On Entire Solutions of an Elliptic System Modeling Phase Separations

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Abstract

We study the qualitative properties of a limiting elliptic system arising in phase separation for Bose-Einstein condensates with multiple states:

$$\begin{cases} \Delta u = uv^2 & \text{in } \mathbb{R}^n, \\ \Delta v = vu^2 & \text{in } \mathbb{R}^n, \\ u, v > 0 & \text{in } \mathbb{R}^n. \end{cases}$$

When $n = 1$, we prove uniqueness of the one-dimensional profile. In dimension 2, we prove that stable solutions with linear growth must be one-dimensional. Then we construct entire solutions in $\mathbb{R}^2$ with polynomial growth $|x|^d$ for any positive integer $d \geq 1$. For $d \geq 2$, these solutions are not one-dimensional. The construction is also extended to multi-component elliptic systems.

Keywords: Stable solutions, elliptic systems, phase separations, Almgren's monotonicity formulae.

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1 Introduction and Main Results

Consider the following two-component Gross-Pitaevskii system

$$-\Delta u + \alpha u^3 + \Lambda v^2 u = \lambda_1 u \qquad \text{in } \Omega, \tag{1.1}$$

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$$-\Delta v+\beta v^3+\Lambda u^2 v=\lambda_2 v \qquad \text{in } \Omega, \tag{1.2}$$

$$u>0, \quad v>0 \qquad \text{in } \Omega, \tag{1.3}$$

$$u=0, \quad v=0 \qquad \text{on } \partial\Omega\,, \tag{1.4}$$

$$\int_\Omega u^2=N_1, \quad \int_\Omega v^2=N_2\,, \tag{1.5}$$

where $\alpha,\beta,\Lambda>0$ and Ω is a bounded smooth domain in $\mathbb{R}^n$. Solutions of (1.1)-(1.5) can be regarded as critical points of the energy functional

$$E_\Lambda(u,v)=\int_\Omega \left(|\nabla u|^2+|\nabla v|^2\right)+\frac{\alpha}{2}u^4+\frac{\beta}{2}v^4+\frac{\Lambda}{2}u^2v^2\,, \tag{1.6}$$

on the space $(u,v)\in H_0^1(\Omega)\times H_0^1(\Omega)$ with constraints

$$\int_\Omega u^2dx=N_1, \int_\Omega v^2dx=N_2. \tag{1.7}$$

The eigenvalues λ_j's are Lagrange multipliers with respect to (1.7). Both eigenvalues $\lambda_j=\lambda_{j,\Lambda}, j=1,2$, and eigenfunctions $u=u_\Lambda, v=v_\Lambda$ depend on the parameter Λ. As the parameter Λ tends to infinity, the two components tend to separate their supports. In order to investigate the basic rules of phase separations in this system one needs to understand the asymptotic behavior of (u_Λ,v_Λ) as $\Lambda\to+\infty$.

We shall assume that the solutions (u_Λ,v_Λ) of (1.1)-(1.5) are such that the associated eigenvalues $\lambda_{j,\Lambda}$'s are uniformly bounded, together with their energies $E_\Lambda(u_\Lambda,v_\Lambda)$. Then, as $\Lambda\to+\infty$, there is weak convergence (up to a subsequence) to a limiting profile (u_∞,v_∞) which formally satisfies

$$\begin{cases}-\Delta u_\infty+\alpha u_\infty^3=\lambda_{1,\infty}u_\infty & \text{in } \Omega_u\,,\\ -\Delta v_\infty+\beta v_\infty^3=\lambda_{2,\infty}v_\infty & \text{in } \Omega_v\,,\end{cases} \tag{1.8}$$

where $\Omega_u=\{x\in\Omega : u_\infty(x)>0\}$ and $\Omega_v=\{x\in\Omega : v_\infty(x)>0\}$ are positivity domains composed of finitely disjoint components with positive Lebesgue measure, and each $\lambda_{j,\infty}$ is the limit of $\lambda_{j,\Lambda}$'s as $\Lambda\to\infty$ (up to a subsequence).

There is a large literature about this type of questions. Effective numerical simulations for (1.8) can be found in [5], [6] and [13]. Chang-Lin-Lin-Lin [13] proved pointwise convergence of (u_Λ,v_Λ) away from the interface $\Gamma\equiv\{x\in\Omega : u_\infty(x)=v_\infty(x)=0\}$. In Wei-Weth [27] the uniform equicontinuity of (u_Λ,v_Λ) is established, while Noris-Tavares-Terracini-Verzini [24] proved the uniform-in-Λ Hölder continuity of (u_Λ,v_Λ). The regularity of the nodal set of the limiting profile has been investigated in [12, 26] and in [16]: it turns out that the limiting pair $(u_\infty(x),v_\infty(x))$ is the positive and negative pair (w^+,w^-) of a solution of the equation $-\Delta w+\alpha(w^+)^3-\beta(w^-)^3=\lambda_{1,\infty}w^+-\lambda_{2,\infty}w^-$.

To derive the asymptotic behavior of (u_Λ,v_Λ) near the interface $\Gamma=\{x\in\Omega : u_\infty(x)=v_\infty(x)=0\}$, one is led to considering the points $x_\Lambda\in\Omega$ such that $u_\Lambda(x_\Lambda)=v_\Lambda(x_\Lambda)=m_\Lambda\to 0$ and $x_\Lambda\to x_\infty\in\gamma\subset\Omega$ as $\Lambda\to+\infty$ (up to a subsequence). Assuming that

$$m_\Lambda^4\Lambda\to C_0>0, \tag{1.9}$$

(without loss of generality we may assume that $C_0 = 1$), then, by blowing up, we find the following nonlinear elliptic system

$$\Delta u = uv^2\,, \quad \Delta v = vu^2\,, \quad u, v > 0 \quad \text{in} \quad \mathbb{R}^n\,. \tag{1.10}$$

Problem (1.10) has been studied in Berestycki-Lin-Wei-Zhao [8], and Noris-Tavares-Terracini-Verzini [24]. It has been proved in [8] that, in the one-dimensional case, (1.9) always holds. In addition, the authors showed the existence, symmetry and nondegeneracy of the solution to one-dimensional limiting system

$$u'' = uv^2, v'' = vu^2, u, v > 0 \text{ in } \mathbb{R}. \tag{1.11}$$

In particular they showed that entire solutions are reflectionally symmetric, i.e., there exists x_0 such that $u(x - x_0) = v(x_0 - x)$. They also established a two-dimensional version of the De Giorgi Conjecture in this framework. Namely, under the growth condition

$$u(x) + v(x) \leq C(1 + |x|), \tag{1.12}$$

all monotone solution is one dimensional.

On the other hand, in [24], it was proved that the linear growth is the lowest possible for solutions to (1.10). In other words, if there exists $\alpha \in (0, 1)$ such that

$$u(x) + v(x) \leq C(1 + |x|)^\alpha, \tag{1.13}$$

then $u, v \equiv 0$.

In this paper we address three problems left open in [8]. First, we prove the uniqueness of (1.11) (up to translations and scaling). This answers the question stated in Remark 1.4 of [8]. Second, we prove that the De Giorgi conjecture still holds in the two dimensional case, when we replace the monotonicity assumption by the stability condition. A third open question of (1.10) is whether all solutions to (1.10) necessarily satisfy the growth bound (1.12). We shall answer this question negatively in this paper.

We first study the one-dimensional problem (1.11). Observe that problem (1.11) is invariant under the translations $(u(x), v(x)) \to (u(x + t), v(x + t)), \forall t \in \mathbb{R}$ and scalings $(u(x), v(x)) \to (\lambda u(\lambda x), \lambda v(\lambda x)), \forall \lambda > 0$. The following theorem classifies all entire solutions to (1.11).

Theorem 1.1. *The solution to (1.11) is unique, up to translations and scaling.*

Next,we want to classify the stable solutions in $\mathbb{R}^2$. We recall that a *stable* solution (u, v) to (1.10) is such that the linearization is weakly positive definite. That is, it satisfies

$$\int_{\mathbb{R}^n} [|\nabla\varphi|^2 + |\nabla\psi|^2 + v^2\varphi^2 + u^2\psi^2 + 4uv\varphi\psi] \geq 0, \qquad \forall \varphi, \psi \in C_0^\infty(\mathbb{R}^n).$$

In [8], it was proved that the one-dimensional solution is stable in $\mathbb{R}^n$. Our first result states that the only stable solution in $\mathbb{R}^2$, among those growing at most linearly, is the one-dimensional family.

Theorem 1.2. *Let (u, v) be a stable solution to (1.10) in $\mathbb{R}^2$. Furthermore, we assume that the growth bound (1.12) holds. Then (u, v) is one-dimensional, i.e., there exists $a \in \mathbb{R}^2, |a| = 1$ such that $(u, v) = (U(a \cdot x), V(a \cdot x))$ where (U, V) are functions of one variable and satisfies (1.11).*

Our third result shows that there are solutions to (1.10) with polynomial growth $|x|^d$ that are not one dimensional. The construction depends on the following harmonic polynomial Φ of degree d:

$$\Phi := \mathrm{Re}(z^d).$$

Note that Φ has some dihedral symmetry; indeed, let us take its d nodal lines $L_1, \cdots, L_d$ and denote the corresponding reflection with respect to these lines by $T_1, \cdots, T_d$. Then there holds

$$\Phi(T_i z) = -\Phi(z). \tag{1.14}$$

The third result of this paper is the following one.

