{R}^N$ $(N \geq 2)$ with radius R, $\varepsilon > 0$ is a small parameter, and p, q, m, s satisfy the following condition:

$$p > 1, q > 0, m > 1, s \geq 0, \frac{qm}{(p-1)(1+s)} > 1.$$

Assume

$$0 < \frac{p-1}{q} < a_\infty \text{ if } N = 2, \text{ and } 0 < \frac{p-1}{q} < 1 \text{ if } N \geq 3$$

where $a_\infty > 1$ whose numerical value is $a_\infty = 1.06119$. We prove that there exists a unique $R_a > 0$ such that for $R \in (R_a, +\infty]$, ($R = +\infty$ corresponds to $\mathbb{R}^N$ case), and for any fixed integer $K \geq 1$, this system has at least one (sometimes two) radially symmetric positive solution $(u_{\varepsilon,K}, v_{\varepsilon,K})$, which concentrate at K spheres $\cup_{j=1}^K \{|x| = r_{\varepsilon,j}\}$, where $r_{\varepsilon,1} > r_{\varepsilon,2} > ... > r_{\varepsilon,K}$ are such that $r_0 - r_{\varepsilon,1} \sim \varepsilon \log \frac{1}{\varepsilon}, r_{\varepsilon,j} - r_{\varepsilon,j} \sim \varepsilon \log \frac{1}{\varepsilon}, j = 2, ..., K$, where $r_0 < R$ is a root of some function $M_R(r)$. This generalizes the results in [20] where a special case $K = 1$ and $\frac{N-2}{N-1} < \frac{p-1}{q} < 1$ was considered.

1. INTRODUCTION

Of concern is the stationary Gierer-Meinhardt system in a ball of $\mathbb{R}^N$:

$$\begin{cases} \varepsilon^2\Delta u - u + \frac{u^p}{v^q} = 0 & \text{in } B_R, \\ \Delta v - v + \frac{u^m}{v^s} = 0 & \text{in } B_R, \\ u > 0, v > 0 & \text{in } B_R, \\ \frac{\partial u}{\partial \nu} = \frac{\partial v}{\partial \nu} = 0 & \text{on } B_R \end{cases} \tag{1.1}$$

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where $\varepsilon > 0$ is a small constant, $\Delta := \sum_{j=1}^N \frac{\partial^2}{\partial x_j \partial x_j}$ denotes the Laplace operator in $\mathbb{R}^N$, ν stands for the unit outer normal to ∂B_R, and the exponents (p, q, m, s) satisfy the following condition:

$$p > 1, q > 0, m > 1, s \geq 0, \frac{qm}{(p-1)(1+s)} > 1. \tag{1.2}$$

In the previous paper [20], the existence of one solution with a single layer concentrating on a $(N-1)-$dimensional sphere was proved, under some restricted conditions on the parameters N, p, q, m, s and the radius R. In this paper, we give a more complete description of parameter space and prove the existence of arbitrarily many clustered layer solutions, which answers a question raised in [20].

Before we state the main results of the paper, let us recall some notation in [20].

We first define two functions: let $J_1(r)$ be the radially symmetric solutions of the following problem

$$J_1^{''} + \frac{N-1}{r} J_1^{'} - J_1 = 0, J^{'}(0) = 0, J_1(0) = 1, J_1 > 0. \tag{1.3}$$

The second function, called $J_2(r)$, satisfies

$$J_2^{''} + \frac{N-1}{r} J_2^{'} - J_2 + \delta_0 = 0, J_2 > 0, J_2(+\infty) = 0, \tag{1.4}$$

where δ_0 is the Dirac measure at 0.

The functions $J_1(r)$ and $J_2(r)$ can be written in terms of modified Bessel's functions. In fact

$$J_1(r) = c_1 r^{\frac{2-N}{2}} I_\nu(r), \quad J_2(r) = c_2 r^{\frac{2-N}{2}} K_\nu(r), \ \nu = \frac{N-2}{2} \tag{1.5}$$

where c_1, c_2 are two positive constants and I_ν, K_ν are modified Bessel functions of order ν. See [page 378, [3]] for details. In the case of $N = 3$, J_1, J_2 can be computed explicitly:

$$J_1 = \frac{\sinh r}{r}, \ J_2(r) = \frac{e^{-r}}{4\pi r}. \tag{1.6}$$

Let $w(y)$ be the unique solution for the following ODE:

$$w^{''} - w + w^p = 0 \text{ in } \mathbb{R}, \ w > 0, \ w(0) = \max_{y \in \mathbb{R}} w(y), w(y) \to 0 \text{ as } |y| \to \infty. \tag{1.7}$$

Let $R > 0$ be a fixed constant. We define

$$J_{2,R}(r) = J_2(r) - \frac{J_2^{'}(R)}{J_1^{'}(R)} J_1(r) \tag{1.8}$$

and a Green's function $G_R(r; r^{'})$

$$G_R^{''} + \frac{N-1}{r} G_R^{'} - G_R + \delta_{r^{'}} = 0, G_R^{'}(0; r^{'}) = 0, G_R^{'}(R; r^{'}) = 0. \tag{1.9}$$

Note that

$$J'_{2,R}(R) = 0, \quad \lim_{R \to +\infty} J_{2,R}(r) = J_2(r). \tag{1.10}$$

It is easy to see that

$$G_R(r; r') = \frac{1}{J'_1(r')J_{2,R}(r') - J_1(r')J'_{2,R}(r')} \begin{cases} J_{2,R}(r')J_1(r), & \text{for } r < r', \\ J_1(r')J_{2,R}(r), & \text{for } r > r'. \end{cases} \tag{1.11}$$

By computing the Wronskian of J_1, J_2, it is easy to see that

$$J'_1(r')J_{2,R}(r') - J_1(r')J'_{2,R}(r') = \frac{1}{c_0(r')^{N-1}} \tag{1.12}$$

for some constant $c_0 \neq 0$. Substituting (1.12) into (1.11), we obtain another formula for $G_R(r; r')$:

$$G_R(r; r') = c_0(r')^{N-1} \begin{cases} J_{2,R}(r')J_1(r), & \text{for } r < r', \\ J_1(r')J_{2,R}(r), & \text{for } r > r'. \end{cases} \tag{1.13}$$

For $t \in (0, R)$, set

$$M_R(t) := \frac{a}{t} + \frac{J'_1(t)}{J_1(t)} + \frac{J'_{2,R}(t)}{J_{2,R}(t)}, \tag{1.14}$$

where a is an important parameter given by

$$a = \frac{(N-1)(p-1)}{q}. \tag{1.15}$$

When $R = +\infty$, $J_{2,+\infty}(r) = J_2(r)$. We denote $G_{+\infty}(r; r')$ as $G(r; r')$ and $M_{+\infty}(t)$ as $M(t)$. That is,

$$G(r; r') = c_0(r')^{N-1} \begin{cases} J_2(r')J_1(r), & \text{for } r < r', \\ J_1(r')J_2(r), & \text{for } r > r', \end{cases} \tag{1.16}$$

$$M(t) := \frac{a}{t} + \frac{J'_1(t)}{J_1(t)} + \frac{J'_2(t)}{J_2(t)}. \tag{1.17}$$

In [20], Ni and Wei proved the following theorem on the existence of layered solutions to (1.1):

Theorem 1.1. *([20].) Let $N \geq 2$ and (p, q, m, s) satisfy (1.2). Assume that there exist two radii $0 < r_1 < r_2 < R$ such that*

$$M_R(r_1)M_R(r_2) < 0. \tag{1.18}$$

Then for ε sufficiently small, problem (1.1) has a solution $(u_{\varepsilon,R}, v_{\varepsilon,R})$ with the following properties:

(1) $u_{\varepsilon,R}, v_{\varepsilon,R}$ are radially symmetric,

(2) $u_{\varepsilon,R}(r) = \xi_{\varepsilon,R}^{\frac{q}{p-1}} w(\frac{r-t_\varepsilon}{\varepsilon})(1+o(1))$,

(3) $v_{\varepsilon,R}(r) = \xi_{\varepsilon,R}(G_R(t_\varepsilon;t_\varepsilon))^{-1}G_R(r;t_\varepsilon)(1+o(1))$, where $G_R(r;t_\varepsilon)$ satisfies (1.9), $\xi_{\varepsilon,R}$ is defined by the following

$$\xi_{\varepsilon,R} = \left(\varepsilon\left(\int_{\mathbb{R}} w^m\right)G_R(t_\varepsilon;t_\varepsilon)\right)^{\frac{(1+s)(p-1)-qm}{qm}} \tag{1.19}$$

and $t_\varepsilon \in (r_1,r_2)$ satisfies $\lim_{\varepsilon\to 0} M_R(t_\varepsilon) = 0$.

Theorem 1.1 reduces the problem of finding one solution to (1.1) to finding a zero of the function $M_R(r) = 0$. A natural question is what is the necessary and sufficient condition for the existence of $M_R(r) = 0$. A second question, which has been posed in [20], is if there are clustered layer solution to (1.1). In this paper, we answer both of these questions completely.

Our first theorem gives a complete classification of the existence of roots of $M_R(r)$, which contains elements of independent interest. We summarize our findings as follows.

Theorem 1.2. *Let a be as given in (1.15). Suppose that $N \geq 3$ first. There are three cases.*

(1.a) *If $N-2 < a < N-1$ then there exists R_0 such that if $R > R_0$ then $M_R(r) = 0$ has exactly two solutions $0 < r_1 < r_2 < R$, and if $R < R_0$, then $M_R(r) = 0$ has no solution. Moreover, for $R > R_0$, $M_R'(r_1) < 0, M_R'(r_2) > 0$.*

(1.b) *If $a \geq N-1$ then $M_R(r) = 0$ has no solution for any R.*

(1.c) *If $a \leq N-2$ then $M_R(r) = 0$ has precisely one solution r_1 for any R and moreover $M_R'(r_1) > 0$.*

Suppose that $N = 2$. Then there exists a number $a_\infty > 1$ whose numerical value is $a_\infty =$ 1.06119 such that one of the following holds:

(2.a) *If $a \in (0, a_\infty)$ then the situation is the same as in case (1.a).*

(2.b) *If $a > a_\infty$ then $M_R(r) > 0$ for any R.*

(2.c) *If $a = a_\infty$ then $M_R(r) > 0$ any $R < \infty$. When $R = \infty$, there exists a number r_0 such that $M_\infty(r_0) = 0 = M_\infty'(r_0)$, and $M_R(r) > 0$ for any $r \neq r_0$.*

From Theorem (1.2), we see that a necessary and sufficient condition for (1.18) holds is the following

(1.20)

$$0 < a < \begin{cases} N-1 & \text{for } N \geq 3, \\ a_\infty & \text{for } N = 2, \end{cases} \quad \text{and } R > R_a = \begin{cases} 0 & \text{if } a \leq N-2, \ N \geq 3, \\ R_0 & \text{if } N-2 < a < N-1, \ N \geq 3, \\ R_0 & \text{if } 0 < a < a_\infty, \ N = 2. \end{cases}$$

Our next result says that under the same condition (1.18) multiple clustered layer solution to (1.1) also exists.

Theorem 1.3. *Assume that (1.20) holds. Then for any given integer $K \geq 1$, there exists $\varepsilon_K > 0$ such that for $\varepsilon < \varepsilon_K$, problem (1.1) has a solution $(u_{\varepsilon,R}, v_{\varepsilon,R})$, with the following properties:*

(1) $u_{\varepsilon,R}, v_{\varepsilon,R}$ are radially symmetric,

(2) $u_{\varepsilon,R}(r) = \sum_{j=1}^K \xi_{\varepsilon,R,j}^{\frac{q}{p-1}} w(\frac{r-t_{\varepsilon,j}}{\varepsilon})(1+o(1))$,

(3) $v_{\varepsilon,R}(r) = \sum_{j=1}^K \xi_{\varepsilon,R,j}(KG_R(t_{\varepsilon,j}; t_{\varepsilon,j}))^{-1} G_R(r; t_{\varepsilon,j})(1+o(1))$, where $G_R(r;t_\varepsilon)$ satisfies (1.9), $\xi_{\varepsilon,R,j}$ is defined by the following

$$\xi_{\varepsilon,R,j} = (\varepsilon \int_{\mathbb{R}} w^m) \sum_{l=1}^K \xi_{\varepsilon,R,l}^{\frac{qm}{p-1}-s} G_R(t_{\varepsilon,l}; t_{\varepsilon,l}) \tag{1.21}$$

and $t_{\varepsilon,j} \in (0,R)$ satisfies $t_{\varepsilon,j} \to r_0$, where $0 < r_0 < R$ is a root of $M_R(r) = 0$ (given by Theorem 1.2) and

$$(1-\delta)\varepsilon \log \frac{1}{\varepsilon} \leq t_{\varepsilon,j} - t_{\varepsilon,j-1} \leq (1+\delta)\varepsilon \log \frac{1}{\varepsilon}, \ j = 2, .., K, \tag{1.22}$$

where $\delta > 0$ is any small number.

