INTERFACE FOLIATION NEAR MINIMAL SUBMANIFOLDS IN RIEMANNIAN MANIFOLDS WITH POSITIVE RICCI CURVATURE

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Abstract. Let $(\mathcal{M},\tilde{g})$ be an N-dimensional smooth compact Riemannian manifold. We consider the singularly perturbed Allen-Cahn equation

$$\varepsilon^2\Delta_{\tilde{g}}u\,+\,(1-u^2)u\,=\,0\quad\text{in }\mathcal{M},$$

where ε is a small parameter. Let $\mathcal{K}\subset\mathcal{M}$ be an $(N-1)$-dimensional smooth minimal submanifold that separates $\mathcal{M}$ into two disjoint components. Assume that $\mathcal{K}$ is non-degenerate in the sense that it does not support non-trivial Jacobi fields, and that $|A_{\mathcal{K}}|^2+\mathrm{Ric}_{\tilde{g}}(\nu_{\mathcal{K}},\nu_{\mathcal{K}})$ is positive along $\mathcal{K}$. Then for each integer $m\geq 2$, we establish the existence of a sequence $\varepsilon=\varepsilon_j\to 0$, and solutions u_ε with m-transition layers near $\mathcal{K}$, with mutual distance $O(\varepsilon|\ln\varepsilon|)$.

Contents

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1. Introduction

In the gradient theory of phase transitions by Allen-Cahn [2], two phases of a material, $+1$ and -1 coexist in a region $\Omega \subset \mathbb{R}^N$ separated by an $(N-1)$-dimensional interface. The phase is idealized as a smooth ε-regularization of the discrete function, which is selected as a critical point of the energy

$$I_\varepsilon(u) = \int_\Omega \frac{\varepsilon}{2}|\nabla u|^2 + \frac{1}{4\varepsilon}(1-u^2)^2,$$

where $\varepsilon > 0$ is a small parameter. While any function with values ± 1 minimizes exactly the second term, the presence of the gradient term conveys a balance in which the interface is selected asymptotically as stationary for perimeter. The energy I_ε may be regarded as an *ε-relaxation* of the surface area: indeed, in [25] it is established that a sequence of local minimizers u_ε, with uniformly bounded energy, must converge in L^1_{loc}-sense to a function of the form $\chi_E - \chi_{E^c}$ so that ∂E locally minimizes perimeter, thus being a (generalized) minimal surface. This is the starting point of the *Γ-convergence theory*, in which the constraint of I_ε to a suitable class of separating-phase functions, converges to the perimeter function of the interface. Indeed, analogous assertions hold true for general families of critical points, and for stronger notions of interface convergence, see [6, 29, 33]. The principle above applies to modeling phase transition phenomena in many contexts: material science, superconductivity, population dynamics and biological pattern formation, see for instance [31] and references therein.

It is natural to consider situations in which phase transitions take place in a manifold rather than in a subset of Euclidean space. In this paper we consider a compact N-dimensional Riemannian manifold $(\mathcal{M}, \tilde{g})$, and want to investigate critical points in $H^1(\mathcal{M})$ of the functional

$$J_\varepsilon(u) = \int_{\mathcal{M}} \frac{\varepsilon}{2}\big|\nabla_{\tilde{g}}\tilde{u}\big|^2 + \frac{1}{4\varepsilon}\big(1-\tilde{u}^2\big)^2,$$

with sharp transitions between -1 and 1 taking place near a $(N-1)$-dimensional minimal submanifolds of $\mathcal{M}$. Critical points of J_ε correspond precisely to classical solutions of the Allen-Cahn equation in $\mathcal{M}$,

$$\varepsilon^2 \Delta_{\tilde{g}}\tilde{u} + \big(1-\tilde{u}^2\big)\tilde{u} = 0 \quad \text{in } \mathcal{M}, \tag{1.1}$$

where $\Delta_{\tilde{g}}$ is the Laplace-Beltrami operator on $\mathcal{M}$.

We let in what follows $\mathcal{K}$ be a minimal $(N-1)$-dimensional embedded submanifold of $\mathcal{M}$, which divides $\mathcal{M}$ into two open components $\mathcal{M}_\pm$. (The latter condition is not needed in some cases.) The *Jacobi operator* $\mathcal{J}$ of $\mathcal{K}$, corresponds to the second variation of N-volume along normal perturbations of $\mathcal{K}$ inside $\mathcal{M}$: given any smooth small function v on $\mathcal{K}$, let us consider the manifold $\mathcal{K}(v)$, the normal graph on $\mathcal{K}$ of the function v, namely the image of $\mathcal{K}$ by the map $p \in \mathcal{K} \mapsto \exp_p\big(v(p)\nu_{\mathcal{K}}(p)\big)$. If $H(v)$ denotes the mean curvature of $\mathcal{K}(v)$, defined as the arithmetic mean of the principal curvatures, then the linear operator $\mathcal{J}$ is the differential of the map $v \mapsto nH(v)$ at $v = 0$. More explicitly, it can be shown that

$$\mathcal{J}\psi = \Delta_{\mathcal{K}}\psi + |A_{\mathcal{K}}|^2\psi + \mathrm{Ric}_{\tilde{g}}(\nu_{\mathcal{K}}, \nu_{\mathcal{K}})\psi, \tag{1.2}$$

where $\Delta_{\mathcal{K}}$ is the Laplace-Beltrami operator on $\mathcal{K}$, $|A_{\mathcal{K}}|^2$ denotes the norm of the second fundamental form of $\mathcal{K}$, $\mathrm{Ric}_{\tilde{g}}$ is the Ricci tensor of $\mathcal{M}$ and $\nu_{\mathcal{K}}$ is a unit normal to $\mathcal{K}$. We will briefly review these concepts in Section 2.

The minimal submanifold $\mathcal{K}$ is said to be *nondegenerate* if the are no nontrivial smooth solutions to the homogeneous problem

$$\mathcal{J}\psi = 0 \quad \text{in } \mathcal{K}. \tag{1.3}$$

This condition implies that $\mathcal{K}$ is isolated as a minimal submanifold of $\mathcal{M}$.

In [28], Pacard and Ritoré assume that $\mathcal{K}$ is non-degenerate and, and proved that there exists a solution u_ε to equation (1.1) with values close to ± 1 inside $\mathcal{M}_\pm$, whose (sharp) 0-level set is a smooth manifold which lies ε-close to $\mathcal{K}$. More precisely, let $w(z) := \tanh\left(\frac{z}{\sqrt{2}}\right)$ be the unique solution of the problem

$$w'' + w - w^3 = 0 \quad \text{in } \mathbb{R},\ w(0) = 0,\ w(\pm\infty) = \pm 1, \tag{1.4}$$

and denote by c_* its total energy, namely

$$c_* := \int_{\mathbb{R}} \frac{1}{2}|w'|^2 + \frac{1}{4}(1-w^2)^2.$$

Then the solution u_ε in [28] resembles near $\mathcal{K}$ the function $w(t/\varepsilon)$, where t is a choice of signed geodesic distance to Γ. In particular

$$J_\varepsilon(u_\varepsilon) \ \to\ c_* \,|\mathcal{K}|\,.$$

In this paper we describe a new phenomenon induced by the presence of positive curvature in the ambient manifold $\mathcal{M}$: in addition to non-degeneracy of $\mathcal{K}$, let us assume that

$$K \ :=\ |A_{\mathcal{K}}|^2 + \mathrm{Ric}_{\tilde{g}}(\nu_{\mathcal{K}}, \nu_{\mathcal{K}}) > 0 \quad \text{on } \mathcal{K}. \tag{1.5}$$

Then, besides the solution by Pacard and Ritoré, there are solutions with *multiple interfaces* collapsing onto $\mathcal{K}$. In fact, given any integer $m \ge 2$, we find a solution u_ε such that $u_\varepsilon^2 - 1$ approaches 0 in $\mathcal{M}_\pm$ as $\varepsilon \to 0$, with zero level set constituted by m smooth components $O(\varepsilon|\log\varepsilon|)$ distant one to each other and to $\mathcal{K}$, and such that

$$J_\varepsilon(u_\varepsilon) \ \to\ m c_* \,|\mathcal{K}|\,.$$

Condition (1.5) is satisfied automatically if the manifold $\mathcal{M}$ has non-negative Ricci curvature. If $N = 2$, K corresponds simply to the Gauss curvature of $\mathcal{M}$ measured along the geodesic $\mathcal{K}$.

The nature of these solutions is drastically different from the single-interface solution by Pacard and Ritoré[28]. They are actually defined only if ε satisfies a *nonresonance condition* in ε. In fact, in the construction ε must remain suitably away from certain values where a shift in Morse index occurs. We expect that the solutions we find have a Morse index $O(|\log\varepsilon|^a)$ for some $a > 0$ as critical points of J_ε, while the single interface solution is likely to have its Morse index uniformly bounded by the index of $\mathcal{K}$ (namely the number of negative eigenvalues of the operator $\mathcal{J}$).

Theorem 1. *Assume that $\mathcal{K}$ is nondegenerate and embedded, and that condition* (1.5) *is satisfied. Then, for each $m \geq 2$, there exists a sequence of values $\varepsilon = \varepsilon_j \to 0$ such that problem* (1.1) *has a solution u_ε such that $u_\varepsilon^2 - 1 \to 0$ uniformly on compact subsets of $\mathcal{M}_\pm$, while near $\mathcal{K}$, we have*

$$u_\varepsilon = \sum_{\ell=1}^{m} w\left(\frac{\tilde z - \varepsilon f_\ell(\tilde y)}{\varepsilon}\right) + \frac{1}{2}\left((-1)^{m-1} - 1\right) + o(1),$$

where $(\tilde y, \tilde z)$ are the Fermi coordinates defined near $\mathcal{K}$ through the exponential map (see Section 2.1), and the functions f_ℓ satisfy

$$f_\ell(\tilde y) = \left(\ell - \frac{m+1}{2}\right)\left[\sqrt{2}\log\frac{1}{\varepsilon} - \frac{1}{\sqrt{2}}\log\log\frac{1}{\varepsilon}\right] + O(1). \tag{1.6}$$

Moreover, when $N = 2$, there exist positive numbers $\nu_1, \ldots, \nu_{m-1}$ such that given $c > 0$ and all sufficiently small $\varepsilon > 0$ satisfying

$$\left|\frac{1}{\log\frac{1}{\varepsilon}} - \frac{\nu_i}{j^2}\right| > \frac{c}{j^3}, \qquad \text{for all} \quad i = 1, \ldots, m-1, \quad j = 1, 2, \ldots. \tag{1.7}$$

a solution u_ε with the above properties exists.

We observe that the same result holds if m is *even* and $\mathcal{M} \setminus \mathcal{K}$ consists of just one component. Thus the condition that $\mathcal{K}$ divides $\mathcal{M}$ into two connected components is not essential in general.

As we will see in the course of the proof, the equilibrium location of the interfaces is asymptotically governed by a small perturbation of the *Jacobi-Toda system*

$$\varepsilon^2\Big(\Delta_{\mathcal{K}} f_j + \big(|A_{\mathcal{K}}|^2 + \mathrm{Ric}_{\tilde g}(\nu_{\mathcal{K}}, \nu_{\mathcal{K}})\big) f_j\Big) - a_0\left[e^{-(f_j - f_{j-1})} - e^{-(f_{j+1} - f_j)}\right] = 0, \tag{1.8}$$

on $\mathcal{K}$, $j = 1, \ldots, m$, with the conventions $f_0 = -\infty$, $f_{m+1} = +\infty$. Heuristically, the interface foliation near $\mathcal{K}$ is possible due to a balance between the interfacial energy, which decreases as the interfaces approach each other, and the fact that the length or area of each individual interface increases as the interface is closer to $\mathcal{K}$ since $\mathcal{M}$ is positively curved near $\mathcal{K}$.

What is unexpected, is the need of a *nonresonance condition* in order to solve the Jacobi-Toda system. A question which is of independent interest is the solvability of the Jacobi-Toda system without the condition (1.5). Similar resonance has been observed the problem of building foliations of a neighborhood of a geodesic by CMC tubes considered in [17, 22]. This has also been the case for (simple) concentration phenomena for various elliptic problems, see [8, 16, 19, 20].

Our result deals with situations in which the minimal submanifold is locally but not globally area minimizing. In fact, since condition (1.5) holds, the Jacobi operator has at least one negative eigenvalue, and near $\mathcal{K}$, $\mathcal{M}$ cannot have parabolic points. In the case of a bounded domain Ω of $\mathbb{R}^2$ under Neumann boundary conditions, a multiple-layer solution near a non-minimizing straight segment orthogonal to the boundary was built in [9]. In ODE cases for the Allen-Cahn equation, clustering interfaces had been previously observed in [7, 26, 27]. No resonance phenomenon is present in those situations, constituting a major qualitative difference with the current setting.

The method consists of linearizing the equation around the approximation

$$u_0(x,z) = \sum_{\ell=1}^{m} w\left(\frac{\tilde z - \varepsilon f_\ell(\tilde y)}{\varepsilon}\right) + \frac{1}{2}\left((-1)^{m-1} - 1\right),$$

and then consider a projected form of the equation which can be solved boundedly after finding a satisfactory linear theory, and then applying the contraction mapping principe. In that process the functions f_j are left as arbitrary functions under some growth constraints. At the last step one gets an equation which can be described as a small perturbation of the Jacobi-Toda system

We do not expect that interface foliation occurs if the limiting interface is a minimizer of the perimeter since in such a case both perimeter of the interfaces and their interactions decrease the energy, so no balance for their equilibrium locations is possible. On the other hand, negative Gauss curvature seems also prevent interface foliation. This is suggested by a version of De Giorgi-Gibbons conjecture for problem (1.1) with $\mathcal{M}$ the hyperbolic space, established in [3].

2. Geometric background and the ansatz

In the first preliminary part of this section, we list some necessary notions from differential geometry: Fermi coordinates near a submanifold of $\mathcal{M}$, minimal submanifold, as well as Laplace-Beltrami and Jacobi operators. We then express the problem in a suitable form, define an approximate solution and estimate its error.

2.1. **Local coordinates.** Let $\mathcal{M}$ be an $N \geq 2$-dimensional smooth compact Riemannian manifold without boundary with given metric $\tilde g$. We assume that $\mathcal{K}$ is an $N-1$ dimensional submanifold of $\mathcal{M}$. For each given point $p \in \mathcal{K}$, $T_p\mathcal{M}$ splits naturally as

$$T_p\mathcal{M} = T_p\mathcal{K} \oplus N_p\mathcal{K},$$

where $T_p\mathcal{K}$ is the tangent space to $\mathcal{K}$ and $N_p\mathcal{K}$ is its normal complement, which spanned respectively by orthonormal bases $\{E_i : i = 1, \cdots, N-1\}$ and $\{E_N\}$. More generally, we have for the tangent and normal bundles over $\mathcal{K}$ the decomposition

$$T\mathcal{M} = T\mathcal{K} \oplus N\mathcal{K}.$$

Let us denote by $\bigtriangledown$ the connection induced by the metric $\tilde g$ and by $\bigtriangledown^N$ the corresponding normal connection on the normal bundle.

Notation: Up to section 2.4, we shall always use the following convention for the indices

$$i, j, k, l \cdots \in \{1, 2, \cdots, N-1\}, \quad a, b, c, \cdots \in \{1, 2, \cdots, N\}.$$

Given $p \in \mathcal{K}$, we use some geodesic coordinates $\tilde y$ centered at p. More precisely, in a neighborhood of p in $\mathcal{K}$, we consider normal geodesic coordinates

$$\tilde y = Y_p(\tilde{\mathrm{y}}) := \exp_p^{\mathcal{K}}(\tilde{\mathrm{y}}_i E_i), \quad \tilde{\mathrm{y}} = (\tilde{\mathrm{y}}_1, \cdots, \tilde{\mathrm{y}}_{N-1}) \in \mathcal{V}, \tag{2.1}$$

where $\exp^{\mathcal{K}}$ is the exponential map on $\mathcal{K}$ and summation over repeated indices is understood. $\mathcal{V}$ is a neighborhood of the origin in $\mathbb{R}^{N-1}$.

This yields the coordinate vector fields $X_i = f_*(\partial_i)$, $i = 1, \cdots, N-1$ where $f(\tilde{\mathrm{y}}) = Y_p(\tilde{\mathrm{y}})$. For any $E \in T_p\mathcal{K}$, the curve

$$s \to \gamma_E(s) = \exp_p^{\mathcal{K}}(sE)$$

is a geodesic in $\mathcal{K}$, so that

$$\bigtriangledown_{X_i} X_j|_p \in N_p\mathcal{K} \quad \text{for any } i,j = 1,\cdots,N-1.$$

We recall that the Christoffel symbols Γ_{ij}^N, $i,j=1,\cdots,N-1$ are given by

$$\nabla_{X_i} X_j|_p \;=\; \Gamma_{ij}^N E_N, \quad \text{i. e.} \quad \Gamma_{ij}^N \;=\; \tilde{g}(\nabla_{X_i} X_j, E_N).$$

We also assume that at p the normal vector E_N is transported parallelly (with respect to $\bigtriangledown^N$) through geodesics $\gamma_E(s)$ from p. This yields a frame field X_N for $N\mathcal{K}$ in a neighborhood of p which satisfies

$$\nabla_{X_i} X_N|_p \in T_p\mathcal{K}, \quad \text{i.e.} \quad \tilde{g}(\nabla_{E_i} E_N, E_N)|_p = 0, \quad i = 1,\cdots,N-1.$$

We define the numbers Γ_{iN}^j, $i,j=1,\cdots,N-1$, by

$$\nabla_{X_i} X_N|_p \;=\; \sum_{j=1}^{N-1} \Gamma_{iN}^j E_j, \quad \text{i.e.} \quad \Gamma_{iN}^j \;=\; \tilde{g}(\nabla_{X_i} X_N, E_j).$$

In a neighborhood of p in $\mathcal{M}$, we choose the *Fermi coordinates* $(\tilde{\mathrm{y}}, \tilde{z})$ on $\mathcal{M}$ defined by

$$\Phi^0(\tilde{\mathrm{y}}, \tilde{z}) = \exp_{Y_p(\tilde{\mathrm{y}})}(\tilde{z}E_N) \quad \text{with } (\tilde{\mathrm{y}}, \tilde{z}) = (\tilde{\mathrm{y}}_1,\cdots,\tilde{\mathrm{y}}_{N-1}, \tilde{z}) \in \mathcal{V} \times \big(-\delta_0, \delta_0\big), \tag{2.2}$$

where $\exp_{Y_p(\tilde{\mathrm{y}})}$ is the exponential map at $Y_p(\tilde{\mathrm{y}})$ in $\mathcal{M}$. We also have corresponding coordinate vector fields

$$X_i = \Phi^0_*(\partial_{\tilde{y}_i}), \quad X_N = \Phi^0_*(\partial_{\tilde{z}}).$$

By construction, $X_N|_p = E_N$.