Theorem 1.3. *For each positive integer $d \geq 1$, there exists a solution (u, v) to problem (1.10), satisfying*

1. *$u - v > 0$ in $\{\Phi > 0\}$ and $u - v < 0$ in $\{\Phi < 0\}$;*
2. *$u \geq \Phi^+$ and $v \geq \Phi^-$;*
3. *$\forall i = 1, \cdots, d$, $u(T_i z) = v(z)$;*
4. *$\forall r > 0$, the Almgren frequency function satisfies*
$$N(r) := \frac{r \int_{B_r(0)} |\nabla u|^2 + |\nabla v|^2 + u^2 v^2}{\int_{\partial B_r(0)} u^2 + v^2} \leq d; \tag{1.15}$$
5. $$\lim_{r \to +\infty} N(r) = d. \tag{1.16}$$

Note that the one-dimensional solution constructed in [8] can be viewed as corresponding to the case $d = 1$. For $d \geq 2$, the solutions of Theorem 1.3 will be obtained by a minimization argument under symmetric variations (φ, ψ) (i.e. satisfying $\varphi \circ T_i = \psi$ for every reflection T_i). The first four claims will be derived from the construction. See Theorem 4.1.

Regarding the claim 5, we note that by Almgren's monotonicity formula, (see Proposition 5.2 below), the Almgren frequency quotient $N(r)$ is increasing in r. Hence $\lim_{r \to +\infty} N(r)$ exists. To understand the asymptotics at infinity of the solutions, one way is to study the blow-down sequence defined by:

$$(u_R(x), v_R(x)) := (\frac{1}{L(R)} u(Rx) \frac{1}{L(R)} v(Rx)),$$

where $L(R)$ is chosen so that

$$\int_{\partial B_1(0)} u_R^2 + v_R^2 = 1.$$

In Section 6, we will prove

Theorem 1.4. *Let (u, v) be a solution of (1.10) such that*

$$d := \lim_{r \to +\infty} N(r) < +\infty.$$

Then d is a positive integer. As $R \to \infty$, (u_R, v_R) defined above (up to a subsequence) converges to (Ψ^+, Ψ^-) uniformly on any compact set of $\mathbb{R}^N$ where Ψ is a homogeneous harmonic polynomial of degree d. If $d = 1$ then (u, v) is asymptotically flat at infinity.

In particular this applies to the solutions found by Theorem 1.3 to yield the following property

Corollary 1.5. *Let (u, v) be a solution of (1.10) given by Theorem 1.3. Then*

$$(u_R(x), v_R(x)) := (\frac{1}{R^d} u(Rx) \frac{1}{R^d} v(Rx))$$

converges uniformly on compact subsets of $\mathbb{R}^2$ to a multiple of (Φ^+, Φ^-), where $\Phi := Re(z^d)$.

Theorem 1.4 roughly says that (u, v) is asymptotic to (Ψ^+, Ψ^-) at infinity for some homogeneous harmonic polynomial. The extra information we have in the setting of Theorem 1.3 is that $\Psi \equiv \Phi = \mathrm{Re}(z^d)$. This can be inferred from the symmetries of the solution (property 3 in Theorem 1.3).

For another elliptic system with a similar form,

$$\begin{cases} \Delta u = uv, u > 0 \text{ in } \mathbb{R}^n, \\ \Delta v = vu, v > 0 \text{ in } \mathbb{R}^n \end{cases} \tag{1.17}$$

the same result has been proved by Conti-Terracini-Verzini in [15]. In fact, their result hold for any dimension $n \geq 1$ and any harmonic polynomial function on $\mathbb{R}^n$. Note however that the problem here is different from (1.17). Actually, equation (1.17) can be reduced to a single equation: indeed, the difference $u - v$ is a harmonic function ($\Delta(u - v) = 0$) and thus we can write $v = u - \Phi$ where Φ is a harmonic function. By restricting to certain symmetry classes, then (1.17) can be solved by sub-super solution method. However, this reduction does not work for system (1.10) that we study here.

For the proof of Theorem 1.3, we first construct solutions to (1.10) in any bounded ball $B_R(0)$ satisfying appropriate boundary conditions:

$$\begin{cases} \Delta u = uv^2, \text{ in } B_R(0), \\ \Delta v = vu^2, \text{ in } B_R(0), \\ u = \Phi^+, v = \Phi^- \text{ on } \partial B_R(0). \end{cases} \tag{1.18}$$

This is done by variational method and using heat flow. The next natural step is to let $R \to +\infty$ and obtain some convergence result. This requires some uniform (in R) upper bound for solutions to (1.18). In order to prove this, we will exploit a new monotonicity formula for symmetric functions (Proposition 5.7). We also need to exclude the possibility of degeneracy, that is that the limit could be 0 or a solution with lower degree such as a one dimensional solution. To this end, we will give some lower bound using the Almgren monotonicity formula.

Lastly, we observe that the same construction works also for a system with many components. Let d be an integer or a half-integer and $2d = hk$ be a multiple of the number of components k, and G denote the rotation of order $2d$. In this way we prove the following result

Theorem 1.6. *There exists a positive solution to the system*

$$\begin{cases} \Delta u_i = u_i \sum_{j\neq i, j=1}^{k} u_j^2, \quad \textit{in} \quad \mathbb{C} = \mathbb{R}^2, i = 1, \dots, k, \\ u_i > 0, i = 1, \dots, k, \end{cases} \tag{1.19}$$

having the following symmetries (here $\overline{z}$ is the complex conjugate of z)

$$\begin{aligned} u_i(z) &= u_i(G^h z), && \textit{on } \mathbb{C}\,, i = 1, \dots, k, \\ u_i(z) &= u_{i+1}(Gz), && \textit{on } \mathbb{C}\,, i = 1, \dots, k, \\ u_{k+1}(z) &= u_1(z), && \textit{on } \mathbb{C} \\ u_{k+2-i}(z) &= u_i(\overline{z}), && \textit{on } \mathbb{C}\,, i = 1, \dots, k. \end{aligned} \tag{1.20}$$

Furthermore,

$$\lim_{r\to\infty} \frac{1}{r^{1+2d}} \int_{\partial B_r(0)} \sum_1^k u_i^2 = b \in (0, +\infty)\ ;$$

and

$$\lim_{r\to\infty} \frac{r \int_{B_r(0)} \sum_1^k |\nabla u_i|^2 + \sum_{i<j} u_i^2 u_j^2}{\int_{\partial B_r(0)} \sum_1^k u_i^2} = d\ .$$

The problem of the full classification of solutions to (1.10) is largely open. In view of our results, one can formulate several open questions.

Open problem 1. We recall from [8] that it is still an open problem to know in which dimension it is true that all monotone solution is one-dimensional. A similar open question is in which dimension it is true that all stable solution is one-dimensional. We refer to [2], [20], [18], [23], and [25] for results of this kind for Allen-Cahn equation.

Open problem 2. Let us recall that in one space dimension, there exists a unique solution to (1.11) (up to translations and scalings). Such solutions have linear growth at infinity and, in the Almgren monotonicity formula, they satisfy

$$\lim_{r\to+\infty} N(r) = 1. \tag{1.21}$$

It is natural to conjecture that, in any space dimension, a solution of (1.10) satisfying (1.21) is actually one dimensional, that is, there is a unit vector a such that $(u(x), v(x)) = (U(a\cdot x), V(a\cdot x))$

for $x \in \mathbb{R}^n$, where (U, V) solves (1.11). However this result seems to be difficult to obtain at this stage.

Open problem 3. A further step would be to prove uniqueness of the (family of) solutions having polynomial asymptotics given by Theorem 1.3 in two space dimension. A more challenging question is to classify all solutions with

$$\lim_{r \to +\infty} N(r) = d. \tag{1.22}$$

Open problem 4. For the Allen-Cahn equation $\Delta u + u - u^3 = 0$ in $\mathbb{R}^2$, solutions similar to Theorem 1.3 was first constructed in [17] for $d = 2$ and in [1] for $d \geq 3$. (However all solutions to Allen-Cahn equation are bounded.) On the other hand, it was also proved in [19] that Allen-Can equation in $\mathbb{R}^2$ admits solutions with multiple fronts. An open question is whether similar result holds for (1.10). Namely, are there solutions to (1.10) such that the set $\{u = v\}$ contains disjoint multiple curves?

Open problem 5. This question is related to extension of Theorem 1.3 to higher dimensions. We recall that for the Allen-Cahn equation $\Delta u + u - u^3 = 0$ in $\mathbb{R}^{2m}$ with $m \geq 2$, saddle-like solutions were constructed in [10] by employing properties of Simons cone. Stable solutions to Allen-Cahn equation in $\mathbb{R}^8$ with non planar level set were found in [23], using minimal cones. We conjecture that all these results should have analogues for (1.10).