In the case when $N-2 < a < N-1$ for $N \geq 3$ and $0 < a < a_\infty$ for $N = 2$, there are **two** *radially symmetric clustered solutions concentrating at two roots of $M_R(r)$, provided that $R > R_0$ and ε is sufficiently small.*

As for bound states, we consider the following elliptic system in $\mathbb{R}^N$:

$$\begin{cases} \varepsilon^2 \Delta u - u + \frac{u^p}{v^q} = 0 & \text{in } \mathbb{R}^N, \\ \Delta v - v + \frac{u^m}{v^s} = 0 & \text{in } \mathbb{R}^N, \\ u, v > 0, \ u, v \to 0 & \text{as } |x| \to +\infty. \end{cases} \tag{1.23}$$

Then we have the following result.

Theorem 1.4. *Let $N \geq 2$ and $a = \frac{(N-1)(p-1)}{q}$ satisfy (1.20). Then for ε sufficiently small, problem (1.23) has a solution $(u_\varepsilon, v_\varepsilon)$ with the following properties:*

(1) $u_\varepsilon, v_\varepsilon$ are radially symmetric,

(2) $u_\varepsilon(r)=\sum_{j=1}^K \xi_{\varepsilon,j}^{\frac{q}{p-1}} w(\frac{r-r_{\varepsilon,j}}{\varepsilon})(1+o(1))$,

(3) $v_\varepsilon(r)=\sum_{j=1}^K \xi_{\varepsilon,j}(KG(r_{\varepsilon,j};r_{\varepsilon,j}))^{-1}G(r;r_{\varepsilon,j})(1+o(1))$, where $\xi_{\varepsilon,j}$ is defined at the following

$$\xi_{\varepsilon,j}=\varepsilon(\int_{\mathbb{R}} w^m)\sum_{j=1}^K \xi_{\varepsilon,j}^{\frac{qm}{p-1}-s} G(r_{\varepsilon,j};r_{\varepsilon,j}) \tag{1.24}$$

and $r_{\varepsilon,j}\to r_0$ with r_0 being the unique root of $M_\infty(r)=0$.

We remark that Theorem 1.2 gives sufficient conditions for the existence of ring-type solutions. We also conjecture that these conditions are necessary in the limit $\varepsilon\to 0$; that is, a solution concentrating on a ring of radius r_0 cannot exist *unless* $M_R(r_0)=0$. In this sense, Theorem 1.2 provides a complete classification of ring-type solutions.

The existence of clustered spikes to (1.23) in one dimensional case has been proved by Chen-M. del Pino-Kowalczyk [4] (using PDE method) and Doelman-Kaper-H. van der Ploeg [7] (using dynamical system method). The existence of multiple spots to (1.23) in two-dimensional case is proved by del Pino-Kowalczyk-Wei [5]. Our result here seems to be first one on the existence of clustered layered solutions for elliptic systems. For single equations, the existence of (single or multiple) layered solutions has been considered by many authors. We refer to [1], [2], [8], [11], [12], [21] and the references therein.

Gierer-Meinhardt system was used in [9] to model head formation of *hydra*, an animal of a few millimeters in length, made up of approximately 100,000 cells of about fifteen different types. The Gierer-Meinhardt system falls within the framework of a theory proposed by Turing [23] in 1952 as a mathematical model for the development of complex organisms from a single cell. We refer to [15] and [16] for background and recent studies on Gierer-Meinhadt system. For the existence and stability of multiple spikes in a bounded domain, we refer to [6], [10], [17], [18], [19], [22], [24], [25], [26] and the references therein.

For simplicity, we only consider the case $s=0$ in (1.1). The general case of $s>0$ can be treated in a similar way as in the last section of [20]. By a rescaling, we will work with the following problem

$$\begin{cases} \varepsilon^2\Delta u-u+\frac{u^p}{v^q}=0 & \text{in } B_R,\\ \Delta v-v+\varepsilon^{-1}u^m=0 & \text{in } B_R,\\ u>0, v>0 & \text{in } B_R,\\ \frac{\partial u}{\partial\nu}=\frac{\partial v}{\partial\nu}=0 & \text{on } B_R \end{cases} \tag{1.25}$$

Our construction is similar to [12] and [20], where we used Liapunov-Schmidt reduction procedure. The main difficulty here is to have good estimates for the interactions between spikes inside a cluster. The basic idea is to write (1.25) into a nonlocal elliptic problem

$$\varepsilon^2 \Delta u - u + \frac{u^p}{(\mathbb{T}[u^m])^q} = 0 \quad \text{in} \ \ B_R, u = u(r), u'(R) = 0 \tag{1.26}$$

where the operator $\mathbb{T}[h]$, for a given function $h \in L^2(B_R)$, is defined as

$$\Delta \mathbb{T}[h] - \mathbb{T}[h] + \varepsilon^{-1} h = 0 \text{ in } B_R, \ \frac{\partial \mathbb{T}[h]}{\partial \nu} = 0 \ \text{ on } \partial B_R(0) \tag{1.27}$$

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2. A Study of $M_R(r)$: proof of Theorem 1.2

In this section, we prove Theorem 1.2. The proof makes use of intricate properties of the two Bessel functions J_1 and J_2. This may be useful in studying other problems involving the Bessel functions.

As the statement indicates, the situation for $N = 2$ is very different from $N \geq 3$. The case $N = 2$ and $a \in (1, a_\infty)$ has no analogue in higher dimensions and is considerably more difficult. For reference, we list here the expansions of J_1 and J_2 for small and big argument (see [3]):

$$\begin{aligned} J_1(r) &\sim A_1 r^{-\frac{N-1}{2}} e^r \left(1 - \frac{(N-1)(N-3)}{8r}\right), \\ J_2(r) &\sim A_2 r^{-\frac{N-1}{2}} e^{-r} \left(1 + \frac{(N-1)(N-3)}{8r}\right) \end{aligned} \quad \text{as} \quad r \to \infty; \tag{2.1}$$

$$J_1(r) \sim B_1 \left(1 + \frac{r^2}{2N}\right), \quad J_2(r) \sim B_2 \left\{ \begin{array}{ll} -\ln(r), & N = 2 \\ \frac{1}{N-2} r^{-N+2}, & N \geq 2 \end{array} \right. , \quad \text{as} \quad r \to 0 \tag{2.2}$$

where A_1, A_2, B_1, B_2 are some positive constants that depend on N but not on r. We start with addressing case (1.a) and a part of case (2.a).

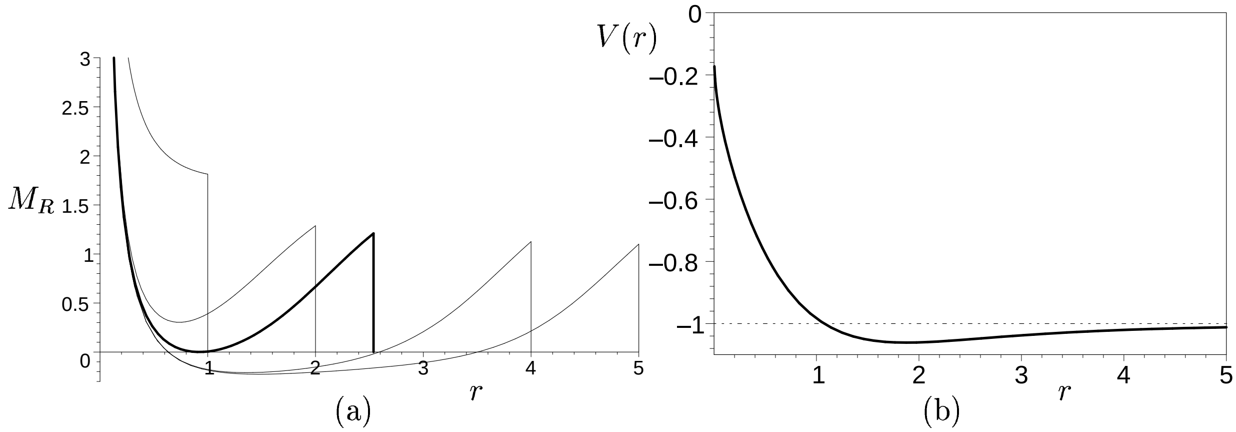


FIGURE 1. (a) The graph of $M_R(r)$ with $N = 3, a = 2.5$ and with $R = 1, R = 2, R = R_0 = 2.538, R = 4, R = 5$. (b) The graph of $V(r)$ (see Lemma 2.2).

Lemma 2.1. *Suppose that*

$$0 < a \leq 1 \quad \text{if} \quad N = 2$$
$$N - 2 < a < N - 1 \quad \text{if} \quad N \geq 3.$$

The system

$$M_R(r) = 0, \quad M_R'(r) = 0 \tag{2.3}$$

has a unique solution $r = r_0$, $R = R_0$ and moreover, if $R > R_0$, then $M_R(r) = 0$ has exactly two solutions $0 < r_1 < r_2 < R$, and if $R < R_0$, then $M_R(r) = 0$ has no solution. Moreover, for $R > R_0$, $M_R'(r_1) < 0, M_R'(r_2) > 0$.

Proof of Lemma 2.1. The proof consists of four steps (see Figure 1.a). In Step 1, we show that $M_R(r)$ is strictly positive on $(0, R)$ for when R is sufficiently small. As R is increased, there are only two ways that a root of $M_R(r)$ can appear or disappear: either through the boundary at $r = R$ or through the presence of a double root (2.3). In Step 2, we rule out the former. In Step 3, we characterize the latter, and also show that the solution to (2.3) is unique. In Step 4 we study M_R for large values of R and complete the proof.

Step 1. In the case $N \geq 3$, using (2.2) and after some algebra we obtain

$$rM_R(r) \sim a - \left(\frac{1 - r_0^N}{\frac{1}{N-2} + N\frac{r_0^{N-2}}{R^2}} \right), \quad r_0 = \frac{r}{R} \in (0, 1)\,; \quad R \ll 1 \tag{2.4}$$

Note that the expression in brackets is a decreasing function of r so that $rM_R(r) \sim a - N + 2$ as $r \to 0$ and hence $rM_R(r)$ attains its minimum at $r = 0$. Since $a > N - 2$, it follows that $M_R(r)$ is always positive for sufficiently small R with $r \in (0, R]$.

For the sub-case $N = 2$, we have:

$$rM_R(r) \sim a - \left(\frac{1 - r_0^2}{\ln R^{-1} + \ln r_0^{-1} + 2R^{-2}} \right), \; r_0 = \frac{r}{R} \in (0,1)\,; \quad R \ll 1.$$

Now the term in brackets is bounded by $\frac{1}{2R^{-1} + \ln R^{-1}} \to 0$ as $R \to 0$ so that M_R is positive for sufficiently small R with $r \in (0, R]$.

Step 2. Since $J_{2,R}(R) = 0$, it follows that $M_R(R) = \frac{a}{R} + \frac{J_1'(R)}{J_1(R)}$. But J_1 is a strictly increasing and positive function so that $M_R(R)$ is always strictly positive.