2.2. **Taylor expansion of the metric.** In this section we will follow the notation and calculations of [17]. By our choice of coordinates and the Gauss Lemma, on $\mathcal{K}$ the metric $\tilde{g}$ splits in the following way,

$$\tilde{g}(p) \;=\; \sum_{i,j=1}^{N-1} \tilde{g}_{ij} \mathrm{d}\tilde{y}_i \otimes \mathrm{d}\tilde{y}_j \;+\; \tilde{g}_{NN} \mathrm{d}\tilde{z} \otimes \mathrm{d}\tilde{z}, \quad p \in \mathcal{K}. \tag{2.3}$$

As usual, the Fermi coordinates above are chosen so that the metric coefficients satisfy

$$\tilde{g}_{ab} = \tilde{g}(X_a,\, X_b) = \delta_{ab} \quad \text{at} \;\; p.$$

Furthermore, $\tilde{g}(X_i,\, X_N) = 0$ in some neighborhood of p in $\mathcal{K}$. Then

$$X_i \tilde{g}(X_j,\, X_N) = \tilde{g}(\bigtriangledown_{X_i} X_j,\, X_N) + \tilde{g}(X_j,\, \bigtriangledown_{X_i} X_N) \quad \text{on } \mathcal{K},$$

yield the identity

$$\Gamma_{ij}^N + \Gamma_{iN}^j = 0 \quad \text{at} \;\; p. \tag{2.4}$$

We denote by $\Gamma_i^j : N\mathcal{K} \to \mathbb{R}$, $i,j = 1,\cdots,N-1$, the 1-forms defined on the normal bundle of $\mathcal{K}$ as

$$\Gamma_i^j(E_N) \;=\; \tilde{g}(\bigtriangledown_{E_i} E_j, E_N). \tag{2.5}$$

The *second fundamental form* $A_{\mathcal{K}} : T\mathcal{K} \times T\mathcal{K} \to N\mathcal{K}$ of the submanifold $\mathcal{K}$ and its corresponding norm are then given by

$$A_{\mathcal{K}}(E_i, E_j) = \Gamma_i^j(E_N) E_N, \quad |A_{\mathcal{K}}|^2 = \sum_{i,j=1}^{N-1} \Big(\Gamma_i^j(E_N)\Big)^2. \tag{2.6}$$

For $X, Y, Z, W \in T\mathcal{M}$, the curvature operator and curvature tensor are respectively defined by the relations

$$R(X,Y,Z) = \bigtriangledown_X \bigtriangledown_Y Z - \bigtriangledown_Y \bigtriangledown_X Z - \bigtriangledown_{[X,Y]} Z, \tag{2.7}$$

$$R(X,Y,Z,W) = \tilde{g}\big(R(Z,W)Y, X\big). \tag{2.8}$$

The *Ricci tensor* of $(\mathcal{M}, \tilde{g})$ is defined by

$$\mathrm{Ric}_{\tilde{g}}(X,Y) = \tilde{g}^{ab} R(X, X_a, Y, X_b). \tag{2.9}$$

We now compute higher order terms in the Taylor expansions of the metric coefficients. The metric coefficients at $q = \Phi^0(0,\tilde{z})$ are given in terms of geometric data at $p = \Phi^0(0,0)$ and $|\tilde{z}| = \mathrm{dist}_{\tilde{g}}(p,q)$, which is expressed by the next lemmas, see Proposition 2.1 in [17] and the references therein.

Lemma 2.1. *At the point $q = \Phi^0(0,\tilde{z})$, the following expansions hold*

$$\bigtriangledown_{X_N} X_N = O(|\tilde{z}|) X_a, \tag{2.10}$$

$$\bigtriangledown_{X_i} X_j = \Gamma_i^j(E_N) X_N + O(|\tilde{z}|) X_a, \quad i,j = 1, \cdots, N-1, \tag{2.11}$$

$$\bigtriangledown_{X_i} X_N = \bigtriangledown_{X_N} X_i = \sum_{j=1}^{N-1} \Gamma_i^j(E_N) X_j + O(|\tilde{z}|) X_a, \quad i = 1, \cdots, N-1. \tag{2.12}$$

Lemma 2.2. *In the above coordinates $(\tilde{y}, \tilde{z})$, for any $i, j = 1, 2, \cdots, N-1$, we have*

$$\begin{aligned} \tilde{g}_{ij}(0,\tilde{z}) \;&=\; \delta_{ij} - 2\Gamma_i^j(E_N)\tilde{z} - R(X_N, X_j, X_N, X_i)|\tilde{z}|^2 \\ &\quad + \sum_{k=1}^{N-1} \Gamma_i^k(E_N)\Gamma_k^j(E_N)|\tilde{z}|^2 + O(|z|^3), \end{aligned} \tag{2.13}$$

$$\tilde{g}_{iN}(0,z) \;=\; O(|\tilde{z}|^2), \tag{2.14}$$

$$\tilde{g}_{NN}(0,z) \;=\; 1 + O(|\tilde{z}|^3). \tag{2.15}$$

2.3. **The Laplace-Beltrami and Jacobi operators.** If $(\mathcal{M}, \tilde{g})$ is an N-dimensional Riemannian manifold, the *Laplace-Beltrami operator* on $\mathcal{M}$ is defined in local coordinates by the formula

$$\Delta_{\mathcal{M}} = \frac{1}{\sqrt{\det \tilde{g}}}\, \partial_a\Big(\sqrt{\det \tilde{g}}\, \tilde{g}^{ab} \partial_b\Big), \tag{2.16}$$

where $\tilde{g}^{ab}$ denotes the inverse of the matrix $(\tilde{g}_{ab})$. Let $\mathcal{K} \subset \mathcal{M}$ be an $(N-1)$-dimensional closed smooth embedded submanifold associated with the metric $\tilde{g}_0$ induced from $(\mathcal{M}, \tilde{g})$. Let $\Delta_{\mathcal{K}}$ be the Laplace-Beltrami operator defined on $\mathcal{K}$.

Let us consider the space $C^\infty(N\mathcal{K})$, which identifies with that of all smooth normal vector fields on $\mathcal{K}$. Since $\mathcal{K}$ is a submanifold of codimension 1, then given a choice of orientation and unit normal vector field along $\mathcal{K}$, denoted by $\nu_{\mathcal{K}} \in N\mathcal{K}$, we can write $\Psi \in C^\infty(N\mathcal{K})$ as $\Psi = \phi \nu_{\mathcal{K}}$, where $\phi \in C^\infty(\mathcal{K})$.

For $\Psi \in C^\infty(N\mathcal{K})$, we consider the one-parameter family of submanifolds $t \to \mathcal{K}_{t,\Psi}$ given by

$$\mathcal{K}_{t,\Psi} \equiv \Big\{ \exp_{\tilde{y}}\big(t\Psi(\tilde{y})\big) : \tilde{y} \in \mathcal{K} \Big\}. \tag{2.17}$$

The first variation formula of the volume functional is defined as

$$\frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0}\mathrm{Vol}(\mathcal{K}_{t,\Psi}) = \int_{\mathcal{K}} <\Psi, \mathbf{h}>_N \,\mathrm{d}V_{\mathcal{K}}, \tag{2.18}$$

where $\mathbf{h}$ is the mean curvature vector of $\mathcal{K}$ in $\mathcal{M}$, $<\cdot,\cdot>_N$ denotes the restriction of $\tilde{g}$ to $N\mathcal{K}$, and $\mathrm{d}V_{\mathcal{K}}$ the volume element of $\mathcal{K}$.

The submanifold $\mathcal{K}$ is said to be *minimal* if it is stationary point for the volume functional, namely if

$$\frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0}\mathrm{Vol}(\mathcal{K}_{t,\Psi}) = 0 \quad \text{for any } \Psi \in C^\infty(N\mathcal{K}), \tag{2.19}$$

or equivalently by (2.18), if the mean curvature $\mathbf{h}$ is identically zero on $\mathcal{K}$. It is a standard fact that if $\Gamma_i^j(E_N)$ is as in (2.5), then

$$\mathcal{K} \text{ is minimal } \iff \sum_{i=1}^{N-1}\Gamma_i^i(E_N) = 0. \tag{2.20}$$

The *Jacobi operator* $\mathcal{J}$ appears in the expression of the second variation of the volume functional for a minimal submanifold $\mathcal{K}$

$$\frac{\mathrm{d}^2}{\mathrm{d}t^2}\Big|_{t=0}\mathrm{Vol}(\mathcal{K}_{t,\Psi}) = -\int_{\Gamma} <\mathcal{J}\Psi, \Psi>_N \,\mathrm{d}V_{\mathcal{K}} \quad \text{for any } \Psi \in C^\infty(N\mathcal{K}), \tag{2.21}$$

and is given by

$$\mathcal{J}\phi = -\Delta_{\mathcal{K}}\phi - \mathrm{Ric}_{\tilde{g}}(\nu_{\mathcal{K}},\nu_{\mathcal{K}})\phi - |A_{\mathcal{K}}|^2\phi, \tag{2.22}$$

where $\Psi = \phi\nu_{\mathcal{K}}$, as has been explained above.

The submanifold $\mathcal{K}$ is said to be *non-degenerate* if the Jacobi operator $\mathcal{J}$ is invertible, or equivalently if the equation $\mathcal{J}\phi = 0$ has only the trivial solution in $C^\infty(\mathcal{K})$

2.4. **Laplace-Beltrami Operator in Stretched Fermi Coordinates.** To construct the approximation to a solution of (1.1), which concentrates near $\mathcal{K}$, after rescaling, in $\mathcal{M}/\varepsilon$, we introduce stretched Fermi coordinates in the neighborhood of the point $\varepsilon^{-1}p \in \varepsilon^{-1}\mathcal{K}$ by

$$\Phi_\varepsilon(\mathtt{y}, z) = \frac{1}{\varepsilon}\Phi^0(\varepsilon\mathtt{y}, \varepsilon z), \quad (\mathtt{y}, z) = (\mathtt{y}_1, \cdots, \mathtt{y}_{N-1}, z) \in \varepsilon^{-1}\mathcal{V} \times \Big(-\frac{\delta_0}{\varepsilon}, \frac{\delta_0}{\varepsilon}\Big). \tag{2.23}$$

Obviously, in $\mathcal{M}_\varepsilon = \varepsilon^{-1}\mathcal{M}$ the new coefficients g_{ab}'s of the Riemannian metric, after rescaling, can be written as

$$g_{ab}(\mathtt{y}, z) = \tilde{g}_{ab}(\varepsilon\mathtt{y}, \varepsilon z), \quad a, b = 1, 2, \cdots, N.$$

Lemma 2.3. *In the above coordinates* $(\mathtt{y}, z)$*, for any* $i, j = 1, 2, \cdots, N-1$*, we have*

$$\begin{aligned} g_{ij}(\mathtt{y}, z) \;&=\; \delta_i^j - 2\varepsilon\Gamma_i^j(E_N)z - \varepsilon^2 R(X_j, X_N, X_N, X_i)|z|^2 \\ &\quad + \varepsilon^2\sum_{k=1}^{N-1}\Gamma_i^k(E_N)\Gamma_k^j(E_N)|z|^2 + O(|\varepsilon z|^3), \end{aligned} \tag{2.24}$$

$$g_{iN}(\mathtt{y}, z) \;=\; O(|\varepsilon z|^2), \tag{2.25}$$

$$g_{NN}(\mathtt{y}, z) \;=\; 1 + O(|\varepsilon z|^3). \tag{2.26}$$

Here $R(\cdot)$ *and* Γ_a^b *are computed at the point* $p \in \mathcal{K}$ *parameterized by* $(0, 0)$*.*

Now we will focus on the expansion of the Laplace-Beltrami operator defined by

$$
\begin{aligned}
\Delta_{\mathcal{M}_\varepsilon} &= \frac{1}{\sqrt{\det g}}\,\partial_a\Big(\,g^{ab}\,\sqrt{\det g}\,\partial_b\,\Big)\\
&= g^{ab}\,\partial_a\partial_b\,+\,(\partial_a g^{ab})\,\partial_b\,+\,\frac{1}{2}\,\partial_a\big(\log\,(\det g)\,\big)\,g^{ab}\,\partial_b.
\end{aligned}
\tag{2.27}
$$

Using the assumption that the submanifold $\mathcal{K}$ is minimal as in formula (2.20), direct computation gives that

$$
\det g = 1-\varepsilon^2 K(\varepsilon y)z^2 + O(\varepsilon^3|z|^3),
$$

where we have, using (2.6) and (2.9), denoted

$$
K = \mathrm{Ric}_{\tilde g}(\nu_{\mathcal{K}},\nu_{\mathcal{K}}) + |A_{\mathcal{K}}|^2. \tag{2.28}
$$

This gives

$$
\log\,(\det g) = -\varepsilon^2 K(\varepsilon y)z^2 + O(\varepsilon^3|z|^3).
$$

Hence, we have the expansion

$$
\Delta_{\mathcal{M}_\varepsilon} = \partial_{zz}\,+\,\Delta_{\mathcal{K}_\varepsilon}\,+\,\varepsilon^2 zK(\varepsilon y)\,\partial_z\,+\,B \tag{2.29}
$$

where the operator B has the form

$$
B\,=\,\varepsilon z\,a^1_{ij}\,\partial_{ij}\,+\,\varepsilon^2 z^2\,a^2_{iN}\,\partial_{iz}\,+\,\varepsilon^3 z^3\,a^3_{NN}\,\partial_{zz}\,+\,\varepsilon^2 z\,b^1_i\partial_i\,+\,\varepsilon^3 z^2\,b^2_N\,\partial_z\,. \tag{2.30}
$$

and all the coefficients are smooth functions defined on a neighborhood of $\mathcal{K}$ in $\mathcal{M}$, evaluated at $(\varepsilon y,\varepsilon z)$.

2.5. **The local approximate solution.** If we set $u(x) := \tilde u(\varepsilon x)$, then problem (1.1) is thus equivalent to

$$
\Delta_{\mathcal{M}_\varepsilon}\,u\,+\,F(u) = 0 \quad \text{ in } \mathcal{M}_\varepsilon, \tag{2.31}
$$

where $F(u)\,\equiv\,u\,-\,u^3$. In the sequel, we denote by $\mathcal{M}_\varepsilon$ and $\mathcal{K}_\varepsilon$ the ε^{-1}-dilations of $\mathcal{M}$ and $\mathcal{K}$.

To define the approximate solution we observe the heteroclinic solution to (1.4) has the asymptotic properties

$$
\begin{aligned}
w(z)\,-\,1 &=\,-2\,e^{-\sqrt{2}\,|z|}\,+\,O\big(e^{-2\sqrt{2}\,|z|}\big), \quad z>1,\\
w(z)\,+\,1 &=\,2\,e^{-\sqrt{2}\,|z|}\,+\,O\big(e^{-2\sqrt{2}\,|z|}\big), \quad z<-1,\\
w'(z) &=\,2\sqrt{2}\,e^{-\sqrt{2}\,|z|}\,+\,O\big(e^{-2\sqrt{2}\,|z|}\big), \quad |z|>1,
\end{aligned}
\tag{2.32}
$$

For a fixed integer $m\geq 2$, we assume that the location of the m phase transition layers are characterized in the coordinate (y,z) defined in (2.23) by the functions $z\,=\,f_j(\varepsilon y)$, $1\leq j\leq m$ with

$$
f_1(\varepsilon y) < f_2(\varepsilon y) < \cdots < f_m(\varepsilon y),
$$

separated one to each other by large distances $O(|\log\varepsilon|)$, and define in coordinates (y,z) the approximation

$$
u_0(y,z) := \sum_{j=1}^{m} w_j\big(z-f_j(\varepsilon y)\big)\,+\,\frac{(-1)^{m-1}-1}{2}, \quad w_j(t) := (-1)^{j-1}w(t), \tag{2.33}
$$

with this definition we have that $u_0(y,z)\approx w_j(z-f_j(\varepsilon y))$ for values of z close to $f_j(\varepsilon y)$.

The functions $f_j : \mathcal{K} \to \mathbb{R}$ will be left as parameters, on which we will assume a set of constraints that we describe next.

Let us fix numbers $p > N$, $M > 0$, and consider functions $h_j \in W^{2,p}(\mathcal{K})$, $j = 1, \ldots, m$, such that

$$\|h_j\|_{W^{2,p}(\mathcal{K})} := \|D^2_{\mathcal{K}} h_j\|_{L^p(\mathcal{K})} + \|D_{\mathcal{K}} h_j\|_{L^p(\mathcal{K})} + \|h_j\|_{L^\infty(\mathcal{K})} \ \le\ M\,. \tag{2.34}$$

For a small $\varepsilon > 0$, we consider the unique number $\rho = \rho_\varepsilon$ with

$$e^{-\sqrt{2}\,\rho} \ = \ \varepsilon^2 \rho. \tag{2.35}$$

We observe that ρ_ε is a large number that can be expanded in ε as

$$\rho_\varepsilon \ = \ \sqrt{2}\log\frac{1}{\varepsilon} - \frac{1}{\sqrt{2}}\log\Big(\sqrt{2}\log\frac{1}{\varepsilon}\Big) + O\Big(\frac{\log\log\frac{1}{\varepsilon}}{\log\frac{1}{\varepsilon}}\Big).$$

Then we assume that the m functions $f_j : \mathcal{K} \to \mathbb{R}$ are given by the relations

$$f_k(\tilde y) \ = \ \left(k - \frac{m+1}{2}\right)\rho_\varepsilon \ + \ h_k(\tilde y), \quad k = 1, \ldots, m, \tag{2.36}$$

so that

$$f_{k+1}(\tilde y) - f_k(\tilde y) \ = \ \rho_\varepsilon \ + \ h_{k+1}(\tilde y) - h_k(\tilde y)\,, \quad k = 1, 2, \ldots, m-1. \tag{2.37}$$

We will use in addition the conventions $h_0 \equiv -\infty$, $h_{m+1} \equiv +\infty$.

Our first goal is to compute the error of approximation in a δ_0/ε neighborhood of $\mathcal{K}_\varepsilon$, namely the quantity:

$$S(u_0) \ \ \equiv \ \ \Delta_{\mathcal{M}_\varepsilon} u_0 \ + \ F(u_0). \tag{2.38}$$

For each fixed ℓ, $1 \le \ell \le m$, this error reproduces a similar pattern on each set of the form

$$A_\ell = \left\{ (y,z) \in \mathcal{K}_\varepsilon \times \big(-\frac{\delta_0}{\varepsilon}, \frac{\delta_0}{\varepsilon}\big) \ / \ |z - f_\ell(\varepsilon y)| \ \le \ \frac{1}{2}\rho_\varepsilon + M \right\}. \tag{2.39}$$

For $(y,z) \in A_\ell$, we write $t = z - f_\ell(\varepsilon y)$ and estimate in this range the quantity $S(u_0)\big(y, t + f_\ell(\varepsilon y)\big)$. We have the validity of the following expression.

Lemma 2.4. *For $\ell \in \{1, \ldots, m\}$ and $(y,z) \in A_\ell$ we have*

$$\begin{aligned} &(-1)^{\ell-1} S(u_0)(y, t + f_\ell) \\ &= 6(1 - w^2(t))\varepsilon^2\rho_\varepsilon \left[e^{-\sqrt{2}(h_\ell - h_{\ell-1})} e^{\sqrt{2}t} \ - \ e^{-\sqrt{2}(h_{\ell+1} - h_\ell)} e^{-\sqrt{2}t} \right] \\ &\quad - \ \varepsilon^2 \Big(\Delta_{\mathcal{K}} h_\ell + (t + f_\ell) K\Big) w'(t) \ + \ \varepsilon^2 |\, \nabla_{\mathcal{K}}\, h_\ell|^2 w''(t) \ + \ (-1)^{\ell-1}\Theta_\ell(\varepsilon y, t)\,. \end{aligned} \tag{2.40}$$

where for some $\tau, \sigma > 0$ we have

$$\|\Theta_\ell(\cdot, t)\|_{L^p(\mathcal{K})} \le C\varepsilon^{2+\tau} e^{-\sigma|t|}.$$

Proof. From (2.29), using that $w_j'' + F(w_j) = 0$, we derive that, for $(y,z) \in A_\ell$

$$
\begin{aligned}
S(u_0) \,&= F\big(u_0(y,z)\big) \,-\, \sum_{j=1}^m F\big(w_j(z - f_j(\varepsilon y)\big) \,+\, \varepsilon^2 \sum_{j=1}^m |\nabla_{\mathcal K}\, h_j(\varepsilon y)|^2 w_j''\big(z - f_j(\varepsilon y)\big) \\
&- \varepsilon^2 \sum_{j=1}^m \Big(\Delta_{\mathcal K} h_j(\varepsilon y) + zK(\varepsilon y)\Big) w_j'(z - f_j(\varepsilon y)) \\
&+ \varepsilon^3 z \left[\, a_{ik}^1(\varepsilon y, \varepsilon z)\, \partial_{ik} h_j(\varepsilon y) \,+\, b_i^1(\varepsilon z, \varepsilon y)\, \partial_i h_j(\varepsilon y) \,\right] w_j'\big(z - f_j(\varepsilon y)\big) \\
&+ \varepsilon^3 \big[\, z^3\, a_{NN}^3(\varepsilon y, \varepsilon z) \,+ z^2\, a_{iN}^2(\varepsilon y, \varepsilon z)\, \partial_i h_j(\varepsilon y)\big]\, w_j''\big(z - f_j(\varepsilon y)\big) \\
&+ \varepsilon^3\, z\, a_{ik}^1(\varepsilon y, \varepsilon z)\, \partial_i h_j(\varepsilon y)\, \partial_k h_j(\varepsilon y)\, w_j''\big(z - f_j(\varepsilon y)\big)\,.
\end{aligned}
\tag{2.41}
$$

Let us consider first the case $2 \le \ell \le m-1$.