2 Uniqueness of solutions in $\mathbb{R}$: Proof of Theorem 1.1

In this section we prove Theorem 1.1. Without loss of generality, we assume that

$$\lim_{x \to +\infty} u(x) = +\infty, \lim_{x \to +\infty} v(x) = 0. \tag{2.1}$$

The existence of such entire solutions has been proved in [8]. By symmetry property of solutions to (1.11) (Theorem 1.3 of [8]), we may consider the following problem

$$\begin{cases} u^{''} = uv^2, v^{''} = vu^2, u, v > 0 \text{ in } \mathbb{R}, \\ \lim_{x \to +\infty} u^{'}(x) = - \lim_{x \to -\infty} v^{'}(x) = a \end{cases} \tag{2.2}$$

where $a > 0$ is a constant. We now prove that there exists a unique solution (u, v) to (2.2), up to translations. We will prove it using the method of moving planes.

First we observe that for any solution (u, v) of (2.2), $u^{''}$ and $v^{''}$ decay exponentially at infinity. Integration shows that as $x \to +\infty$, $|u^{'}(x) - a|$ decays exponentially. (See also [8].) This implies the existence of a positive constant A such that

$$|u(x) - ax^+| + |v(x) - ax^-| \leq A. \tag{2.3}$$

Moreover, the limits

$$\lim_{x\to+\infty}(u(x)-ax^+), \lim_{x\to-\infty}(v(x)-ax^-)$$

exist.

Now assume (u_1, v_1) and (u_2, v_2) are two solutions of (2.2). For $t > 0$, denote

$$u_{1,t}(x) := u_1(x+t), v_{1,t}(x) := v_1(x+t).$$

We want to prove that there exists an optimal t_0 such that for all $t \geq t_0$,

$$u_{1,t}(x) \geq u_2(x), v_{1,t}(x) \leq v_2(x) \quad \text{in} \quad \mathbb{R}. \tag{2.4}$$

Then we will show that when $t = t_0$ these inequalities are identities. This will imply the uniqueness result.

Without loss of generality, assume (u_1, v_1) and (u_2, v_2) satisfy the estimate (2.3) with the same constant A.

Step 1. For $t \geq \frac{16A}{a}$ (A as in (2.3)), (2.4) holds.

Firstly, in the region $\{x \geq -t + \frac{2A}{a}\}$, by (2.3) we have

$$u_{1,t}(x) \geq a(x+t) - A \geq ax^+ + A \geq u_2(x); \tag{2.5}$$

while in the region $\{x \leq -t + \frac{2A}{a}\}$, we have

$$v_{1,t}(x) \leq a(x+t)^- + A \leq ax^- - A \leq v_2(x). \tag{2.6}$$

On the interval $\{x < -t + \frac{2A}{a}\}$, we have

$$\begin{cases} u_{1,t}^{''} = u_{1,t}v_{1,t}^2 \leq u_{1,t}v_2^2, \\ u_2^{''} = u_2 v_2^2. \end{cases} \tag{2.7}$$

With the right boundary conditions

$$u_{1,t}(-t+\frac{2A}{a}) \geq u_2(-t+\frac{2A}{a}), \lim_{x\to-\infty} u_{1,t}(x) = \lim_{x\to-\infty} u_2(x) = 0,$$

a direct application of the maximum principle implies

$$\inf_{\{x<-t+\frac{2A}{a}\}} (u_{1,t} - u_2) \geq 0.$$

By the same type of argument also show that

$$\sup_{\{x>-t+\frac{2A}{a}\}} (v_{1,t} - v_2) \leq 0.$$

Therefore, we have shown that for $t \geq \frac{16A}{a}$, $u_{1,t} \geq u_2$ and $v_{1,t} \leq v_2$.

Step 2. We now decrease the t to an optimal value when (2.4) holds

$$t_0 = \inf\{t^{'} \mid \text{such that} \ \ (2.4) \ \ \text{holds} \ \ \text{for all} \ \ t \geq t^{'}\}.$$

Thus t_0 is well defined by Step 1. Since $-(u_{1,t_0} - u_2)^{''} + v_{1,t_0}^2(u_{1,t_0} - u_2) \geq 0$, $-(v_2 - v_{1,t_0})^{''} + u_{1,t_0}^2(v_2 - v_{1,t_0}) \geq 0$, by the strong maximum principle, either

$$u_{1,t_0}(x) \equiv u_2(x), v_{1,t_0}(x) \equiv v_2(x) \ \ \text{in} \ \ \mathbb{R},$$

or

$$u_{1,t_0}(x) > u_2(x), v_{1,t_0}(x) < v_2(x) \ \ \text{in} \ \ \mathbb{R}. \tag{2.8}$$

Let us argue by contradiction that (2.8) holds. By the definition of t_0, there exists a sequence of $t_k < t_0$ such that $\lim\limits_{k\to+\infty} t_k = t_0$ and either

$$\inf_{\mathbb{R}}(u_{1,t_k} - u_2) < 0, \tag{2.9}$$

or

$$\sup_{\mathbb{R}}(v_{1,t_k} - v_2) > 0.$$

Let us only consider the first case.

Define $w_{1,k} := u_{1,t_k} - u_2$ and $w_{2,k} := v_2 - v_{1,t_k}$. Direct calculations show that they satisfy

$$\begin{cases} -w_{1,k}^{''} + v_{1,t_k}^2 w_{1,k} = u_2(v_2 + v_{1,t_k})w_{2,k} \ \ \text{in} \ \ \mathbb{R}, \\ -w_{2,k}^{''} + u_{1,t_k}^2 w_{2,k} = v_2(u_2 + u_{1,t_k})w_{1,k} \ \ \text{in} \ \ \mathbb{R}. \end{cases} \tag{2.10}$$

We use the auxiliary function $g(x) = \log(|x| + 3)$ as in [14]. Note that

$$g \geq 1, \quad g^{''} < 0 \ \ \text{in} \ \ \{x \neq 0\}.$$

Define $\widetilde{w}_{1,k} := w_{1,k}/g$ and $\widetilde{w}_{2,k} := w_{2,k}/g$. For $x \neq 0$ we have

$$\begin{cases} -\widetilde{w}_{1,k}^{''} - 2\dfrac{g^{'}}{g}\widetilde{w}_{1,k}^{'} + [v_{1,t_k}^2 - \dfrac{g^{'}}{g}]\widetilde{w}_{1,k} = u_2(v_2 + v_{1,t_k})\widetilde{w}_{2,k}, \ \ \text{in} \ \ \mathbb{R}, \\ -\widetilde{w}_{2,k}^{''} - 2\dfrac{g^{'}}{g}\widetilde{w}_{2,k}^{'} + [u_{1,t_k}^2 - \dfrac{g^{'}}{g}]\widetilde{w}_{2,k} = v_2(u_2 + u_{1,t_k})\widetilde{w}_{1,k}, \ \ \text{in} \ \ \mathbb{R}. \end{cases} \tag{2.11}$$

By definition, $w_{1,k}$ and $w_{2,k}$ are bounded in $\mathbb{R}$, and hence

$$\widetilde{w}_{1,k}, \widetilde{w}_{2,k} \to 0 \ \ \text{as} \ \ |x| \to \infty.$$

In particular, in view of (2.9), we know that $\inf_{\mathbb{R}}(\widetilde{w}_{1,k}) < 0$ is attained at some point $x_{k,1}$.

Note that $|x_{k,1}|$ must be unbounded, for if $x_{k,1} \to x_\infty, t_k \to t_0$, then $w_{1,k}(x_{k,1}) \to u_{1,t_0}(x_\infty) - u_2(x_\infty) = 0$. But this violates the assumption (2.8).

Since $|x_{k,1}|$ is unbounded, at $x = x_{k,1}$ there holds

$$\widetilde{w}_{1,k}'' \geq 0 \quad \text{and} \quad \widetilde{w}_{1,k}' = 0.$$

Substituting this into the first equation of (2.11), we get

$$[v_{1,t_k}(x_{k,1})^2 - \frac{g''(x_{k,1})}{g(x_{k,1})}]\widetilde{w}_{1,k}(x_{k,1}) \geq u_2(x_{k,1})(v_2(x_{k,1}) + v_{1,t_k}(x_{k,1}))\widetilde{w}_{2,k}(x_{k,1}) \tag{2.12}$$

which implies that $\widetilde{w}_{2,k}(x_{k,1}) < 0$. Thus we also have $\inf_{\mathbb{R}} \widetilde{w}_{2,k} < 0$. Assume it is attained at $x_{k,2}$. Same argument as before shows that $|x_{k,2}|$ must also be unbounded. Similar to (2.12), we have

$$[u_{1,t_k}(x_{k,2})^2 - \frac{g''(x_{k,2})}{g(x_{k,2})}]\widetilde{w}_{2,k}(x_{k,2}) \geq v_2(x_{k,2})(u_2(x_{k,2}) + u_{1,t_k}(x_{k,2}))\widetilde{w}_{1,k}(x_{k,2}). \tag{2.13}$$

