VORTEX SOLUTIONS OF THE HIGH-κ HIGH-FIELD GINZBURG-LANDAU MODEL WITH AN APPLIED CURRENT

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Abstract. In this paper, we study the single-vortex solutions of a two-dimensional high-κ high-field Ginzburg-Landau model of superconductivity with a constant applied current. Under a nondegeneracy condition and for appropriate ranges of the applied magnetic field and applied current, we construct some special solutions which, up to a constant shift of phase in time, are the stationary solutions of the model equation. Our result provides partial justification to the existence of a critical applied current which is the one important step towards a rigorous mathematical characterization of the interactions between the quantized vortices and applied electric current.

1. Introduction. Quantized vortices have a long history that begins with the studies of superconductors. It is well known that Type II superconductors are characterized by the existence of the vortex state, which consists of many normal filaments embedded in a superconductor material. Each filament carries with it a quantized amount of magnetic flux with circulating supercurrent, which is thus named a vortex.

It is important to understand the features of the vortex state, since it is in this state that most superconductors are utilized in applications [26]. The motion of vortices is of particular interest, since this motion dissipates energy and results in an effective resistivity. More specifically, in the vortex state, an applied current generally exerts a Lorentz force on each vortex. The motion of vortices due to the Lorentz force induces an electric field, and thus produces electrical resistance. In practice, many important properties of the superconducting state can be preserved when the Lorentz force can be effectively balanced by vortex pinning forces. The latter can take on various forms, such as those from doping or spatial inhomogeneities.

Quantized vortex phenomena in superconductors have been extensively studied theoretically and computationally within the phenomenological Ginzburg-Landau (GL) model [13, 26]. In particular, the time dependent Ginzburg-Landau (TDGL) model may be used as a prototype model for the study of vortex-current interactions. In recent years, much progress have been made on the mathematical studies of the vortex state in the Ginzburg-Landau models, see for example, [1, 4, 5, 8, 15, 16, 20, 21, 24]. We refer to additional references given in the recent lecture notes [11] and the monograph [20]. The computational studies given in [12] strongly suggest that it is equally enlightening to study an even simpler system: the so-called high-κ high-field (HKHF) model [10, 12] given by

$$\frac{\partial\psi_0}{\partial t} + i\Phi_a\psi_0 + (i\nabla + \boldsymbol{A}_0)^2\psi_0 + \frac{1}{\epsilon^2}(|\psi_0|^2 - 1)\psi_0 = 0 \qquad \text{in } \Omega, \tag{1.1}$$

$$(i\nabla + \boldsymbol{A}_0)\psi_0 \cdot \nu = 0 \qquad \text{on } \partial\Omega\,, \tag{1.2}$$

$$\psi_0(\mathbf{x}, 0) = \varphi(\mathbf{x}) \qquad \text{in } \Omega\,. \tag{1.3}$$

Here, $\boldsymbol{A}_0$ is a time-independent magnetic potential which, for a given applied magnetic field, can be solved from the Maxwell equation. The scalar electric potential Φ_a,

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meanwhile, can be determined from $\nabla\Phi_a$ being a constant applied current, that is, $\nabla\Phi_a(x_1, x_2) = (0, J)^T$ for some constant applied current J. The parameter ε may be viewed as a rescaled coherence length, a measure of the vortex core size which is very small in practice in comparison with the sample size.

The above HKHF model was first studied, in the time-independent case and without the applied current, in [6]. The derivation of the time-dependent version and with the applied current, was given in [12]. Beyond the rigorous derivation of HKHF from the TDGL and the basic well-posedness theory [12], there exists very little rigorous mathematical analysis on the solutions of the HKHF equation. Interestingly, with the presence of Φ_a and $\boldsymbol{A}_0$, the HKHF model preserves much of the physics described by the original full time-dependent Ginzburg-Landau model and retains much richer dynamics than the special case of $J = 0$ [11]. In practical terms, the physical validity of the HKHF model is more restrictive than the full time-dependepent Ginzburg-Landau model in that the sample domain under consideration should be proportional to the penetration depth. This would hinder its applicability to the study of certaint problems related to type I superconductors where the spatial domain size is often far bigger than the penetration depth of the applied magnetic field [13]. Yet, the HKHF model remains a good simplied model for problems related to the vortex state in type-II superconductors since such phenomena are often on the scale of the coherence length. Moreover, for $J = 0$, the HKHF system (1.1-1.3) is a gradient flow of a simplified Ginzburg Landau free energy (with the induced magnetic field being equal to the applied field). It is then expected that asymptotically the time dependent solutions will approach to some steady state solution [16]. For an appropriate applied magnetic field, such steady state solutions may contain a single vortex, that is, an isolated zero for the complex order parameter [16]. Notice that the square of the magnitude of the complex order parameter represents the local density of the superconducting carriers, or the *Cooper pairs*. Moreover, in the dimensionless form considered here, the material is in the superconducting phase where the magnitude is close to one, and it is in the normal phase where $|u| \simeq 0$. The locations of quantized vortices are defined by positITIONS where the order parameter vanishes.

For small enough J, solutions of the HKHF system (1.1-1.3) with a single stationary vortex have been shown to exist numerically, see Fig. 1.1 for two contour plots of the magnitude of numerical solutions computed on the unit square domain for magnetic field near the lower critical value [11]. A constant current is applied along the vertical axis direction which results in the horizontal stationary shift of the vortex center under the influence of the Lorentz force in the direction perpendicular to the applied current, away from the center of the domain. Further numerical experiments in [12] suggest that for large enough J, there are solutions to (1.1-1.3) representing periodic motion of vortices and the subsequent collapse to the normal state for even greater applied current.