We begin with the term

$$
F\big(u_0(y, t + f_\ell)\big) \,-\, \sum_{j=1}^m F\Big(\, w_j(t + f_\ell - f_j)\,\Big), \quad |t| < \frac{\rho_\varepsilon}{2}.
$$

Since

$$
w(s) = \pm\big(1 - 2e^{-\sqrt{2}|s|}\big) \,+\, O\big(e^{-2\sqrt{2}|s|}\big) \quad \text{as } s \to \pm\infty,
$$

we find that for $j < \ell$,

$$
w\big(t + f_\ell - f_j\big) - 1 = -2e^{-\sqrt{2}(f_\ell - f_j)} e^{-\sqrt{2}t} \,+\, O\left(e^{-2\sqrt{2}|t + f_\ell - f_j|}\right), \tag{2.42}
$$

while for $j > \ell$,

$$
w\big(t + f_\ell - f_j\big) + 1 = 2e^{-\sqrt{2}(f_j - f_\ell)} e^{\sqrt{2}t} \,+\, O\left(e^{-2\sqrt{2}|t + f_\ell - f_j|}\right). \tag{2.43}
$$

Now, since

$$
F(w) = w(1 - w^2), \quad |t| < \frac{\rho_\varepsilon}{2} + O(1),
$$

$$
|f_\ell - f_j| = |\ell - j|\rho_\varepsilon + O(1), \quad e^{-\sqrt{2}\rho_\varepsilon} = \varepsilon^2 \rho_\varepsilon,
$$

we find that if $|j - \ell| \ge 2$ and $0 < \sigma < \sqrt{2}$, then for some $\tau > 0$,

$$
|F(w_j(t + f_\ell - f_j))| \,\le\, Ce^{-\sqrt{2}|t + f_\ell - f_j|} \,\le\, \varepsilon^{2+\tau}\, e^{-\sigma|t|}. \tag{2.44}
$$

On the other hand, for certain numbers $s_1, s_2 \in (0,1)$ we have

$$
F\big(w(t + f_\ell - f_{\ell-1})\big) = F'(1)\, a_1, + \frac{1}{2} F''(1 + s_1 a_1)\, a_1^2, \tag{2.45}
$$

$$
F\big(w(t + f_\ell - f_{\ell+1})\big) = F'(1)\, a_2 + \frac{1}{2} F''(1 - s_2 a_2)\, a_2^2. \tag{2.46}
$$

where

$$
a_1 := w(t + f_\ell - f_{\ell-1}) - 1, \quad a_2 := w(t + f_\ell - f_{\ell+1}) + 1.
$$

Now, we find

$$
(-1)^{\ell-1} u_0 = w(t) - a_1 - a_2 - a_3, \quad a_3 = O\left(\max_{|j-\ell|\ge 2} e^{-\sqrt{2}|t + f_\ell - f_j|}\right).
$$

Thus for some $s_3 \in (0,1)$,

$$(-1)^{\ell-1} F(u_0) \;= F(w) \,-\, F'(w)\,(a_1+a_2) \,+\, \frac{1}{2} F''(w - s_3(a_1+a_2))(a_1+a_2)^2$$

$$+\, O\left(\max_{|j-\ell|\geq 2} e^{-\sqrt{2}|t+f_\ell-f_j|} \right). \tag{2.47}$$

Combining relations (2.44)-(2.47) and using that

$$F'(1) - F'(w) = 3(1-w^2), \quad |a_1| + |a_2| = O(e^{-\sqrt{2}\frac{\rho_\varepsilon}{2}}) = O(e^{-\sqrt{2}|t|}),$$

we obtain

$$\begin{aligned}
&(-1)^{\ell-1}\Big(F(u_0) - \sum_{j=1}^m F\big(w_j(t+f_\ell-f_j)\big) \Big) \\
&= \, 3(1-w^2)\,(a_1+a_2) \\
&\quad + \, \frac{1}{2}\Big[F''(1-s_2a_2) - F''(w - s_3(a_1+a_2))\Big]\,(a_1^2+a_2^2) \,+\, O\left(\varepsilon^{2+\tau} e^{-\sigma|t|}\right) \\
&= \, 3(1-w^2)\,(a_1+a_2) \,+\, O\left(\varepsilon^{2+\tau} e^{-\sigma|t|}\right).
\end{aligned}$$

Hence, recalling relations (2.37), (2.35), the definitions of a_1, a_2 and the asymptotic expansions (2.42), (2.43) for $j=\ell-1$ and $j=\ell+1$, we find

$$F(u_0) - \sum_{j=1}^m F(w_j(t-f_j)) \;=$$

$$6(-1)^{\ell-1}(1-w^2(t))\varepsilon^2\rho_\varepsilon \, \left[e^{-\sqrt{2}(h_\ell - h_{\ell-1})} e^{\sqrt{2}t} \,-\, e^{-\sqrt{2}(h_{\ell+1}-h_\ell)} e^{-\sqrt{2}t} \right] \,+\, \theta_\ell. \tag{2.48}$$

where $\theta_\ell = O\left(\varepsilon^{2+\tau} e^{-\sigma|t|}\right)$.

Substituting (2.48) in expression (2.41) we then find

$$\begin{aligned}
&(-1)^{\ell-1} S(u_0)(y, t+f_\ell) \\
&= \, 6(1-w^2(t))\varepsilon^2\rho_\varepsilon \, \left[e^{-\sqrt{2}(h_\ell-h_{\ell-1})} e^{\sqrt{2}t} \,-\, e^{-\sqrt{2}(h_{\ell+1}-h_\ell)} e^{-\sqrt{2}t} \right] \\
&\quad - \, \varepsilon^2 \Big(\Delta_{\mathcal{K}} h_\ell + (t+f_\ell)K \Big) w'(t) + \varepsilon^2 |\nabla_{\mathcal{K}}\, h_\ell|^2 w''(t) \\
&\quad + \, (-1)^{\ell-1}\Theta_\ell(\varepsilon y, t)\,.
\end{aligned} \tag{2.49}$$

Here we have denoted

$$
\begin{aligned}
\Theta_\ell(\varepsilon y,t) \,=\, \theta_\ell(\varepsilon y,t) - \varepsilon^2 \sum_{j\neq\ell} \Big(\Delta_{\mathcal K} h_j + (t+f_\ell)K\Big) w_j'(t+f_\ell-f_j) \\
+\, \varepsilon^2 \sum_{j\neq\ell} |\nabla_{\mathcal K}\, h_j|^2 w_j''(z-f_j) \\
+\, \varepsilon^3 (-1)^{\ell-1} z\, a_{ik}^1\, \partial_i h_j\, \partial_k h_j\, w_j''(t+f_\ell-f_j) \\
+\, \varepsilon^3 (t+f_\ell) \left[\, a_{ik}^1\, \partial_{ik} h_j \,+\, b_i^1\, \partial_i h_j \,\right] w_j'(t+f_\ell-f_j) \\
+\, \varepsilon^3 \left[\, z^3\, a_{NN}^3 \,+ z^2\, a_{iN}^2\, \partial_i h_j\right] w_j''(t+f_\ell-f_j),
\end{aligned}
\tag{2.50}
$$

where the coefficients are understood to be evaluated at εy or $\big(\varepsilon y, \varepsilon(t+f_\ell(\varepsilon y))\big)$.

While this expression has been obtained assuming $2\le \ell\le m-1$, we see that it also holds for $\ell=m$, $\ell=1$. The cases $\ell=1$ and $\ell=m$ are dealt similarly. The only difference is that the term $\left[\, e^{-\sqrt2 (h_\ell-h_{\ell-1})} e^{\sqrt2 t} \,-\, e^{-\sqrt2(h_{\ell+1}-h_\ell)} e^{-\sqrt2 t}\,\right]$ gets respectively replaced by

$$
-\, e^{-\sqrt2 h_1} e^{-\sqrt2 t}\,,\ \ell=1 \quad \text{ and } \quad e^{-\sqrt2 h_m} e^{\sqrt2 t}\,,\ \ell=m. \tag{2.51}
$$

2.6. **Size of the error.** Examining expression (2.50) we see that the error in the considered region is made up by terms that can be bounded by a power of ε times a factor with exponential decay in t. We introduce the following norm for a function $g(y,t)$ defined on $\mathcal K_\varepsilon\times\mathbb R$. Let $\sigma>0$, $1<p\le+\infty$. We set

$$
\|g\|_{p,\sigma} = \sup_{(y,t)\in\mathcal K_\varepsilon\times\mathbb R} e^{\sigma|t|}\|g\|_{L^p\left(B((y,t),1)\right)}. \tag{2.52}
$$

We want to consider the error associated to points in the set A_ℓ as a function defined in the entire space $\mathcal K_\varepsilon\times\mathbb R$. To do so, we consider a smooth cut-off function $\zeta(s)$ with $\zeta(s)=1$ for $s<1$ and $\zeta(s)=0$ for $s>2$ and define

$$
\zeta_\varepsilon(t) = \zeta\big(|t| - \frac{\rho_\varepsilon}{2} - 2M\big).
$$

We extend the error as follows. Let us set

$$
\begin{aligned}
S_\ell(u_0) \,:=\, 6\big(1-w^2(t)\big)\varepsilon^2\rho_\varepsilon \left[\, e^{-\sqrt2 (h_\ell-h_{\ell-1})} e^{\sqrt2 t} \,-\, e^{-\sqrt2(h_{\ell+1}-h_\ell)} e^{-\sqrt2 t}\,\right] \zeta_\varepsilon(t) \\
-\, \varepsilon^2 \Big(\Delta_{\mathcal K} h_\ell + (t+f_\ell)K\Big) w'(t) \,+\, \varepsilon^2 |\nabla_{\mathcal K_\varepsilon}\, h_\ell|^2 w''(t) \\
+\, (-1)^{\ell-1}\,\zeta_\varepsilon(t)\,\Theta_\ell(\varepsilon y,t)\,,
\end{aligned}
\tag{2.53}
$$

where the cut-off expressions are understood to be zero outside the support of ζ_ε. We see that

$$
(-1)^{\ell-1} S(u_0)(\varepsilon y, t+f_\ell) \,=\, S_\ell(u_0)(y,t) \quad \text{for all} \quad (y,t)\in A_\ell.
$$

The following lemma on the accuracy of the error is readily checked.

Lemma 2.5. *For a given $0<\sigma<\sqrt2$ and any $p>1$ we have the estimates*

$$
\|S_\ell(u_0)\|_{p,\sigma} \le C\,\varepsilon^{2-\tau}, \quad \|\zeta_\varepsilon\,\Theta_\ell\|_{p,\sigma} \le C\,\varepsilon^{3-\tau}, \tag{2.54}
$$

where τ is any number with $\tau > \frac{1}{2\sqrt2}\sigma$ and $\tau > \frac{N-1}{p}$.

Proof. The proof amounts to a straightforward verification of the bound term by term. Let us consider for instance

$$E_1 = 6\big(1 - w^2(t)\big)\varepsilon^2\rho_\varepsilon \left[e^{-\sqrt{2}(h_\ell - h_{\ell-1})} e^{\sqrt{2}t} - e^{-\sqrt{2}(h_{\ell+1} - h_\ell)} e^{-\sqrt{2}t} \right] \zeta_\varepsilon(t).$$

Then for $|t| \le \frac{\rho_\varepsilon}{2}$ we get

$$|E_1| \le C\varepsilon^2 |\log\varepsilon| \le Ce^{-\sqrt{2}\rho_\varepsilon} \le e^{-\sigma\frac{\rho_\varepsilon}{2}} e^{-(\sqrt{2}-\frac{1}{2}\sigma)\rho_\varepsilon} \le e^{-\sigma|t|} \varepsilon^{2-\tau},$$

where $\tau > \frac{1}{2\sqrt{2}}\sigma$. This implies $\|E_1\|_{p,\sigma} \le C\varepsilon^{2-\tau}$ for any $1 < p \le +\infty$. Now, let us consider the term

$$E_2(y,t) = \varepsilon^2 \, \Delta_{\mathcal{K}} h_\ell(\varepsilon y) \, w'(t).$$

Then for any $\sigma \le \sqrt{2}$ we have

$$\begin{aligned} e^{\sigma|t|} \|E_2\|_{L^p(B((t,y),1)} &\le C\varepsilon^2 \|\Delta_{\mathcal{K}} h_\ell(\varepsilon\tilde{y})\|_{L^p(B(y,1)} \\ &\le C\varepsilon^{2-\frac{N-1}{p}} \|\Delta_{\mathcal{K}} h_\ell\|_{L^p(\mathcal{K})} \le C\varepsilon^{2-\frac{N-1}{p}}. \end{aligned}$$

The rest of the terms are dealt similarly, being in fact roughly at least ε times smaller than those above. □

Very important for subsequent developments is the Lipschitz character of the error in the parameter function $\mathtt{h} = (h_1, \dots, h_N)$. Let us write $S_j(\mathtt{h})$ to emphasize the dependence on this function. We have

Lemma 2.6. *Let us assume that the vector-valued functions* $\mathtt{h}_1$, $\mathtt{h}_2$ *satisfy the constraints (2.34). We have the validity of the following Lipschitz conditions.*

$$\|S_j(\mathtt{h}^1) - S_j(\mathtt{h}^2)\|_{p,\sigma} \le C\varepsilon^{2-\tau} \|\mathtt{h}^1 - \mathtt{h}^2\|_{W^{2,p}(\mathcal{K})},$$

$$\|\zeta_\varepsilon \, \Theta_\ell(\mathtt{h}^1) - \zeta_\varepsilon \, \Theta_\ell(\mathtt{h}^2)\|_{p,\sigma} \le C\varepsilon^{3-\tau} \|\mathtt{h}^1 - \mathtt{h}^2\|_{W^{2,p}(\mathcal{K})},$$

for $\tau > \frac{1}{2\sqrt{2}}\sigma$ *and* $\tau > \frac{N-1}{p}$.

Proof. Again the proof consists in establishing the bound for each of its individual terms, more precisely, we need to bound now for instance $\partial_{\partial_i h_j} S_j(\mathtt{h})$. Since the dependence on this object, and as well on second derivatives comes in linear or quadratic way, always multiplied by exponentially decaying factors and small powers of ε, the desired result directly follows. The dependence on the values of the functions h_j appears in a more nonlinear fashion, however smooth and exponentially decaying. We omit the details. The complete arguments are rather similar to those in the proof of Corollary 5.1 of [10]. □

2.7. **The global approximation.** The approximation u_0 is so far defined only in a neighborhood of $\mathcal{K}_\varepsilon$ in $\mathcal{M}_\varepsilon$, where the local Fermi coordinates make sense. Let us assume that m is an odd number. In this case we require that $\mathcal{K}_\varepsilon$ separates $\mathcal{M}_\varepsilon$ into two components that we denote $\mathcal{M}_\varepsilon^-$ and $\mathcal{M}_\varepsilon^+$.

Let us use the convention that the normal to $\mathcal{K}_\varepsilon$ points in the direction of $\mathcal{M}_\varepsilon^+$. Let us consider the function $\mathbb{H}$ defined in $\mathcal{M}_\varepsilon \setminus \mathcal{K}_\varepsilon$ as

$$\mathbb{H}(x) := \begin{cases} 1 & \text{if } x \in \mathcal{M}_\varepsilon^+, \\ -1 & \text{if } x \in \mathcal{M}_\varepsilon^-. \end{cases} \tag{2.55}$$

Then our approximation $u_0(x)$ approaches $\mathbb{H}(x)$ at an exponential rate $O(e^{-\sqrt{2}|t|})$ as $|t|$ increases. The global approximation we will use consists simply of interpolating u_0 with $\mathbb{H}$ sufficiently well-inside $\mathcal{M}_\varepsilon \setminus \mathcal{K}_\varepsilon$ through a cut-off in $|z|$. Let $\mathcal{N}_\delta$ be the set of points in $\mathcal{M}_\varepsilon$ that have Fermi coordinates (y,z) well-defined and $|z| < \frac{\delta}{\varepsilon}$. with some positive constant $\delta < \delta_0/10$.

Let $\eta(s)$ be a smooth cut-off function with $\eta(s) = 1$ for $s < 1$ and $= 0$ for $s > 2$ and define

$$\eta_\delta(x) := \begin{cases} \eta(\,|z| - \frac{\delta}{\varepsilon}) & \text{if } x \in \mathcal{N}_\delta\,, \\ 0 & \text{if } x \notin \mathcal{N}_\delta. \end{cases} \tag{2.56}$$

Then we let our global approximation $\mathtt{w}(x)$ be simply defined as

$$\mathtt{w} := \eta_\delta u_0 + (1-\eta_\delta)\mathbb{H}, \tag{2.57}$$

where $\mathbb{H}$ is given by (2.55) and $\mathtt{w}$ is just understood to be $\mathbb{H}(x)$ outside $\mathcal{N}_\delta$.

Since $\mathbb{H}$ is an exact solution in $\mathbb{R}^N \setminus \mathcal{M}_\varepsilon$, the global error of approximation is simply computed as

$$S(\mathtt{w}) = \Delta\mathtt{w} + F(\mathtt{w}) = \eta_\delta S(u_0) + E, \tag{2.58}$$

where

$$E = 2\nabla\eta_\delta\nabla u_0 + \Delta\eta_\delta(u_0 - \mathbb{H}) + F\big(\eta_\delta u_0 + (1-\eta_\delta)\mathbb{H}\big) - \eta_\delta F(u_0)\,.$$

Observe that E has exponential size $O(e^{-\frac{c}{\varepsilon}})$ inside its support, and hence the contribution of this error to the entire error is essentially negligible.

If m is even, we simply define

$$\mathtt{w} := \eta_\delta u_0 + (1-\eta_\delta)(-1). \tag{2.59}$$

In this case there is no need that $\mathcal{K}$ separates $\mathcal{M}$ into two components.