Observe that

$$\widetilde{w}_{2,k}(x_{k,2}) = \inf_{\mathbb{R}} \widetilde{w}_{2,k} \leq \widetilde{w}_{2,k}(x_{k,1}),$$
$$\widetilde{w}_{1,k}(x_{k,1}) = \inf_{\mathbb{R}} \widetilde{w}_{1,k} \leq \widetilde{w}_{1,k}(x_{k,2}).$$

Substituting these into (2.12) and (2.13), we obtain

$$\widetilde{w}_{1,k}(x_{k,1}) \geq \frac{u_2(x_{k,1})[v_2(x_{k,1}) + v_{1,t_k}(x_{k,1})]}{v_{1,t_k}(x_{k,1})^2 - \frac{g''(x_{k,1})}{g(x_{k,1})}} \frac{v_2(x_{k,2})[u_2(x_{k,2}) + u_{1,t_k}(x_{k,2})]}{u_{1,t_k}(x_{k,2})^2 - \frac{g''(x_{k,2})}{g(x_{k,2})}} \widetilde{w}_{1,k}(x_{k,1}). \tag{2.14}$$

Since $\tilde{w}_{1,k}(x_{k,1}) < 0$, we conclude from (2.14) that

$$\frac{u_2(x_{k,1})[v_2(x_{k,1}) + v_{1,t_k}(x_{k,1})]}{v_{1,t_k}(x_{k,1})^2 - \frac{g''(x_{k,1})}{g(x_{k,1})}} \frac{v_2(x_{k,2})[u_2(x_{k,2}) + u_{1,t_k}(x_{k,2})]}{u_{1,t_k}(x_{k,2})^2 - \frac{g''(x_{k,2})}{g(x_{k,2})}} \geq 1 \tag{2.15}$$

where $|x_{k,1}| \to +\infty, |x_{k,2}| \to +\infty$. This is impossible since $\frac{g''(x)}{g(x)} \sim -\frac{1}{|x|^2 \log(|x|+3)}$ as $|x| \to +\infty$, and we also use the decaying as well as the linear growth properties of u and v at ∞.

We have thus reached a contradiction, and the proof of Theorem 1.1 is thereby completed.

3 Stable solutions: Proof of Theorem 1.2

In this section, we prove Theorem 1.2. The proof follows an idea from Berestycki-Caffarelli-Nirenberg [7]-see also Ambrosio-Cabré [2] and Ghoussoub-Gui [20]. First, by the stability, we have the following

Lemma 3.1. *There exist a constant $\lambda \geq 0$ and two functions $\varphi > 0$ and $\psi < 0$, smoothly defined in $\mathbb{R}^2$ such that*

$$\begin{cases} \Delta\varphi = v^2\varphi + 2uv\psi - \lambda\varphi, \\ \Delta\psi = 2uv\varphi + v^2\psi - \lambda\psi. \end{cases} \tag{3.1}$$

Proof. For any $R < +\infty$ the stability assumption reads

$$\lambda(R) := \min_{\varphi,\psi \in H_0^1(B_R(0))\setminus\{0\}} \frac{\int_{B_R(0)} |\nabla\varphi|^2 + |\nabla\psi|^2 + v^2\varphi^2 + u^2\psi^2 + 4uv\varphi\psi}{\int_{B_R(0)} \varphi^2 + \psi^2} \geq 0.$$

It's well known that the corresponding minimizer is the first eigenfunction. That is, let (φ_R, ψ_R) realizing $\lambda(R)$, then

$$\begin{cases} \Delta\varphi_R = v^2\varphi_R + 2uv\psi_R - \lambda(R)\varphi_R, \text{ in } B_R(0), \\ \Delta\psi_R = 2uv\varphi_R + v^2\psi_R - \lambda(R)\psi_R, \text{ in } B_R(0), \\ \varphi_R = \psi_R = 0 \text{ on } \partial B_R(0). \end{cases} \tag{3.2}$$

By possibly replacing (φ_R, ψ_R) with $(|\varphi_R|, -|\psi_R|)$, we can assume $\varphi_R \geq 0$ and $\psi_R \leq 0$. After a normalization, we also assume

$$|\varphi_R(0)| + |\psi_R(0)| = 1. \tag{3.3}$$

$\lambda(R)$ is decreasing in R, thus uniformly bounded as $R \to +\infty$. Let

$$\lambda := \lim_{R\to+\infty} \lambda(R).$$

The equation for φ_R and $-\psi_R$ (both of them are nonnegative functions) forms a cooperative system, thus by the Harnack inequality ([3] or [9]), φ_R and ψ_R are uniformly bounded on any compact set of $\mathbb{R}^2$. By letting $R \to +\infty$, we can obtain a converging subsequence and the limit (φ, ψ) satisfies (3.1).

We also have $\varphi \geq 0$ and $\psi \leq 0$ by passing to the limit. Hence

$$-\Delta\varphi + (v^2 - \lambda)\varphi \geq 0.$$

Applying the strong maximum principle, either $\varphi > 0$ strictly or $\varphi \equiv 0$. If $\varphi \equiv 0$, substituting this into the first equation in (3.1), we obtain $\psi \equiv 0$. This contradicts the normalization condition (3.3). Thus, it holds true that $\varphi > 0$ and similarly $\psi < 0$. □

Fix a unit vector ξ. Differentiating the equation (1.10) yields the following equation for (u_ξ, v_ξ)

$$\begin{cases} \Delta u_\xi = v^2 u_\xi + 2uvv_\xi, \\ \Delta v_\xi = 2uvu_\xi + v^2 v_\xi. \end{cases} \tag{3.4}$$

Let

$$w_1 = \frac{u_\xi}{\varphi}, w_2 = \frac{v_\xi}{\psi}.$$

Direct calculations using (3.1) and (3.4) show

$$\begin{cases} \operatorname{div}(\varphi^2\nabla w_1) = 2uv\varphi\psi(w_2 - w_1) + \lambda\varphi^2 w_1, \\ \operatorname{div}(\varphi^2\nabla w_2) = 2uv\varphi\psi(w_1 - w_2) + \lambda\psi^2 w_2. \end{cases}$$

For any $\eta \in C_0^\infty(\mathbb{R}^2)$, testing these two equations with $w_1\eta^2$ and $w_2\eta^2$ respectively, we obtain

$$\begin{cases} -\int \varphi^2|\nabla w_1|^2\eta^2 - 2\varphi^2 w_1\eta\nabla w_1\nabla\eta = \int 2uv\varphi\psi(w_2-w_1)w_1\eta^2 + \lambda\varphi^2 w_1\eta^2, \\ -\int \psi^2|\nabla w_2|^2\eta^2 - 2\psi^2 w_2\eta\nabla w_2\nabla\eta = \int 2uv\varphi\psi(w_1-w_2)w_2\eta^2 + \lambda\psi^2 w_2\eta^2. \end{cases}$$

Adding these two and applying the Cauchy-Schwarz inequality, we infer that

$$\int \varphi^2|\nabla w_1|^2\eta^2 + \psi^2|\nabla w_2|^2\eta^2 \leq 16\int \varphi^2 w_1^2|\nabla\eta|^2 + \psi^2 w_2^2|\nabla\eta|^2 \leq 16\int (u_\xi^2 + v_\xi^2)|\nabla\eta|^2. \tag{3.5}$$

Here we have taken away the positive term in the right hand side and used the fact that

$$2uv\varphi\psi(w_2-w_1)w_1\eta^2 + 2uv\varphi\psi(w_1-w_2)w_2\eta^2 = -2uv\varphi\psi(w_1-w_2)^2\eta^2 \geq 0,$$

because $\varphi > 0$ and $\psi < 0$.

On the other hand, testing the equation $\Delta u \geq 0$ with $u\eta^2$ (η as above) and integrating by parts, we get

$$\int |\nabla u|^2\eta^2 \leq 16\int u^2|\nabla\eta|^2.$$

The same estimate also holds for v. For any $r > 0$, take $\eta \equiv 1$ in $B_r(0)$, $\eta \equiv 0$ outside $B_{2r}(0)$ and $|\nabla\eta| \leq 2/r$. By the linear growth of u and v, we obtain a constant C such that

$$\int_{B_r(0)} |\nabla u|^2 + |\nabla v|^2 \leq Cr^2. \tag{3.6}$$

Now for any $R > 0$, in (3.5), we take η to be

$$\eta(z) = \begin{cases} 1, & x \in B_R(0), \\ 0, & x \in B_{R^2}(0)^c, \\ 1 - \frac{\log(|z|/R)}{\log R} & x \in B_{R^2}(0)\setminus B_R(0). \end{cases}$$

With this η, we infer from (3.5)

$$\begin{aligned} & \int_{B_R(0)} \varphi^2|\nabla w_1|^2 + \psi^2|\nabla w_2|^2 \\ \leq\ & \frac{C}{(\log R)^2}\int_{B_{R^2}(0)\setminus B_R(0)} \frac{1}{|z|^2}(|\nabla u|^2 + |\nabla v|^2) \\ \leq\ & \frac{C}{(\log R)^2}\int_R^{R^2} r^{-2}(\int_{\partial B_r(0)} |\nabla u|^2 + |\nabla v|^2)dr \\ =\ & \frac{C}{(\log R)^2}\int_R^{R^2} r^{-2}(\frac{d}{dr}\int_{B_r(0)} |\nabla u|^2 + |\nabla v|^2)dr \end{aligned}$$

$$
\begin{aligned}
&= \frac{C}{(\log R)^2}[r^{-2}\int_{\partial B_r(0)} |\nabla u|^2 + |\nabla v|^2)|_R^{R^2} + 2\int_R^{R^2} r^{-3}(\int_{B_r(0)} |\nabla u|^2 + |\nabla v|^2)dr] \\
&\leq \frac{C}{\log R}.
\end{aligned}
$$

By letting $R \to +\infty$, we see $\nabla w_1 \equiv 0$ and $\nabla w_2 \equiv 0$ in $\mathbb{R}^2$. Thus, there is a constant c such that

$$
(u_\xi, v_\xi) = c(\varphi, \psi).
$$

Because ξ is an arbitrary unit vector, from this we actually know that after changing the coordinates suitably,

$$
u_y \equiv 0, v_y \equiv 0 \quad \text{in } \mathbb{R}^2.
$$

That is, u and v depend on x only and they are one dimensional.