Step 3. Let $f_R(r) = aJ_1(r)J_2(r) + r(J_1J_2)'$, where $J_1 = I(r)$, $J_{2,R} = J_1(r) - J_2(r)\frac{K'(R)}{I'(R)}$. Then (2.3) is equivalent to

$$f_R(r) = 0, f_R'(r) = 0 \tag{2.5}$$

Let (2.5) hold. Then we have

$$r(J_1J_{2,R})' = -a(J_1J_{2,R})$$

$$\frac{J_{2,R}'}{J_{2,R}} = -\frac{a}{r} - \frac{J_1'}{J_1(r)} \tag{2.6}$$

and hence

$$\begin{aligned} rf_R'(r) &= r(a+1)(J_1J_{2,R})' + r^2(J_1J_{2,R})'' \\ &= -a(a+1)(J_1J_{2,R}) + r^2(J_1''J_{2,R} + 2J_1'J_{2,R}' + J_{2,R}''J_1) \\ &= -a(a+1)(J_1J_{2,R}) + r^2(2J_1J_{2,R} + 2J_1'J_{2,R}' - \frac{N-1}{r}(J_1J_{2,R})') \\ &= (-a^2 - a + 2r^2)(J_1J_{2,R}) - (N-1)r(J_1J_{2,R})' + 2r^2J_1'J_{2,R}' \\ &= (a(N-1) - a^2 - a + 2r^2)(J_1J_{2,R}) + 2r^2J_1'J_{2,R}' \end{aligned}$$

Using (2.6), we have

$$-\frac{rJ_1f_R'(r)}{J_2, R} = (a^2 - a(N-2) - 2r^2)J_1^2 - 2r^2J_1J_1'(-\frac{a}{r} - \frac{J_1'}{J_1}) = 0$$

We obtain

$$g(r) := (a^2 - a(N-2) - 2r^2)J_1^2 + 2raJ_1J_1' + 2r^2J_1'^2 = 0. \tag{2.7}$$

We now study the roots of the function $g(r)$. Using the equation for $J_1'' = -\frac{1}{r}J_1 + J_1$ and after some computations, we have that $g(r)$ satisfies

$$rg' + r^2 Cg = J_1^2(B - Ar^2) \tag{2.8}$$

where the constants A, B and C satisfy

$$A = 4\,(N-1-a)\,, \qquad B = (2N-a-4)(a+2-N)a, \qquad C = 2N-4-a \tag{2.9}$$

Note that for $N \geq 3, A, B, C > 0$, while for $N = 2, C < 0, B < 0, A > 0$.

Let us consider the case $N \geq 3$ first. If $N \geq 3$, then $J_1^2(B - Ar^2) > 0$ for $r < r^*$ and $J_1^2(B - Ar^2) < 0$ for $r > r^*$, where

$$r^* = \sqrt{\frac{B}{A}}. \tag{2.10}$$

Observe that

$$g(0) = a(a - N + 2) > 0.$$

On the other hand using the big argument expansions (2.1) after some algebra we obtain

$$g(r) \sim (-2(N-1) + 2a)A_0^2 r^{-(N-1)} e^{2r} \text{ as } r \to +\infty. \tag{2.11}$$

Thus $g(r) = 0$ admits at least one root. Let r_1 be the first root of g. Then $g'(r_1) \leq 0$, which implies that $r_1 \geq r^*$. We show that $r_1 > r^*$. In fact, suppose that $r_1 = r^*$. Then $g(r^*) = g'(r^*) = 0$ and $g''(r^*) \geq 0$. On the other hand, differentiating (2.8) gives

$$rg'' + (r^2C + 1)g' + 2rCg = 2J_1J_1'(B - Ar^2) - 2ArJ^2$$

and hence

$$r^*g''(r^*) = -2Ar^*J_1^2(r^*) < 0$$

which contradicts the fact that $g''(r^*) \geq 0$. Hence $r_1 > r^*$. Suppose that g admits more than one root. Let r_2 be the first root of g with $r_2 > r_1 \geq r^*$. Then $g'(r_2) \geq 0$ which is a contradiction since $r_2 g'(r_2) = J_1^2(B - Ar_2^2) < 0$.

In the case $N = 2$, we have $B - Ar^2 < 0$, which implies that $rg' + Cr^2 g < 0$. Thus $g(r)$ can have at most one root. Moreover, $g(0) = a^2 > 0$ and if $0 < a < 1$ then (2.11) shows that g is negative for large r so that g admits exactly one root. In the sub-case $a = 1,\ N = 2$, the equation (2.11) is insufficient to determine the behavior of g at ∞, but a more accurate expansion using

$$J_1(r) \sim A_0 r^{-1/2} e^r \left(1 + \frac{1}{8}\frac{1}{r} + \frac{9}{128}\frac{1}{r^2} + \dots\right), \quad r \to \infty$$

yields

$$g(r) \sim -A_0^2 \frac{1}{r^2} e^{2r} \to -\infty \text{ as } r \to \infty$$

and so g still admits exactly one root.

Step 4. For sufficiently large R, and with $r \ll R$ we have $M_R(r) \sim M_\infty(r) = \frac{a}{r} + \frac{J_1'(r)}{J_1(r)} + \frac{J_2'(r)}{J_2(r)}$.

Expanding for small r using (2.2) we obtain

$$M_\infty(r) \sim (a + 2 - N)\frac{1}{r}, \quad r \to 0$$

so that $M_\infty(r) \to \infty$ as $r \to 0$. For large r with $1 \ll r \ll R$, we use the asymptotic expansions (2.1) to obtain

$$M_\infty(r) \sim \frac{a+1-N}{r}, \quad r \to \infty$$

This shows that $M_R(r)$ admits at least one root $r_1 \ll R$ when $a < N-1$ and R is sufficiently large. In the case $a = 1, N = 2$, a more careful expansion (see [20]) shows that M_∞ is still negative for large r so the conclusion is unchanged. Using continuity and applying steps 1,2 and 3 shows that $M_R(r)$ has exactly two roots $0 < r_1 < r_2 < R$ whenever $R > R_0$. Since M_R is positive for small r and by Step 2, it follows that $M_R'(r_1) < 0$ and $M_R'(r_2) > 0$. □

Next we address the more difficult case $N = 2, a > 1$. We first study the case $R = \infty$ in Lemma 2.2. We then show in Lemma 2.3 how to reduce the more general case of arbitrary R to the case $R = \infty$.

Lemma 2.2. *Suppose that $N = 2, R = \infty$. Then the system*

$$M_\infty(r; a) = 0 = M_\infty'(r; a)$$

has a unique solution $r = r_\infty$, $a = a_\infty > 1$. If $a < a_\infty$ then $M_\infty(r)$ has at least one root and if $a > a_\infty$ then $M_\infty(r) > 0$ for all r.

Proof of Lemma 2.2. Let

$$V(r) = \frac{r(J_1 J_2)'}{J_1 J_2}.$$

The statement of the lemma is equivalent to showing that the equation $V'(r) = 0$ has a unique solution $r = r_\infty$, and that moreover the minimum of V is attained at $r = r_\infty$. The number a_∞ is then given by $a_\infty = -V(r_\infty)$. For reference, the graph of V is shown on Figure 1.b.

Step 1: *We show that* $(J_1 J_2)' < 0$. Define a function $u(r) = (J_1 J_2)'$. After some algebra we find that

$$u_{rr} + \frac{3}{r} u_r - 4u = 2\frac{J_1 J_2}{r} \tag{2.12}$$

Now the right hand side is positive and using (2.2) and (2.1) we find that u is negative for small or large r. Then by maximum principle, u is negative for all r.

Step 2: *Let*

$$W(r) = 1 - \frac{J_1' J_2'}{J_1 J_2}. \tag{2.13}$$

We claim that $W \in (1,2)$. Now from (2.2) and (2.1) we see that $W(0) = 1$ and $W(\infty) = 2$. So to prove that $W(r) \in (1,2)$ it suffices to show that w never crosses 1 or 2. Clearly $W(r) \neq 1$ since $J_1' J_2'$ is nonzero. So it remains to show that the function $v = J_1 J_2 + J_1' J_2'$ is never zero. After some algebra we obtain

$$v_{rr} + \frac{3}{r} v_r - 4v = \frac{(J_1 J_2)'}{r}.$$

Now the right hand side is negative by Step 1, and $v \to +\infty$ as $r \to 0^+$, $v \to 0^+$ as $r \to \infty$. Therefore by maximum principle, v must be strictly positive for all r.

Step 3: *We show that there exists a number* r^* *such that* $V > -1$ *if* $r < r^*$ *and* $V < -1$ *if* $r > r^*$ *and moreover,* V *has no minimum for* $r \in (0, r^*]$. First note that $V \to -1^-$ as $r \to \infty$ and $V \to 0$ as $r \to 0$. Thus there exists r^* such that $V(r^*) = -1$. After substituting $u = V r^{-1} J_1 J_2$ where $u = (J_1 J_2)'$ satisfies (2.12), we obtain

$$V_{rr} + \frac{1}{r} V_r + \frac{2}{r} V_r V - 2VW = 4 \tag{2.14}$$

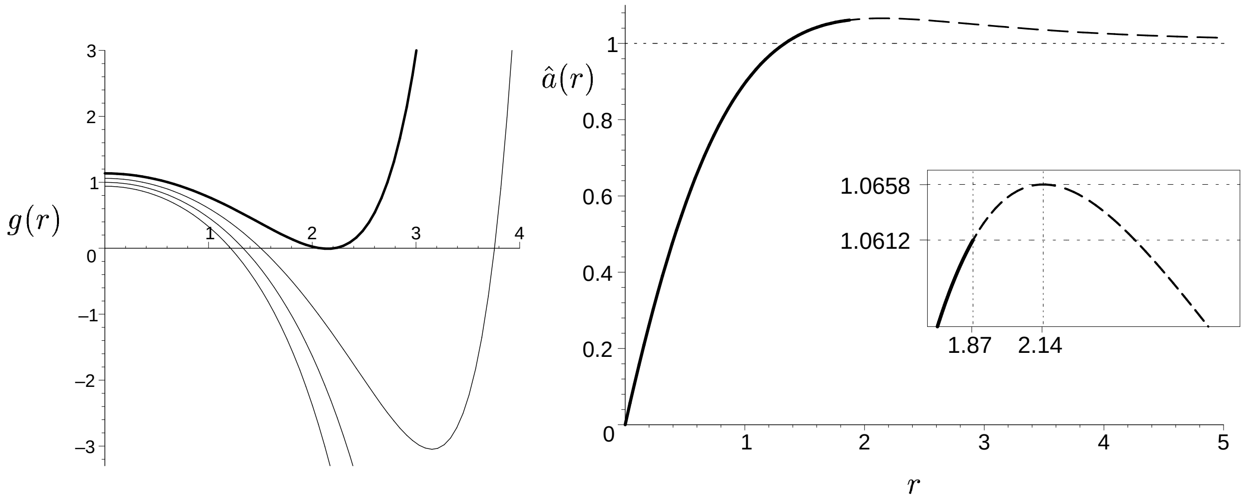
where W is defined in (2.13). Now suppose that $V_r = 0$. Then

$$V_{rr} = 2(2 + VW).$$

But $W \in (1,2)$ so that $V_{rr} > 0$ whenever $V \geq -1$ with $V_r = 0$. This shows that V has no interior maximum whenever $V \geq -1$. It immediately follows that r^* is unique.

Step 4: $W(r)$ *is increasing for* $r > r^*$. Since $W(r) \to 2^-$ as $r \to \infty$, it suffices to show that $W'(r) \neq 0$. Suppose not. After some algebra, we see that $W' = 0$ implies that $V = \frac{2(1-W)}{W}$ and hence $V \in (-1, 0)$ since $W \in (1,2)$ from Step 2. But this contradicts Step 3 since we assumed that $r > r^*$.

Step 5: Suppose that V has a maximum. By Step 3, it must be located at $r = r_M > r^*$ with $V(r_M) < -1$. Since $V \to 1^-$ as $V \to \infty$, this implies that V must have an inflection point $r_i > r_M$ with $V(r_i) < -1$, $V'(r_i) > 0$ and with $V''(r_i) = 0$. Moreover, choose r_i to be the first



(a) (b)

FIGURE 2. (a) The graph of $g(r)$ for $a = 0.9,\ 1,\ 1.03$ and $a = a_\infty = 1.06119$. (b) The graph of $\hat{a}(r)$. The insert shows the magnification of $\hat{a}(r)$ near its maximum.

such inflection point to the right of r_M. From (2.14) we then obtain:

$$V_r\left(r_i\right) = \frac{2r_i(2 + V\left(r_i\right) W\left(r_i\right))}{1 + 2V\left(r_i\right)}.$$

Now V is decreasing inside $[r_M, r_i]$ whereas W is increasing on this interval by Step 4. It follows that $2+V(r_i)W(r_i) < 2+V(r_M)W\left(r_M\right) < 0$. Moreover, $V\left(r_i\right) < -1$ implies that $1+2V\left(r_i\right) < 0$ so that $V_r\left(r_i\right) > 0$, a contradiction.