3. The gluing procedure

Once the global approximation $\mathtt{w}(x)$ in (2.57) or (2.59) has been built, we then want to find a solution to the full problem of the form

$$u(x) = \mathtt{w}(x) + \varphi(x), \quad x \in \mathcal{M}_\varepsilon$$

where $\varphi(x)$ is a small function. Thus φ must satisfy

$$\Delta_{\mathcal{M}_\varepsilon}\varphi + F'(\mathtt{w})\varphi = -S(\mathtt{w}) - N(\varphi) \quad \text{in } \mathcal{M}_\varepsilon \tag{3.1}$$

where

$$N(\varphi) = F(\mathtt{w}+\varphi) - F(\mathtt{w}) - F'(\mathtt{w})\varphi.$$

We shall look for a solution of the form

$$u(x) = \sum_{j=1}^{m} \zeta_{j2}(x)\tilde{\phi}_j(y,z) + \psi(x),$$

where the functions $\tilde\phi_j$ are defined in the entire space $\mathcal{K}_\varepsilon\times\mathbb{R}$. Then the equation is equivalent to

$$\sum_{j=1}^m \zeta_{j2}\Big[\Delta_{\mathcal{M}_\varepsilon}\tilde\phi_j \,+\, F'(\mathtt{w})\tilde\phi_j \,+\, \zeta_{j1}(F'(\mathtt{w})+2)\psi \,+\, \zeta_{j1}N(\psi+\tilde\phi_j) \,+\, S(\mathtt{w})\Big]$$
$$+\sum_{j=1}^m\Big[2\big\langle\nabla_{\mathcal{M}_\varepsilon}\zeta_{j2},\nabla_{\mathcal{M}_\varepsilon}\tilde\phi_j\big\rangle+\tilde\phi_j\Delta_{\mathcal{M}_\varepsilon}\zeta_{j2}\Big] \,+\, \Delta\psi \,-\, \Big(2-F'(\mathtt{w})\big(1-\sum_{j=1}^m\zeta_{j1}\big)\Big)\psi$$
$$+\,\big(1-\sum_{j=1}^m\zeta_{j1}\big)\,N\Big(\psi+\sum_{i=1}^m\zeta_{i2}\tilde\phi_i\Big) \,+\, \Big(1-\sum_{j=1}^m\zeta_{j2}\Big)\,S(\mathtt{w}) \,=\, 0 \quad \text{in } \mathcal{M}_\varepsilon.$$

This system will be satisfied if the $(m+1)$-tuple $(\tilde\phi_1,\dots,\tilde\phi_m,\psi)$ solves the system

$$\Delta_{\mathcal{M}_\varepsilon}\tilde\phi_j \,+\, F'(\mathtt{w})\tilde\phi_j \,+\, \zeta_{j1}\big(F'(\mathtt{w})+2\big)\psi \,+\, \zeta_{j1}N\big(\psi+\tilde\phi_j\big) \,+\, S(\mathtt{w}) \,=\, 0, \tag{3.2}$$

for $|z|<C|\log\varepsilon|$, $j=1,\dots,m$, and

$$-\,\Delta_{\mathcal{M}_\varepsilon}\psi \,+\, 2\psi \,=\, \mathcal{Q}(\psi,x) \quad \text{in } \mathcal{M}_\varepsilon, \tag{3.3}$$

where we have denoted

$$\mathcal{Q}(\psi,x) := \Big(1-\sum_{j=1}^m\zeta_{j1}\Big)\left\{\,\big[2+F'(\mathtt{w})\big]\,\psi \,+\, N\big(\psi+\sum_{i=1}^m\zeta_{i2}\tilde\phi_i\big)\,\right\} \tag{3.4}$$
$$+\,\big(1-\sum_{j=1}^m\zeta_{j2}\big)S(\mathtt{w}) \,+\, \sum_{j=1}^m\Big[\,2\big\langle\nabla_{\mathcal{M}_\varepsilon}\zeta_{j2},\nabla_{\mathcal{M}_\varepsilon}\tilde\phi_j\big\rangle+\tilde\phi_j\Delta_{\mathcal{M}_\varepsilon}\zeta_{j2}\Big].$$

The gluing procedure consists in solving equation (3.3) for ψ in terms of a given $\tilde\phi=(\tilde\phi_1,\dots,\tilde\phi_m)$ chosen arbitrary but sufficiently small, and then substituting the result in equation (3.2). Let us assume the following constraints on the $\tilde\phi_j$'s:

$$\tilde\phi_j(y,z) \;=:\; \phi_j\big(y,z-f_j(\varepsilon y)\big), \quad \|\phi_j\|_{2,p,\sigma}\le 1 \quad \text{for all} \quad j=1,\dots m. \tag{3.5}$$

Lemma 3.1. *Given functions ϕ_j and $\mathtt{h}$ satisfying respectively constraints (3.5) and (2.34), there exists a unique solution $\psi=\Psi(\phi,\mathtt{h})$ to equation (3.3) with*

$$\|\psi\|_\infty\le C(\varepsilon^{4-\tau}+\varepsilon^{2-\tau}\|\phi\|_{2,p,\sigma}),$$

for a small $\tau>0$. In addition the operator Ψ satisfies the Lipschitz condition

$$\|\Psi(\phi^1,\mathtt{h}^1)-\Psi(\phi^2,\mathtt{h}^2)\|_\infty \,\le\, C\varepsilon^{2-\tau}\big[\,\|\phi^1-\phi^2\|_{2,p,\sigma}+\|\mathtt{h}_1-\mathtt{h}_2\|_{2,p}\,\big]. \tag{3.6}$$

Proof. Let us consider first the linear equation

$$-\,\Delta_{\mathcal{M}_\varepsilon}\psi+2\psi=E(x) \quad \text{in } \mathcal{M}_\varepsilon. \tag{3.7}$$

We claim that if we set

$$\|E\|_{p,0}=\sup_{x\in M_\varepsilon}\|E\|_{L^p(B(x,1))},$$

then problem (3.7) has, for all small $\varepsilon>0$, a unique bounded solution $\psi=\mathfrak{A}(E)$, which in addition satisfies

$$\|D_{\mathcal{M}_\varepsilon}\psi\|+\|\psi\|_\infty \,\le\, C\|E\|_{p,0},$$

provided that $p > m$. To prove this claim, it suffices to establish the a priori estimate in L^∞-norm. If that was not true, there would be sequences $\varepsilon = \varepsilon_n$, ψ_n, E_n, with $\|E_n\|_{p,0} \to 0$, $\|\psi_n\|_\infty = 1$ such that

$$-\Delta_{\mathcal{M}_\varepsilon} \psi_n + 2\psi_n = E_n \quad \text{in } \mathcal{M}_\varepsilon.$$

Using local normal coordinates around a point $p_n \in \mathcal{M}_\varepsilon$ where $|\psi_n(p_n)| = 1$, the same procedure as in the proof of the a priori estimate in Proposition 4.1 leads us to local convergence of ψ_n to a nontrivial bounded solution of

$$-\Delta_{\mathbb{R}^N} \psi + 2\psi = 0 \quad \text{in } \mathbb{R}^N,$$

and a contradiction is reached.

To solve equation (3.3) we write it in fixed point form as

$$\psi = \mathfrak{A}(\mathcal{Q}(\psi, \cdot)). \tag{3.8}$$

In the region where the functions $(1 - \sum_i \zeta_{1i})\zeta_{2j}$, $\nabla_{\mathcal{M}_\varepsilon} \zeta_{2j}$, $\Delta_{\mathcal{M}_\varepsilon} \zeta_{2j}$ are supported we have, thanks to (3.5),

$$|\tilde{\phi}(x)| + |\nabla_{\mathcal{M}_\varepsilon} \tilde{\phi}(x)| \;\le\; C\, e^{-\sigma|\xi_j - \xi_{j-1}|}\, \|\phi_j\|_{2,p,\sigma} \;\le\; \varepsilon^{2-\tau} \|\phi_j\|_{2,p,\sigma},$$

for a small $\tau > 0$. We also notice that

$$|\mathcal{Q}(0,x)| \le C\varepsilon^{4-\tau}\Big(|D^2_{\mathcal{K}}\mathtt{h}(\varepsilon y)| + |D_{\mathcal{K}}\mathtt{h}(\varepsilon y)| + |\mathtt{h}(\varepsilon y)| + 1\Big) \;+\; \varepsilon^{2-\tau} \sum_{j=1}^m \|\phi_j\|_{2,p,\sigma}.$$

We observe then that

$$\|\mathcal{Q}(0,\cdot)\|_{0,p} \le C\varepsilon^{4-\tau-\frac{N}{p}} \|h\|_{W^{2,p}(\mathcal{K})} + \varepsilon^{2-\tau}\|\phi\|_{2,p,\sigma} \le C(\varepsilon^{4-\tau} + \varepsilon^{2-\tau}\|\phi\|_{2,p,\sigma}) \;.$$

We check next the Lipschitz character of this operator, not just in ψ , but also in the rest of its arguments. Let us write $\mathcal{Q} = \mathcal{Q}(\psi, \mathtt{h}, \phi)$ and assume

$$\|\phi\|_{2,p,\sigma} \le 1, \quad \|\psi\|_{2,p,\sigma} \le \beta\varepsilon^{2-\tau}, \quad \|\mathtt{h}\|_{W^{2,p}(\mathcal{K})} \le M. \tag{3.9}$$

We consider $(\psi^l, \phi^l, \mathtt{h}^l)$, $l = 1, 2$, satisfying (3.9), and denote $\mathcal{Q}^l = \mathcal{Q}(\psi^l, \phi^l, \mathtt{h}^l)$. We will show that

$$\begin{aligned} &\|\mathcal{Q}(\psi^1, \phi^1, \mathtt{h}^1) - \mathcal{Q}(\psi^2, \phi^2, \mathtt{h}^2)\|_{0,p} \\ &\quad \le\; C\varepsilon^{2-\tau}\big[\, \|\psi_1 - \psi_2\|_\infty + \|\phi^1 - \phi^2\|_{2,p,\sigma} + \|\mathtt{h}_1 - \mathtt{h}_1\|_{2,p} \,\big]. \end{aligned} \tag{3.10}$$

Let us observe that for $(\psi, \phi, \mathtt{h})$ satisfying (3.9),

$$\mathcal{Q}(x) = \mathcal{Q}(\psi, \phi, \mathtt{h})(x) \;=\; \mathcal{Q}\Big(\psi(x), \mathtt{h}(x), D\mathtt{h}(x), D^2\mathtt{h}(x), \phi(x), D\phi(x), x\Big).$$

We decompose

$$\begin{aligned} \mathcal{Q}(x) \;=\; &\underbrace{\Big(1 - \sum_{j=1}^m \zeta_{j2}\Big) S(\mathtt{w})}_{Q_1} \;+\; \underbrace{\sum_{j=1}^m \Big[\, 2\big\langle \nabla_{\mathcal{M}_\varepsilon} \zeta_{j2}, \nabla_{\mathcal{M}_\varepsilon} \tilde{\phi}_j \big\rangle + \tilde{\phi}_j \Delta_{\mathcal{M}_\varepsilon} \zeta_{j2} \Big]}_{Q_2} \\ &\underbrace{+ \Big(1 - \sum_{j=1}^m \zeta_{j1}\Big) \left(\big[2 + F'(\mathtt{w})\big]\, \psi \;+\; N\Big(\psi + \sum_{i=1}^m \zeta_{i2} \tilde{\phi}_i\Big) \right)}_{Q_3} \;. \end{aligned} \tag{3.11}$$

Then we find

$$\partial_\psi \mathcal{Q} = \partial_\psi \mathcal{Q}_3 = \Big(1 - \sum_{j=1}^m \zeta_{j1}\Big) \left\{ \big[2 + F'(\mathtt{w})\big] \,+\, N'\Big(\psi + \sum_{i=1}^m \zeta_{i2}\tilde{\phi}_i\Big) \right\}$$

where $N'(s) = F'(\mathtt{w} + s) - F'(\mathtt{w}) = O(|s|)$. It follows that $\partial_\psi \mathcal{Q} = O(\varepsilon^{2-\tau})$ in the considered range for the parameters. Now,

$$\partial_{h_k} \mathcal{Q}_3 = \Big(1 - \sum_{j=1}^m \zeta_{j1}\Big) \left\{ F''(\mathtt{w})\,\psi \,+\, N'\Big(\psi + \sum_{i=1}^m \zeta_{i2}\tilde{\phi}_i\Big) \right\} w'(z - f_j(\varepsilon y)),$$

since $\partial_{h_k}\mathtt{w} = w'\big(z - f_j(\varepsilon y)\big)$. Thus $\partial_{h_k}\mathcal{Q}_3 = O(\varepsilon^{4-\tau})$. We also have $\partial_{h_k}\mathcal{Q}_1 = O(\varepsilon^{4-\tau})$ and

$$\partial_{D^l h_k}\mathcal{Q} = \partial_{D^l h_k}\mathcal{Q}_1 = \Big(1 - \sum_{j=1}^m \zeta_{j2}\Big)\partial_{D^l h_k} S(\mathtt{w}) = O(\varepsilon^{4-\tau}), \quad l = 1, 2.$$

Finally,

$$\partial_{\phi_j}\mathcal{Q} = \partial_{\phi_j}\mathcal{Q}_3 = \Delta_{\mathcal{M}_\varepsilon}\zeta_{j2} = O(1), \quad \partial_{D\phi_j}\mathcal{Q} = 2\nabla\zeta_{j2} = O(1).$$

As a conclusion, using the mean value formula and the facts

$$|\phi^1 - \phi^2| + |D\phi^1 - D\phi^2| \le e^{-\sigma|t|}\|\phi^1 - \phi^2\|_{2,p,\sigma},$$

$$\|D^2\mathtt{h}^1(\varepsilon y) - D^2\mathtt{h}^2(\varepsilon y)\|_{0,p} \le C\varepsilon^{-\frac{m}{p}}\|D^2\mathtt{h}^1 - D^2\mathtt{h}^2\|_{L^p(\mathcal{K})},$$

we readily find the validity of (3.10). In particular, we obtain that for $\|\psi_l\|_\infty \le \beta\varepsilon^{2-\tau}$, $l = 1, 2$, and

$$\|\mathcal{Q}(\psi_1, \phi, \mathtt{h}) - \mathcal{Q}(\psi_2, \phi, \mathtt{h})\|_{p,0} \le C\varepsilon^{2-\tau}\|\psi_1 - \psi_2\|_\infty. \tag{3.12}$$

Thus, from the contraction mapping principle, we find that for certain $\beta > 0$ large and fixed, problem (3.8) has a unique solution $\psi = \Psi(\phi, \mathtt{h})$ such that

$$\|\Psi(\phi, \mathtt{h})\|_\infty \le C(\varepsilon^3 + \varepsilon^{2-\tau}\|\phi\|_{2,p,\sigma}). \tag{3.13}$$

The Lipschitz dependence of Ψ (3.6) in its arguments follows immediately from (3.12) and the fixed point characterization (3.8). □

Now, assuming that $\|\phi\|_{2,p,\sigma}$ is in the considered range, we substitute $\psi = \Psi(\phi, \mathtt{h})$ in (3.2) and then obtain that

$$\varphi = \Psi(\phi, \mathtt{h}) + \sum_{j=1}^m \zeta_{j2}\tilde{\phi}_j, \qquad \tilde{\phi}_j(y, z) = \phi_j(y, z - f_j(\varepsilon y)),$$

solves problem (3.1) if the vector $\phi = (\phi_1, \dots, \phi_j)$ satisfies the system of equations

$$\Delta_{\mathcal{M}_\varepsilon}\tilde{\phi}_j + F'(\mathtt{w})\tilde{\phi}_j + \zeta_{j1}\big(F'(\mathtt{w}) + 2\big)\psi + \zeta_{j1}N\big(\Psi(\phi, \mathtt{h}) + \tilde{\phi}_j\big) + \zeta_{j1}S(\mathtt{w}) = 0, \tag{3.14}$$

in the support of ζ_{j2}. We want to extend these equations to the entire $\mathcal{K}_\varepsilon \times \mathbb{R}$. We recall that in (y, z) coordinates we can write

$$\Delta_{\mathcal{M}_\varepsilon} = \partial^2_{zz} + \Delta_{\mathcal{K}_\varepsilon} - \varepsilon^2 z K(\varepsilon y)\partial_z + B,$$

where B is a small operator given by (2.30). It is convenient to rewrite equations (3.14) in terms of the functions ϕ_j defined as

$$\phi_j(y, t) = \tilde{\phi}_j(y, t + f_j(\varepsilon y)).$$

We find in coordinates (y,t),

$$\Delta_{\mathcal{M}_\varepsilon}\tilde\phi_j \,=\, \partial^2_{tt}\phi_j \,+\, \Delta_{\mathcal{K}_\varepsilon}\phi_j \,+\, B^1_j\phi_j \,+\, B^2_j\phi_j, \tag{3.15}$$

where

$$\begin{aligned} B^1_j\phi \;&= \varepsilon^2\big|\nabla_{\mathcal{K}}h_j(\varepsilon y)\big|^2\,\partial^2_{tt}\phi \,-\, \varepsilon^2\Delta_{\mathcal{K}}h_j(\varepsilon y)\,\partial_t\phi \\ &\quad -\, \varepsilon^2 K(\varepsilon y)\,\big(t+f_j(\varepsilon y)\big)\partial_t\phi \,-\, 2\varepsilon\big\langle\nabla_{\mathcal{K}}h_j(\varepsilon y)\,,\,\nabla_{\mathcal{K}_\varepsilon}\partial_t\phi\big\rangle, \end{aligned}$$

and, expressed in local coordinates $(\mathtt{y},t)$, $y=Y_p(\mathtt{y})$, the operator $B^2_j\phi$ becomes

$$\begin{aligned} B^2_j\phi \,&= \varepsilon^3\big(t+f_j(\varepsilon\mathtt{y})\big)^3\, a^3_{NN}\,\partial_{tt} \,+\, \varepsilon^2\,\big(t+f_j(\varepsilon\mathtt{y})\,\big)\, b^1_i\,\big(\partial_i\phi-\varepsilon\partial_i h_j(\varepsilon\mathtt{y})\,\partial_t\phi\,\big) \\ &\quad +\,\varepsilon\,\big(t+f_j(\varepsilon\mathtt{y})\big)\,a^1_{il}\,\Big[\,\partial_{il}\phi-2\varepsilon\partial_l h_j(\varepsilon\mathtt{y})\,\partial_{it}\phi \\ &\qquad\qquad -\,\varepsilon^2\partial_{il}h_j(\varepsilon\mathtt{y})\,\partial_t\phi \,+\, \varepsilon^2\partial_i h_j\partial_l h_j(\varepsilon\mathtt{y})\,\partial_{tt}\phi\,\Big] \\ &\quad +\,\varepsilon^2\big(t+f_j(\varepsilon\mathtt{y})\big)^2\,a^2_{iN}\,\big(\partial_{it}\phi-\varepsilon\partial_i h_j(\varepsilon\mathtt{y})\,\partial_{tt}\phi\,\big) \,+\, \varepsilon^3\,\big(t+f_j(\varepsilon\mathtt{y})\big)^2\, b^2_N\,\partial_t\,, \end{aligned} \tag{3.16}$$

with coefficients evaluated at $(\varepsilon\mathtt{y},\varepsilon t+\varepsilon f_j(\varepsilon\mathtt{y}))$. The difference between the operators B^1_j and B^2_j is that the expression for B^1_j actually makes sense for all (y,t), while B^2_j does only up to $|t|<\delta/\varepsilon$. We set $\chi_0(t)=\zeta(|t|-10\log\varepsilon)$, where, we recall, $\zeta(\tau)=1$ for $\tau<1$ and $=0$ for $\tau>2$. Then we extend the operator $B^1_j+B^2_j$ to entire space (y,z) setting

$$B_j := B^1_j + \chi_0 B^2_j.$$

Let us relabel

$$\chi_{js}(y,t) \,:=\, \zeta_{js}(t+f_j(\varepsilon y)) \,=\, \zeta(|t+h_j(\varepsilon y)|-d_\varepsilon-s),$$

and denote

$$\Psi_j(\phi,\mathtt{h})(y,t) \,:=\, \Psi(\phi,\mathtt{h})(y,t+f_j(\varepsilon y)),$$

$$S_j(\mathtt{h})(y,t) \,:=\, \chi_{j3}S(\mathtt{w})(y,t+f_j), \tag{3.17}$$

(observe that this is the same S_j introduced in (2.53))

$$\mathtt{w}_j := \mathtt{w}(y,t+f_j) = w(t)+\theta_j,$$

where $\theta_j(y,t)=O(\varepsilon^{-2+\tau})$. We have

$$\begin{aligned} &\partial^2_{zz}\tilde\phi_j + \Delta_{\mathcal{K}_\varepsilon}\tilde\phi_j + F'(w_j)\tilde\phi_j \,+\, \zeta_{j3}\,B\tilde\phi_j + \zeta_{j3}\,\big(F'(\mathtt{w})-F'(w_j)\big)\tilde\phi_j \\ &+\zeta_{j1}\,\Big[\,\big(F'(\mathtt{w})+2\big)\Psi(\phi,\mathtt{h}) + N\big(\Psi(\phi,\mathtt{h})+\tilde\phi_j\big) + S(\mathtt{w})\Big] = 0 \quad \text{in } \mathcal{K}_\varepsilon\times\mathbb{R}, \end{aligned} \tag{3.18}$$

where $w_j(y,z)=w(z-f_j(\varepsilon y))$. Finally, we recast equations (3.18) as

$$\partial^2_{tt}\phi_j + \Delta_{\mathcal{K}_\varepsilon}\phi_j + F'(w(t))\tilde\phi_j + S_j(\mathtt{h}) + \mathtt{N}_j(\phi,\mathtt{h}) \,=0 \quad \text{in } \mathcal{K}_\varepsilon\times\mathbb{R}, \tag{3.19}$$

for all $j = 1, \dots, m$, where

$$\mathtt{N}_j(\phi, \mathtt{h}) \;=\; \mathtt{B}_j(\phi_j) \;+\; \chi_{j1}\Big[\big(F'(\mathtt{w}_j) + 2\big)\Psi_j(\phi, \mathtt{h}) + N\big(\Psi_j(\phi, \mathtt{h}) + \phi_j\big)\Big], \tag{3.20}$$

with

$$\mathtt{B}_j(\phi_j) = \chi_{j3}\Big[B_j\phi_j + \big(F'(\mathtt{w}_j) - F'(w)\big)\phi_j\Big], \quad B_j \text{ given by (3.16)}. \tag{3.21}$$

We will concentrate in what follows in solving system (3.19). We will do this in two steps: 1. solving a projected version of the problem, carrying $\mathtt{h}$ as a parameter, and 2. finding $\mathtt{h}$ such that the solution of this projected problem is an actual solution of (3.19). We consider then the system, for all $j = 1, \dots, m$

$$\begin{gathered} \partial^2_{tt}\phi_j \;+\; \Delta_{\mathcal{K}_\varepsilon}\phi_j \;=\; -S_j(\mathtt{h}) - \mathtt{N}_j(\phi, \mathtt{h}) \;+\; c_j(y)w'(t) \quad \text{ in } \mathcal{K}_\varepsilon \times \mathbb{R}, \\ \int_{\mathbb{R}} \phi_j(y,t)\, w'(t)\, \mathrm{d}t = 0 \quad \text{on } \mathcal{K}_\varepsilon\,, \quad c_j(y) = \frac{\int_{\mathbb{R}} \big(S_j(\mathtt{h}) + \mathtt{N}_j(\phi, \mathtt{h})\big)\, w'\, \mathrm{d}t}{\int_{\mathbb{R}} w'^2 \mathrm{d}t}\,. \end{gathered} \tag{3.22}$$

To solve it we need a suitable invertibility theory for the linear operator involved in the above equation. We do this next.