4 Existence in bounded balls

In this section we first construct a solution (u, v) to the problem

$$
\begin{cases} \Delta u = uv^2 & \text{in} \quad B_R(0), \\ \Delta v = vu^2 & \text{in} \quad B_R(0), \end{cases} \tag{4.1}
$$

satisfying the boundary condition

$$
u = \Phi^+, v = \Phi^- \quad \text{on } \partial B_R(0) \subset \mathbb{R}^2. \tag{4.2}
$$

More precisely, we prove

Theorem 4.1. *There exists a solution (u_R, v_R) to problem (4.1), satisfying*

1. $u_R - v_R > 0$ *in* $\{\Phi > 0\}$ *and* $u_R - v_R < 0$ *in* $\{\Phi < 0\}$*;*
2. $u_R \geq \Phi^+$ *and* $v_R \geq \Phi^-$*;*
3. $\forall i = 1, \cdots, d$, $u_R(T_i z) = v_R(z)$*;*
4. $\forall r \in (0, R)$,
$$
N(r; u_R, v_R) := \frac{r\int_{B_r(0)} |\nabla u_R|^2 + |\nabla v_R|^2 + u_R^2 v_R^2}{\int_{\partial B_r(0)} u_R^2 + v_R^2} \leq d.
$$

Proof. Let us denote $\mathcal{U} \subset H^1(B_R(0))^2$ the set of pairs satisfying the boundary condition (4.2), together with conditions $(1, 2, 3)$ of the statement of the Theorem (with the strict inequality $<$ replaced by $\leq$, and so now $\mathcal{U}$ is a closed set).

The desired solution will be a minimizer of the energy functional

$$E_R(u,v) := \int_{B_R(0)} |\nabla u|^2 + |\nabla v|^2 + u^2 v^2$$

over $\mathcal{U}$. Existence of at least one minimizer follows easily from the direct method of the Calculus of Variations. To prove that the minimizer also satisfies equation (4.1), we use the heat flow method. More precisely, we consider the following parabolic problem

$$\begin{cases} U_t - \Delta U = -UV^2, & \text{in } [0,+\infty) \times B_R(0), \\ V_t - \Delta V = -VU^2, & \text{in } [0,+\infty) \times B_R(0), \end{cases} \tag{4.3}$$

with the boundary conditions $U = \Phi^+$ and $V = \Phi^-$ on $(0,+\infty) \times \partial B_R(0)$ and initial conditions in $\mathcal{U}$.

By the standard parabolic theory, there exists a unique local solution (U,V). Then by the maximum principle, $0 \le U \le \sup_{B_R(0)} \Phi^+$, $0 \le V \le \sup_{B_R(0)} \Phi^-$, hence the solution can be extended to a global one, for all $t \in (0,+\infty)$. By noting the energy inequality

$$\frac{d}{dt} E_R(U(t),V(t)) = -\int_{B_R(0)} |\frac{\partial U}{\partial t}|^2 + |\frac{\partial V}{\partial t}|^2 \tag{4.4}$$

and the fact that $E_R \ge 0$, standard parabolic theory implies that for any sequence $t_i \to +\infty$, there exists a subsequence of t_i such that $(U(t_i), V(t_i))$ converges to a solution (u,v) of (4.1).

Next we show that $\mathcal{U}$ is positively invariant by the parabolic flow. First of all, by the symmetry of initial and boundary data, $(V(t,T_i z), U(t,T_i z))$ is also a solution to the problem (4.3). By the uniqueness of solutions to the parabolic system (4.3), (U,V) inherits the symmetry of (Φ^+,Φ^-). That is, for all $t \in [0,+\infty)$ and $i = 1,\cdots,d$,

$$U(t,z) = V(t,T_i z).$$

This implies

$$U - V = 0 \quad \text{on} \quad \{\Phi = 0\}.$$

Thus, in the open set $D_R := B_R(0) \cap \{\Phi > 0\}$, we have, for any initial datum $(u_0,v_0) \in \mathcal{U}$,

$$\begin{cases} (U-V)_t - \Delta(U-V) = UV(U-V), & \text{in } [0,+\infty) \times D_R(0), \\ U - V \ge 0, & \text{on } [0,+\infty) \times \partial D_R(0), \\ U - V \ge 0, & \text{on } \{0\} \times D_R(0). \end{cases} \tag{4.5}$$

The strong maximum principle implies $U - V > 0$ in $(0,+\infty) \times D_R(0)$. By letting $t \to +\infty$, we obtain that the limit satisfies

$$u - v \ge 0 \quad \text{in} \quad D_R(0). \tag{4.6}$$

(u, v) also has the symmetry, $\forall i = 1, \cdots, d$

$$u(T_i z) = v(z).$$

Similar to (4.5), noting (4.6), we have

$$\begin{cases} -\Delta(u - v) \geq 0, & \text{in } D_R(0), \\ u - v = \Phi^+, & \text{on } \partial D_R(0). \end{cases} \tag{4.7}$$

Comparing with Φ^+ on $D_R(0)$, we obtain

$$u - v > \Phi^+ > 0, \quad \text{in } D_R(0). \tag{4.8}$$

Because $u > 0$ and $v > 0$ in $B_R(0)$, we in fact have

$$u > \Phi^+, \quad \text{in } B_R(0). \tag{4.9}$$

In conclusion, (u, v) satisfies conditions $(1, 2, 3)$ in the statement of the theorem.

Let (u_R, v_R) be a minimizer of E_R over $\mathcal{U}$. Now we consider the parabolic equation (4.3) with the initial condition

$$U(x, t) = u_R(x), V(x, t) = v_R(x). \tag{4.10}$$

By (4.4), we deduce that

$$E_R(u_R, v_R) \leq E_R(U, V) \leq E_R(u_R, v_R)$$

and hence $(U(x, t), V(x, t)) \equiv (u_R(x), v_R(x))$ for all $t \geq 0$. By the arguments above, we see that (u_R, v_R) satisfies (4.1)and conditions $(1, 2, 3)$ in the statement of the theorem.

In order to prove (4), we firstly note that, as (u_R, v_R) minimizes the energy and $(\Phi^+, \Phi^-) \in \mathcal{U}$, there holds

$$\int_{B_R(0)} |\nabla u_R|^2 + |\nabla v_R|^2 + u_R^2 v_R^2 \leq \int_{B_R(0)} |\nabla \Phi|^2.$$

Now by the Almgren monotonicity formula (Proposition 5.2 below) and the boundary conditions, $\forall r \in (0, R)$, we derive

$$N(r; u_R, v_R) \leq N(R; u_R, v_R) \leq \frac{R \int_{B_R(0)} |\nabla \Phi|^2}{\int_{\partial B_R(0)} |\Phi|^2} = d.$$

This completes the proof of Theorem 4.1.

□

Let us now turn to the system with many components. In a similar way we shall prove the existence on bounded sets. Let d be an integer or a half-integer and $2d = hk$ be a multiple of the number of components k, and G denote the rotation of order $2d$. Take the fundamental domain F of the rotations group of degree $2d$, that is $F = \{z \in \mathbb{C} : \theta = \arg(z) \in (-\pi/2d, \pi/2d)\}$.

$$\Psi(z) = \begin{cases} r^d \cos(d\theta) & \text{if } z \in \cup_{i=0}^{h-1} G^{ik}(F), \\ 0 & \text{otherwise in } \mathbb{C}. \end{cases} \tag{4.11}$$

Note that $\Psi(z)$ is positive whenever it is not zero. Next we construct a solution $(u_1, \dots, u_k)$ to the system

$$\Delta u_i = u_i \sum_{j \neq i, j=1}^{k} u_j^2, \quad \text{in} \quad B_R(0), i = 1, \dots, k \tag{4.12}$$

satisfying the symmetry and boundary condition (here $\overline{z}$ is the complex conjugate of z)

$$\begin{cases} u_i(z) = u_i(G^h z), & \text{on } B_R(0)\,, i = 1, \dots, k, \\ u_i(z) = u_{i+1}(Gz), & \text{on } B_R(0)\,, i = 1, \dots, k, \\ u_{k+2-i}(z) = u_i(\overline{z}), & \text{on } B_R(0)\,, i = 1, \dots, k, \\ u_{k+1}(z) = u_1(z), & \text{on } B_R(0), \end{cases} \tag{4.13}$$

$$u_{i+1}(z) = \Psi(G^i(z)), \qquad \text{on } \partial B_R(0)\,, i = 0, \dots, k-1. \tag{4.14}$$

More precisely, we prove the following.