Since V has no maximum, it follows that it has only one minimum and this completes the proof of the lemma. □

Lemma 2.3. *Suppose that $N = 2$. Define*

$$a_\infty \equiv \min_{r\in(0,\infty)} \frac{r\left(J_1 J_2\right)'}{J_1 J_2}. \tag{2.15}$$

Given any $a \in \left(1, a_\infty\right)$, there exists a unique $r_0 = r_0\left(a\right)$ and a unique $R_0 = R_0\left(a\right)$ such that $M_{R_0}\left(r_0\right) = 0 = M'_{R_0}\left(r_0\right)$ and $M_{R_0}\left(r\right) > 0$ whenever $r \neq r_0$. If $R > R_0$ then $M_R\left(r\right)$ has exactly two roots $0 < r_1 < r_2 < R$ with $M'_R(r_1) < 0, M'_R(r_2) > 0$. If $R < R_0$, then $M_R(r) = 0$ has no solution.

Proof of Lemma 2.3. As in the proof of Lemma 2.1, recall that the system $M_R(r) = 0 = M_R'(r)$ is equivalent to solving $M_R(r) = 0,\ g(r) = 0$ where

$$g(r;a) = (a^2 - 2r^2)J_1^2 + 2raJ_1J_1' + 2r^2J_1'^2$$

and g satisfies

$$rg' - r^2ag = -J_1^2\left(a^3 - 4(a-1)r^2\right). \tag{2.16}$$

The goal is to show that the system $M_R = 0 = g$ has a unique solution for any given $a < a_\infty$. To this end, $\hat{a}(r_0)$ be the solution to $g(r_0; \hat{a}(r_0)) = 0$ so that

$$\hat{a}(r) = \frac{-rJ_1' + r\sqrt{2J_1^2 - J_1'^2}}{J_1}. \tag{2.17}$$

Note that $g(0) = a^2 > 0$. Now since the right hand side of (2.16) has one root, it follows that g has at most two roots. Now from (2.11) it follows that for $a < 1$, $g(\infty) \to -\infty$ whereas for $a > 1$, $g(\infty) \to +\infty$. Hence g has precisely two roots if $a > 1$ and if a is sufficiently close to 1. Moreover g is an increasing function of a, so as a is increased, the two roots move towards each other until they disappear at the fold-point bifurcation at some $a = a_M \geq a_\infty > 1$ (see Figure 2.a). This proves that the graph of $\hat{a}$ has the shape as shown on Figure 2.b; more precisely, it has a maximum at $a_M = \hat{a}(r_M)$, and has no other local min/max. In what follows, we will show that when $g(r) = 0$, $M_R(r) = 0$ has a solution only if $r < r_\infty \leq r_M$, where r_∞ is such that $a_\infty = \hat{a}(r_\infty)$. This makes $\hat{a}$ invertible on $r \in (0, r_\infty)$ and shows the uniqueness of the solution to $M_R = 0 = g$.

Step 1. *There exists a unique r_∞ such that if $r < r_\infty$, then the equation*

$$M_R(r; \hat{a}(r)) = 0 \tag{2.18}$$

has a unique solution R=R_0; and if $r > r_\infty$ then this equation has no solution for R. Note that (2.18) is equivalent to solving

$$-\frac{J_1'(R)}{J_2'(R)} = F(r) := \frac{J_2(r)}{J_1(r)\,a(r)}\left(\hat{a}(r) + \frac{(J_1(r)\,J_2(r))'\,r}{J_1(r)\,J_2(r)}\right)$$

In particular, the solutions exist if and only if $F(r)$ is positive. We need to show that $F(r)$ has a unique root. Note that if $F(r) = 0$ then $R = \infty$. Now in this case, $M_\infty(r) = 0 = M_\infty'(r)$. It follows from Lemma 2.2 that $r = r_\infty$ and $\hat{a}(r) = \hat{a}(r_\infty) = a_\infty$. This shows that the root of $F(r)$ is unique. Moreover, F is positive for small r so that $F > 0$ for any $r < r_\infty$ and $F < 0$ for any $r > r_\infty$.

Step 2. *There are no solutions to* $M_R(r,a) = 0$ *if* $a > a_\infty$. Note that $M_R(r;a)$ is a decreasing function of R. Moreover by the definition of a_∞, $M_\infty(r;a_\infty) \geq 0$. This shows that $M_R(r;a) > 0$ for any $a > a_\infty$.

Step 3. *We show that* $r_\infty \leq r_M$. Now there are at most two solutions to the equation $a_\infty = a(r)$ (see Figure 2.b). We claim that $r = r_\infty$ is the leftmost solution. Suppose not. Then $r > r_M$, $a_\infty < a_M$ and by Step 2 there are no solutions to $M_R(r;a_M) = 0$. But this this contradicts Step 1.

Step 4. Fix $a \in (0, a_\infty)$. By Step 1 of Lemma 2.2, $M_R(r;a)$ is strictly positive for sufficiently small R. By the definition of a_∞, $M_\infty(r;a)$ has a root when $a = a_\infty$. Since $M_R(r;a)$ is an increasing function of a, it follows that $M_R(r;a)$ has a root for sufficiently large R when $a < a_\infty$. So there exists R_0,r_0 such that

$$M_{R_0}(r_0;a) = 0 = M'_{R_0}(r_0;a) \tag{2.19}$$

but $M_{R_0}(r;a) > 0$ for all $r \neq r_0$. Suppose that there is another pair of numbers R_1,r_1 that has the property (2.19). By Step 3, it follows that $r_1 < r_\infty$. But then $r_1 = r_0$ by monotonicity of $\hat{a}$ on $(0, r_\infty)$. It follows by monotonicity of J_1'/J_2' that $R_1 = R_0$. This shows that the solution to (2.19) is unique whenever $a \in (0, a_\infty)$.

Step 5. Note that M_R has no roots if R is sufficiently small (see Lemma 2.1, Step 1). and $M_R(R) > 0$. Since the solution to 2.19 is unique, the proof is complete by showing that $M_R(r)$ admits a root $r = r_0 < R$ for $a \in (0, a_\infty)$ and with R sufficiently large. Now $M_R(r)$ is positive when r is small, and from the definition (2.15) of $a_{\inf}$, the minimum of $M_\infty(r)$ is negative. So $M_\infty(r)$ has a root, and therefore so does $M_R(r)$ with R sufficiently large.This competes the proof of Lemma 2.3. □

Proof of Theorem 1.2. We are now ready to finish the proof of Theorem 1.2. Case 1.a and 2.a with $a \leq 1$ are covered by Lemma 2.2. Cases 2.a with $a \in (1, a_c)$ as well as Cases 2.b and 2.c are covered by Lemma 2.3 so only the cases 1.b and 1.c remain.

Consider first the case $a < N-2$, $N \geq 3$. With g as in Lemma 2.1, we have $g(0) = a(a-(N-2)) < 0$ and $g \to -\infty$ as $r \to \infty$. Moreover recall (2.8) that g satisfies $rg' + r^2Cg = J_1^2(B - Ar^2)$ but here, $C > 0$, $B < 0$, $A > 0$. Hence $g' < 0$ whenever $g = 0$, so g has no roots. It follows that $M_R(r) = M_R'(r) = 0$ has no solutions for any R. But for small values of R, it is easy to see using (2.4) that $M_R(r)$ has precisely one root $r = r_0$ with $M_R'(r_0) = 0$.

Since $M_R(R) > 0$ for all R (see Lemma 2.2 Step 2), it follows by continuity that $M_R(r)$ has precisely one root for all R.

Lastly the case $a \geq N-1$. Then $g(0) > 0$; $rg' + r^2Cg = J_1^2(B - Ar^2)$ with $A \leq 0,\ B > 0$. Thus $g' > 0$ whenever $g = 0$ which implies that g no roots. Since $M_R(r)$ has no roots for small R, it therefore follows that $M_R(r)$ has no roots for any M_R. This completes the proof of Theorem 1.2. ☐

3. Approximate Solutions and a Linear Problem

The rest of the sections are devoted to proving Theorem 1.3. The present section contains some preliminaries. We first define approximate solutions and then we study a linear problem, which forms the foundation of the finite dimension reduction.

Suppose that the assumption (1.20) holds. By Theorem 1.2, there exists $R_a \geq 0$ such that for $R > R_a$, the equation $M_R(t) = 0$ has at least one root $r_0 \in (0, R)$. Furtheremore, $M_R'(r_0) \neq 0$. Let us fix such r_0 throughout the rest of the paper. Let us define

$$A(r) = \frac{J_1'}{J_1} > 0,\ B(r) = \frac{J_{2,R}'}{J_{2,R}} < 0. \tag{3.1}$$

Choose two fixed numbers $R_1 \in (0, r_0), R_2 \in (r_0, R)$. Let $\chi(s)$ be a function such that $\chi(s) = 1$ for $s \in [R_1, \frac{r_0+R_2}{2}]$ and $\chi(s) = 0$ for $s < \frac{R_1}{2}$ or $s > R_2$. Fix $t \in (R_1, R_2)$. We set

$$\mathbb{I}_\varepsilon := (0, \frac{R}{\varepsilon}), \quad \mathbb{I}_{\varepsilon,t} := (-\frac{t}{\varepsilon}, \frac{R-t}{\varepsilon}) \tag{3.2}$$

and then define

$$w_{\varepsilon,t}(y) := w(y - \frac{t}{\varepsilon})\chi(\varepsilon y), y \in \mathbb{I}_\varepsilon \tag{3.3}$$

We introduce the following set

(3.4)
$$\Lambda = \left\{ \mathbf{t} = (t_1, ..., t_K) \middle| |\frac{1}{K}\sum_{j=1}^{K} t_j - r_0| \leq \varepsilon^{\frac{\tau}{2}},\ (1-\delta)\varepsilon \log\frac{1}{\varepsilon} \leq t_{j+1} - t_j \leq (1+\delta)\varepsilon \log\frac{1}{\varepsilon}, j = 1, ..., K-1, \right\}$$

where $\tau > 0$ is to be chosen later. For $\mathbf{t} \in \Lambda$, we define

$$w_{\varepsilon,\mathbf{t}}(r) = \sum_{j=1}^{K} w_{\varepsilon,t_j}(r). \tag{3.5}$$

Then we have

$$t_j = O(\varepsilon|\ln\varepsilon|), j = 1, ..., N,\ |t_i - t_j| \geq (1-\delta)|i-j|\varepsilon \log\frac{1}{\varepsilon}. \tag{3.6}$$

The choice of the approximated location of the concentration points comes from the computations carried out in the proof of formula (4.11).

As in [12] and [20], for $u, v \in H^1(B_{\frac{R}{\varepsilon}})$, we equip them with the following scalar product:

$$(u,v)_\varepsilon = \int_{\mathbb{I}_\varepsilon} (u^{'}v^{'} + uv)(y)^{N-1}dy \tag{3.7}$$

(which is equivalent to the inner product of $H^1(B_R)$).

Then orthogonality to the function $w^{'}_{\varepsilon,t_j}$ with respect to this scalar product is equivalent to the orthogonality to the function

$$Z_{\varepsilon,t_j} = w^{'''}_{\varepsilon,t_j} + \frac{(N-1)}{y} w^{''}_{\varepsilon,t_j} - w^{'}_{\varepsilon,t_j} \tag{3.8}$$

in $L^2(\mathbb{I}_\varepsilon)$, equipped with the following scalar product

$$< u, v >_\varepsilon = \int_{\mathbb{I}_\varepsilon} (uv)(y)^{N-1}dy \tag{3.9}$$

(which is equivalent to the inner product of $L^2(B_R)$).

Let

$$\mathcal{C}[\phi] = \frac{qm}{p-1}\left(\sum_{l=1}^{K} \xi_{\varepsilon,l}^{\frac{qm}{p-1}-1} G_R(t_l;t_l)\int_{\mathbb{I}_\varepsilon} w^{m-1}_{\varepsilon,t_l}\phi\right) \tag{3.10}$$

where $\xi_{\varepsilon,j}$ is defined at (4.2).