Our goal here is to rigorously construct the computationally observed stationary single-vortex solutions for suitable ranges of the applied current and applied field. We begin with an investigation of time-periodic solutions of the form $\psi_0(x,t) = e^{-id_\varepsilon t}u(x)$ of the HKHF equation, where d_ε is an *unknown* constant depending only on ε. Thus, $u = u(x)$ satisfies

$$\begin{cases} (i\nabla + \boldsymbol{A}_0)^2 u + \dfrac{1}{\varepsilon^2}(|u|^2 - 1)u + i\Phi_a u = id_\varepsilon u & \text{in } \Omega, \\ (i\nabla + \boldsymbol{A}_0)u \cdot \nu = 0 & \text{on } \partial\Omega. \end{cases} \tag{1.4}$$

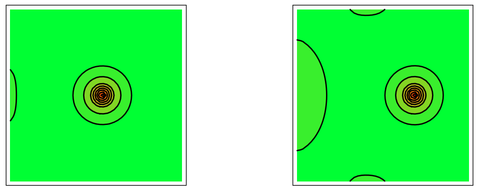

FIG. 1.1. *Contour plots of $|u|$ numerically solved in a unit square domain with two different values of the applied current [11].*

The linear in time but constant in space shift of the phase $(-d_\epsilon t)$ of the solution ψ_0 is connected to the freedom in specifying the scalar potential Φ_a up to a scalar constant. The latter is a consequence of the gauge invariance of the original TDGL model [9, 12] which, in turn, implies that the time-dependent solution of the HKHF model is allowed to vary up to a global-in-space and linear-in-time shift of its phase. Thus, the solution u of (1.4) can be effectively viewed as a stationary solution of the time-dependent HKHF model in the sense of gauge equivalence. We illustrate later that for some special geometric domain Ω (such as *disks* and *rectangles*) and special functions Φ_a, it is possible to specify the exact value of the d_ε to be $\frac{1}{|\Omega|}\Phi_a$ due to symmetry properties. In general, we have $d_\epsilon \to \frac{1}{|\Omega|}\Phi_a$ as $\epsilon \to 0$.

In earlier works on the mathematical analysis of the vortex state within the GL framework, such as the pioneering works on the study of vortex solutions given in [4] and [5] and direct construction of multiple vortex solutions in [18], the appearance of vortices is assured due to the boundary condition imposed artificially. Later, the rigorous connection of the vortex nucleation due to an applied magnetic field was made in studies like [3, 16, 21]. The approach we take in this study differs from the above, mainly due to the fact that the equation (1.4) is *not variational*. Instead, we follow the technique introduced in a recent work [19] to provide the constuction of single-vortex solutions to the equation (1.4) with a non-zero applied current. The construction of vortex solutions given in [19] was made for the problem

$$\varepsilon^2\Delta u + (1-|u|^2)u = 0 \qquad \text{in } \Omega \tag{1.5}$$

with Neumann or Dirichlet boundary conditions, using a reduction under the assumption that Ω has a non-trivial topology. By assuming some appropriate *nondegenerate conditions*, we provide, in this work, a more general construction of vortex solutions via a non-variational reduction for problems like (1.4).

To put our work in a broader context, we note that given the technological interests of studying vortex interactions with the applied current, there has been numerous calls for in-depth mathematical analysis of the various issues related to the time-dependent G-L models [10, 11]. Indeed, there is now growing attention on the subject, see, for instance, the recent studies on the effect of applied current or voltage using a one-dimensional TDGL models [2, 22, 23] and the study of finite time vortex motion using the two dimensional TDGL with an approximate boundary current condition [25]. In this work, we take a different angle and construct a single-vortex solution by following the approach in [19]. We thus see that, if the applied current is sufficiently weak, the geometric pinning force can counter the Lorentz force generated by the applied current and prevents the vortex from moving across the sample. As a consequence of such force balance, the stationary location of the vortex also varies with the applied current. While there could be an appreciable stationary shift in the vortex position for relatively large values of the applied current, the shift reduces to zero in the limit that the applied current diminishes to zero together with the vortex core size. It can be further illustrated that the shift happens naturally along the direction of the Lorentz force which is perpendicular to the applied current. Putting together, we see that the results of this paper provide a partial justification to the existence of a critical applied current for generating vortex motion. Moreover, the results offer new insight into the rich vortex dynamics of the HKHF model, which in turn shed light on the current driven dynamics of the full TDGL model.

To make the presentation of the key findings in more clear manner, the main results are summarized in Section 2. The rest of the papers is organized as follows. In Section 3, we introduce the approximate solution. In Section 4, an error estimate is given. In Section 5 and Section 6, we study the linear problem and the nonlinear problem respectively. In Section 7, we prove the theorem 2.2 while in Section 8, we sketch the proofs of Theorems 2.3 and 2.4. Finally, connections of the results presented here to other studies are discussed in Section 9.

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2. Main results.

We now present the main results of the paper in this section. First, we need to introduce some definitions and assumptions.