Theorem 4.2. *For every $R > 0$, there exists a solution $(u_{1,R}, \dots, u_{k,R})$ to the system (4.12) with symmetries (4.13) and boundary conditions (4.14), satisfying,*

$$N(r) := \frac{r \int_{B_r(0)} \sum_1^k |\nabla u_{i,R}|^2 + \sum_{i<j} u_{i,R}^2 u_{j,R}^2}{\int_{\partial B_r(0)} \sum_1^k u_{i,R}^2} \leq d, \ \forall r \in (0, R).$$

Proof. Let us denote by $\mathcal{U} \subset H^1(B_R(0))^k$ the set of pairs satisfying the symmetry and boundary condition (4.13), (4.14). The desired solution will be the minimizer of the energy functional

$$\int_{B_r(0)} \sum_1^k |\nabla u_{i,R}|^2 + \sum_{i<j} u_{i,R}^2 u_{j,R}^2$$

over $\mathcal{U}$. Once more, to deal with the constraints, we may take advantage of the positive invariance of the associated heat flow:

$$\begin{cases} \dfrac{\partial U_i}{\partial t} - \Delta U_i = -U_i \displaystyle\sum_{j \neq i} U_j^2, \quad \text{in} \quad [0, +\infty) \times B_R(0), \end{cases} \tag{4.15}$$

which can be solved under conditions (1.20), (4.14) and initial conditions in $\mathcal{U}$. Thus, the minimizer of the energy $(u_{1,R}, \dots, u_{k,R})$ solves the differential system. In addition, using the test function $(\Psi_1, \dots, \Psi_k)$, where $\Psi_i = \Psi \circ G^{i-1}$, $i = 1, \dots, k$, we have

$$\int_{B_R(0)} \sum_1^k |\nabla u_{i,R}|^2 + \sum_{i<j} u_{i,R}^2 u_{j,R}^2 \leq k \int_{B_R(0)} |\nabla \Psi|^2.$$

Now by the Almgren monotonicity formula below (Proposition 5.2) and the boundary conditions, we get

$$N(r) \leq N(R) \leq \frac{R\int_{B_R(0)} |\nabla \Psi|^2}{\int_{\partial B_R(0)} |\Psi|^2} = d, \ \forall r \in (0, R).$$

□

In order to conclude the proof of Theorems 4.1 and 4.2, we need to find upper and lower bounds for the solutions, uniform with respect to R on bounded subsets of $\mathbb{C}$. That is, we will prove that for any $r > 0$, there exists positive constants $0 < c(r) < C(r)$ (independent of R) such that

$$c(r) < \sup_{B_r(0)} u_R \leq C(r). \tag{4.16}$$

Once we have this estimate, then by letting $R \to +\infty$, a subsequence of (u_R, v_R) will converge to a solution (u, v) of problem (1.10), uniformly on any compact set of $\mathbb{R}^2$. It is easily seen that properties (1), (2), (3) and (4) in Theorem 4.1 can be derived by passing to the limit, and we obtain the main results stated in Theorem 1.3 and 1.6. It then remains to establish the bound (4.16). In the next section, we shall obtain this estimate by using the monotonicity formula.

5 Monotonicity formula

Let us start by stating some monotonicity formulae for solutions to (1.10), for any dimension $n \geq 2$. The first two are well-known and we include them here for completeness. But we will also require some refinements.

Proposition 5.1. *For $r > 0$ and $x \in \mathbb{R}^n$,*

$$E(r) = r^{2-n} \int_{B_r(x)} \sum_1^k |\nabla u_i|^2 + \sum_{i<j} u_i^2 u_j^2$$

is nondecreasing in r.

For a proof, see [12]. The next statement is an Almgren-type monotonicity formula with remainder.

Proposition 5.2. *For $r > 0$ and $x \in \mathbb{R}^n$, let us define*

$$H(r) = r^{1-n} \int_{\partial B_r(x)} \sum_1^k u_i^2.$$

Then

$$N(r; x) := \frac{E(r)}{H(r)}$$

is nondecreasing in r. In addition there holds

$$\int_0^r \frac{2\int_{B_s}\sum_{i<j}^k u_i^2u_j^2}{\int_{\partial B_s}\sum_1^k u_i^2}ds \le N(r)\ . \tag{5.1}$$

Proof. For simplicity, take x to be the origin 0 and let $k=2$. We have

$$H(r) = r^{1-n}\int_{\partial B_r} u^2+v^2\ , \qquad E(r) = r^{2-n}\int_{B_r}|\nabla u|^2+|\nabla v|^2+u^2v^2\ .$$

Then, direct calculations show that

$$\frac{d}{dr}H(r) = 2r^{1-n}\int_{B_r}|\nabla u|^2+|\nabla v|^2+2u^2v^2. \tag{5.2}$$

By the proof of Proposition 5.1, we have

$$\frac{d}{dr}E(r) = 2r^{2-n}\int_{\partial B_r}[u_r^2+v_r^2]+2r^{1-n}\int_{B_r}u^2v^2. \tag{5.3}$$

With these two identities, we obtain

$$\begin{aligned}\frac{d}{dr}\frac{E}{H}(r) &= \frac{H[2r^{2-n}\int_{\partial B_r}(u_r^2+v_r^2)+2r^{1-n}\int_{B_r}u^2v^2]-E[2r^{1-n}\int_{\partial B_r}uu_r+vv_r]}{H^2}\\ &\ge \frac{2r^{3-2n}\int_{\partial B_r}(u^2+v^2)\int_{\partial B_r}(u_r^2+v_r^2)-2r^{3-2n}\left[\int_{\partial B_r}uu_r+vr_r\right]^2}{H^2}+\\ &\qquad\qquad + \frac{2r^{1-n}\int_{B_r}u^2v^2}{H} \ge \frac{2r^{1-n}\int_{B_r}u^2v^2}{H}.\end{aligned}$$

Here we have used the following inequality

$$E(r) \le \int_{B_r}|\nabla u|^2+|\nabla v|^2+2u^2v^2 = \int_{\partial B_r}uu_r+vr_r.$$

Hence this yields monotonicity of the Almgren quotient. In addition, by integrating the above inequality we obtain

$$\int_{r_0}^r \frac{2\int_{B_s}u^2v^2}{\int_{\partial B_s}u^2+v^2}ds \le N(r)\ .$$

□

If $x=0$, we simply denote $N(r;x)$ as $N(r)$. Assuming an upper bound on $N(r)$, we establish a doubling property by the Almgren monotonicity formula.

Proposition 5.3. *Let $R > 1$ and let $(u_1, \dots, u_k)$ be a solution of* (1.19) *on B_R. If $N(R) \le d$, then for any $1 < r_1 \le r_2 \le R$*

$$\frac{H(r_2)}{H(r_1)} \le e^d \frac{r_2^{2d}}{r_1^{2d}}. \tag{5.4}$$

Proof. For simplicity of notation, we expose the proof for the case of two components. By direct calculation using (5.2), we obtain

$$\begin{aligned}
\frac{d}{dr} \log \left[r^{1-n} \left(\int_{\partial B_r(0)} u^2 + v^2 \right) \right] &= \frac{2 \int_{B_r} |\nabla u|^2 + |\nabla v|^2 + 2u^2 v^2}{\int_{\partial B_r(0)} u^2 + v^2} \\
&\le \frac{2N(r)}{r} + \frac{2 \int_{B_r} u^2 v^2}{\int_{\partial B_r(0)} u^2 + v^2} \\
&\le \frac{2d}{r} + \frac{2 \int_{B_r} u^2 v^2}{\int_{\partial B_r(0)} u^2 + v^2}
\end{aligned}$$

Thanks to (5.1), by integrating, we find that, if $r_1 \le r_2 \le 2r_0$ then

$$\frac{H(r_2)}{H(r_1)} \le e^d \frac{r_2^{2d}}{r_1^{2d}}. \tag{5.5}$$

□

An immediate consequence of Proposition 5.3 is the lower bound on bounded sets for the solutions found in Theorems 4.1 and 4.2.

Proposition 5.4. *Ler $(u_{1,R}, \dots, u_{k,R})$ be a family of solutions to* (1.19) *such that $N(R) \le d$ and $H(R) = CR^{2d}$. Then, for every fixed $r < R$, there holds*

$$H(r) \ge Ce^{-d} r^{2d}.$$

Another byproduct of the monotonicity formula with the remainder (5.1) is the existence of the limit of $H(r)/r^{2d}$.