Then we consider the following problem: for $h \in L^2(B_{\frac{R}{\varepsilon}}) \cap L^\infty(\mathbb{I}_\varepsilon)$ being given, find a function ϕ satisfying

$$\begin{cases} \mathbb{L}_{\varepsilon,\mathbf{t}}[\phi] := \Delta\phi - \phi + pw^{p-1}_{\varepsilon,\mathbf{t}}\phi - (p-1)\mathcal{C}[\phi]w^p_{\varepsilon,\mathbf{t}} = h + \sum_{j=1}^K c_j Z_{\varepsilon,t_j}, \\ \phi^{'}(0) = \phi^{'}(\frac{R}{\varepsilon}) = 0, < \phi, Z_{\varepsilon,t_j} >_\varepsilon = 0, j = 1, ..., K, \end{cases} \tag{3.11}$$

for some constants $c_1, ..., c_K$.

Let $0 < \mu < \frac{1}{10}\min(p-1, m-1, 1)$ be a small number such that lemma 5.1 of [20] holds. For every function $\phi : \mathbb{I}_\varepsilon \to \mathbb{R}$ define

$$\|\phi\|_* = \|e^{\mu \min_{j=1}^K < y - \frac{t_j}{\varepsilon} >}\phi(y)\|_{L^\infty(\mathbb{I}_\varepsilon)} \tag{3.12}$$

where $< y >= (1+y^2)^{\frac{1}{2}}$.

Since $\frac{N-1}{y}w''_{\varepsilon,t} = O(\varepsilon)e^{-|y-\frac{t}{\varepsilon}|}$, we obtain

(3.13)

$$Z_{\varepsilon,t}(y) = w'''(y - \frac{t}{\varepsilon}) - w'(y-\frac{t}{\varepsilon}) + O(\varepsilon)e^{-\mu< y - \frac{t}{\varepsilon}>} = -pw^{p-1}(y-\frac{t}{\varepsilon})w'(y-\frac{t}{\varepsilon}) + O(\varepsilon)e^{-\mu<y-\frac{t}{\varepsilon}>}$$

uniformly for $t \in [R_1, R_2]$.

The following proposition provides a priori estimates of ϕ satisfying (3.11).

Proposition 3.1. *Let (ϕ, c) satisfy (3.11). Then for ε sufficiently small, $\mathbf{t} \in \Lambda$, we have*

$$\|\phi\|_* \leq C\|h\|_* \tag{3.14}$$

where C is a positive constant depending on R, N, p, K only.

Proof. The proof is similar to Proposition 3.1 of [12] or Proposition 5.1 of [20]. We prove the inequality by contradiction. Arguing by contradiction there exists sequences $\varepsilon_k \to 0, \mathbf{t}_k \in \Lambda$ and a sequence of functions $\phi_{\varepsilon_k, \mathbf{t}_k}$ satisfying (3.11) such that the following holds:

$$\|\phi_{\varepsilon_k,\mathbf{t}_k}\|_* = 1, \|h_k\|_* = o(1), \int_{\mathbb{I}_{\varepsilon_k, t_{j,k}}} \phi_{\varepsilon_k,\mathbf{t}_k} Z_{\varepsilon_k, t_{j,k}}(y)^{N-1} dy = 0.$$

For simplicity of notation, we drop the subindex k.

Multiplying the first equation of (3.11) by w'_{ε,t_j} and integrating over $\mathbb{I}_\varepsilon$, we obtain that

$$\sum_{l=1}^K c_l \int_{\mathbb{I}_\varepsilon} Z_{\varepsilon,t_l} w'_{\varepsilon,t_j} = -\int_{\mathbb{I}_\varepsilon} h w'_{\varepsilon,t_j} + \int_{\mathbb{I}_\varepsilon} (\mathbb{L}_\varepsilon[\phi_{\varepsilon,\mathbf{t}}]) w'_{\varepsilon,t_j}. \tag{3.15}$$

The left hand side of (3.15) equals $\sum_{l=1}^K c_l(-\int_{\mathbb{R}} p w^{p-1}(w')^2 \delta_{lj} + o(1))$ because of (3.13). The first term on the right hand side of (3.15) can be estimated by

$$\int_{\mathbb{I}_\varepsilon} h w'_{\varepsilon,t_j} = O(\|h\|_*)$$

where we have used the fact that w is exponentially decaying.

The last term equals

$$\int_{\mathbb{I}_\varepsilon} (\mathbb{L}_\varepsilon[\phi_{\varepsilon,\mathbf{t}}]) w'_{\varepsilon,t_j} = \int_{\mathbb{I}_{\varepsilon,t_j}} \left[\phi''_{\varepsilon,\mathbf{t}} + \frac{\varepsilon(N-1)}{t_j + \varepsilon z} \phi'_{\varepsilon,\mathbf{t}} - \phi_{\varepsilon,\mathbf{t}} + p w^{p-1}_{\varepsilon,t_j} \phi_{\varepsilon,\mathbf{t}} \right] w'_{\varepsilon,t_j}$$

$$+p \int_{\mathbb{I}_\varepsilon} (w^{p-1}_{\varepsilon,\mathbf{t}} - w^{p-1}_{\varepsilon,t_j}) \phi w'_{\varepsilon,t_j} - (p-1)\mathcal{C}(\phi) \int_{\mathbb{I}_\varepsilon} w^p_{\varepsilon,\mathbf{t}} w'_{\varepsilon,t_j}$$

Similar to the estimates in the proof of Proposition 5.2 of [20], we obtain that

$$\sum_{j=1}^K |c_j| = O(\|h\|_*) + o(\|\phi_{\varepsilon,\mathbf{t}}\|_*), \quad \|h + \sum_{j=1}^K c_j Z_{\varepsilon,t_j}\|_* = o(1). \tag{3.16}$$

Next we set

$$\hat{\phi}_j(y) := \phi_{\varepsilon,\mathbf{t}}(y - \frac{r_j}{\varepsilon}), j = 1, ..., K.$$

Since $\|\phi_{\varepsilon,\mathbf{t}}\|_* = 1$, we see that $\hat{\phi}_j(y) \to \hat{\phi}_{0,j}$ locally in any compact interval of $\mathbb{R}$. Furthermore, we see that

$$\mathcal{C}(\phi) \to \mu_0 \frac{\sum_{j=1}^K \int_{\mathbb{R}} w^{m-1} \hat{\phi}_{0,j}}{\int_{\mathbb{R}} w^m} \tag{3.17}$$

where

$$\mu_0 = \lim_{\varepsilon \to 0} \frac{qm}{p-1} \left(\sum_{l=1}^K \xi_{\varepsilon,l}^{\frac{qm}{p-1}-1} G_R(t_l; t_l) \right) \int_{\mathbb{R}} w^m = \frac{1}{K} \frac{qm}{p-1} \tag{3.18}$$

by Lemma 4.1.

Therefore, $\hat{\phi}_{0,j}$ satisfies

$$L[\hat{\phi}_{0,j}] - \mu_0(p-1) \frac{\sum_{j=1}^K \int_{\mathbb{R}} w^{m-1} \hat{\phi}_{0,j}}{\int_{\mathbb{R}} w^m} w^p = 0, j = 1, ..., K. \tag{3.19}$$

Summing up the above equality, we obtain

$$L[\sum_{j=1}^K \hat{\phi}_{0,j}] - K\mu_0(p-1) \frac{\sum_{j=1}^K \int_{\mathbb{R}} w^{m-1} \hat{\phi}_{0,j}}{\int_{\mathbb{R}} w^m} w^p = 0,$$

which by (3) of Lemma 3.1 of [20] implies that $\sum_{j=1}^K \hat{\phi}_{0,j} = cw'$, since $\mu_0 K \neq 1$. This then yields that $\int_{\mathbb{R}} w^{m-1} \sum_{j=1}^K \hat{\phi}_{0,j} = 0$. So $L[\hat{\phi}_{0,j}] = 0$ and hence $\hat{\phi}_{0,j} = \alpha_j w', j = 1, ..., K$ for some constant α_j.

On the other hand, taking the limit in $\int_{\mathbb{I}_\varepsilon} Z_{\varepsilon,t_j} \phi_{\varepsilon,\mathbf{t}} = 0$ gives $\int_{\mathbb{R}} w^{p-1} w' \hat{\phi}_{0,j} = 0$. Thus $\alpha_j = 0$ and $\hat{\phi}_{j,k} \to 0$ locally in $\mathbb{R}$. This then implies that

$$\mathcal{C}(\phi) = o(1) \tag{3.20}$$

and

$$\|w_{\varepsilon,\mathbf{t}}^{p-1} \phi_{\varepsilon,\mathbf{t}}\|_* = \sup_{y \in \mathbb{I}_{\varepsilon,t_j}} |e^{\mu < y - \frac{t_j}{\varepsilon} >} w_{\varepsilon,t_j}^{p-1}(y) \phi_{\varepsilon,\mathbf{t}}(y)| = o(1). \tag{3.21}$$

The rest of the proof is exactly the same as in those of Proposition 5.1 of [20]. We omit the details. □

Similarly, we have

Proposition 3.2. *There exists an $\varepsilon_0 > 0$ such that for any $\varepsilon < \varepsilon_0$, $\mathbf{t} \in \Lambda$, given any $h \in L^2(\mathbb{I}_\varepsilon) \cap L^\infty(\mathbb{I}_\varepsilon)$, there exists a unique pair $(\phi, c_1, ..., c_K)$ such that the following hold:*

$$\mathbb{L}_{\varepsilon,\mathbf{t}}[\phi] = h + \sum_{j=1}^K c_j Z_{\varepsilon,t_j}, \tag{3.22}$$

$$\phi^{'}(0) = \phi^{'}(\frac{R}{\varepsilon}) = 0, \ <\phi, Z_{\varepsilon,t_j}>_\varepsilon = 0, j = 1, ..., K \tag{3.23}$$

Moreover, we have

$$\|\phi\|_* \leq C\|h\|_*. \tag{3.24}$$

4. STUDY OF THE OPERATOR $\mathbb{T}[h]$

In this section, we study the operator $\mathbb{T}[h]$, defined at (1.27), where we choose h to be

$$h = \left(\sum_{j=1}^{K} \xi_{\varepsilon,j}^{\frac{q}{p-1}} w_{\varepsilon,t_j}(\frac{r}{\varepsilon}) + \phi\right)^m, \ \|\phi\|_* = O(\varepsilon^\sigma) \tag{4.1}$$

where $\xi_{\varepsilon,j}$ is chosen such that

$$\mathbb{T}\left[\sum_{j=1}^{K} \xi_{\varepsilon,j}^{\frac{q}{p-1}} w_{\varepsilon,t_j}(\frac{r}{\varepsilon})\right](t_j) = \xi_{\varepsilon,j}, j = 1, ..., K \tag{4.2}$$

and $\mathbf{t} = (t_1, ..., t_K) \in \Lambda$. We first have

Lemma 4.1. *Let $\mathbf{t} \in \Lambda$. Then (4.2) has a unique solution $\xi_{\varepsilon,j} = \xi_{\varepsilon,j}(\mathbf{t})$ such that*

$$\xi_{\varepsilon,j} = \left(K(\int_{\mathbb{R}} w^m) G(r_0; r_0)\right)^{\frac{p-1-qm}{qm}} + O(\varepsilon|\log\varepsilon|), j = 1, ..., K \tag{4.3}$$

The proof of Lemma 4.1 will be given at the end of the section.