4. The auxiliary linear projected problem

Crucial for later purposes is a solvability theory for the following linear problem:

$$\begin{gathered} \partial^2_{tt}\phi \;+\; \Delta_{\mathcal{K}_\varepsilon}\phi \;=\; g(y,t) \;+\; c(y)w'(t) \quad \text{ in } \mathcal{K}_\varepsilon \times \mathbb{R}, \\ \int_{\mathbb{R}} \phi(y,t)\, w'(t)\, \mathrm{d}t = 0 \quad \text{for all} \quad y \in \mathcal{K}_\varepsilon\,, \quad c(y) = -\frac{\int_{\mathbb{R}} g(y,t) w' \mathrm{d}t}{\int_{\mathbb{R}} w'^2 \mathrm{d}t}\,. \end{gathered} \tag{4.1}$$

We have the following result.

Proposition 4.1. *Given $p > m$ and $0 < \sigma < \sqrt{2}$, there exists a constant $C > 0$ such that for all sufficiently small $\varepsilon > 0$ the following holds. Given g with $\|g\|_{p,\sigma} < +\infty$, then Problem* (4.1) *has a unique solution ϕ with $\|\phi\|_{\infty,\sigma} < +\infty$, which in addition satisfies*

$$\|D^2\phi\|_{p,\sigma} + \|D\phi\|_{\infty,\sigma} + \|\phi\|_{\infty,\sigma} \;\le\; C\|g\|_{p,\sigma}\,. \tag{4.2}$$

The main fact needed is that the one-variable solution w of (1.4) is *nondegenerate* in $L^\infty(\mathbb{R}^m)$ in the sense that the linearized operator

$$L(\phi) = \Delta_y\phi + \partial^2_{tt}\phi + F'(w(t))\phi, \quad (y,t) \in \mathbb{R}^{N-1} \times \mathbb{R},$$

is such that the following property holds.

Lemma 4.1. *Let ϕ be a bounded, smooth solution of the problem*

$$L(\phi) = 0 \quad \text{in } \mathbb{R}^{N-1} \times \mathbb{R}. \tag{4.3}$$

Then $\phi(y,t) = Cw'(t)$ for some $C \in \mathbb{R}$.

Proof. This fact is by now standard, so we only sketch the proof. The one-dimensional operator $L_0(\psi) = \psi'' + F'(w)\psi$ is such that $L_0(w') = 0$ and $w' > 0$, hence 0 is its least eigenvalue. Using this, it is easy to show that there is a constant $\gamma > 0$ such that whenever $\int_{\mathbb{R}} \psi w' = 0$ with $\psi \in H^1(\mathbb{R})$ we have that

$$\int_{\mathbb{R}} \left(|\psi'|^2 - F'(w)\psi^2 \right) \mathrm{d}t \;\geq\; \gamma \int_{\mathbb{R}} \left(|\psi'|^2 + |\psi|^2 \right) \mathrm{d}t. \tag{4.4}$$

Let ϕ be a bounded solution of equation (4.3). Since $F'(w(t)) \sim -2$ for all large $|t|$ an application of the maximum principle shows that if $0 < \sigma < \sqrt{2}$ and $t_0 > 0$ is large then

$$|\phi(y,t)| \;\leq\; C\|\phi\|_\infty e^{-\sigma|t|} \quad \text{if } |t| > t_0\,.$$

On the other hand, the function

$$\bar{\phi}(y,t) \;=\; \phi(y,t) \;-\; \frac{w'(t)}{\int_{\mathbb{R}} w'^2} \int_{\mathbb{R}} w'(\zeta)\,\phi(y,\zeta)\,\mathrm{d}\zeta,$$

also satisfies $L(\bar{\phi}) = 0$ and, in addition,

$$\int_{\mathbb{R}} w'(t)\,\bar{\phi}(y,t)\,\mathrm{d}t = 0 \quad \text{for all} \quad y \in \mathbb{R}^{N-1}. \tag{4.5}$$

Now, the function

$$\varphi(y) := \int_{\mathbb{R}} \left|\bar{\phi}(y,t)\right|^2 \mathrm{d}t,$$

is well defined and smooth. We compute

$$\Delta_y \varphi(y) \;=\; 2\int_{\mathbb{R}} \Delta_y \bar{\phi} \cdot \bar{\phi}\,\mathrm{d}t \;+\; 2\int_{\mathbb{R}} \left|\nabla_y \bar{\phi}\right|^2 \mathrm{d}t,$$

and hence

$$\begin{aligned} 0 &= \int_{\mathbb{R}} \left(L(\bar{\phi}) \cdot \bar{\phi} \right) \\ &= \frac{1}{2}\Delta_y \varphi - \int_{\mathbb{R}} \left|\nabla_y \bar{\phi}\right|^2 \mathrm{d}t - \int_{\mathbb{R}} \left(|\bar{\phi}_t|^2 - F'(w)\bar{\phi}^2 \right) \mathrm{d}t\,. \end{aligned} \tag{4.6}$$

From (4.5) and (4.4), we then get $\frac{1}{2}\Delta_y\varphi - \gamma\varphi \geq 0$. Since φ is bounded, it must be zero. In particular this implies that the bounded function

$$g(y) = \int_{\mathbb{R}} w_\zeta(\zeta)\,\phi(y,\zeta)\,\mathrm{d}\zeta$$

is harmonic and bounded, hence a constant. We conclude that $\phi(y,t) = Cw'(t)$, as desired. ☐

Proof of Proposition 4.1: We begin by proving a priori estimates. Let $0 < \sigma < \sqrt{2}$. We first claim that there exists a constant $C > 0$ such that for all small ε and every solution ϕ to Problem (4.1) with $\|\phi\|_{\infty,\nu,\sigma} < +\infty$ and right hand side g satisfying $\|g\|_{p,\sigma} < +\infty$ we have

$$\|D^2\phi\|_{p,\sigma} + \|D\phi\|_{\infty,\sigma} + \|\phi\|_{\infty,\sigma} \leq C\|g\|_{p,,\sigma}. \tag{4.7}$$

To establish this fact, it clearly suffices to consider the case $c(y) \equiv 0$. By local elliptic estimates, it is enough to show that

$$\|\phi\|_{\infty,\sigma} \leq C\|g\|_{p,\sigma}. \tag{4.8}$$

Let us assume by contradiction that (4.8) does not hold. Then we have sequences $\varepsilon = \varepsilon_n \to 0$, g_n with $\|g_n\|_{p,\sigma} \to 0$, ϕ_n with $\|\phi_n\|_{\infty,\sigma} = 1$ such that

$$\begin{aligned} \partial_{tt}\phi_n + \Delta_{\mathcal{K}_\varepsilon}\phi_n + F'(w(t))\phi_n = g_n \quad \text{in } \mathcal{K}_\varepsilon \times \mathbb{R}, \\ \int_{\mathbb{R}} \phi_n(y,t)\, w'(t)\, \mathrm{d}t = 0 \quad \text{for all} \quad y \in \mathcal{K}_\varepsilon\,. \end{aligned} \tag{4.9}$$

Then we can find points $(p_n, t_n) \in \mathcal{K}_\varepsilon \times \mathbb{R}$ such that

$$e^{-\sigma|t_n|}\,|\phi_n(p_n,t_n)| \geq \frac{1}{2}.$$

We will consider different possibilities. Let us consider the local coordinates for $\mathcal{K}_{\varepsilon_n}$ around p_n,

$$Y_{p_n,\varepsilon_n}(\mathrm{y}) = \varepsilon_n^{-1} Y_{\varepsilon_n p_n}(\varepsilon_n \mathrm{y}), \quad |\mathrm{y}| < \frac{1}{\varepsilon_n},$$

where $Y_p(\mathrm{y})$ is given by (2.1). Let us assume first that $|t_n| \leq C$. Then, the Laplace-Beltrami operator of $\mathcal{K}_{\varepsilon_n}$ takes locally the form

$$\Delta_{\mathcal{K}_{\varepsilon_n}} \; = \; a^0_{ij}(\varepsilon_n \mathrm{y})\partial_{ij} + \varepsilon_n b^0_j(\varepsilon_n \mathrm{y})\partial_j$$

where

$$a^0_{ij}(\varepsilon_n \mathrm{y}) = \delta_{ij} + o(1), \quad b^0_i(\varepsilon_n \mathrm{y}) = O(1).$$

Thus

$$a^0_{ij}\partial_{ij}\tilde\phi_n + \varepsilon_n b^0_j \partial_j \tilde\phi_n + \partial_{tt}\tilde\phi_n + F'(w(t))\tilde\phi_n = \tilde g_n(\mathrm{y},t), \quad |\mathrm{y}| < \frac{1}{\varepsilon},$$

where $\tilde g_n(\mathrm{y},t) := g_n\big(Y_n(\varepsilon \mathrm{y}), t\big)$. We observe that this expression is valid for y inside the domain $\varepsilon^{-1}\mathcal{U}_k$ which is expanding to entire $\mathbb{R}^{N-1}$. Since $\tilde\phi_n$ is bounded, and $\tilde g_n \to 0$ in $L^p_{loc}(\mathbb{R}^N)$, we obtain local uniform $W^{2,p}$-bounds. Hence we may assume, passing to a subsequence, that $\tilde\phi_n$ converges uniformly in the compact subsets of $\mathbb{R}^N$ to a function $\tilde\phi(\mathbf{y},t)$ that satisfies

$$\Delta_{\mathbb{R}^{N-1}}\tilde\phi + \partial_{tt}\tilde\phi + F'(w(t))\tilde\phi = 0\,.$$

Thus $\tilde\phi$ is non-zero and bounded. Hence Lemma 4.1 implies that, necessarily, $\tilde\phi(\mathbf{y},t) = Cw'(t)$. On the other hand, we have

$$0 = \int_{\mathbb{R}} \tilde\phi_n(\mathbf{y},t)\, w'(t)\, \mathrm{d}t \longrightarrow \int_{\mathbb{R}} \tilde\phi(\mathbf{y},t)\, w'(t)\, \mathrm{d}t \qquad \text{as } n \to \infty.$$

Hence, necessarily $\tilde\phi \equiv 0$. But $|\tilde\phi_n(0,t_n)| \geq \frac{1}{2}$, and t_n was bounded, the local uniform convergence implies $\tilde\phi \neq 0$. We have reached a contradiction.

Now, if t_n is unbounded, say, $t_n \to +\infty$, the situation is similar. The variation is that we define now

$$\tilde\phi_n(\mathbf{y},t) = e^{\sigma(t_n+t)}\phi_n(\mathbf{y}, t_n+t), \quad \tilde g_n(\mathbf{y},t) = e^{\sigma(t_n+t)} g_n(\mathbf{y}, t_n+t).$$

Then $\tilde\phi_n$ is uniformly bounded, and $\tilde g_n \to 0$ in $L^p_{loc}(\mathbb{R}^N)$. Now $\tilde\phi_n$ satisfies

$$\begin{aligned} a^0_{ij}(\varepsilon_n \mathbf{y})\,\partial_{ij}\tilde\phi_n \;+\; \partial_{tt}\tilde\phi_n \;+\; \varepsilon_n b_j(\mathrm{y}_n + \varepsilon_n \mathbf{y})\,\partial_j\tilde\phi_n \\ -2\sigma\,\partial_t\tilde\phi_n \;+\; F'\big(w(t+t_n) + \sigma^2\big)\,\tilde\phi_n \;=\; \tilde g_n. \end{aligned}$$

We fall into the limiting situation

$$\Delta_{\mathbb{R}^{N-1}}\tilde{\phi} + \partial^2_{tt}\tilde{\phi} - 2\sigma\,\partial_t\tilde{\phi} - (2-\sigma^2)\,\tilde{\phi} = 0 \quad \text{in } \mathbb{R}^N \tag{4.10}$$

with $\tilde{\phi} \neq 0$ bounded. The maximum principle implies that $\tilde{\phi} \equiv 0$. We obtain a contradiction that proves the validity of (4.7).

It remains to prove existence of a solution ϕ of problem (4.1) with $\|\phi\|_{\infty,\sigma} < +\infty$. We assume first that g has compact support. For such a g, Problem (4.1) has a variational formulation. Let

$$\mathcal{H} = \left\{ \phi \in H^1_0(\mathcal{K}_\varepsilon \times \mathbb{R}) \;/\; \int_{\mathbb{R}} \phi(y,t)\, w'(t)\,\mathrm{d}t = 0 \quad \text{for all} \quad y \in \mathcal{K}_\varepsilon \right\}.$$

$\mathcal{H}$ is a closed subspace of $H^1_0(\mathcal{K}_\varepsilon \times \mathbb{R})$, hence a Hilbert space when endowed with its natural norm,

$$\|\phi\|^2_{\mathcal{H}} = \int_{\mathcal{K}_\varepsilon}\int_{\mathbb{R}} \Big(|\partial_t\phi|^2 + |\nabla_{\mathcal{K}_\varepsilon}\phi|^2 - F'(w(t))\,\phi^2 \Big)\,\mathrm{d}V_{\mathcal{K}_\varepsilon}\,\mathrm{d}t\ .$$

ϕ is then a weak solution of Problem (4.1) if $\phi \in \mathcal{H}$ and satisfies

$$\begin{aligned} a(\phi,\psi) &:= \int_{\mathcal{K}_\varepsilon\times\mathbb{R}} \Big(\nabla_{\mathcal{K}_\varepsilon}\phi\cdot\nabla_{\mathcal{K}_\varepsilon}\psi \,-\, F'(w(t))\,\phi\,\psi \Big)\,\mathrm{d}V_{\mathcal{K}_\varepsilon}\,\mathrm{d}t \\ &= -\int_{\mathcal{K}_\varepsilon\times\mathbb{R}} g\,\psi\,\mathrm{d}V_{\mathcal{K}_\varepsilon}\,\mathrm{d}t \quad \text{for all} \quad \psi \in \mathcal{H}. \end{aligned}$$

It is standard to check that a weak solution of Problem (4.1) is also classical provided that g is regular enough. Let us observe that because of the orthogonality condition defining $\mathcal{H}$ we have that

$$\gamma \int_{\mathcal{K}_\varepsilon\times\mathbb{R}} \psi^2\,\mathrm{d}V_{\mathcal{K}_\varepsilon}\,\mathrm{d}t \;\le\; a(\psi,\psi) \quad \text{for all} \quad \psi \in \mathcal{H}.$$

Hence the bilinear form a is coercive in $\mathcal{H}$, and existence of a unique weak solution follows from Riesz's theorem. If g is regular and compactly supported, ψ is also regular. Local elliptic regularity implies in particular that ϕ is bounded. Since for some $t_0 > 0$, the equation satisfied by ϕ is

$$\Delta\phi + F'(w(t))\,\phi = c(y)w'(t), \quad |t| > t_0, \quad y \in \mathcal{K}_\varepsilon, \tag{4.11}$$

and $c(y)$ is bounded, then enlarging t_0 if necessary, we see that for $\sigma < \sqrt{2}$, a suitable barrier argument shows that $|\phi| \le Ce^{-\sigma|t|}$, hence $\|\phi\|_{p,\sigma} < +\infty$. From (4.7) we obtain that

$$\|D^2\phi\|_{p,\sigma} + \|D\phi\|_{\infty,\sigma} + \|\phi\|_{\infty,\sigma} \le C\|g\|_{p,\sigma}. \tag{4.12}$$

Now, for an arbitrary $\|g\|_{p,\sigma} < +\infty$ we consider a sequence of compactly supported approximations uniformly controlled in $\|\ \|_{p,\sigma}$ (thus inheriting corresponding control on the approximate solutions). Passing to a subsequence if necessary, we obtain local convergence to a solution $\tilde{\phi}$ to the full problem which respects the estimate (4.2). This concludes the proof. $\square$

5. Solving the nonlinear projected problem

To solve Problem (3.22) and for the subsequent step of adjusting $\mathtt{h}$ so that the quantities $c_l(y)$ are all identically zero, it is important to keep track of the Lipschitz character of the operators involved in this equation. We have the following result.

Lemma 5.1. *There is a constant $C>0$ such that for all $\mathtt{h}^l$ satisfying (3.9) and all ϕ^l with $\|\phi^l\|_{2,p,\sigma}\le \varepsilon^{2-\tau}$, $l=1,2$ we have*

$$\|N_j(\phi^1,\mathtt{h}^1)-N_j(\phi^2,\mathtt{h}^2)\|_{p,\sigma} \ \le\ C\varepsilon^{4-\tau}\|\mathtt{h}_1-\mathtt{h}_2\|_{W^{2,p}(\mathcal{K})}+\varepsilon^{2-\tau}\|\phi^1-\phi^2\|_{2,p,\sigma}, \tag{5.1}$$

$$\|S_j(\mathtt{h}^1)-S_j(\mathtt{h}^2)\|_{p,\sigma} \ \le\ C\varepsilon^{3-\tau}\|\mathtt{h}_1-\mathtt{h}_2\|_{W^{2,p}(\mathcal{K})}. \tag{5.2}$$

Proof. We have to check the Lipschitz character of the operators $\mathtt{N}_j(\phi,\mathtt{h})$ in (3.20) in the norm $\|\ \|_{p,\sigma}$. Let us consider each of the terms in formula (3.20). Let us consider first the operator $\mathtt{B}_j\phi_j$ in (3.21). On ϕ and $\mathtt{h}$ we assume

$$\|\phi\|_{2,p,\sigma}\le\varepsilon^{2-\tau},\qquad \|\mathtt{h}\|_{W^{2,p}(\mathcal{K})}\le M. \tag{5.3}$$

This operator has the form in local coordinates

$$\mathtt{B}_j\phi_j(y,t)=\mathtt{B}\big(h_j,\partial_{ik}h_j,\partial_i h_j,\phi_j,\mathtt{y},t\big).$$

Let us consider the operator $B_j\phi_j$ in (3.16). We see that the explicit dependence on h_j comes only from the coefficients a_{ik} and b_i, more precisely on smooth functions of the form $a\big(\varepsilon\mathtt{y},\varepsilon t+\varepsilon f_j(\varepsilon\mathtt{y})\big)$, $f_j=\xi_j+h_j$, so that $\partial_{h_j}a=O(\varepsilon)$. We also find

$$\partial_{h_j}\chi_{j3}=O(1),\quad \partial_{h_j}F'(\mathtt{w}_j)=\sum_{k\ne j}w'\big(t-(f_k-f_j)\big)=O(\varepsilon^{2-\tau}).$$

Taking these facts into account we then find that for arbitrarily small $\tau>0$,

$$\partial_{h_j}\mathtt{B}_j\phi_j\ =\ O(\varepsilon^{1-\tau})D^2_{\mathcal{M}_\varepsilon}\phi_j\ +\ O(\varepsilon^{2-\tau})D_{\mathcal{M}_\varepsilon}\phi_j\ +\ O(\varepsilon^{2-\tau})\phi_j,$$

and hence

$$\|\partial_{h_j}\mathtt{B}_j\phi_j\|_{0,p,\sigma}\le C\varepsilon^{1-\tau}\|\phi\|_{2,p,\sigma}\,.$$

Observe that we have as well that

$$\|\mathtt{B}_j\phi_j\|_{0,p,\sigma}\le C\varepsilon^{1-\tau}\|\phi\|_{2,p,\sigma}\,.$$

Let us consider the dependence on the derivatives of h. We easily check that

$$\partial_{D_{\mathcal{K}}h}\mathtt{B}_j\phi_j=O(\varepsilon)D^2\phi,\quad \partial_{D^2_{\mathcal{K}}h}\mathtt{B}_j\phi_j=O(\varepsilon)D\phi_j.$$