Corollary 5.5. *Let $R > 1$ and let $(u_1, \dots, u_k)$ be a solution of* (1.19) *on $\mathbb{C}$ such that $\lim_{r \to +\infty} N(r) \le d$, then there exists*

$$\lim_{r \to +\infty} \frac{H(r)}{r^{2d}} < +\infty\ . \tag{5.6}$$

Now we prove the optimal lower bound on the growth of the solution. To this aim, we need a fine estimate on the asymptotics of the lowest eigenvalue as the competition term diverges. The following result is an extension of Theorem 1.6 in [8], where the estimate was proved in case of two components.

Theorem 5.6. *Let d be a fixed integer and let us consider*

$$\mathcal{L}(d,\Lambda)=\min\left\{\int_0^{2\pi}\sum_i^d|u_i'|^2+\Lambda\sum_{i<j}^d u_i^2u_j^2 \;\middle|\; \begin{array}{l}\int_0^{2\pi}\sum_i u_i^2=1,\ \ u_{i+1}(x)=u_i(x-2\pi/d),\\ u_1(-x)=u_1(x)\ ,u_{d+1}=u_1\end{array}\right\}. \quad (5.7)$$

Then, there exists a constant C such that for all $\Lambda>1$ we have

$$d^2-C\Lambda^{-1/4}\le\mathcal{L}(d,\Lambda)\le d^2\ . \quad (5.8)$$

Proof. Any minimizer $(u_{1,\Lambda},\dots,u_{d,\Lambda})$ solves the system of ordinary differential equations

$$u_i^{''}=\Lambda u_i\sum_{j\neq i}u_j^2-\lambda u_i\ ,\qquad i=1,\dots,d, \quad (5.9)$$

together with the associated energy conservation law

$$\sum_1^d(u_i^{'})^2+\lambda u_i^2-\Lambda\sum_{i<j}^d u_i^2u_j^2=h\ . \quad (5.10)$$

Note that the Lagrange multiplier satisfies

$$\lambda=\int_0^{2\pi}\sum_i^d|u_i'|^2+2\Lambda\sum_{i<j}^d u_i^2u_j^2=\mathcal{L}(d,\Lambda)+\int_0^{2\pi}\Lambda\sum_{i<j}^d u_i^2u_j^2\ .$$

As $\Lambda\to\infty$, we see convergence of the eigenvalues $\lambda\simeq\mathcal{L}(d,\Lambda)\to d^2$, together with the energies $h\to 2d^2$. Moreover, the solutions remain bounded in Lipschitz norm and converge in Sobolev and Hölder spaces (see [8] for more details). Now, let us focus on the interval $I=(a,a+2\pi/d)$ where the i-th component is active. The symmetry constraints imply

$$u_{i-1}(a)=u_i(a)\ ,u_{i-1}^{'}(a)=-u_i^{'}(a)\ ,$$
$$u_{i+1}(a+2\pi/d)=u_i(a+2\pi/d)\ ,u_{i+1}^{'}(a+2\pi/d)=-u_i^{'}(a+2\pi/d)$$

We observe that there is interaction only with the two prime neighboring components, while the others are exponentially small (in Λ) on I. Close to the endpoint a, the component u_i is increasing and convex, while u_{i-1} is decreasing and again convex. Similarly to [8] we have that

$$u_i(a)=u_{i-1}(a)\simeq K\Lambda^{-1/4}\ ,u_i'(a)=-u_{i-1}^{'}(a)\simeq H=(h+K)/2\ . \quad (5.11)$$

Hence, in a right neighborhood of a, there holds $u_i(x)\ge u_i(a)$, and therefore, as $u_{i-1}^{''}\ge\Lambda u_i^2(a)u_{i-1}$, from the initial value problem (5.11) we infer

$$u_{i-1}(x)\le Cu_i(a)e^{-\Lambda^{1/2}u_i(a)(x-a)}\ ,\forall x\in[a,b].$$

On the other hand, on the same interval we have

$$u_i(x) \le u_i(a) + C(x-a)\ ,\forall x \in [a,b].$$

(here and below C denotes a constant independent of Λ). Consequently, there holds

$$\Lambda \int_I u_{i-1}^2 u_i^2 + u_{i-1}^3 u_i + u_{i-1} u_i^2 \le C\Lambda^{-1/2} u_i(a)^{-1} \simeq C\Lambda^{-1/4}\ . \tag{5.12}$$

In particular, this yields

$$\mathcal{L}(d,\Lambda) \ge \lambda - C\Lambda^{-1/4}\ . \tag{5.13}$$

In order to estimate λ, let us consider $\widehat{u}_i = \left(u_i - \sum_{j=i\pm1} u_j\right)^+$. Then, as $u_i(a) = u_{i-1}(a)$ and $u_i(a+2\pi/d) = u_{i+1}(a+2\pi/d)$, $\widehat{u}_i \in H_0^1(I)$. By testing the differential equation for $u_i - \sum_{j=i\pm1} u_j$ with $\widehat{u}_i$ on I we find

$$\int_I |\widehat{u}_i'|^2 \le \lambda \int_I |\widehat{u}_i|^2 + C\Lambda^{-1/4}\ ,$$

where in the last term we have majorized all the integrals of mixed fourth order monomials with (5.12). As $|I| = 2\pi/d$, using Poincaré inequality and (5.13) we obtain the desired estimate on $\mathcal{L}(d,\Lambda)$. □

We are now ready to apply the estimate from below on $\mathcal{L}$ to derive a lower bound on the energy growth. We recall that there holds

$$\widehat{E}(r) := \int_{B_r(x)} \sum_1^k |\nabla u_i|^2 + 2\sum_{i<j} u_i^2 u_j^2 = \int_{\partial B_r(x)} \sum_1^k u_i \frac{\partial u_i}{\partial r}$$

Proposition 5.7. *Let $(u_{1,R},\dots,u_{k,R})$ be a solution of (1.19) having the symmetries (1.20) on B_R. There exists a constant C (independent of R) such that for all $1 \le r_1 \le r_2 \le R$ there holds*

$$\frac{\widehat{E}(r_2)}{\widehat{E}(r_1)} \ge C\frac{r_2^{2d}}{r_1^{2d}} \tag{5.14}$$

Proof. Let us compute,

$$\begin{aligned}\frac{d}{dr}\log\left(r^{-2d}\widehat{E}(r)\right) &= -\frac{2d}{r} + \frac{\int_{\partial B_r(x)} \sum_1^k |\nabla u_i|^2 + 2\sum_{i<j} u_i^2 u_j^2}{\int_{\partial B_r(x)} \sum_1^k u_i \frac{\partial u_i}{\partial r}} \\ &= -\frac{2d}{r} + \frac{\int_{\partial B_r(x)} \sum_1^k \left(\frac{\partial u_i}{\partial r}\right)^2 + \frac{1}{r^2}\left[\sum_1^k \left(\frac{\partial u_i}{\partial \theta}\right)^2 + 2r^2 \sum_{i<j} u_i^2 u_j^2\right]}{\int_{\partial B_r(x)} \sum_1^k u_i \frac{\partial u_i}{\partial r}}\end{aligned}$$

$$= -\frac{2d}{r} + \frac{\int_0^{2\pi} \sum_1^k \left(\frac{\partial u_i}{\partial r}\right)^2 + \frac{1}{r^2}\left[\sum_1^k \left(\frac{\partial u_i}{\partial \theta}\right)^2 + 2r^2 \sum_{i<j} u_i^2 u_j^2\right]}{\int_0^{2\pi} \sum_1^k u_i \frac{\partial u_i}{\partial r}}$$

Now we use Theorem 5.6 and we continue the chain of inequalities:

$$\frac{d}{dr} \log\left(r^{-2d}\widehat{E}(r)\right) \geq -\frac{2d}{r} + \frac{\int_0^{2\pi} \sum_1^k \left(\frac{\partial u_i}{\partial r}\right)^2 + \frac{\mathcal{L}(d, 2r^2)}{r^2} \int_0^{2\pi} \sum_1^k u_i^2}{\int_0^{2\pi} \sum_1^k u_i \frac{\partial u_i}{\partial r}}$$

$$\geq -\frac{2d - 2\sqrt{\mathcal{L}(d, 2r^2)}}{r} \geq -\frac{C}{r^{3/2}} , \quad (5.15)$$

where in the last line we have used Hölder inequality. By integration we easily obtain the assertion. □

A direct consequence of the above inequalities is the non vanishing of the quotient E/r^{2d}:

Corollary 5.8. *Let $R > 1$ and let $(u_1, \dots, u_k)$ be a solution of (1.19) on $\mathbb{C}$ satisfying 1.20: then there exists*

$$\lim_{r\to+\infty} \frac{\widehat{E}(r)}{r^{2d}} = b \in (0, +\infty] . \quad (5.16)$$

If, in addition, $\lim_{r\to+\infty} N(r) \leq d$, then we have that $b < +\infty$ and

$$\lim_{r\to+\infty} N(r) = d, \quad \text{and} \quad \lim_{r\to+\infty} \frac{E(r)}{r^{2d}} = b . \quad (5.17)$$

Proof. Note that (5.16) is a straightforward consequence of the monotonicity formula (5.15). To prove (5.17), we first notice that

$$\lim_{r\to+\infty} \frac{E(r)}{r^{2d}} = \lim_{r\to+\infty} N(r)\frac{H(r)}{r^{2d}}.$$