Let us decompose

$$h = \left(\sum_{j=1}^{K} \xi_{\varepsilon,j}^{\frac{q}{p-1}} w_{\varepsilon,t_j}(\frac{r}{\varepsilon}) + \phi\right)^m = h_1 + h_2 + h_3$$

where

$$h_1 = \sum_{j=1}^{K} \xi_{\varepsilon,j}^{\frac{qm}{p-1}} \left(w_{\varepsilon,t_j}(\frac{r}{\varepsilon})\right)^m$$

$$h_2 = \left(\sum_{j=1}^{K} \xi_{\varepsilon,j}^{\frac{q}{p-1}} w_{\varepsilon,t_j}(\frac{r}{\varepsilon})\right)^m - \sum_{j=1}^{K} \xi_{\varepsilon,j}^{\frac{qm}{p-1}} \left(w_{\varepsilon,t_j}(\frac{r}{\varepsilon})\right)^m$$

$$h_3 = \left(\sum_{j=1}^{K} \xi_{\varepsilon,j}^{\frac{q}{p-1}} w_{\varepsilon,t_j}(\frac{r-t_j}{\varepsilon}) + \phi\right)^m - \left(\sum_{j=1}^{K} \xi_{\varepsilon,j}^{\frac{q}{m-1}} w_{\varepsilon,t_j}(\frac{r-t_j}{\varepsilon})\right)^m$$

According to (1.27), we have

$$\mathbb{T}[h](r^{'}) = \varepsilon^{-1} \int_0^R G_R(r^{'}; r) h(r) dr$$

$$= E_1 + E_2 + E_3$$

where $E_j = \mathbb{T}[h_j], j = 1,2,3$. Our remaining task is to compute $\mathbb{T}[h_i](t_j + \varepsilon z) - \mathbb{T}[h](t_j)$.

Similar to the proof of Lemma 6.1 of [20], we have

$$\mathbb{T}[\xi_{\varepsilon,j}^{\frac{qm}{p-1}} w_{\varepsilon,t_j}(\frac{r}{\varepsilon})](t_j+\varepsilon z) = \xi_{\varepsilon,j}^{\frac{qm}{p-1}} G_R(t_j;t_j) \int_{\mathbb{R}} w^m (1+\varepsilon\rho_j(z) + O(\varepsilon^2(1+|z|^2)) \tag{4.4}$$

where $\rho_j(z)$ is defined by

(4.5)

$$\rho_j(z) = \frac{1}{\int_{\mathbb{R}} w^m}\left[\frac{J_{2,R}'(t_j)}{J_{2,R}(t_j)}\left(z\int_{-\infty}^z w^m + \int_z^\infty yw^m(y)dy\right) + \frac{J_1'(t_j)}{J_1(t_j)}\left(z\int_z^\infty w^m + \int_{-\infty}^z yw^m(y)dy\right)\right].$$

For $l \neq j$, we have

(4.6)

$$\mathbb{T}[\xi_{\varepsilon,l}^{\frac{qm}{p-1}} w_{\varepsilon,t_l}(\frac{r}{\varepsilon})](t_j+\varepsilon z) = \xi_{\varepsilon,l}^{\frac{qm}{p-1}} G_R(t_l;t_l) \int_{\mathbb{R}} w^m (1+\varepsilon\rho_l(\frac{t_j-t_l}{\varepsilon}+z) + O(\sum_{j\neq l}|t_j-t_l|^2 + \varepsilon^2|z|^2)).$$

Therefore from the definition of (4.5) we have

$$\mathbb{T}[h_1](t_j+\varepsilon z) = \sum_{l=1}^K \xi_{\varepsilon,l}^{\frac{qm}{p-1}} G_R(t_l;t_l)\int_{\mathbb{R}} w^m\left(1+\varepsilon\rho(\frac{t_j-t_l}{\varepsilon}+y) + O(\varepsilon^2+\varepsilon^2|z|^2)\right). \tag{4.7}$$

For E_2, we have

$$\mathbb{T}[h_2](t_j+\varepsilon z) - \mathbb{T}[h_2](t_j) = O(|y|\int_0^R \sum_{i\neq j} w_{\varepsilon,t_i}^{m-1} w_{\varepsilon,t_j}) = O(\varepsilon^2|\log\varepsilon||z|)$$

For E_3, we have

$$E_3 = \varepsilon^{-1}\int_0^R G_R(r';r)[m\sum_{l=1}^K \xi_{\varepsilon,l}^{\frac{q(m-1)}{p-1}} w_{\varepsilon,t_l}^{m-1}\phi]dy + O(\varepsilon^\sigma\|\phi\|_* + \|\phi\|_*^{1+\tau})$$

$$= m\sum_{l=1}^K \xi_{\varepsilon,l}^{\frac{q(m-1)}{p-1}} G_R(t_l;t_l)\int_{\mathbb{I}_\varepsilon} w_{\varepsilon,t_l}^{m-1}\phi + O(\varepsilon^\sigma\|\phi\|_* + \|\phi\|_*^{1+\tau})$$

where the constant τ is defined by

$$\tau = \min(1, p-1, m-1). \tag{4.8}$$

On the other hand, for $|r' - r_0| \geq \frac{\delta}{4}$, we use

$$|\phi(\frac{r}{\varepsilon})| \leq C\|\phi\|_* e^{-\mu<\frac{r-t_j}{\varepsilon}>}. \tag{4.9}$$

This implies, by the same estimates as in [20],

$$\mathbb{T}[h](r') \geq C \tag{4.10}$$

Summarizing all the estimates, we have obtained the following lemma:

Lemma 4.2. *(1) For* $|r^{'} - r_0| < \frac{\delta}{4}, r^{'} = t_j + \varepsilon z$*, we have*

(4.11)
$$\mathbb{T}[(\sum_{l=1}^{K} \xi_{\varepsilon,l}^{\frac{q}{p-1}} w_{\varepsilon,t_l} + \phi)^m](t_j + \varepsilon z) = \sum_{l=1}^{K} \xi_{\varepsilon,l}^{\frac{qm}{p-1}} G_R(t_l;t_l) \int_{\mathbb{R}} w^m \left(1 + \varepsilon\rho(\frac{t_j - t_l}{\varepsilon} + z) + O(\varepsilon^2 + \varepsilon^2|z|^2)\right)$$
$$+ m \sum_{l=1}^{K} \xi_{\varepsilon,l}^{\frac{q(m-1)}{p-1}} G_R(t_l;t_l) \int_{\mathbb{I}_\varepsilon} w_{\varepsilon,t_l}^{m-1}\phi + O(\varepsilon^\sigma \|\phi\|_* + \|\phi\|_*^{1+\tau})$$

(2) For $|r^{'} - r_0| \geq \frac{\delta}{4}$*, we then have*

$$\mathbb{T}[(\sum_{l=1}^{K} \xi_{\varepsilon,l}^{\frac{q}{p-1}} w_{\varepsilon,t_l} + \phi)^m](r^{'}) \geq C. \tag{4.12}$$

Finally, we prove Lemma 4.1.

Proof of Lemma 4.1: From the computations above, we obtain that

$$\mathbb{T}[(\sum_{l=1}^{K} \xi_{\varepsilon,j}^{\frac{q}{p-1}} w_{\varepsilon,t_l})^m](t_j) = \sum_{l=1}^{K} \xi_{\varepsilon,l}^{\frac{qm}{p-1}} G_R(t_l;t_l) \int_{\mathbb{R}} w^m \left(1 + O(\varepsilon|\log\varepsilon|)\right).$$

Thus equation (4.2) becomes

$$\xi_{\varepsilon,j} = \sum_{l=1}^{K} \xi_{\varepsilon,l}^{\frac{qm}{p-1}} G_R(t_l;t_l) \int_{\mathbb{R}} w^m \left(1 + O(\varepsilon|\log\varepsilon|)\right), j = 1, ..., K. \tag{4.13}$$

As $\varepsilon \to 0$, $\xi_{\varepsilon,j} \to \xi_j$. We obtain

$$\xi_j = \sum_{l=1}^{K} \xi_l^{\frac{qm}{p-1}} G_R(r_0;r_0) \int_{\mathbb{R}} w^m \tag{4.14}$$

which admits a solution $\xi_j = \xi_0$ where ξ_0 satisfies

$$\xi_0^{1-\frac{qm}{p-1}} = K G_R(r_0;r_0) \int_{\mathbb{R}} w^m \tag{4.15}$$

Now we search for a solution to (4.13) with $\xi_{\varepsilon,j} = \xi_0 + \hat{\xi}_{\varepsilon,j}$ where $\hat{\xi}_{\varepsilon,j} = o(1)$. Then we have

$$\hat{\xi}_{\varepsilon,j} - \frac{qm}{p-1} \xi_0^{\frac{qm}{p-1}-1} (K G_R(r_0;r_0) \int_{\mathbb{R}} w^m) \sum_{l=1}^{K} \hat{\xi}_{\varepsilon,l} + O(\varepsilon \log\varepsilon) \tag{4.16}$$

The matrix on the left hand side of (4.16) is nondegenerate since $\frac{qm}{p-1} \neq 1$. This, together with the implicit function theorem, proves Lemma 4.1. $\square$

5. A Nonlinear Problem

In this section, we solve the following system of equations for (ϕ, β):

$$(W+\phi)'' + \frac{\varepsilon(N-1)}{y}(W+\phi)' - (W+\phi) + \frac{(W+\phi)^p}{(\mathbb{T}[(W+\phi)^m](\varepsilon y))^q} = \sum_{j=1}^{K} \beta_j Z_{\varepsilon,t_j}, \tag{5.1}$$

$$\phi'(0) = \phi'(\frac{R}{\varepsilon}) = 0, \ \int_{\mathbb{I}_\varepsilon} \phi Z_{\varepsilon,t_j} y^{N-1} dy = 0, j = 1, ..., K, \tag{5.2}$$

where, from now on, we use the following notation:

$$W := \sum_{j=1}^{K} \xi_{\varepsilon,j}^{\frac{q}{p-1}} w_{\varepsilon,t_j}, j = 1, ..., K \tag{5.3}$$

and we recall the definition of $\xi_{\varepsilon,j}$ at (4.2). Note that if $\beta_j = 0$, then we have solved (1.1).

The main result in this section is to show the following proposition:

Proposition 5.1. *For* $\mathbf{t} \in \Lambda$ *and* ε *sufficiently small, there exists a unique pair* $(\phi_{\varepsilon,\mathbf{t}}, \beta_{\varepsilon,1}(\mathbf{t}), ..., \beta_{\varepsilon,K}(\mathbf{t}))$ *satisfying (5.1)-(5.2). Furthermore,* $(\phi_{\varepsilon,\mathbf{t}}, \beta_{\varepsilon,1}(\mathbf{t}), ..., \beta_{\varepsilon,K}(\mathbf{t}))$ *is continuous in* t *and we have the following estimate*

$$\|\phi_{\varepsilon,\mathbf{t}}\|_* \leq \varepsilon^\sigma \tag{5.4}$$

where $\sigma \in (\frac{1}{1+\tau}, 1)$ *is a constant, and* τ *is defined by (4.8).*

Proof: The proof is similar to that of Proposition 6.1 of [20], by writing (5.1) in the following form:

$$\mathbb{L}_{\varepsilon,\mathbf{t}}[\phi] = \mathbb{E}_\varepsilon + \mathbb{M}_\varepsilon[\phi] + \sum_{j=1}^{K} \beta_j Z_{\varepsilon,t_j}. \tag{5.5}$$

Here

$$\mathbb{E}_\varepsilon = -W'' + W - \frac{(N-1)}{y} W' - \frac{W^p}{(\mathbb{T}[W^m])^q} \tag{5.6}$$

and $\mathbb{M}_\varepsilon[\phi]$ is given by

$$\mathbb{M}_\varepsilon[\phi] = \frac{(W+\phi)^p}{(\mathbb{T}[(W+\phi)^m])^q} - \frac{W^p}{(\mathbb{T}[W^m])^q} - p w_{\varepsilon,\mathbf{t}}^{p-1} \phi + \mathcal{C}(\phi) w_{\varepsilon,\mathbf{t}}^p \tag{5.7}$$

Note that

$$\mathbb{E}_\varepsilon = -W'' + W - \frac{(N-1)}{y} W' - \frac{W^p}{(\mathbb{T}[W^m])^q}$$

$$= \sum_{j=1}^{K} \frac{1-N}{y} \xi_{\varepsilon,j} w'_{\varepsilon,t_j} + \sum_{j=1}^{K} \xi_{\varepsilon,j}^{\frac{q}{p-1}} \left(w_{\varepsilon,t_j}^p - \frac{\xi_{\varepsilon,j}^q}{(\mathbb{T}[W^m])^q} \right) + \frac{\sum_{j=1}^K \xi_{\varepsilon,j}^{\frac{qp}{p-1}} w_{\varepsilon,t_j}^p - W^p}{(\mathbb{T}[W^m])^q} \tag{5.8}$$

Using Lemma 4.2, we obtain

$$\|E_\varepsilon\|_* \leq C\varepsilon. \tag{5.9}$$

For $\mathbb{M}_\varepsilon$, we note that

$$\mathbb{M}_\varepsilon[\phi] = \frac{(W+\phi)^p - W^p - pW^{p-1}\phi}{(\mathbb{T}[(W+\phi)^m])^q}$$

$$-\left(\frac{W^p}{(\mathbb{T}[w_{\varepsilon,t}^m])^q} - \frac{W^p}{(\mathbb{T}[(W+\phi)^m])^q} - \mathcal{C}(\phi) \frac{W^p}{(\mathbb{T}[(W+\phi)^m])^q} \right)$$

$$-pW^{p-1}\phi \left(\frac{1}{(\mathbb{T}[W^m])^q} - \frac{1}{(\mathbb{T}[(W+\phi)^m])^q} \right)$$

$$-p(w_{\varepsilon,\mathbf{t}}^{p-1} - \frac{W^{p-1}}{(\mathbb{T}[(W+\phi)^m])^q}) + \mathcal{C}[\phi](\frac{W^p}{(\mathbb{T}[(W+\phi)^m])^q} - w_{\varepsilon,\mathbf{t}}^p)$$

Using Lemma 4.2, we see that

$$\|\mathbb{M}_\varepsilon[\phi]\|_* \leq C\left(\varepsilon\|\phi\|_* + \|\phi\|_*^{1+\tau} \right). \tag{5.10}$$

The rest of the proof is similar to that of [20].