Definition 2.1. *Let $\mathbf{\Psi}(\xi)$: $\Omega \to \mathbb{R}^N$ be a continuous vector-valued function, where $\Omega \subset \mathbb{R}^N$ is an open set. We say that $\xi_0 \in \Omega$ is a stable zero point of $\mathbf{\Psi}$ if*

i) $\mathbf{\Psi}(\xi_0) = 0$;

ii) there is a neighborhood $\mathcal{U}$ of ξ_0 such that

$$\mathbf{\Psi}(\xi) \neq 0 \qquad \forall\ \xi \in \partial\mathcal{U},$$

and

$$\deg(\mathbf{\Psi}, \mathcal{U}, 0) \neq 0,$$

where deg *denotes the Brouwer degree.*

We remark that a nondegenerate zero of $\mathbf{\Psi}$ is a stable zero. Here a zero ξ_0 of $\mathbf{\Psi}$ is called *nondegenerate* if $\mathbf{\Psi}(\xi_0) = 0, \det(\nabla\Psi(\xi_0)) \neq 0$. If $\mathbf{\Psi}(x) = \nabla\psi(x)$ for some scalar function ψ, then any local minimum or local maximum points of ψ are stable zeroes of $\mathbf{\Psi}$.

Throughout the paper, the functions $\boldsymbol{A}_0$ and Φ_a are assumed to be smooth over $\overline{\Omega}$. We postpone the definition of $\mathbf{\Psi}(\xi)$ to Section 3. Now we state our main results on the existence of single-vortex solutions to (1.4) in general domains.

Theorem 2.1. *Assume that $\boldsymbol{A}_0$ and Φ_a are independent of ε. If $\xi_0 \in \Omega$ is a stable zero point of $\mathbf{\Psi}$ defined in (3.6), then for small ε, there exist a constant d_ε and a single-vortex solution u_ε satisfying (1.4). Moreover, the vortex is degree $+1$ (or -1) and centered at ξ_ε such that $\xi_\varepsilon \to \xi_0$, and the constant $d_\varepsilon \to \frac{1}{|\Omega|}\int_\Omega \Phi_a$ as $\varepsilon \to 0$.*

In the above, both $\boldsymbol{A}_0$ and Φ_a are independend on ε so that we may see the effect of nonvanishing applied magnetic field and applied current when the vortex core size diminishes. In fact, we can also deal with the case that $\boldsymbol{A}_0$ and Φ_a do depend on ε. In this case, $\boldsymbol{\Psi}$ depends upon ε too, which is thus denoted as $\boldsymbol{\Psi}^\varepsilon$ (the specific form is to be given later in the discussion). To give a proper definition of the degree, we need a stronger non-degeneracy condition as follows.

Definition 2.2. *Let $\boldsymbol{\Psi}^\varepsilon(\xi)$: $\Omega \to \mathbb{R}^N$ be a smooth vector-valued function, where $\Omega \subset \mathbb{R}^N$ is an open set. We say that ξ_0^ε is a uniformly non-degenerate inner zero point of $\boldsymbol{\Psi}^\varepsilon$ if*

i) $\mathrm{dist}(\xi_0^\varepsilon, \partial\Omega) \geq C$ *(independent of ε);*

ii) $\boldsymbol{\Psi}^\varepsilon(\xi_0^\varepsilon) = 0$;

iii) there is a constant C independent of ε such that all the eigenvalues of $\nabla\boldsymbol{\Psi}^\varepsilon(\xi_0^\varepsilon)$ satisfy

$$|\lambda^\varepsilon(\nabla\boldsymbol{\Psi}^\varepsilon(\xi_0^\varepsilon))| \geq C.$$

Let

$$\|\boldsymbol{A}_0\| = \max\left\{\|A_{01}(x)\|_{C^{1,\beta}(\overline{\Omega})}, \|A_{02}(x)\|_{C^{1,\beta}(\overline{\Omega})}\right\},$$

where $0 < \beta < 1$ is an arbitrary constant and $\|\cdot\|_{C^{1,\beta}}$ is the usual Hölder norm. We then have the following theorem.

Theorem 2.2. *Assume that $\boldsymbol{A}_0$ and Φ_a satisfy*

$$\|\boldsymbol{A}_0\| = O(\varepsilon^{-\alpha}), \quad \|\Phi_a\|_{C^\beta(\overline{\Omega})} = O(\varepsilon^{-\alpha}) \qquad \text{for some } 0 \leq \alpha < \frac{1}{6}. \tag{2.1}$$

Then for small ε, if ξ_0^ε is a uniformly non-degenerate inner zero point of $\boldsymbol{\Psi}^\varepsilon(\xi)$ defined in (3.6) below, there exists a constant d_ε and a single-vortex solution u_ε satisfying (1.4). Moreover, the vortex is degree $+1$ (or -1) and centered at ξ_ε such that $|\xi_\varepsilon - \xi_0^\varepsilon| = O(\varepsilon^{\frac{2}{3}-4\alpha}|\log\varepsilon|^2)$, and the constant $\left|d_\varepsilon - \frac{1}{|\Omega|}\int_\Omega \Phi_a\right| = O(\varepsilon^{1-\alpha})$ as $\varepsilon \to 0$.