As a conclusion we find that, emphasizing the dependence on $\mathtt{h}$ of the operator $\mathtt{B}_j$,

$$\|\mathtt{B}_j(\phi^1,\mathtt{h}^1)-\mathtt{B}_j(\phi^2,\mathtt{h}^2)\|_{0,p}\le\varepsilon^{1-\tau}\|\phi^1-\phi^2\|_{2,p,\sigma}+\varepsilon^{3-\tau}\,\|\mathtt{h}^1-\mathtt{h}^2\|_{W^{2,p}(\mathcal{K})}. \tag{5.4}$$

Let us consider the remaining operator in $\mathtt{N}_j$,

$$\mathcal{N}(\phi,\mathtt{h}):=\chi_{j1}\ \big[\,(F'(\mathtt{w}_j)+2)\Psi_j(\phi,\mathtt{h})+N(\Psi_j(\phi,\mathtt{h})+\phi_j)\,\big]\,.$$

We write it as

$$\mathcal{N}(\phi,\mathtt{h})(y,t)=\tilde{\mathcal{N}}(\phi,\psi,\mathtt{h},y,t),\quad \psi=\Psi_j(\phi,\mathtt{h}),$$

and recall from Lemma 3.1 that $\|\psi\|_\infty=O(\varepsilon^{4-\tau})$. Observe first that

$$\partial_\psi\tilde{\mathcal{N}}=\chi_{j1}\ \big[\,(F'(\mathtt{w}_j)+2)+N'(\psi+\phi_j)\,\big]=O(\varepsilon e^{-\sigma|t|}),$$

$$\partial_{\phi_j}\tilde{\mathcal{N}}=\chi_{j1}\,N'(\psi+\phi_j)=O(|\psi|+|\phi_j|)=O(\varepsilon^{2-\tau}).$$

In addition, we also check that

$$\partial_{\mathtt{h}} \tilde{N} = O(|\psi| e^{-\sigma|t|} + |\phi|^2) = O(\varepsilon^{4-\tau}).$$

Using these estimates, and writing $\psi^l = \Psi_j(\phi^l, \mathtt{h}^l)$ we find

$$\begin{aligned} \|\mathcal{N}(\phi^1, \mathtt{h}^1) - \mathcal{N}(\phi^2, \mathtt{h}^2)\|_{p,\sigma} \ &= \ \|\tilde{N}(\phi^1, \psi^l, \mathtt{h}^1, \cdot) - \tilde{N}(\phi^2, \psi^2, \mathtt{h}^2, \cdot)\|_{p,\sigma} \\ &\le \ C\varepsilon \|\psi^1 - \psi^2\|_\infty + C\varepsilon^{2-\tau}\|\phi^1 - \phi^2\|_{2,p,\sigma} + C\varepsilon^{4-\tau}\|\mathtt{h}^1 - \mathtt{h}^2\|_\infty . \end{aligned}$$

Recalling now, (3.6), and combining this with estimate (5.4) we arrive to the desired result. The proof of (5.1) is concluded. The proof of estimate (5.2) is similar, taking into account the explicit form of the error. $\square$

Proposition 5.1. *Given* $\mathtt{h}$ *satisfying (2.34), problem (3.22) has a unique solution* $\phi = \Phi(\mathtt{h})$ *with* $\|\phi\|_{2,p,\sigma} \le \varepsilon^{2-\tau}$*. Moreover, we have the validity of the Lipschitz conditions*

$$\|\Phi(\mathtt{h}^1) - \Phi(\mathtt{h}^2)\|_{2,p,\sigma} \ \le \ C\,\varepsilon^{2-\tau}\,\|\mathtt{h}^1 - \mathtt{h}^2\|_{W^{2,p}(\mathcal{K})}. \tag{5.5}$$

In addition, we have that

$$\|\mathtt{N}_j(\Phi(\mathtt{h}^1), \mathtt{h}^1) - \mathtt{N}_j(\Phi(\mathtt{h}^2), \mathtt{h}^2)\|_{2,p,\sigma} \le C\,\varepsilon^{4-\tau}\,\|\mathtt{h}^1 - \mathtt{h}^2\|_{W^{2,p}(\mathcal{K})}. \tag{5.6}$$

Proof. Let $T(g)$ be the operator defined as the solution of (4.1) predicted by Proposition 4.1. Then we find a solution to (3.22) if we solve the fixed point problem for $\phi = (\phi_1, \ldots, \phi_N)$

$$\phi_j = \mathfrak{B}_j(\phi, \mathtt{h}) := -T\big(S_j(\mathtt{h}) + \mathtt{N}_j(\phi, \mathtt{h})\big) \quad \text{for all} \quad j = 1, \ldots, N. \tag{5.7}$$

We will check that the operator $\mathfrak{B}(\phi, \mathtt{h}) = \big(\mathfrak{B}_1(\phi, \mathtt{h}), \ldots, \mathfrak{B}_N(\phi, \mathtt{h})\big)$ is a contraction mapping in ϕ in a ball for the norm $\|\ \|_{2,p,\sigma}$. We will do more, checking as well the Lipschitz dependence in $\mathtt{h}$. Using the above lemma we conclude that the operator $\mathfrak{B}$ is a contraction mapping on the region $\|\phi\|_{2,p,\sigma} \le \varepsilon^{2-\tau}$. Now, using (3.13),

$$|\mathfrak{B}(0)| \ \le \ \chi_{j1}\Big(\big(F'(\mathtt{w}_j) + 2\big)|\Psi(0)| + |\Psi(0)|^2 + |S_j(\mathtt{w})|\Big).$$

Thus

$$|\mathfrak{B}(0)| \ \le \ \varepsilon^{4-\tau} e^{-\sigma|t|} + \chi_{j1}\varepsilon^7 + C\varepsilon^2 |D^2_{\mathcal{K}} h(\varepsilon y)| e^{-\sigma|t|} + C\varepsilon^2 e^{-\sigma|t|},$$

and hence

$$\|\mathfrak{B}(0)\|_{p,\sigma} \le C\varepsilon^{2-\tau}.$$

As a conclusion, we can apply the contraction mapping principle, and find a unique solution ϕ of problem (5.7) such that

$$\|\phi\|_{2,p,\sigma} \le \beta\varepsilon^{2-\tau},$$

for a suitably large choice of β.

6. The Jacobi-Toda system

Once problem (3.22) has been solved by $\phi = \Phi(\mathtt{h})$, according to Proposition 5.1, the remaining task is to find an $\mathtt{h}$ such that for all $\ell = 1, \dots, m$, we have

$$I_\ell(y) = \int_{\mathbb{R}} \big(S_\ell(\mathtt{h}) + \mathtt{N}_j(\Phi(\mathtt{h}), \mathtt{h})\big)\, w' \,\mathrm{d}t \, = 0 \quad \text{for all} \quad y \in \mathcal{K}_\varepsilon. \tag{6.1}$$

Using the definition of S_ℓ in (3.17), expansion (2.53), Lemma 2.5 and the definition of $\mathtt{N}_j(\Phi(\mathtt{h}), \mathtt{h})$, we get

$$\begin{aligned} \varepsilon^{-2} I_\ell(\varepsilon^{-1} y) \, &= b_1 \left(\Delta_{\mathcal{K}} h_\ell + K(y) f_\ell, \right) \, - \, b_2 \rho_\varepsilon \, \left[e^{-\sqrt{2}(h_\ell - h_{\ell-1})} - e^{-\sqrt{2}(h_{\ell+1} - h_\ell)} \right] \\ &\quad + \, \theta_\ell(\mathtt{h}), \end{aligned} \tag{6.2}$$

where θ_ℓ is a small operator:

$$\|\theta_\ell(\mathtt{h})\|_{L^p(\mathcal{K})} \, = \, O(\varepsilon^{1-\tau}),$$

for any $\tau > \frac{N-1}{p}$, uniformly on $\mathtt{h}$. The constants b_1, b_2 are given by

$$b_1 = \int_{\mathbb{R}} w'(t)^2 \,\mathrm{d}t, \quad b_2 = \int_{\mathbb{R}} 6(1 - w^2(t)) e^{\sqrt{2}t} w'(t) \mathrm{d}t = \int_{\mathbb{R}} 6(1 - w^2(t)) e^{-\sqrt{2}t} w'(t) \mathrm{d}t.$$

Part I: Recall the relation in (2.37)

$$f_\ell(y) = \Big(\ell - \frac{m+1}{2}\Big) \rho_\varepsilon + h_\ell(y).$$

Since we want that the functions h_ℓ make the quantities I_ℓ as small as possible, it is reasonable to find first an $\mathtt{h}$ such that the equations, for $\ell = 1, \dots m$,

$$b_1 \left(\Delta_{\mathcal{K}} h_\ell + K(y) f_\ell, \right) \, - \, b_2 \rho_\varepsilon \, \left[e^{-\sqrt{2}(h_\ell - h_{\ell-1})} - e^{-\sqrt{2}(h_{\ell+1} - h_\ell)} \right] = 0, \tag{6.3}$$

be approximately satisfied. We set

$$R_\ell(\mathtt{h}) := \sigma \Big(\Delta_{\mathcal{K}} h_\ell + K(y) f_\ell, \Big) \, - \, \left[e^{-\sqrt{2}(h_\ell - h_{\ell-1})} - e^{-\sqrt{2}(h_{\ell+1} - h_\ell)} \right], \tag{6.4}$$

where

$$\sigma := \sigma_\varepsilon = \rho_\varepsilon^{-1} b_1 b_2^{-1} \sim (\log \frac{1}{\varepsilon})^{-1} \ ,$$

and

$$\mathbf{R}(\mathtt{h}) := \begin{bmatrix} R_1(\mathtt{h}) \\ \vdots \\ R_m(\mathtt{h}) \end{bmatrix}. \tag{6.5}$$

We would like to find a solution $\mathtt{h}$ to the system $\mathbf{R}(\mathtt{h}) = 0$. To this end, we find first a convenient representation of the operator $\mathbf{R}(\mathtt{h})$. Let us consider the auxiliary variables

$$\mathtt{v} := \begin{bmatrix} \bar{\mathtt{v}} \\ v_m \end{bmatrix}, \quad \bar{\mathtt{v}} = \begin{bmatrix} v_1 \\ \vdots \\ v_{m-1} \end{bmatrix},$$

defined in terms of $\mathtt{h}$ as

$$v_\ell = h_{\ell+1} - h_\ell \quad \text{with } \ell = 1, \dots, m-1, \quad v_m = \sum_{\ell=1}^m h_\ell,$$

with the conventions $v_0 = v_{m+1} = +\infty$ and define the operators

$$\mathbf{S}(\mathtt{v}) := \begin{bmatrix} \bar{\mathbf{S}}(\bar{\mathtt{v}}) \\ S_m(v_m) \end{bmatrix}, \quad \bar{\mathbf{S}}(\bar{\mathtt{v}}) = \begin{bmatrix} S_1(\bar{\mathtt{v}}) \\ \vdots \\ S_{m-1}(\bar{\mathtt{v}}) \end{bmatrix}.$$

where we have setted

$$S_\ell(\mathtt{v}) := R_{\ell+1}(\mathtt{h}) - R_\ell(\mathtt{h}) \ =$$

$$\sigma\Big(\Delta_{\mathcal{K}} v_\ell + K(y)(\rho_\varepsilon + v_\ell)\Big) + \begin{cases} e^{-\sqrt{2}v_2} - 2e^{-\sqrt{2}v_1} & \text{if} \quad \ell = 1, \\ e^{-\sqrt{2}v_{\ell+1}} - 2e^{-\sqrt{2}v_\ell} + e^{-\sqrt{2}v_{\ell-1}} & \text{if} \quad 1 < \ell < m-1, \\ -2e^{-\sqrt{2}v_{m-1}} + e^{-\sqrt{2}v_{m-2}} & \text{if} \quad \ell = m-1, \end{cases}$$

and

$$S_m(v_m) := \sum_{\ell=1}^{m} R_\ell(\mathtt{h}) = \sigma\Big(\Delta_{\mathcal{K}} v_m + K(y) v_m\Big).$$

Then the operators $\mathbf{R}$ and $\mathbf{S}$ are in correspondence through the formula

$$\mathbf{S}(\mathtt{v}) \ = \ \mathbf{B}\,\mathbf{R}\left(\mathbf{B}^{-1}\mathtt{v}\right), \tag{6.6}$$

where $\mathbf{B}$ is the constant, invertible $N \times N$ matrix

$$\mathbf{B} \ = \ \begin{bmatrix} -1 & 1 & 0 & \cdots & 0 \\ 0 & -1 & 1 & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & -1 & 1 \\ 1 & \cdots & 1 & 1 & 1 \end{bmatrix}, \tag{6.7}$$

and then the system $\mathbf{R}(\mathtt{h}) = 0$ is equivalent to $\mathbf{S}(\mathtt{v}) = 0$, which setting $\beta = b_2 b_1^{-1}$ decouples into

$$\bar{\mathbf{S}}(\bar{\mathtt{v}}) \ = \ \sigma\Big[\Delta_{\mathcal{K}}\bar{\mathtt{v}} + K(y)\bar{\mathtt{v}}\Big] \ + \ \beta K(y) \begin{bmatrix} 1 \\ \vdots \\ 1 \end{bmatrix} \ + \ \bar{\mathbf{S}}_0(\bar{\mathtt{v}}) \ = \ 0, \tag{6.8}$$

$$S_m(v_m) \ = \ \sigma\left(\Delta_{\mathcal{K}} v_m + K(y) v_m\right) \ = \ 0, \tag{6.9}$$

where

$$\bar{\mathbf{S}}_0(\bar{\mathtt{v}}) := -\mathbf{C} \begin{bmatrix} e^{-\sqrt{2}v_1} \\ \vdots \\ e^{-\sqrt{2}v_{m-1}} \end{bmatrix}, \quad \mathbf{C} \ = \ \begin{bmatrix} 2 & -1 & 0 & \cdots & 0 \\ -1 & 2 & -1 & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & -1 & 2 & -1 \\ 0 & \cdots & & -1 & 2 \end{bmatrix}. \tag{6.10}$$

In system (6.8)-(6.9), the second relation and our non-degeneracy assumption force $v_m = 0$. Thus we look for a solution $\mathtt{v} = (\bar{\mathtt{v}}, 0)$ of the system, where $\bar{\mathtt{v}}$ satisfies (6.8). Rather than finding an exact solution $\bar{\mathtt{v}}$ of $\bar{\mathbf{S}}(\bar{\mathtt{v}}) = 0$ we will find a good approximation. More precisely, by means of a simple iterative procedure, we will find for each $k \geq 1$ a function $\bar{\mathtt{v}}^k$ with the property that

$$\bar{\mathbf{S}}(\bar{\mathtt{v}}^k) = O(\sigma^k). \tag{6.11}$$

Let us find a function $\bar{\mathtt{v}}^1$ with the desired property (6.11) for $k = 1$. We consider the vector $\bar{\mathtt{v}}^1(y)$ defined by the relations

$$\bar{\mathbf{S}}_0(\bar{\mathtt{v}}^1) \ = \ -\mathbf{C} \begin{bmatrix} e^{-\sqrt{2}v_1^1} \\ \vdots \\ e^{-\sqrt{2}v_{m-1}^1} \end{bmatrix} = -\beta K(y) \begin{bmatrix} 1 \\ \vdots \\ 1 \end{bmatrix} .$$

We compute explicitly

$$v_\ell^1(y) = \frac{1}{\sqrt{2}} \log \left[\frac{\beta}{2} K(y)\,(m-\ell)\,\ell \right], \ell = 1, \dots, m-1, \tag{6.12}$$

and get from (6.8)

$$\bar{\mathbf{S}}(\bar{\mathtt{v}}^1) \ = \ \sigma \left[\Delta_{\mathcal{K}} \bar{\mathtt{v}}^1 + K(y) \bar{\mathtt{v}}^1 \right] \ = \ O(\sigma).$$

This approximation can be improved to any order in powers of σ, as the following lemma states.

Lemma 6.1. *Given $k \geq 1$, there exists a function of the form*

$$\bar{\mathtt{v}}^k(y, \sigma) = \bar{\mathtt{v}}^1(y) + \sigma \xi_k(y, \sigma),$$

where $\bar{\mathtt{v}}^1(y)$ is defined by (6.12), $\xi_1 \equiv 0$, and ξ_k is smooth on $\mathcal{K} \times [0, \infty)$, such that

$$\bar{\mathbf{S}}(\bar{\mathtt{v}}^k) = O(\sigma^k)$$

as $\sigma \to 0$, uniformly on $\mathcal{K}$. In particular,

$$\mathtt{h}^k := \mathbf{B}^{-1} \begin{bmatrix} \bar{\mathtt{v}}^k \\ 0 \end{bmatrix},$$

with $\mathbf{B}$ is given by (6.7), solves approximately system (6.3) in the sense that

$$\mathbf{R}(\mathtt{h}^k) = O(\sigma^k).$$

Proof. In order to find a subsequent improvement of approximation beyond $\mathtt{v}^1$, we set $\bar{\mathtt{v}}^2 = \bar{\mathtt{v}}^1 + \omega_1$. Let us expand

$$\bar{\mathbf{S}}(\bar{\mathtt{v}}^1 + \omega) \ = \ \sigma \Big[\Delta_{\mathcal{K}} \omega + K(y) \omega \Big] + \sigma \big(\Delta_{\mathcal{K}} \mathtt{v}^1 + K(y) \mathtt{v}^1 \big) + D\bar{\mathbf{S}}_0(\bar{\mathtt{v}}^1)\omega + \mathbf{N}(\omega), \tag{6.13}$$

where

$$D\bar{\mathbf{S}}_0(\bar{\mathtt{v}}^1) \ = \ \sqrt{2}\mathbf{C} \begin{bmatrix} e^{-\sqrt{2}v_1^1} & 0 & \cdots & 0 \\ 0 & e^{-\sqrt{2}v_2^1} & \cdots & 0 \\ \vdots & & \ddots & \vdots \\ 0 & 0 & \cdots & e^{-\sqrt{2}v_{m-1}^1} \end{bmatrix}$$

$$
= \frac{\beta}{\sqrt{2}}\,K(y)\begin{bmatrix}
2a_1 & -a_2 & 0 & & \cdots & 0\\
-a_1 & 2a_2 & -a_3 & & \cdots & 0\\
0 & -a_2 & 2a_3 & & \cdots & 0\\
\vdots & & \ddots & \ddots & \ddots & \vdots\\
0 & & \cdots & -a_{m-3} & 2a_{m-2} & -a_{m-1}\\
0 & & \cdots & & -a_{m-2} & 2a_{m-1}
\end{bmatrix}, \tag{6.14}
$$

with

$$
a_\ell = (m-\ell)\,\ell, \quad \ell = 1,\dots,m-1, \tag{6.15}
$$

and

$$
\mathbf{N}(\bar{\mathrm{v}}) = \frac{\beta}{2}\,\mathbf{C}\begin{bmatrix}
a_2(\,e^{-\sqrt{2}v_1^0} - 1 + \sqrt{2}\,v_1^0\,)\\
\vdots\\
a_m(\,e^{-\sqrt{2}v_m^0} - 1 + \sqrt{2}\,v_m^0\,)
\end{bmatrix}.
$$

The matrix $D\bar{\mathbf{S}}_0(\bar{\mathrm{v}}^1)$ is clearly invertible. Let us consider the unique solution $\omega_1 = O(\sigma)$ of the linear system

$$
D\bar{\mathbf{S}}_0(\bar{\mathrm{v}}^1)\omega_1 \;=\; -\sigma\big(\Delta_{\mathcal{K}}\bar{\mathrm{v}}^1 + K(y)\bar{\mathrm{v}}^1\big) \;=\; O(\sigma), \tag{6.16}
$$

and define $\bar{\mathrm{v}}^2 = \bar{\mathrm{v}}^1 + \omega_1$. Then from (6.13) we have

$$
\bar{\mathbf{S}}(\bar{\mathrm{v}}^2) \;=\; \sigma\big(\Delta_{\mathcal{K}}\omega_1 + K(y)\omega_1\big) + \mathbf{N}(\omega_1) \;=\; O(\sigma^2). \tag{6.17}
$$

Next we define $\bar{\mathrm{v}}^3 = \bar{\mathrm{v}}^2 + \omega_2$ where $\omega_2 = O(\sigma^2)$ is the unique solution of

$$
-\,D\bar{\mathbf{S}}_0(\bar{\mathrm{v}}^1)\omega_2 \;=\; \sigma\big(\Delta_{\mathcal{K}}\omega_1 + K(y)\omega_1\big) + \mathbf{N}(\omega_1). \tag{6.18}
$$