So the limit of $E(r)/r^{2d}$ exists finite. Now we use (5.1)

$$\int_0^{+\infty} \frac{2\int_{B_s} \sum_{i<j}^k u_i^2 u_j^2}{\int_{\partial B_s} \sum_1^k u_i^2} ds < +\infty$$

and we infer

$$\liminf_{r\to+\infty} \frac{r\int_{B_r} \sum_{i<j}^k u_i^2 u_j^2}{\int_{\partial B_r} \sum_1^k u_i^2} = 0.$$

Next, using Corollary 5.5 we can compute

$$\liminf_{r\to+\infty} \frac{\int_{B_r}\sum_{i<j}^k u_i^2 u_j^2}{r^{2d}} = \liminf_{r\to+\infty} \frac{\int_{B_r}\sum_{i<j}^k u_i^2 u_j^2}{H(r)} \frac{H(r)}{r^{2d}} = 0,$$

and finally

$$\liminf_{r\to+\infty} \frac{\widehat{E}(r) - E(r)}{r^{2d}} = 0; .$$

Was the limit of $N(r)$ strictly less that d, the growth of $H(r)$ would be in contradiction with that of $E(r)$. ☐

Now we can combine the upper and lower estimates to obtain convergence of the approximating solutions on compact sets and complete the proof of Theorems 1.6

Proof of Theorem 1.6. Let $(u_{1,R},\dots,u_{k,R})$ be a family of solutions to (1.19) such that $N_R(R) \le d$ and $H_R(R) = CR^{2d}$. Since $H_R(R) = CR^{2d}$, then, by Proposition 5.3 we deduce that, for every fixed $1 < r < R$, there holds

$$H_R(r) \ge Ce^{-d} r^{2d} \ .$$

Assume first that there holds a uniform bound for some $r > 1$,

$$H_R(r) \le C \ . \tag{5.18}$$

Then $H_R(r)$ and $E_R(r)$ are uniformly bounded on R. This implies a uniform bound on the $H^1(B_r)$ norm. As the components are subharmonic, standard elliptic estimates (Harnack inequality) yield actually a $\mathcal{C}^2$ bound on $B_{r/2}$, which is independent on R. Note that, by Proposition 5.4, $H_R(r)$ is bounded away from zero, so the weak limit cannot be zero. By the doubling Property 5.3 the uniform bound on $H_R(r_2) \le Cr_2^{2d}$ holds for every $r_2 \in \mathbb{R}$ larger than r. Thus, a diagonal procedure yields existence of a nontrivial limit solution of the differential system, defined on the whole of $\mathbb{C}$. It is worthwhile noticing that this solution inherits all the symmetries of the approximating solutions together with the upper bound on the Almgren's quotient. Finally, from Corollary 5.5 and 5.8 infer the limit

$$\lim_{r\to+\infty} \frac{H(r)}{r^{2d}} = \lim_{r\to+\infty} \frac{1}{N(r)}, \lim_{r\to+\infty} \frac{E(r)}{r^{2d}} = \frac{b}{d} \in (0,+\infty) \ . \tag{5.19}$$

Let us now show that $H_R(r)$ is uniformly bounded with respect to R for fixed r. We argue by contradiction and assume that, for a sequence $R_n \to +\infty$, there holds

$$\lim_{n\to+\infty} H_{R_n}(r) = +\infty \ . \tag{5.20}$$

Denote $u_{i,n} = u_{i,R_n}$ and H_n, E_n, N_n the corresponding functions. Note that, as E_n is bounded, we must have $N_n(r) \to 0$. For each n, let $\lambda_n \in (0,r)$ such that

$$\lambda_n^2 H_n(\lambda_n) = 1$$

(such λ_n exist right because of (5.20)) and scale

$$\tilde{u}_{i,n}(z) = \lambda_n u_{i,n}(\lambda_n z) \ , \quad |z| < R_n/\lambda_n \ .$$

Note that the $(\tilde{u}_{i,n})_i$ still solve system (1.19) on the disk $B(0, R_n/\lambda_n)$ and enjoy all the symmetries (1.20). Let us denote $\tilde{H}_n$, $\tilde{E}_n$, $\tilde{N}_n$ the corresponding quantities. We have

$$\begin{aligned}\tilde{H}_n(1) &= \lambda_n^2 H_n(\lambda_n) = 1,\\ \tilde{E}_n(1) &= \lambda_n^2 E_n(\lambda_n) \to 0\\ \tilde{N}_n(1) &= N_n(\lambda_n) \to 0\end{aligned}$$

In addition there holds $\tilde{N}_n(s) \le d$ for $s < R_n/\lambda_n$. By the compactness argument exposed above, we can extract a subsequence converging in the compact-open topology of $\mathcal{C}^2$ to a nontrivial symmetric solution of (1.19) with Almgren quotient vanishing constantly. Thus, such solution should be a nonzero constant in each component, but constant solution are not compatible with the system of PDE's (1.19) . □

6 Asymptotics at infinity

We now come to the proof of Theorem 1.4. Note that by Proposition 5.3, the condition on $N(r)$ implies that u and v have a polynomial growth. (In fact, with more effort we can show the reverse also holds. Namely, if u and v have polynomial growth, then $N(r)$ approaches a positive integer as $r \to +\infty$. We leave out the proof.)

Recall the blow down sequence is defined by

$$(u_R(x), v_R(x)) := (\frac{1}{L(R)} u(Rx), \frac{1}{L(R)} v(Rx)),$$

where $L(R)$ is chosen so that

$$\int_{\partial B_1(0)} u_R^2 + v_R^2 = \int_{\partial B_1(0)} \Phi^2. \tag{6.1}$$

For the solutions in Theorem 1.3, by (5.19), we have

$$L(R) \sim R^d. \tag{6.2}$$

We will now analyze the limit of (u_R, v_R) as $R \to +\infty$.

Because for any $r \in (0, +\infty)$, $N(r) \le d$, (u, v) satisfies Proposition 5.3 for any $r \in (1, +\infty)$. After rescaling, we see that Proposition 5.3 holds for (u_R, v_R) as well. Hence, there exists a constant $C > 0$, such that for any R and $r \in (1, +\infty)$,

$$\int_{\partial B_r(0)} u_R^2 + v_R^2 \le C e^d r^d. \tag{6.3}$$

Next, (u_R, v_R) satisfies the equation

$$\begin{cases} \Delta u_R = L(R)^2 R^2 u_R v_R^2, \\ \Delta v_R = L(R)^2 R^2 v_R u_R^2, \\ u_R, v_R > 0 \ \text{ in } \ \mathbb{R}^2. \end{cases} \tag{6.4}$$

Here we need to observe that, by (6.2),

$$\lim_{R\to+\infty} L(R)^2 R^2 = +\infty.$$

By (6.3), as $R \to +\infty$, u_R and v_R are uniformly bounded on any compact set of $\mathbb{R}^2$. Then by the main result in [16], [24] and [26], there is a harmonic function Ψ defined in $\mathbb{R}^2$, such that (a subsequence of) $(u_R, v_R) \to (\Psi^+, \Psi^-)$ in H^1 and in Hölder spaces on any compact set of $\mathbb{R}^2$. By (6.1),

$$\int_{\partial B_1(0)} \Psi^2 = \int_{\partial B_1(0)} \Phi^2,$$

so Ψ is nonzero. Because $L(R) \to +\infty$, $u_R(0)$ and $v_R(0)$ goes to 0, hence

$$\Psi(0) = 0. \tag{6.5}$$

After rescaling in Proposition 5.2, we obtain a corresponding monotonicity formula for (u_R, v_R),

$$N(r; u_R, v_R) := \frac{r \int_{B_r(0)} |\nabla u_R|^2 + |\nabla v_R|^2 + L(R)^2 R^2 u_R^2 v_R^2}{\int_{\partial B_r(0)} u_R^2 + v_R^2} = N(Rr)$$

is nondecreasing in r. By (4) in Theorem 1.3 and from Corollary 5.8,

$$N(r; u_R, v_R) \le d = \lim_{r\to+\infty} N(r; u_R, v_R) \ \ , \forall\, r \in (0, +\infty). \tag{6.6}$$

In [16], it's also proved that $(u_R, v_R) \to (\Psi^+, \Psi^-)$ in H^1_{loc} and for any $r < +\infty$,

$$\lim_{R\to+\infty} \int_{B_r(0)} L(R)^2 R^2 u_R^2 v_R^2 = 0.$$

After letting $R \to +\infty$ in (6.6), we get

$$N(r; \Psi) := \frac{r \int_{B_r(0)} |\nabla \Psi|^2}{\int_{\partial B_r(0)} \Psi^2} = \lim_{R\to+\infty} N(r; u_R, v_R) = \lim_{R\to+\infty} N(Rr) = d. \tag{6.7}$$

In particular, $N(r; \Psi)$ is a constant for all $r \in (0, +\infty)$. So Ψ is a homogeneous polynomial of degree d. Actually the number d is the vanishing order of Ψ at 0, which must therefore be a positive integer. Now it remains to prove that $\Psi \equiv \Phi$: this is easily done by exploiting the symmetry conditions on Ψ (point (3) of Theorem 1.3).

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