Remark 2.1. *In the above two theorems, if $\int_\Omega \Phi_a \neq 0$, then the constant $d_\varepsilon \neq 0$. In general even if $\int_\Omega \Phi_a = 0$, the constant d_ε may be nonzero. This is strikingly different from the results in [19].*

Next, we consider several specific cases with $\int_\Omega \Phi_a = 0$ and $d_\varepsilon = 0$. More precisely, assume that Ω is the unit disk D in $\mathbb{R}^2$ and $\Phi_a = Jx_2$ for some small constant applied current $J \neq 0$, then we may consider the steady state solution $u = u(x)$ of (1.1)-(1.3) that satisfies

$$\begin{cases} (i\nabla + \boldsymbol{A}_0)^2 u + \dfrac{1}{\varepsilon^2}(|u|^2 - 1)u + i\Phi_a u = 0 & \text{in } D, \\ (i\nabla + \boldsymbol{A}_0)u \cdot \nu = 0 & \text{on } \partial D. \end{cases} \tag{2.2}$$

Theorem 2.3. *Assume $\boldsymbol{A}_0 = 0$ and $\Phi_a(x) = Jx_2$ for some nonzero constant J independent of ε. Then for small enough ε, there exists a single-vortex solution of (2.2) with degree $+1$ or -1. Moreover, the location of the vortex has a slight nonzero shift in x_1 axis from the origin.*

Remark 2.2. *Theorem 2.3 is also true for domains being symmetric in x-axis and y-axis. The proof is similar by employing the symmetries.*

Finally, we consider two examples with $\boldsymbol{A}_0$ given respectively by

$$\boldsymbol{A}_0(x) = \frac{H}{2}(-x_2, x_1), \tag{2.3}$$

or

$$\boldsymbol{A}_0(x) = \left(-\frac{H}{2}x_2, \frac{H}{2}x_1 - \frac{J}{2}x_1^2\right), \tag{2.4}$$

for a magnetic field H close to the lower critical field H_{C_1}, i.e. $H \sim |\log \varepsilon|/2$ [16, 20, 21]. The vector potential given in (2.4) satisfies in particular the Maxwell equation for the corresponding electric potential $\Phi_a = Jx_2$ [12].

Theorem 2.4. *Assume that $\Phi_a = Jx_2$, J is finite and independent of ε, and $\boldsymbol{A}_0$ satisfies (2.3) or (2.4) with $H \sim |\log \epsilon|/2$. Then for small enough ε, there exists a single-vortex solution of (2.2) with degree $+1$ or -1.*

Moreover, for any fixed small ε and any $J \neq 0$, the location of the vortex has a slight nonzero shift on x_1 axis from origin, but the shift approaches to 0 as $\varepsilon \to 0$.

Remark 2.3. *In Theorem 2.4, since $H \sim |\log \varepsilon|/2$ is very large, its impact on the location of the vortex overshadows that of J. The distance of the vortex to the origin is on the order of $O(\frac{J}{H})$, which still approaches to 0 as $\varepsilon \to 0$. This is different from the case in Theorem 2.3.*

Remark 2.4. *In Theorem 2.4, if $H \sim |\log \varepsilon|/2$ is very large, we can also allow J to be large so that*

$$J < cH \tag{2.5}$$

where $c < 2$ is any fixed number. In this case, the distance of the vortex to the origin is on the order of $O(\frac{J}{H})$. The proof is similar to that of Theorem 2.4 by using Theorem 2.2.

Theorems 2.1 and 2.2 can also be extended to multi-vortex. For simplicity, we only deal with single-vortex case.

Our main idea of proving Theorem 2.1 is the *finite-dimensional Liapunov-Schmidt reduction method*. In [19], del Pino, Kowalczyk and Musso performed a similar reduction method for the variational problem (1.5). Our analysis here utilizes the ansatz $u = v(y) = \eta(V_0 + iV_0\psi) + (1-\eta)V_0 e^{i\psi}$ similar to that given in [19] and we adopt a degree-theoretic approach to provide a global construction of the solution. We note that Pacard and Riviere [18] has also developed a reduction theory for Ginzburg-Landau equation which also works for non-variational problems. Their approach takes a different ansatz with vortices glued together and is based on an implicit function theorem. Our results here illustrate that the approach developed in [19] for variational problems can be extended to non-variational problems as well.

Since d_ϵ can be replaced by $\hat{d}_\epsilon + (\int_\Omega \Phi_a)/|\Omega|$ with a change of notation, we may assume that, with no loss of generality,

$$\int_\Omega \Phi_a = 0 \tag{2.6}$$

in the rest of the paper.

In what follows we just deal with the case that $\boldsymbol{A}_0$ and Φ_a are dependent on ε. As for the independent case, we put $\alpha = 0$ directly.

The rest of the paper is organized as follows: The ansatz and preliminary estimates are given in Section 3 and Section 4 respectively. In Section 5, we develop the necessary projected linear theory and the nonlinear projected problem is solved in Section 6. Section 7 contains the proofs of Theorem 2.1 and Theorem 2.2. Finally we give a sketch proof of Theorem 2.3 and Theorem 2.4 in Section 8.

For convenience of notation, we denote $x^{\perp} = (-x_2, x_1)$ for any vector x, and $\bar{w}$ the conjugate of the complex-valued w. Moreover, we let C denote a generic constant independent of ε which can take various values.