Then from (6.13) we get

$$
\bar{\mathbf{S}}(\bar{\mathrm{v}}^3) \;=\; \sigma\big(\Delta_{\mathcal{K}}\bar{\omega}_2 + K(y)\bar{\omega}_2\big) + \mathbf{N}(\omega_1+\omega_2) - \mathbf{N}(\omega_1) \;=\; O(\sigma^2). \tag{6.19}
$$

In general, we define inductively, for $k \geq 3$, $\bar{\mathrm{v}}^{k+1} = \bar{\mathrm{v}}^k + \omega_k$ where ω_k is the unique solution of the linear system

$$
\begin{aligned}
-D\bar{\mathbf{S}}_0(\bar{\mathrm{v}}^1)\,\omega_k \;&=\; \sigma\big(\Delta_{\mathcal{K}}\omega_{k-1} + K(y)\omega_{k-1}\big) \;+\; \mathbf{N}(\omega_1+\cdots+\omega_{k-1})\\
&\qquad -\; \mathbf{N}(\omega_1+\cdots+\omega_{k-2}).
\end{aligned} \tag{6.20}
$$

Then clearly $\omega_k = O(\sigma^k)$. Let us estimate the size of $\bar{\mathbf{S}}(\bar{\mathrm{v}}^{k+1})$. From (6.13) we have

$$
\bar{\mathbf{S}}(\bar{\mathrm{v}}^{k+1}) \;=\; \sigma\big(\Delta_{\mathcal{K}}\bar{\mathrm{v}}^1 + K(y)\bar{\mathrm{v}}^1\big) \;+\; \Big[\sigma(\Delta_{\mathcal{K}} + K) + D\mathbf{S}_0(\bar{\mathrm{v}}^1) + \mathbf{N}\Big]\,\Big(\,\sum_{i=1}^{k}\omega_i\,\Big).
$$

Now, using (6.16), (6.18) and (6.20) we get

$$
\begin{aligned}
&\Big[\sigma(\Delta_{\mathcal K}+K)+D\mathbf{S}_0(\bar{\mathtt v}^1)\Big]\,\Big(\sum_{i=1}^{k}\omega^i\Big)\\
&\quad= \sigma(\Delta_{\mathcal K}\bar{\mathtt v}^1+K\bar{\mathtt v}^1)\,+\,D\bar{\mathbf S}_0(\bar{\mathtt v}^1)\omega^1\,+\,\sigma(\Delta_{\mathcal K}\omega_k+K\omega_k)\\
&\qquad+\sum_{i=2}^{k}\big[\sigma(\Delta_{\mathcal K}\omega_{i-1}+K\omega_{i-1})+D\bar{\mathbf S}_0(\bar{\mathtt v}^1)\omega_i\big]\\
&\quad= -\mathbf N(\omega^1)-\sum_{i=3}^{k}[\mathbf N(\omega_1+\cdots+\omega_{i-1})-\mathbf N(\omega_1+\cdots+\omega_{i-2})]\\
&\quad= -\mathbf N(\omega_1+\cdots+\omega_k)\,.
\end{aligned}
$$

Hence,

$$
\begin{aligned}
\bar{\mathbf S}(\bar{\mathtt v}^{k+1}) &= \sigma(\Delta_{\mathcal K}\omega_k+K\omega_k)\,+\,\mathbf N(\omega_1+\cdots+\omega_{k-1}+\omega_k)\,-\,\mathbf N(\omega_1+\cdots+\omega_{k-1})\\
&= O(\sigma^{k+1}). \qquad (6.21)
\end{aligned}
$$

Finally, the functions $\xi_1\equiv 0$ and

$$\xi_k := \sigma^{-1}(\omega_1+\cdots+\omega_{k-1}),\quad k\ge 2,$$

clearly satisfy the conclusions of the lemma, and the proof is concluded. □

Part II: The question now, is how to use the approximation $\mathtt h^k$ just constructed to find an exact $\mathtt h$ solution to system (6.1). This system takes the form

$$\mathbf R(\mathtt h)=g, \tag{6.22}$$

where g is a small function, actually a small nonlinear operator in $\mathtt h$. For the moment we will think of g as a fixed function. Since the operator $\mathbf R$ decouples as in (6.6) when expressed in terms of $\mathbf S$, it is more convenient to consider the equivalent problem

$$\mathbf S(\mathtt v)=g, \tag{6.23}$$

which, according to expressions (6.8) and (6.9), decouples as

$$\bar{\mathbf S}(\bar{\mathtt v})\,=\,\sigma\Big[\Delta_{\mathcal K}\bar{\mathtt v}+K(y)\bar{\mathtt v}\Big]\,+\,\beta K(y)\begin{bmatrix}1\\ \vdots\\ 1\end{bmatrix}\,+\,\bar{\mathbf S}_0(\bar{\mathtt v})\,=\,\bar g, \tag{6.24}$$

$$\mathbf S_m(v_m)\,=\,\sigma\,(\Delta_{\mathcal K}v_m+K(y)v_m)\,=\,g_m. \tag{6.25}$$

Equation (6.25) has a unique solution v_m for any given function g_m, thanks to the nondegeneracy assumption. Therefore we will concentrate in solving Problem (6.24), for a small given $\bar g$. Let us write

$$\bar{\mathtt v}=\bar{\mathtt v}^k+\omega,$$

where $\bar{\mathtt v}^k$ is the approximation in Lemma 6.1. We express (6.24) in the form

$$\tilde L_\sigma(\omega)\,:=\,-\sigma\Big[\Delta_{\mathcal K}\omega+K(y)\omega\Big]-D\bar{\mathbf S}_0(\bar{\mathtt v}^k)\omega\,=\,\bar{\mathbf S}(\bar{\mathtt v}^k)+\mathbf N_1(\omega)+\bar g, \tag{6.26}$$

where

$$\mathbf{N}_1(\omega) := \bar{\mathbf{S}}_0(\bar{\mathrm{v}}^k + \omega) - \bar{\mathbf{S}}_0(\bar{\mathrm{v}}^k) - D\bar{\mathbf{S}}_0(\bar{\mathrm{v}}^k)\omega\,, \tag{6.27}$$

and $\mathbf{S}_0$ is the operator in (6.10).

The desired solvability theory will be a consequence of a suitable invertibility statement for the linear operator $\tilde{L}_\sigma$. Thus we consider the equation

$$\tilde{L}_\sigma(\omega) = \tilde{g} \quad \text{in } \mathcal{K}. \tag{6.28}$$

This operator is vector valued. It is convenient to express it in self-adjoint form by replacing the matrix $D\bar{\mathbf{S}}_0(\bar{\mathrm{v}}^k)$ with a symmetric one. We recall that we have

$$D\bar{\mathbf{S}}_0(\bar{\mathrm{v}}^k) \;=\; \sqrt{2}\mathbf{C}\begin{bmatrix} e^{-\sqrt{2}v_1^k} & 0 & \cdots & 0 \\ 0 & e^{-\sqrt{2}v_2^k} & \cdots & 0 \\ \vdots & & \ddots & \vdots \\ 0 & 0 & \cdots & e^{-\sqrt{2}v_{m-1}^k} \end{bmatrix},$$

where the matrix $\mathbf{C}$ is given in (6.10). $\mathbf{C}$ is symmetric and positive definite. Indeed, a straightforward computation yields that its eigenvalues are explicitly given by

$$1,\ \frac{1}{2}, \ldots, \frac{m-1}{m}.$$

We consider the symmetric, positive definite square root matrix of $\mathbf{C}$ and denote it by $\mathbf{C}^{\frac{1}{2}}$. Then setting

$$\omega := \mathbf{C}^{\frac{1}{2}}\psi, \quad g := \mathbf{C}^{-\frac{1}{2}}\tilde{g},$$

we see that equation (6.28) becomes

$$L_\sigma(\psi) := -\sigma\Delta_{\mathcal{K}}\psi - \mathbf{A}(y,\sigma)\psi \;=\; g \quad \text{in } \mathcal{K}, \tag{6.29}$$

where $\mathbf{A}$ is the symmetric matrix

$$\mathbf{A}(y,\sigma) \;=\; \sigma K(y)\,\mathbf{I}_{m-1} + \sqrt{2}\,\mathbf{C}^{\frac{1}{2}}\begin{bmatrix} e^{-\sqrt{2}v_1^k} & 0 & \cdots & 0 \\ 0 & e^{-\sqrt{2}v_2^k} & \cdots & 0 \\ \vdots & & \ddots & \vdots \\ 0 & 0 & \cdots & e^{-\sqrt{2}v_{m-1}^k} \end{bmatrix}\mathbf{C}^{\frac{1}{2}}. \tag{6.30}$$

Since

$$\mathrm{v}^k = \mathrm{v}^1(y) + \sigma\xi^k(y,\sigma),$$

we have that A is smooth in its variables and

$$\mathbf{A}(y,0) \;=\; \frac{\beta}{\sqrt{2}}\,K(y)\,\mathbf{C}^{\frac{1}{2}}\begin{bmatrix} a_1 & 0 & \cdots & 0 \\ 0 & a_2 & \cdots & 0 \\ \vdots & & \ddots & \vdots \\ 0 & 0 & \cdots & a_{m-1} \end{bmatrix}\mathbf{C}^{\frac{1}{2}}. \tag{6.31}$$

where $a_\ell = \ell(m-\ell)$. In particular, $\mathbf{A}(y,\sigma)$ has uniformly positive eigenvalues whenever σ is sufficiently small.

Our main result concerning uniform solvability of Problem (6.29) is the following.

Proposition 6.1. *There exists a sequence of values $\sigma = \sigma_\ell \to 0$ such that L_σ is invertible. More precisely, for any $g \in L^2(\mathcal{K})$ there exists a unique solution $\psi = L_\sigma^{-1} g \in H^1(\mathcal{K})$ to equation (6.29). This solution satisfies*

$$\sigma \|D^2_{\mathcal{K}}\psi\|_{L^2(\mathcal{K})} + \|\psi\|_{L^2(\mathcal{K})} \ \le\ C\sigma^{-\frac{N-1}{2}} \|g\|_{L^2(\mathcal{K})}. \tag{6.32}$$

Moreover, if $p > N-1$, there exist $C, \nu > 0$ such that the solution satisfies besides the estimate

$$\|D^2_{\mathcal{K}}\psi\|_{L^p(\mathcal{K})} + \|D_{\mathcal{K}}\psi\|_{L^\infty(\mathcal{K})} + \|\psi\|_{L^\infty(\mathcal{K})} \ \le\ C\sigma^{-\frac{N-1}{2}-\nu} \|g\|_{L^p(\mathcal{K})}.$$

In addition, for $N = 2$, we have the existence of positive numbers $\nu_1, \nu_2, \ldots, \nu_{m-1}$ such that for all small σ with

$$|\nu_i \sigma - j^2| > c\sigma^{-\frac{1}{2}} \quad \textit{for all} \quad j \ge 1, \quad i = 1, \ldots, m-1,$$

for some $c > 0$, then $\psi = L_\sigma^{-1} g$ exists and estimate (6.32) holds.

We postpone the proof of this result in the last section. Assuming its validity, we will use it to derive a solvability statement for the Problem (6.22), and to conclude the corresponding solvability of system (6.1), hence that of Theorem 1.

7. Solving system (6.1): Conclusion of the proof of Theorem 1

7.1. **Solving Problem (6.22).** Here we refer to objects and notation introduced in the previous section.

Because of the definition of L_σ, the statement of Proposition 6.1 holds as well for the operator $\tilde{L}_\sigma$ in equation (6.28). Choosing σ as in the proposition, we write this equation as the fixed point problem

$$\omega \ =\ T(\omega) := \tilde{L}_\sigma^{-1} \left(\bar{\mathbf{S}}(\bar{\mathrm{v}}^k) + \bar{g} + \mathbf{N}_1(\omega) \right). \tag{7.1}$$

By construction, we have that

$$\|\bar{\mathbf{S}}(\bar{\mathrm{v}}^k)\|_{L^p(\mathcal{K})} \le C\sigma^k.$$

On the other hand, if $\|\omega\|_{L^\infty(\mathcal{K})} \le \delta$, with δ sufficiently small we also have that

$$\|\mathbf{N}_1(\omega)\|_{L^\infty(\mathcal{K})} \le C\delta^2,$$

and in this region

$$\|\mathbf{N}_1(\omega_1) - \mathbf{N}_1(\omega_2)\|_{L^\infty(\mathcal{K})} \le C\delta \|\omega_1 - \omega_2\|_{L^\infty(\mathcal{K})}.$$

We observe then that, for ν as in Proposition 6.1,

$$\|T(\omega_1) - T(\omega_1)\|_{W^{2,p}(\mathcal{K})} \ \le\ C\delta\sigma^{-\frac{N-1}{2}-\nu} \|\omega_1 - \omega_2\|_{L^\infty(\mathcal{K})},$$

and

$$\|T(\omega)\|_{W^{2,p}} \ \le\ C\sigma^{-\frac{N-1}{2}-\nu} \left(\sigma^k + \delta^2 + \|g\|_{L^p(\mathcal{K})}\right).$$

Thus if we choose

$$k > 2\Big(\frac{N-1}{2} + \nu\Big),$$

and g with

$$\|g\|_{L^p(\mathcal{K})} \le \sigma^k,$$

then the choice $\delta = \mu\sigma^{k-\frac{N-1}{2}-\nu}$ with μ large and fixed yields, thanks to the contraction mapping principle, the existence of a unique solution ω to Problem (7.1), with

$$\|\omega\|_{W^{2,p}(\mathcal{K})} \leq \mu\sigma^{k-\frac{N-1}{2}-\nu}.$$

Let us call $\omega =: \Omega(g)$. Then, in addition Ω satisfies the Lipschitz condition

$$\|\Omega(g_1) - \Omega(g_2)\|_{W^{2,p}(\Omega)} \;\leq\; C\,\sigma^{-\frac{N-1}{2}-\nu}\,\|g_1 - g_2\|_{L^p(\mathcal{K})}$$

for all g_1, g_2 with $\|g\|_{L^p(\mathcal{K})} \leq \sigma^k$. It follows that the equation (6.23)

$$\mathbf{S}(\mathtt{v}) = g \tag{7.2}$$

can be solved under these conditions. In the form

$$\mathtt{v} = V(g) := \begin{bmatrix} \bar{\mathtt{v}}^k + \Omega(\bar{g}) \\ \sigma^{-1}(\Delta_{\mathcal{K}} + K)^{-1} g_m \end{bmatrix},$$

and therefore the equation $\mathbf{R}(\mathtt{h}) = \tilde{g}$ can be solved for any given small $\tilde{g} \in L^p(\mathcal{K})$ by means of the correspondence

$$\mathbf{S}(\mathtt{v}) \;=\; \mathbf{B}\,\mathbf{R}\left(\mathbf{B}^{-1}\mathtt{v}\right).$$

This yields the following result.

Lemma 7.1. *Given $k > 2(\frac{N-1}{2} + \nu)$, then for all sufficiently small σ satisfying the statement of Proposition 6.1, and all functions $\tilde{g}$ with*

$$\|\tilde{g}\|_{L^p(\Omega)} \leq \sigma^k,$$

there exists a solution of the equation

$$\mathbf{R}(\mathtt{h}) = \tilde{g}, \tag{7.3}$$

of the form

$$\mathtt{h} = \mathtt{h}^k + H(\tilde{g}),$$

where the operator H satisfies

$$\|H(\tilde{g}_1)\|_{W^{2,p}(\Omega)} \;\leq\; C\,\sigma^{k-\frac{N-1}{2}-\nu},$$

and

$$\|H(\tilde{g}_1) - H(\tilde{g}_2)\|_{W^{2,p}(\Omega)} \;\leq\; C\,\sigma^{-\frac{N-1}{2}-\nu}\,\|\tilde{g}_1 - \tilde{g}_2\|_{L^p(\mathcal{K})}.$$

□

7.2. **Proof of Theorem 1.** We need to prove the existence of $\mathtt{h}$ satisfying System (6.1). According to expansion (6.2), we have that

$$\frac{\sigma}{b_2}\varepsilon^{-2} I_\ell(\varepsilon^{-1}y) \;=\; \mathbf{R}(\mathtt{h}) \;-\; G(\mathtt{h}), \tag{7.4}$$

where

$$G(\mathtt{h}) = -\sigma\frac{\sigma}{b_2}\,\theta_\ell(y),$$

and θ_ℓ is the remainder in (6.2). We will estimate this operator. We have that

$$\theta_\ell(y) = \underbrace{(-1)^{\ell-1}\varepsilon^{-2}\int_{\mathbb{R}} \zeta_\varepsilon \Theta_\ell(\mathtt{h})(\varepsilon^{-1}y,t) w'(t)\,\mathrm{d}t}_{Q_1(\mathtt{h})} + \underbrace{\varepsilon^{-2}\int_{\mathbb{R}} \mathtt{N}_\ell(\Phi(\mathtt{h}),\mathtt{h})(\varepsilon^{-1}y,t)\,w'\,\mathrm{d}t}_{Q_2(\mathtt{h})}$$

where Θ_ℓ is described in (2.53). We have, using Lemma 2.5,

$$\|Q_1(\mathtt{h})\|_{L^p(\mathcal{K})} \le \varepsilon^{\frac{N-1}{p}-2} \int_{\mathbb{R}} \|\zeta_\varepsilon \Theta_\ell(\mathtt{h})(\cdot,t)\|_{L^p(\mathcal{K}_\varepsilon)}\, w'(t)\,\mathrm{d}t$$

$$\le C\varepsilon^{-2}\|\zeta_\varepsilon\Theta_\ell(\mathtt{h})(\cdot,t)\|_{p,\sigma} \le C\varepsilon^{1-\tau}.$$

And similarly, using Lemma 2.6 we get

$$\|Q_1(\mathtt{h}^1) - Q_1(\mathtt{h}^2)\|_{L^p(\mathcal{K})} \le C\varepsilon^{1-\tau}\|\mathtt{h}^1 - \mathtt{h}^2\|_{W^{2,p}(\mathcal{K})}.$$

whenever the vector-valued functions $\mathtt{h}_1$, $\mathtt{h}_2$ satisfy constraints (2.34). A similar argument, using the estimates for the operator $N_j(\Phi(\mathtt{h}),\mathtt{h})$ in Proposition 5.1 yields

$$\|Q_2(\mathtt{h})\|_{L^p(\mathcal{K})} \le C\varepsilon^{2-\tau}, \quad \|Q_2(\mathtt{h}^1) - Q_2(\mathtt{h}^2)\|_{L^p(\mathcal{K})} \le C\varepsilon^{2-\tau}\|\mathtt{h}^1 - \mathtt{h}^2\|_{W^{2,p}(\mathcal{K})}.$$

As a consequence, the operator $G(\mathtt{h})$ satisfies

$$\|G(\mathtt{h})\|_{L^p(\mathcal{K})} \le C\varepsilon^{2-\tau}, \quad \|G(\mathtt{h}^1) - G(\mathtt{h}^2)\|_{L^p(\mathcal{K})} \le C\varepsilon^{1-\tau}\|\mathtt{h}^1 - \mathtt{h}^2\|_{W^{2,p}(\mathcal{K})}, \tag{7.5}$$

Thus we need to solve the system

$$\mathbf{R}(\mathtt{h}) = G(\mathtt{h}), \tag{7.6}$$

which can be rewritten in the form

$$\mathbf{R}(\mathtt{h}^k + \mathtt{q}) = G(\mathtt{h}^k + \mathtt{q}). \tag{7.7}$$

We use the operator $H(g)$ defined in Lemma 7.1, and look for a solution of (7.7) by solving

$$\mathtt{h} = H(G(\mathtt{h}^k + \mathtt{q})) =: D(\mathtt{q}), \tag{7.8}$$

for a sufficiently large k, in the region

$$\mathcal{R} = \Big\{ \mathtt{q} \in W^{2,p}(\mathcal{K}) \ / \ \|\mathtt{q}\|_{W^{2,p}(\mathcal{K})} \le \mu\sigma^{k-\frac{N-1}{2}-\nu} \Big\},$$

for a sufficiently large μ. From Lemma 7.1 and (7.5), we get that

$$\|D(\mathtt{q}_1) - D(\mathtt{q}_2)\|_{W^{2,p}(\mathcal{K})} \le C\,\sigma^{-\frac{N-1}{2}-\nu}\,\|G(\mathtt{h}^k + \mathtt{q}_1) - G(\mathtt{h}^k + \mathtt{q}_2)\|_{L^p(\mathcal{K})}$$

$$\le C\,\sigma^{-\frac{N-1}{2}-\nu}\,\varepsilon^{\tau}\|\mathtt{q}_1 - \mathtt{q}_2\|_{W^{2,p}(\mathcal{K})},$$

Hence D is a contraction mapping in $\mathcal{R}$. Besides we have

$$\|D(0)\|_{W^{2,p}(\Omega)} \le C\,\sigma^{k-\frac{N-1}{2}-\nu}.$$

From here it follows the existence of a fixed point $\mathtt{q} = O(\sigma^{k-\frac{N-1}{2}-\nu})$ for Problem (7.8), and hence $\mathtt{h} = \mathtt{h}^k + \mathtt{q}$ satisfies constraints (2.34) and solves System (6.1). This concludes the proof of the theorem. $\square$

8. Inverting the linearized Jacobi-Toda operator

In this section we will prove Proposition 6.1. The first part of the result holds in larger generality. Actually the properties we will use in the matrix function $\mathbf{A}(y,\sigma)$ are its symmetry, its smooth dependence in its variables on $\mathcal{K}\times[0,\sigma_0)$, and the fact that for certain numbers $\gamma_\pm>0$, we have

$$\gamma_-|\xi|^2\le \xi^T\mathbf{A}(y,\sigma)\xi\le\gamma_+|\xi|^2 \quad\text{for all}\quad \xi\in\mathbb{R}^{m-1},\ (y,\sigma)\in\mathcal{K}\times[0,\sigma_0). \tag{8.1}$$

Most of the work in the proof consists in finding the sequence σ_ℓ such that 0 lies suitably away from the spectrum of L_{σ_ℓ}, when this operator is regarded as self-adjoint in $L^2(\mathcal{K})$. The result will be a consequence of various considerations on the asymptotic behavior of the small eigenvalues of L_σ as $\sigma\to 0$. The general scheme below has already been used in related settings, see [19, 20, 21, 16, 17], using the theory of smooth and analytic dependence of eigenvalues of families of Fredholm operators due to T. Kato[12]. Our proof relies only on elementary considerations on the variational characterization of the eigenvalues of L_σ and Weyl's asymptotic formula.