□

6. The reduced problem

In this section we solve the reduced problem and establish our main existence result given by Theorem 1.3. In particular, we prove that

Proposition 6.1. *For ε sufficiently small, $\beta_\varepsilon(\mathbf{t})$ is continuous in $\mathbf{t}$ and we have*

$$\beta_{\varepsilon,j}(\mathbf{t}) = b_0\varepsilon \left[\frac{a}{t_j} + \frac{1}{K}(A(t_j) + B(t_j)) + \frac{2(j-1)}{K} B(t_j) + \frac{2(K-j)}{K} A(t_j) \right] \tag{6.1}$$

$$+c_0 \sum_{l\neq j} \frac{t_j - t_l}{|t_j - t_l|} e^{-\frac{|t_j - t_l|}{\varepsilon}} + O(\varepsilon^{\sigma(1+\tau)}), j = 1, ..., K$$

for some generic constants $b_0, c_0 > 0$.

From Proposition 6.1, we can finish the proof of Theorem 1.3.

Proof of Theorem 1.3: We need to find a $\mathbf{t} \in \Lambda$ such that $\beta_{\varepsilon,j}(\mathbf{t}) = 0$. To this end, we use a degree argument.

Consider a new vector field $\mathbf{F} = (\mathbf{F}_1, ..., \mathbf{F}_K)$ as

$$\mathbf{F}_j(\mathbf{t}) = b_0\varepsilon\left[\frac{a}{t_j} + \frac{1}{K}(A(t_j) + B(t_j)) + \frac{2(j-1)}{K}B(t_j) + \frac{2(K-j)}{K}A(t_j)\right]$$
$$+c_0 \sum_{|l-j|=1} \frac{t_j - t_l}{|t_j - t_l|} e^{-\frac{|t_j - t_l|}{\varepsilon}}, j = 1, ..., K$$

For $\mathbf{t} \in \Lambda$, we see that

$$\beta_{\varepsilon,j}(\mathbf{t}) = \mathbf{F}_j(\mathbf{t}) + O(\varepsilon^{1+\tau})$$

Now we consider a homotopy of β_ε and $\mathbf{F}$:

$$\hat{F}_j(\mathbf{t}) = \alpha\beta_{\varepsilon,j}(\mathbf{t}) + (1-\alpha)\mathbf{F}_j(\mathbf{t}) = \mathbf{F}_j(\mathbf{t}) + O(\varepsilon^{1+\tau}) \tag{6.2}$$

where $\alpha \in [0,1]$. We claim that for $\mathbf{t} \in \partial\Lambda, \hat{F}(\mathbf{t}) \neq 0$. In fact, suppose for some $\mathbf{t} \in \partial\Lambda, \hat{F}(\mathbf{t}) = 0$. By definition, we have either $\sum_{j=1}^K t_j = \varepsilon^{\frac{\tau}{4}}$ or $t_j - t_{j-1} = \varepsilon(1-\delta)\log\frac{1}{\varepsilon}$ for some j or $t_j - t_{j-1} = \varepsilon(1+\delta)\log\frac{1}{\varepsilon}$ for some j.

In the first case, we have

(6.3)
$$0 = \sum_{l=1}^K \hat{F}_l(\mathbf{t}) = b_0\varepsilon \sum_{j=1}^K \left[\frac{a}{t_j} + \frac{1}{K}(A(t_j) + B(t_j)) + \frac{2(j-1)}{K}B(t_j) + \frac{2(K-j)}{K}A(t_j)\right] + O(\varepsilon^{1+\tau})$$
$$= Kb_0\varepsilon M_R'(r_0)(\frac{\sum_{j=1}^K t_j}{K} - t_0) + O(\varepsilon^{1+\tau})$$

which is impossible since $M_R'(r_0) \neq 0$.

To consider the second and the third case, we need to compute

$$\sum_{l=1}^j \hat{F}_l(\mathbf{t}) = b_0\varepsilon \sum_{l=1}^j \left[\frac{a}{t_l} + \frac{1}{K}(A(t_l) + B(t_l)) + \frac{2(l-1)}{K}B(t_l) + \frac{2(K-l)}{K}A(t_l)\right] - c_0 e^{\frac{t_j - t_{j+1}}{\varepsilon}} \tag{6.4}$$
$$= b_0\varepsilon\left[j\frac{a}{r_0} + \frac{j}{K}(A(r_0) + B(r_0)) + \frac{j(j-1)}{K}B(r_0) + \frac{2Kj - j(j+1)}{K}A(r_0)\right] - c_0 e^{\frac{t_j - t_{j+1}}{\varepsilon}} + O(\varepsilon^{1+\tau})$$

So

$$\sum_{l=1}^j \hat{F}_l(\mathbf{t}) = b_0\varepsilon\frac{j(K-j)}{K}(A(r_0) - B(r_0)) - c_0 e^{\frac{t_j - t_{j+1}}{\varepsilon}} + O(\varepsilon^{1+\delta}). \tag{6.5}$$

In the second case, we then have

$$0=\sum_{l=1}^{j}\hat{F}_l(\mathbf{t})=O(\varepsilon)-e^{\frac{t_{j-1}-t_j}{\varepsilon}}\le -C\varepsilon^{1-\delta} \tag{6.6}$$

which is a contradiction.

In the third case, we then have

$$0=\sum_{l=1}^{j}\hat{F}_l(\mathbf{t})=b_0\varepsilon\frac{j(K-j)}{K}(A(r_0)-B(r_0))+O(\varepsilon^{1+\delta}). \tag{6.7}$$

This is impossible again since

$$A(r_0)-B(r_0)=\frac{J_1^{'}}{J_1}-\frac{J_{2,R}^{'}}{J_{2,R}}>0. \tag{6.8}$$

By degree theory, we have

$$deg\ (\beta_\varepsilon,\Lambda)=deg\ (\mathbf{F},\Lambda)$$

Next we show that $\mathbf{F}$ has only one zero in Λ and the zero is *nondegenerate*. To this end, we consider another vector field:

$$\bar{F}_j(\mathbf{t})=\sum_{l=1}^{j}\mathbf{F}_l(\mathbf{t})$$

Then

$$deg\ (\beta_\varepsilon,\Lambda)=deg\ (\mathbf{F},\Lambda)=deg\ (\bar{F},\lambda)$$

We first claim that, any zero of $\bar{F}$ in Λ (denoted by $\mathbf{t}^0$) must satisfy

$$t_j^0-t_{j-1}^0=\varepsilon\log\frac{1}{\varepsilon}+\varepsilon A_j+o(\varepsilon),\ \frac{1}{K}\sum_{j=1}^{K}t_j^0=t_0+O(\varepsilon^\tau) \tag{6.9}$$

where A_j is some constant. In fact, this follows from (6.3) and (6.5).

Now we show that $\mathbf{t}^0$ is unique and nondegenerate. To this end, we compute the Jacobian $\mathbf{J}:=\nabla\bar{F}$. It is easy to see that $\mathbf{Js}=0$ if and only if

$$\varepsilon\sum_{l=1}^{j}a_ls_ls_j-(s_{j+1}-s_j)=0, j=1,...,K-1, \varepsilon\sum_{l=1}^{N}a_ls_l=0 \tag{6.10}$$

where

$$a_l=-\frac{a}{(t_l^0)^2}+\frac{1}{K}(A^{'}(t_l^0)+B^{'}(t_l^0))+\frac{2(l-1)}{K}B^{'}(t_l^0)+\frac{2(K-l)}{K}A^{'}(t_l^0)$$

Since $\sum_{l=1}^{N}a_l=M_R^{'}(r_0)+o(1)\neq 0$, we see that (6.10) implies then $s_1=...=s_K=0$.

The above argument shows that $\hat{F}$ and hence $\mathbf{F}$ has a unique and nondegenerate zero in Λ. Therefore $deg\ (\mathbf{F}, \Lambda) \neq 0$. So $deg\ (\beta_\varepsilon, \Lambda) \neq 0$. A zero of β_ε, denoted by t_ε, is guaranteed, which produces a solution $u_\varepsilon = W + \phi_{\varepsilon,\mathbf{t}_\varepsilon}$ to (5.1)-(5.2). It is easy to verify that u_ε satisfies all the properties of Theorem 1.3.

□

We now **prove Proposition 6.1**. Observe that $\phi_{\varepsilon,\mathbf{t}}$ satisfies (5.5).

Multiplying equation (5.5) by w'_{ε,t_j} and integrating over $\mathbb{I}_\varepsilon$, we obtain

(6.11)
$$\beta_{\varepsilon,j}(\mathbf{t})\int_{\mathbb{I}_\varepsilon} Z_{\varepsilon,t_j}w'_{\varepsilon,t_j} + \sum_{k\neq j} O(\varepsilon^\sigma|\beta_{\varepsilon,k}|) = \int_{\mathbb{I}_\varepsilon} \mathbb{L}_{\varepsilon,\mathbf{t}}[\phi_{\varepsilon,\mathbf{t}}]w'_{\varepsilon,t_j} + \int_{\mathbb{I}_\varepsilon}(-\mathbb{E}_\varepsilon w'_{\varepsilon,t_j}) + \int_{\mathbb{I}_\varepsilon}(-\mathbb{M}_\varepsilon[\phi_{\varepsilon,\mathbf{t}}]w'_{\varepsilon,t_j}).$$

The left hand of of (6.11) can be computed by (3.13):

$$\sum_{l=1}^{K}\beta_{\varepsilon,l}(\mathbf{t})\int_{\mathbb{I}_\varepsilon} Z_{\varepsilon,t_l}w'_{\varepsilon,t_j} = -\beta_{\varepsilon,j}(\mathbf{t})\int_{\mathbb{R}}((p-1)w^{p-1}(w')^2) + O(\varepsilon\sum_{l\neq j}|\beta_{\varepsilon,l}(\mathbf{t})|). \tag{6.12}$$

We estimate each term on the right hand side of (6.11). For the first term, we use integration by parts:

$$\int_{\mathbb{I}_\varepsilon}\mathbb{L}_{\varepsilon,\mathbf{t}}[\phi_{\varepsilon,\mathbf{t}}]w'_{\varepsilon,t_j} = \int_{\mathbb{I}_\varepsilon}\left[\phi''_{\varepsilon,\mathbf{t}} - \phi_{\varepsilon,\mathbf{t}} + pw^{p-1}_{\varepsilon,\mathbf{t}}\phi_{\varepsilon,\mathbf{t}}\right]w'_{\varepsilon,t_j} + O(\varepsilon^{1+\sigma})$$
$$= \int_{\mathbb{I}_\varepsilon}\left[w'''_{\varepsilon,t_j} - w'_{\varepsilon,t_j} + pw^{p-1}_{\varepsilon,t_j}w'_{\varepsilon,t_j}\right]\phi_{\varepsilon,\mathbf{t}} = O(\varepsilon^{1+\sigma}).$$