3. Ansatz. We first introduce the standard single-vortex solutions $w_{\pm}(z)$ of respective degrees $+1$ and -1 in the plane, of the equation

$$-\Delta w + (|w|^2 - 1)w = 0 \qquad \text{in } \mathbb{R}^2,$$

which has the form

$$w_{\pm}(z) = S(r_0)e^{\pm i\theta_0}$$

with (r_0, θ_0) being the usual polar coordinates and $S(r_0)$ the unique solution to

$$\begin{cases} S'' + \dfrac{S'}{r_0} - \dfrac{S}{r_0^2} + (1 - |S|^2)S = 0 \qquad & \text{in } (0, \infty), \\ S(0) = 0, \quad S(\infty) = 1. \end{cases} \tag{3.1}$$

It is well known, (see e.g. [7]), that $S'(0) > 0$ and

$$S(r_0) = 1 - \frac{1}{2r_0^2} + O(\frac{1}{r_0^4}) \qquad \text{as } r_0 \to \infty. \tag{3.2}$$

In general, $w_{\pm}(z)$ does not satisfy the boundary condition that $(i\nabla + \boldsymbol{A}_0)u \cdot \nu = 0$ on ∂D. To this end, we need to add a phase function $\varphi(x)$ so that both the equation and the boundary conditions are simultaneously satisfied. In this sense, our choice of ansatz provides a global construction. Our first approximation to a solution of (1.4) can thus be written as

$$U_0(x) = S\left(\frac{|x - \xi|}{\varepsilon}\right) e^{i[\theta(x) + \varphi(x)]}, \tag{3.3}$$

where $\theta(x) = \theta_0(x - \xi)$ and, by (2.6), the phase function $\varphi(x)$ is the unique real solution of

$$\begin{cases} \Delta\varphi - (\nabla \cdot \boldsymbol{A}_0) - \Phi_a = 0 \qquad & \text{in } \Omega, \\ \dfrac{\partial\varphi}{\partial\nu} = -\dfrac{\partial\theta}{\partial\nu} + \boldsymbol{A}_0 \cdot \nu & \text{on } \partial\Omega, \\ \displaystyle\int_{\Omega} \varphi = 0. \end{cases} \tag{3.4}$$

Remark 3.1. *Similar construction can be carried out for a solution with degree -1 just by replacing θ with $-\theta$ in (3.3) and then in (3.4) correspondingly.*

We assume that ξ stays in the following configuration set

$$\Lambda = \Big\{ \xi \in \Omega : \ \mathrm{dist}(\xi, \partial\Omega) > \delta \Big\}.$$

Then since

$$\nabla \theta(x) = \frac{(x-\xi)^{\perp}}{|x-\xi|^2} \in C^{\infty}(\partial\Omega),$$

in addition to the condition (2.1), it is easy to see from (3.4), by standard Schauder estimates, that

$$\|\varphi\|_{C^{2,\beta}(\overline{\Omega})} = O(\varepsilon^{-\alpha}). \tag{3.5}$$

Here α and β are defined in (2.1).

Throughout the paper, we define

$$\mathbf{\Psi}_0^{\varepsilon}(x,\xi) = \nabla\varphi(x,\xi) - \boldsymbol{A}_0(x),$$

and

$$\mathbf{\Psi}^{\varepsilon}(\xi) = \mathbf{\Psi}_0^{\varepsilon}(\xi,\xi). \tag{3.6}$$

Remark 3.2. *If $\boldsymbol{A}_0$ and Φ_a are independent of ε, so is $\mathbf{\Psi}^{\varepsilon}$ obviously. Thus we use $\mathbf{\Psi}$ to replace $\mathbf{\Psi}^{\varepsilon}$ for the sake of convenience.*

Remark 3.3. *In the end, the vortex point ξ_0^{ϵ} satisfies $\mathbf{\Psi}^{\epsilon}(\xi_0^{\epsilon}) \sim 0$. The fact that Ψ^{ϵ} governs the location of the vortex comes from mathematical estimations carried out later. Physically this means that the combination of the domain Green's function, representing the geometric pinning force and barrier provided by the magnetic field, and the electric current, representing the effect of Lorentz force, determines the vortex location.*

Finally in this section, we introduce the set-up of our problem and an overall strategy of solving the problem.

Denote $v(y) = u(\varepsilon y)$. Obviously u satisfies (1.4) if and only if v is a solution to

$$\begin{cases} (i\nabla + \varepsilon \widetilde{\boldsymbol{A}}_0)^2 v + (|v|^2 - 1)v + i\varepsilon^2 \widetilde{\Phi}_a v = i\varepsilon^2 d_{\varepsilon} v & \text{in } \Omega_{\varepsilon}, \\ (i\nabla + \varepsilon \widetilde{\boldsymbol{A}}_0) v \cdot \nu = 0 & \text{on } \partial\Omega_{\varepsilon}, \end{cases} \tag{3.7}$$

where $\widetilde{\boldsymbol{A}}_0(y) = \boldsymbol{A}_0(\varepsilon y)$, $\widetilde{\Phi}_a(y) = \Phi_a(\varepsilon y)$ and $\Omega_{\varepsilon} = \Omega/\varepsilon$. We shall set in what follows

$$V_0(y) = U_0(\varepsilon y) = S(|y-\xi'|) e^{i\left[\tilde{\theta}(y) + \tilde{\varphi}(y)\right]},$$

where $\xi' = \xi/\varepsilon$, $\tilde{\theta}(y) = \theta(\varepsilon y)$ and $\tilde{\varphi}(y) = \varphi(\varepsilon y)$. Let $\tilde{\eta}$: $\mathbb{R} \to \mathbb{R}$ be a smooth cut-off function such that $\tilde{\eta}(s) = 1$ for $s \le 1$ and $\tilde{\eta}(s) = 0$ for $s \ge 2$. Define

$$\eta(y) = \tilde{\eta}(|y - \xi'|).$$

Following the ideas in [19], we shall look for the solutions of (3.7) with the form

$$v(y) = \eta(V_0 + iV_0\psi) + (1-\eta)V_0 e^{i\psi}, \tag{3.8}$$

where ψ is small away from the vortex and possibly unbounded nearby, whereas $iV_0\psi$ is always bounded.