As in the above mentioned works, the assertion holds not only along a sequence, but actually for all values of σ inside a sequence of disjoint intervals centered at the σ_ℓ's of width $O(\sigma_\ell^{\frac{N-1}{2}})$. The corresponding assertion for $N=2$ can be made much more precise.

Thus, we consider the eigenvalue problem

$$L_\sigma\phi=\lambda\phi\quad\text{in }\mathcal{K}. \tag{8.2}$$

For each $\sigma>0$ the eigenvalues are given by a sequence $\lambda_j(\sigma)$, characterized by the Courant-Fisher formulas

$$\lambda_j(\sigma)=\sup_{dim(M)=j-1}\ \inf_{\phi\in M^\perp\setminus\{0\}}Q_\sigma(\phi,\phi)=\inf_{dim(M)=j}\ \sup_{\phi\in M\setminus\{0\}}Q_\sigma(\phi,\phi), \tag{8.3}$$

where

$$Q_\sigma(\phi,\phi)=\frac{\int_{\mathcal{K}}L_\sigma\phi\cdot\phi}{\int_{\mathcal{K}}|\phi|^2}=\frac{\int_{\mathcal{K}}\sigma|\nabla\phi|^2-\phi^T\mathbf{A}(y,\sigma)\phi}{\int_{\mathcal{K}}|\phi|^2}.$$

We have the validity of the following result.

Lemma 8.1. *There is a number $\sigma_*>0$ such that for all $0<\sigma_1<\sigma_2<\sigma_*$ and all $j\ge 1$ the following inequalities hold.*

$$(\sigma_2-\sigma_1)\frac{\gamma_-}{2\sigma_2^2}\le\sigma_2^{-1}\lambda_j(\sigma_2)-\sigma_1^{-1}\lambda_j(\sigma_1)\le 2(\sigma_2-\sigma_1)\frac{\gamma_+}{\sigma_1^2}. \tag{8.4}$$

In particular, the functions $\sigma\in(0,\sigma_*)\mapsto\lambda_j(\sigma)$ are continuous.

Proof. Let us consider small numbers $0<\sigma_1<\sigma_2$. We observe that for any ϕ with $\int_{\mathcal{K}}|\phi|^2=1$ we have

$$\sigma_1^{-1}Q_{\sigma_1}(\phi,\phi)-\sigma_2^{-1}Q_{\sigma_2}(\phi,\phi)=-\int_{\mathcal{K}}\phi^T(\sigma_1^{-1}\mathbf{A}(y,\sigma_1)-\sigma_2^{-1}\mathbf{A}(y,\sigma_2))\phi$$

$$=(\sigma_1-\sigma_2)\int_{\mathcal{K}}\phi^T(\sigma^{-2}\mathbf{A}(y,\sigma)-\sigma^{-1}\partial_\sigma\mathbf{A}(y,\sigma))\phi, \tag{8.5}$$

for some $\sigma \in (\sigma_1, \sigma_2)$. From the assumption (8.1) on the matrix A we then find that

$$\sigma_1^{-1} Q_{\sigma_1}(\phi,\phi) + (\sigma_2-\sigma_1)\frac{\gamma_-}{2\sigma_2^2} \le \sigma_2^{-1} Q_{\sigma_2}(\phi,\phi) \le \sigma_1^{-1} Q_{\sigma_1}(\phi,\phi) + 2(\sigma_2-\sigma_1)\frac{\gamma_+}{\sigma_1^2}.$$

From here, and formulas (8.3), inequality (8.4) follows. □

Corollary 8.1. *There exists a number $\delta > 0$ such that for any $\sigma_2 > 0$ and j such that*

$$\sigma_2 + |\lambda_j(\sigma_2)| < \delta,$$

and any σ_1 with $\frac{1}{2}\sigma_2 \le \sigma_1 < \sigma_2$, we have that

$$\lambda_j(\sigma_1) \ < \ \lambda_j(\sigma_2).$$

Proof. Let us consider small numbers $0 < \sigma_1 < \sigma_2$ such that $\sigma_1 \ge \frac{\sigma_2}{2}$. Then from (8.4) we find that

$$\lambda_j(\sigma_1) \le \lambda_j(\sigma_2) + (\sigma_2-\sigma_1)\frac{1}{\sigma_2}[\lambda_j(\sigma_2) - \gamma\frac{\sigma_1}{\sigma_2}],$$

for some $\gamma > 0$. From here the desired result immediately follows. □

8.1. **Proof of Proposition 6.1, general N.** Let us consider the numbers $\bar\sigma_\ell := 2^{-\ell}$ for large $\ell \ge 1$. We will find a sequence of values $\sigma_\ell \in (\bar\sigma_{\ell+1}, \bar\sigma_\ell)$ as in the statement of the lemma.

We define

$$\digamma_\ell = \big\{\sigma \in (\bar\sigma_{\ell+1}, \bar\sigma_\ell) : \ \ker L_\sigma \ne \{0\}\big\}. \tag{8.6}$$

If $\sigma \in \digamma_\ell$ then for some j we have that $\lambda_j(\sigma) = 0$. It follows that $\lambda_j(\bar\sigma_{l+1}) < 0$. Indeed, let us assume the opposite. Then, given $\delta > 0$, the continuity of λ_j implies the existence of $\tilde\sigma$ with $\frac{1}{2}\sigma \le \tilde\sigma < \sigma$ and $0 \le \lambda_j(\tilde\sigma) < \delta$. If δ is chosen as in Corollary 8.1, and ℓ is so large that $2^{-\ell} < \delta$, we obtain a contradiction.

As a conclusion, we find that for all large ℓ

$$\operatorname{card}(\digamma_\ell) \le N(\bar\sigma_{\ell+1}), \tag{8.7}$$

where $N(\sigma)$ denotes the number of negative eigenvalues of problem (8.2). We estimate next this number for small σ. Let us consider $a_+ > 0$ such that

$$\xi^T \mathbf{A}(y,\sigma)\xi \le a_+|\xi|^2 \quad \text{for all} \quad \xi \in \mathbb{R}^{m-1},\ (y,\sigma) \in \mathcal{K}\times[0,\sigma_0),$$

and the operator

$$L_\sigma^+ = -\Delta_{\mathcal{K}} - \frac{a^+}{\sigma}. \tag{8.8}$$

Let $\lambda_j^+(\sigma)$ denote its eigenvalues. From the Courant-Fisher characterization we see that $\lambda_j^+(\sigma) \le \lambda_j(\sigma)$. Hence $N(\sigma) \le N_+(\sigma)$, where the latter quantity designates the number of negative eigenvalues of L_σ^+.

Let us denote by μ_j the eigenvalues of $-\Delta_{\mathcal{K}}$. Then Weyl's asymptotic formula for eigenvalues of the Laplace-Beltrami operator, see for instance [5, 15, 23], asserts that for a certain constant $C_{\mathcal{K}} > 0$,

$$\mu_j = C_{\mathcal{K}} j^{\frac{2}{N-1}} \ + \ o(j^{\frac{2}{N-1}}) \quad \text{as } j \to +\infty. \tag{8.9}$$

Using the fact that $\lambda_j^+(\sigma) = \mu_j - \frac{a^+}{\sigma}$ and (8.9) we then find that

$$N_+(\sigma) = C\sigma^{-\frac{N-1}{2}} + o(\sigma^{-\frac{N-1}{2}}) \quad \text{as } \sigma \to 0. \tag{8.10}$$

As a conclusion, using (8.7) we find

$$\mathrm{card}\,(\mathcal{F}_\ell) \le N(\bar\sigma_{\ell+1}) \le C\bar\sigma_{\ell+1}^{-\frac{N-1}{2}} \le C2^{\ell\frac{N-1}{2}}. \tag{8.11}$$

Hence there exists an interval $(a_\ell, b_\ell) \subset (\bar\sigma_{\ell+1}, \bar\sigma_\ell)$ such that $a_\ell, b_\ell \in \mathcal{F}_\ell$, $\mathrm{Ker}\,(L_\sigma) = \{0\}$, $\sigma \in (a_\ell, b_\ell)$ and

$$b_\ell - a_\ell \ge \frac{\bar\sigma_\ell - \bar\sigma_{\ell+1}}{\mathrm{card}\,(\mathcal{F}_\ell)} \ge C\bar\sigma_\ell^{1+\frac{N-1}{2}}. \tag{8.12}$$

Let

$$\sigma_\ell := \frac{1}{2}(b_\ell + a_\ell).$$

We will analyze the spectrum of L_{σ_ℓ}. If some $c > 0$, and all j we have

$$|\lambda_j(\sigma_\ell)| \ge c\bar\sigma_\ell^{\frac{N-1}{2}}, \tag{8.13}$$

then we have the validity of the existence assertion and estimate (6.32). Assume the opposite, namely that for some j we have $|\lambda_j(\sigma_\ell)| \le \delta\sigma_\ell^{\frac{N-1}{2}}$, with δ arbitrarily small. Let us assume first that $0 < \lambda_j(\sigma_\ell) < \delta\sigma_\ell^{\frac{N-1}{2}}$. Then we have from Lemma 8.1,

$$\lambda_j(a_\ell) \le \lambda_j(\sigma_\ell) - (\sigma_\ell - a_\ell)\frac{1}{\sigma_\ell}[\lambda_j(\sigma_\ell) + \gamma\frac{a_\ell}{2\sigma_\ell}].$$

Hence, (8.12) and (8.13) imply that

$$\lambda_j(a_\ell) \le \delta\sigma_\ell^{\frac{N-1}{2}} - C\frac{\bar\sigma_\ell}{2\sigma_\ell}\,\bar\sigma_\ell^{\frac{N-1}{2}}\Big[\lambda_j(\sigma_\ell) + \frac{\gamma_- a_\ell}{2\sigma_\ell}\Big] < 0,$$

if δ was chosen a priori sufficiently small. It follows that $\lambda_j(\sigma)$ must vanish at some $\sigma \in (a_\ell, \sigma_\ell)$, and we have thus reached a contradiction with the choice of the interval (a_ℓ, b_ℓ).

The case $-\delta\sigma_\ell^{\frac{N-1}{2}} < \lambda_j(\sigma_\ell) < 0$ is handled similarly. In that case we get $\lambda_j(b_\ell) > 0$. The proof of existence and estimate (6.32) is thus complete.

Let us consider now a number $p > N-1$. Now we want to estimate the inverse of L_{σ_ℓ} in Sobolev norms. The equation satisfied by $\psi = L_{\sigma_\ell}^{-1}g$ has the form

$$\Delta_{\mathcal{K}}\psi = O(\sigma^{-1})[\psi + g]$$

for $\sigma = \sigma_\ell$. Then from elliptic estimates we get

$$\|\psi\|_{W^{2,q}(\mathcal{K})} \le C\sigma^{-1}[\|\psi\|_{L^q(\mathcal{K})} + \|g\|_{L^q(\mathcal{K})}] \tag{8.14}$$

Using this for $q = 2$ and estimate (6.32) we obtain

$$\|\psi\|_{W^{2,2}(\mathcal{K})} \le C\sigma^{-1}[\|\psi\|_{L^2(\mathcal{K})} + \|g\|_{L^2(\mathcal{K})}] \le C\sigma^{-\frac{N}{2}-1}\|g\|_{L^p(\mathcal{K})}.$$

From Sobolev's embedding we then find

$$\|\psi\|_{L^q(\mathcal{K})} \le C\sigma^{-\frac{N-1}{2}-1}\|g\|_{L^p(\mathcal{K})}.$$

for any $1 < q \le \frac{2(N-1)}{N-5}$ if $N > 5$, and any $q > 1$ if $N \le 5$. If $q = p$ is admissible in this range, the estimate follows from (8.14). If not, we apply it for $q = \frac{2(N-1)}{N-5}$, and then Sobolev's embedding yields

$$\|\psi\|_{L^s(\mathcal{K})} \le C\sigma^{-\frac{N-1}{2}-2}\|g\|_{L^p(\mathcal{K})}.$$

for any $1 < s \le \frac{2(N-1)}{N-6}$ if $N > 6$, and any $s > 1$ if $N \le 6$. Iterating this argument, we obtain the desired estimate after a finite number of steps. The proof of the first part of the proposition is concluded. $\square$

8.2. **The case $N = 2$. Conclusion of the proof.** We consider now the problem of solving system (6.29) when $N = 2$. We consider first the problem of solving

$$-\sigma\Delta_{\mathcal{K}}\psi - \mathbf{A}(y,0)\psi \;=\; g \quad \text{in } \mathcal{K}. \tag{8.15}$$

A main observation is the following: the linear system (8.15) can be decoupled: If $\Lambda_1, \dots, \Lambda_{m-1}$ denote the eigenvalues of the matrix

$$\mathbf{Q} := \mathbf{C}^{\frac12} \begin{bmatrix} a_1 & 0 & \cdots & 0 \\ 0 & a_2 & \cdots & 0 \\ \vdots & & \ddots & \vdots \\ 0 & 0 & \cdots & a_{m-1} \end{bmatrix} \mathbf{C}^{\frac12},$$

which coincide with those of

$$\begin{bmatrix} 2a_1 & -a_2 & 0 & & \cdots & 0 \\ -a_1 & 2a_2 & -a_3 & & \cdots & 0 \\ 0 & -a_2 & 2a_3 & & \cdots & 0 \\ \vdots & & \ddots & \ddots & \ddots & \vdots \\ 0 & & \cdots & -a_{m-3} & 2a_{m-2} & -a_{m-1} \\ 0 & & \cdots & & -a_{m-2} & 2a_{m-1} \end{bmatrix},$$

then system (8.15) expressed in coordinates associated to eigenfunctions of $\mathbf{Q}$ decouples into $m-1$ equations of the form

$$-\sigma\Delta_{\mathcal{K}}\psi_j - \frac{\beta}{\sqrt{2}}\Lambda_j\, K(y)\,\psi_j \;=\; g_j \quad \text{in } \mathcal{K}, \quad j = 1, \dots, m-1. \tag{8.16}$$

When $N = 2$ this problem reduces to an ODE. $\mathcal{K}$ is then a geodesic of $\mathcal{M}$ and $K(y)$ will simply be Gauss curvature measured along $\mathcal{K}$. Using y as arclength coordinate, and dropping the index j, Equations (8.16) take the generic form

$$\begin{gathered} -\sigma\psi'' - \mu\, K(y)\,\psi = g \quad \text{in } (0,\ell), \\ \psi(0) \;=\; \psi(\ell), \quad \psi'(0) \;=\; \psi'(\ell), \end{gathered} \tag{8.17}$$

where μ is given and fixed, and ℓ is the total length of $\mathcal{K}$.

For this problem to be uniquely solvable, we need that $\mu\sigma^{-1}$ differs from the eigenvalues $\lambda = \lambda_j$ of the problem

$$\begin{gathered} -\varphi'' = \lambda\, K(y)\,\varphi \quad \text{in } (0,\ell), \\ \varphi(0) \;=\; \varphi(\ell), \quad \varphi'(0) \;=\; \varphi'(\ell). \end{gathered} \tag{8.18}$$

More precisely, in such a case we have that the solution of (8.17) satisfies

$$\|\psi\|_{L^2(\mathcal{K})} \le \frac{\sigma^{-1}}{\min_j |\lambda_j - \sigma^{-1}\mu|}\|g\|_{L^2(\mathcal{K})}. \tag{8.19}$$

Now, we restate Problem (8.18) using the following Liouville transformation:

$$\ell_0 = \int_0^\ell \sqrt{K(y)}\,\mathrm{d}\,y, \qquad t = \frac{\pi}{\ell_0}\int_0^y \sqrt{K(\theta)}\,\mathrm{d}\,\theta,\ t\in[0,\pi),$$
$$\Psi(y) = K(y)^{-\frac14}, \qquad e(t) = \varphi(y)/\,\Psi(y), \qquad q(t) = \frac{\ell_0^2\,\Psi''(y)}{\pi^2\,\Psi^2(y)\,K(y)}.$$

Equation (8.18) then becomes

$$-e'' - q(t)\,e = \frac{\ell_0^2}{\pi^2}\,\lambda\,e \quad \text{ in } (0,\pi),\ e(0) \,=\, e(\pi),\ e'(0) = e'(\pi).$$

A result in [14] shows that, as $j\to\infty$ we have

$$\lambda_j = \frac{4\pi^2 j^2}{\ell_0^2} \,+\, O(j^{-2}).$$

Hence, if for some $c>0$ we have that

$$\left|\sigma^{-1}\mu - \frac{4\pi^2 j^2}{\ell_0^2}\right| \,>\, c\sigma^{-\frac12} \quad \text{for all} \quad j\geq 1,$$

and σ is sufficiently small, then the problem will be solvable, and thanks to (8.19), we will have the estimate

$$\|\psi\|_{L^2(\mathcal{K})} \leq C\sigma^{-\frac12}\|g\|_{L^2(\mathcal{K})}, \tag{8.20}$$

for the unique solution of Problem (8.17). It follows that, under these conditions System (8.16) is uniquely solvable and its solution $\psi = -(\sigma\Delta_{\mathcal{K}}\psi + \mathbf{A}(y,0))^{-1}g$ satisfies estimate (8.20).

Now, for σ as above, we can write system (6.29) in the fixed point form in $L^2(\mathcal{K})$,

$$\psi + T(\psi) = -(\sigma\Delta_{\mathcal{K}}\psi + \mathbf{A}(y,0))^{-1}g, \quad \psi\in L^2(\mathcal{K}), \tag{8.21}$$

where

$$T(\psi) := (\sigma\Delta_{\mathcal{K}}\psi + \mathbf{A}(y,0))^{-1}[(\mathbf{A}(y,\sigma) - \mathbf{A}(y,0))\psi]\,.$$

We observe that, as an operator in $L^2(\mathcal{K})$, $\|T\| = O(\sigma^{\frac12})$. Thus, for small σ, Problem (8.21) is uniquely solvable, and satisfies (8.20). Finally, for the L^p case, we argue with the same bootstrap procedure of §8.1.

The proof of the proposition is complete. □

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