The last term in (6.11) gives, using (5.10),

$$\int_{\mathbb{I}_\varepsilon}\mathbb{M}_{\varepsilon,t}[\phi_{\varepsilon,\mathbf{t}}]w'_{\varepsilon,t_j} = O(\varepsilon^{1+\sigma}+\varepsilon^{\sigma(1+\tau)}) = O(\varepsilon^{\sigma(1+\tau)}). \tag{6.13}$$

It remains to compute the second term at the right hand side of (6.11): by (5.8)

$$\int_{\mathbb{I}_\varepsilon}(-E_\varepsilon)w'_{\varepsilon,t_j} = \xi_{\varepsilon,j}^{\frac{q}{p-1}}\int_{\mathbb{I}_{\varepsilon,t_j}}\frac{\varepsilon(N-1)}{t_j+\varepsilon y}(w'_{\varepsilon,t_j})^2 + O(\varepsilon^2)$$
$$+\int_{\mathbb{I}_\varepsilon}\xi_{\varepsilon,j}^{\frac{q}{p-1}}w^p_{\varepsilon,t_j}(1-\frac{\xi^q_{\varepsilon,j}}{(\mathbb{T}[W])^q})w'_{\varepsilon,t_j}$$
$$+\int_{\mathbb{I}_\varepsilon}\frac{W^p - \sum_{l=1}^K\xi^{\frac{qp}{p-1}}w^p_{\varepsilon,t_l}}{(\mathbb{T}[w^m_{\varepsilon,t}])^q}w'_{\varepsilon,t_j} + O(\varepsilon^2).$$

It is easy to see that

$$\int_{\mathbb{I}_{\varepsilon,t}}\frac{\varepsilon(N-1)}{t_j+\varepsilon y}(w'_{\varepsilon,t_j})^2 = \varepsilon\frac{N-1}{t_j}\int_{\mathbb{R}}(w')^2 + O(\varepsilon^2). \tag{6.14}$$

Then we estimate, using Lemma 4.2,

$$\int_{\mathbb{I}_\varepsilon} w'_{\varepsilon,t_j} w^p_{\varepsilon,t_j} \left(\frac{1-(\mathbb{T}[W^m])^q}{(\mathbb{T}[W])^q}\right) = -q\varepsilon \sum_{l=1}^{K} \left(\int_{\mathbb{R}} w' w^p \rho\left(y + \frac{t_j - t_l}{\varepsilon}\right) + O(\varepsilon) \right). \tag{6.15}$$

By the computation in [20], we have

$$\int_{\mathbb{R}} w^m \int_{\mathbb{R}} w' w^p \rho(y) = -\frac{1}{2(p+1)} \left(\frac{J'_{2,R}(t)}{J_{2,R}(t)} + \frac{J'_1(t)}{J_1(t)} \right) \int_{\mathbb{R}} w^{p+1}(y) \int_R w^m. \tag{6.16}$$

On the other hand, for $l < j, t_l < t_j$, we have

$$\rho\left(y + \frac{t_j - t_l}{\varepsilon}\right) = \frac{J'_1(t_l)}{J_1(t_l)}\left(y + \frac{t_j - t_l}{\varepsilon}\right) + O(\varepsilon|y|) \tag{6.17}$$

and hence

$$\int_{\mathbb{R}} w' w^p \rho\left(y + \frac{t_j - t_l}{\varepsilon}\right) = \frac{J'_1(t_l)}{J_1(t_l)} \int_{\mathbb{R}} y w' w^p = -\left(\int_{\mathbb{R}} \frac{w^{p+1}}{p+1}\right) \frac{J'_1(t_l)}{J_1(t_l)} + O(\varepsilon) \tag{6.18}$$

$$-\left(\int_{\mathbb{R}} \frac{w^{p+1}}{p+1}\right) \frac{J'_1(t_j)}{J_1(t_j)} + O(\varepsilon|\log\varepsilon|)$$

Similarly, for $l > j, t_l > t_j$, we have

$$\int_{\mathbb{R}} w' w^p \rho\left(y + \frac{t_j - t_l}{\varepsilon}\right) = -\left(\int_{\mathbb{R}} \frac{w^{p+1}}{p+1}\right) \frac{J'_{2,R}(t_j)}{J_{2,R}(t_j)} + O(\varepsilon) \tag{6.19}$$

Finally, we have

$$\int_{\mathbb{I}_\varepsilon} \frac{W^p - \sum_{l=1}^K \xi_{\varepsilon,l}^{\frac{qp}{p-1}} w^p_{\varepsilon,t_l}}{(\mathbb{T}[W^m])^q} w'_{\varepsilon,t_j}$$

$$= \int_{\mathbb{I}_\varepsilon} p w^{p-1}_{\varepsilon,t_j} \sum_{l\neq j} \xi_{\varepsilon,l}^{\frac{q}{p-1}} w_{\varepsilon,t_l} w'_{\varepsilon,t_j} + O(\varepsilon^2)$$

$$= -\xi_{\varepsilon,j}^{\frac{q}{p-1}} \int_{\mathbb{I}_\varepsilon} w^p_{\varepsilon,t_j} \sum_{l\neq j} w'_{\varepsilon,t_l} + O(\varepsilon^2)$$

$$= \sum_{l\neq j} \frac{t_j - t_l}{|t_j - t_l|} e^{-\frac{|t_j - t_l|}{\varepsilon}} \left(\int_{\mathbb{R}} w^p e^{-y}\right) + O(\varepsilon^2) \tag{6.20}$$

Combining all together, we arrive at

$$-\beta_{\varepsilon,j}(\mathbf{t}) \xi_0^{-\frac{q}{p-1}} \int_{\mathbb{R}} ((p-1) w^{p-1} (w')^2) = \varepsilon \frac{N-1}{t_j} \int_{\mathbb{R}} (w')^2$$

$$+\varepsilon \frac{q}{2(p+1)} \int_{\mathbb{R}} w^{p+1} \left(\frac{J'_{2,R}(t_j)}{J_{2,R}(t_j)} + \frac{J'_1(t_j)}{J_1(t_j)} \right)$$

$$+\varepsilon\sum_{l>j}\frac{q}{(p+1)}\int_{\mathbb{R}}w^{p+1}(\frac{J_{2,R}'(t_j)}{J_{2,R}(t_j)})+\varepsilon\sum_{l<j}\frac{q}{(p+1)}\int_{\mathbb{R}}w^{p+1}(\frac{J_1'(t_j)}{J_1(t_j)})$$

$$+\sum_{l\neq j}\frac{t_j-t_l}{|t_j-t_l|}e^{-\frac{|t_j-t_l|}{\varepsilon}}(\int_{\mathbb{R}}w^pe^{-y})+O(\varepsilon^{\sigma(1+\tau)})$$

(6.21)

$$=\varepsilon b_0\left[\frac{a}{t_j}+\frac{1}{K}(A(t_j)+B(t_j))+\frac{2(j-1)}{K}B(t_j)+\frac{2(K-j)}{K}A(t_j)\right]+c_0\sum_{l\neq j}\frac{t_j-t_l}{|t_j-t_l|}e^{-\frac{|t_j-t_l|}{\varepsilon}}+O(\varepsilon^{\sigma(1+\tau)})$$

where $b_0>0, c_0>0$.

This proves the Proposition.

□

References

[1] A.Ambrosetti, A. Malchiodi and W.-M. Ni, Singularly perturbed elliptic equations with symmetry: existence of solutions concentrating on spheres, Part I, *Comm. Math. Phys.* 235 (2003), no. 3, 427-466.

[2] A.Ambrosetti, A. Malchiodi and W.-M. Ni, Singularly perturbed elliptic equations with symmetry: existence of solutions concentrating on spheres, Part II, *Indiana Univ. Math. J.* 53 (2004), no. 2, 297-329.

[3] M. Abramowitz and I.A. Stegun, *Handbook of Mathematical Functions with Formulas, Graphs, and Mathematical Tables*, Washington, National Bureau of Standards Applied Mathematics, 1964.

[4] X. Chen, M. del Pino and M. Kowalczyk, The Gierer-Meinhardt system: the breaking of symmetry of homoclinics and multi-bump ground states, *Comm. Contemp. Math.* , 3 (3), (2001) 419–439.

[5] M. del Pino, M. Kowalczyk and J. Wei, Multi-bump ground states of the Gierer-Meinhardt system in $\mathbb{R}^2$, *Ann. Inst. H. Poincaré Anal. Non Linéaire* 20(2003), 53-85.

[6] A. Doelman, R. A. Gardner and T.J. Kaper, Large stable pulse solutions in reaction-diffusion equations, *Indiana Univ. Math. J* 50 (1), (2001) 443–507.

[7] A. Doelman, T.J. Kaper, and H. van der Ploeg, Spatially periodic and aperiodic multi-pulse patterns in the one-dimensional Gierer-Meinhardt equation, *Methods Appl. Anal.* 8(2001), 387-414.

[8] E.N. Dancer and S. Yan, Multi-layer solutions for an elliptic problem, *J. Diff. Eqns.* 194 (2003), 382-405.

[9] A. Gierer and H. Meinhardt, A theory of biological pattern formation, *Kybernetik (Berlin)* 12 (1972), 30-39.

[10] D. Iron, M. Ward, and J. Wei, The stability of spike solutions to the one-dimensional Gierer-Meinhardt model, *Phys. D* 150 (2001), 25-62.

[11] A. Malchiodi, M. Montenegro, Boundary concentration phenomena for a singularly perturbed elliptic problem, *Comm. Pure Appl. Math.* 55(2002), 1507-1568.

[12] A. Malchiodi, W. M. Ni, J. Wei. Multiple clustered layer solutions for semilinear Neumann problems on a ball, *Ann. Inst. H. Poincaré Anal. Non Linéaire* 22(2005), 143-163.

[13] H. Meinhardt, The algorithmic beauty of sea shells, Springer, Berlin, Heidelberg, 2nd edition, 1998.

[14] H. Meinhardt, Models of Biological Pattern Formation, Acad. Press, London, 1982.

[15] W.-M. Ni, Diffusion, cross-diffusion, and their spike-layer steady states, *Notices of Amer. Math. Soc.* 45 (1998), 9-18.

[16] W.-M. Ni, *Qualitative properties of solutions to elliptic problems.* Stationary partial differential equations. Vol. I, 157-233, Handb. Differ. Equ., North-Holland, Amsterdam, 2004.

[17] W.-M. Ni and I. Takagi, Point-condensation generated by a reaction-diffusion system in axially symmetric domains, *Japan J. Industrial Appl. Math.* 12 (1995), 327-365.
[18] W.-M. Ni, I. Takagi and E. Yanagida, Stability analysis of point-condensation solutions to a reaction-diffusion system proposed by Gierer-Meinhardt, *Tohoku Math. J.*, to appear.
[19] W.-M. Ni, I. Takagi and E. Yanagida, Stability of least energy patterns of the shadow system for an activator-inhibitor model, *Japan J. Industrial Appl. Math.* 18(2)(2001), 259-272.
[20] W.-M. Ni and J. Wei, On positive solutions concentrating on spheres for the Gierer-Meinhardt system, *J. Diff. eqns.* 221(2006), 158-189.
[21] K. Nakashima and K. Tanaka, Clustering layers and boundary layers in spatially inhomogeneous phase transition problems, *Ann. Inst. H. Poincaré Anal. Non Linéaire* 20 (2003), no. 1, 107-143.
[22] I. Takagi, Point-condensation for a reaction-diffusion system, *J. Diff. Eqns.* 61 (1986), 208-249.
[23] A. M. Turing, The chemical basis of morphogenesis, *Phil. Trans. Roy. Soc. Lond. B* 237 (1952), 37-72.
[24] J. Wei and M. Winter, On the two-dimensional Gierer-Meinhardt system with strong coupling, *SIAM J.Math.Anal.* 30 (1999), 1241-1263.
[25] J. Wei and M. Winter, On multiple spike solutions for the two-dimensional Gierer-Meinhardt system: the strong coupling case, *J. Diff. Eqns.* 178(2002), 478-518.
[26] J. Wei and M. Winter, Spikes for the two-dimensional Gierer-Meinhardt system: the weak coupling case, *J. Nonlinear Science* 6(2001), 415-458.

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