Remark 3.4. *When $|y-\xi^{'}|\le 1$, we have $v(y)=V_0+iV_0\psi$ and when $|y-\xi^{'}|\ge 2$, we have $v(y)=V_0e^{i\psi}$. This choice of ansatz helps us to effectively represent and analyze the nonlinear term, that is, outside the vortex, $|v|^2=Se^{-\psi_2}$ where $\psi=\psi_1+i\psi_2$ with ψ_1 and ψ_2 being real-valued. If instead we set $v=V_0+\phi$ with ϕ small, then the nonlinear term $|v|^2v$ becomes very complicated.*

Setting

$$\phi = iV_0\psi, \tag{3.9}$$

we require that ϕ is bounded (and smooth) near the vortex. Direct computation shows that ψ should satisfy

$$\begin{cases} \mathcal{L}^\varepsilon(\psi) = R + N(\psi) & \text{in } \Omega_\varepsilon, \\ \dfrac{\partial\psi}{\partial\nu} = F & \text{on } \partial\Omega_\varepsilon. \end{cases} \tag{3.10}$$

Here

$$\mathcal{L}^\varepsilon(\psi) = \Delta\psi + 2\left(\frac{\nabla V_0}{V_0} - i\varepsilon\widetilde{\boldsymbol{A}}_0\right)\nabla\psi - 2i|V_0|^2\psi_2 - \frac{\eta}{\eta+(1-\eta)e^{i\psi}}\frac{E}{V_0}\psi, \tag{3.11}$$

$$\begin{aligned} N(\psi) = {} & \frac{1}{iV_0[\eta+(1-\eta)e^{i\psi}]}\Big\{\Delta\eta V_0(e^{i\psi}-1-i\psi) + 2\nabla\eta\nabla V_0(e^{i\psi}-1-i\psi) \\ & - 2\nabla\eta\varepsilon\widetilde{\boldsymbol{A}}_0 iV_0(e^{i\psi}-1-i\psi) + 2i\nabla\eta\nabla\psi(e^{i\psi}-1) \\ & - 2\eta|V_0|^2\psi_2 iV_0\psi - \eta(1-\eta)^2|V_0|^2(e^{-2\psi_2}-1+2\psi_2)iV_0\psi \\ & + \eta^2(2-\eta)|V_0|^2|\psi|^2 iV_0\psi\Big\} - i\eta(2-\eta)|V_0|^2|\psi|^2 \\ & - i\frac{(1-\eta)e^{i\psi}}{\eta+(1-\eta)e^{i\psi}}(\nabla\psi)^2 - i(1-\eta)^2|V_0|^2(e^{-2\psi_2}-1+2\psi_2), \end{aligned} \tag{3.12}$$

$$R(y) = (iV_0)^{-1}E(y), \tag{3.13}$$

$$F(y) = iV_0^{-1}\frac{\partial V_0}{\partial\nu} + \varepsilon\boldsymbol{A}_0\cdot\nu, \tag{3.14}$$

where

$$E(y) = (i\nabla + \varepsilon\widetilde{\boldsymbol{A}}_0)^2V_0 + (|V_0|^2-1)V_0 + i\varepsilon^2\widetilde{\Phi}_aV_0 - i\varepsilon^2d_\varepsilon V_0. \tag{3.15}$$

In the above computations, we have already used the equation (3.4).

In the rest of the paper, we proceed to solve (3.10). This will be done in two steps:

Step 1: First, we fix the vortex position ξ and the scalar constant d and solve a projected nonlinear problem for ψ

$$\begin{cases} \mathcal{L}^\varepsilon(\psi) = & R + N(\psi) + c_0\epsilon^2\chi_{\Omega_\epsilon\setminus B(\xi',\delta/\epsilon)} \\ & + \sum_\ell c_\ell \frac{1}{iw(y-\xi')}\chi_{\{r<1/2\}}\frac{\partial w}{\partial y_\ell}(y-\xi') \quad \text{in } \Omega_\varepsilon, \\ \dfrac{\partial\psi}{\partial\nu} = & F \text{ on } \partial\Omega_\varepsilon, \\ \displaystyle\int_{\Omega_\varepsilon\setminus B(\xi',\delta/\varepsilon)} |V_0|^2\psi_1 = 0, & \quad \mathrm{Re}\int_{|z|<1}\hat{\phi}\bar{w}_{z_\ell} = 0, l=1,2. \end{cases} \tag{3.16}$$

Here c_0, c_ℓ are coefficients (Larange multipliers) which depend on ξ and d continuously.

Step 2: Then we use a degree argument to solve the reduced finite-dimensional problem $c_0 = c_\ell = 0, \ell = 1, 2$ for ξ.

Remark 3.5. *Pacard and Riviere [18] developed a reduction theory for Ginzburg-Landau equation by using Hölder spaces. The difference between their approach and del Pino-Kowalczyk-Musso's approach has two parts: First the ansatzes are different. Pacard and Riviere expand the solution simply as $u = V_0 + \phi$ while del Pino, Kowalczyk and Musso expand the solution u as $u = v(y) = \eta(V_0 + iV_0\psi) + (1-\eta)V_0 e^{i\psi}$. Secondly the reduced problems are solved differently. Del Pino, Kowalczyk and Musso use the variational reduction method to reduce the problem to a finite dimensional variational problem and make use of topological methods. Pacard and Riviere uss degree-theoretic method to solve the reduced problem. In summary, both approaches are almost identical in the reduction step, and they only differ in the reduced problem step. The approach in [19] is variational (can be used for variational problems), while the approach in [18] is more analytical (can be used for variational and non-variational problems as well).*

4. Preliminary Estimates. In this section we estimate the error term $E(y)$ defined at (3.15) and boundary term $F(y)$ defined at (3.14).

Lemma 1. *There exists a constant C, depending on δ such that for small ε and all points $\boldsymbol{\xi} \in \Lambda$, we have*

$$\|E\|_{C^1(|y-\xi'|<3)} \leq C\varepsilon^2|d_\varepsilon| + C\varepsilon^{1-\alpha}. \tag{4.1}$$

Moreover, we have that $E(y) = iV_0\Big[-\varepsilon^2 d_\varepsilon + R_1(y) + iR_2(y)\Big]$ with R_1, R_2 real-valued and

$$|R_1(y)| \leq C\varepsilon^{1-\alpha}\frac{1}{|y-\xi'|^3}, \tag{4.2}$$

$$|R_2(y)| \leq C\varepsilon^{1-\alpha}\frac{1}{|y-\xi'|} + C\varepsilon^{2-2\alpha} \tag{4.3}$$

if $|y-\xi'| > 1$. Finally, we have $F(y) = iF_2(y)$ where F_2 is real-valued and

$$\|F_2\|_{L^\infty(\partial\Omega_\varepsilon)} + \varepsilon^{-1}\|\nabla F_2\|_{L^\infty(\partial\Omega_\varepsilon)} \leq C\varepsilon^3.$$

Proof. Straightforward computation gives

$$\nabla V_0(y) = V_0(y)\left\{\frac{\nabla S(|y-\xi'|)}{S(|y-\xi'|)} + i(\nabla\tilde{\theta} + \nabla\tilde{\varphi})\right\}, \tag{4.4}$$

and

$$\begin{aligned}\Delta V_0(y) = iV_0\Bigg\{&(\Delta\tilde{\theta} + \Delta\tilde{\varphi}) + 2\frac{\nabla S(|y-\xi'|)}{S(|y-\xi'|)}(\nabla\tilde{\theta} + \nabla\tilde{\varphi})\\ &-i\frac{\Delta S(|y-\xi'|)}{S(|y-\xi'|)} + i(\nabla\tilde{\theta} + \nabla\tilde{\varphi})^2\Bigg\}.\end{aligned}$$

Thus, since $\tilde{\theta}$ is harmonic, we have

$$
\begin{aligned}
&(i\nabla + \varepsilon\widetilde{\boldsymbol{A}}_0)^2 V_0 \\
&= iV_0\Big\{ -\Delta\tilde{\varphi} - 2\frac{\nabla S(|y-\xi'|)}{S(|y-\xi'|)}(\nabla\tilde{\theta} + \nabla\tilde{\varphi} - \varepsilon\widetilde{\boldsymbol{A}}_0) + \varepsilon(\nabla\cdot\widetilde{\boldsymbol{A}}_0) \\
&\qquad + i\frac{\Delta S(|y-\xi'|)}{S(|y-\xi'|)} - i(\nabla\tilde{\theta} + \nabla\tilde{\varphi})^2 + 2i\varepsilon\widetilde{\boldsymbol{A}}_0(\nabla\tilde{\theta} + \nabla\tilde{\varphi}) - i\varepsilon^2|\widetilde{\boldsymbol{A}}_0|^2 \Big\}.
\end{aligned}
$$

Using (3.4) and the fact that $\nabla S(|y-\xi'|)\nabla\tilde{\theta} = 0$, we easily get

$$
\begin{aligned}
&(i\nabla + \varepsilon\widetilde{\boldsymbol{A}}_0)^2 V_0 + i\varepsilon^2\widetilde{\Phi}_a V_0 - i\varepsilon^2 d_\varepsilon V_0 \\
&= \ iV_0\Big\{ -\varepsilon^2 d_\varepsilon - 2\frac{\nabla S(|y-\xi_j'|)}{S(|y-\xi_j'|)}(\nabla\tilde{\varphi} - \varepsilon\widetilde{\boldsymbol{A}}_0) \\
&\qquad + i\frac{\Delta S(|y-\xi'|)}{S(|y-\xi'|)} - i|\nabla\tilde{\theta}|^2 - 2i\nabla\tilde{\theta}(\nabla\tilde{\varphi} - \varepsilon\widetilde{\boldsymbol{A}}_0) - i|\nabla\tilde{\varphi} - \varepsilon\widetilde{\boldsymbol{A}}_0|^2 \Big\}.
\end{aligned}
$$

Observe that $|\nabla\tilde{\theta}| = \frac{1}{|y-\xi'|}$ and

$$
(|V_0|^2 - 1)V_0 = \left[S^2(|y-\xi'|) - 1\right] V_0.
$$

The above estimates allow us to conclude that

$$
\begin{aligned}
E = iV_0\Big\{ &-\varepsilon^2 d_\varepsilon - 2\frac{\nabla S(|y-\xi'|)}{S(|y-\xi'|)}(\nabla\tilde{\varphi} - \varepsilon\widetilde{\boldsymbol{A}}_0) \\
&-2i\nabla\tilde{\theta}(\nabla\tilde{\varphi} - \varepsilon\widetilde{\boldsymbol{A}}_0) - i|\nabla\tilde{\varphi} - \varepsilon\widetilde{\boldsymbol{A}}_0|^2 \Big\}.
\end{aligned} \tag{4.5}
$$

From (4.5), the desired estimate (4.2) follows. Recalling that $S(r) \sim Cr$ as $r \to 0$, (4.1) also holds.

On $\partial\Omega_\varepsilon$, by (3.4) and (4.4), we have

$$
F(y) = i\frac{1}{S(|y-\xi'|)}\frac{\partial S(|y-\xi'|)}{\partial\nu} := iF_2(y).
$$

Since now $|y-\xi'| > \frac{\delta}{\varepsilon}$, it is easy to see that

$$
F_2 = O(\varepsilon^3).
$$

Direct calculation also shows that

$$
\nabla F_2(y) = O(\varepsilon^4).
$$

The proof is concluded.

Using the form of the ansatz in the region $|y-\xi'| > 2$, we see that (3.10) takes the simple form

$$
\mathcal{L}^\varepsilon(\psi) = -R - i(\nabla\psi)^2 - i|V_0|^2(e^{-2\psi_2} - 1 + 2\psi_2). \tag{4.6}
$$

In terms of the real part and the imaginary part, (4.6) becomes

$$\Delta\psi_1 + 2\frac{\nabla S(|y-\xi'|)}{S(|y-\xi'|)}\nabla\psi_1 - 2\left(\nabla\tilde{\theta} + \nabla\tilde{\varphi} - \varepsilon\widetilde{\boldsymbol{A}}_0\right)\nabla\psi_2 \\ -2\nabla\psi_1\nabla\psi_2 - \varepsilon^2 d_\varepsilon + R_1 = 0 \tag{4.7}$$

and

$$\Delta\psi_2 + 2\frac{\nabla S(|y-\xi'|)}{S(|y-\xi'|)}\nabla\psi_2 + 2\left(\nabla\tilde{\theta} + \nabla\tilde{\varphi} - \varepsilon\widetilde{\boldsymbol{A}}_0\right)\nabla\psi_1 \\ -2|V_0|^2\psi_2 + |\nabla\psi_1|^2 - |\nabla\psi_2|^2 + |V_0|^2(e^{-2\psi_2} - 1 + 2\psi_2) + R_2 = 0,$$

where $R_1(y)$, $R_2(y)$ are defined in Lemma 1 and, as defined below (3.7),

$$V_0(y) = w(y-\xi')e^{i\tilde{\varphi}}. \tag{4.8}$$

To study the ansatz near the point ξ', it is more convenient to do this in the translated variable $z = y - \xi'$. We define the function $\hat{\phi}(z)$ such that

$$\hat{\phi}(z) = iw(z)\psi(z+\xi'), \tag{4.9}$$

which implies, from (3.9) and (4.8),

$$\phi(y) = e^{i\tilde{\varphi}}\hat{\phi}(y-\xi').$$

We shall write Problem (3.10) in terms of the function $\hat{\phi}$. Let us consider the operator L^ε defined by

$$L^\varepsilon(\hat{\phi})(z) = iw(z)\mathcal{L}^\varepsilon(\psi)(z+\xi').$$

Thus $\hat{\phi}$ should satisfy

$$L^\varepsilon(\hat{\phi}) = \widehat{R} + \widehat{N}(\hat{\phi}).$$

Here explicitly, designating that $V_1(z) = V_0(z+\xi')$, $\widehat{\boldsymbol{A}}_0(z) = \widetilde{\boldsymbol{A}}_0(z+\xi')$, $\widehat{\Phi}_a(z) = \widetilde{\Phi}_a(z+\xi')$, $\hat{\varphi}(z) = \tilde{\varphi}(z+\xi')$,

$$L^\varepsilon(\hat{\phi}) = L^0(\hat{\phi}) + 2i\left(\nabla\hat{\varphi} - \varepsilon\widehat{\boldsymbol{A}}_0\right)\nabla\hat{\phi} \\ -2i\left(\nabla\hat{\varphi} - \varepsilon\widehat{\boldsymbol{A}}_0\right)\frac{\nabla w}{w}\hat{\phi} - \frac{\tilde{\eta}(|z|)}{\tilde{\eta} + (1-\tilde{\eta})e^{\frac{\hat{\phi}}{w}}}\frac{E_1}{V_1}\hat{\phi} \tag{4.10}$$

where L^0 is the linear operator defined by

$$L^0(\hat{\phi}) = \Delta\hat{\phi} + (1 - |w(z)|^2)\hat{\phi} - 2\,\mathrm{Re}(\bar{w}\hat{\phi})w(z),$$

and E_1 is given by

$$E_1(z) = (i\nabla + \varepsilon\widehat{\boldsymbol{A}}_0)^2 V_1 + (|V_1|^2 - 1)V_1 + i\varepsilon^2\widehat{\Phi}_a V_1 - i\varepsilon^2 d_\varepsilon V_1.$$

The term $\widehat{R}$ is

$$\widehat{R}(z) = iw(z)R(z+\xi') = e^{-i\hat{\varphi}}E_1(z), \tag{4.11}$$

while the nonlinear term $\widehat{N}(\hat{\phi})$ is given by

$$\widehat{N}(\hat{\phi}) = iwN(\psi)(z+\xi'). \tag{4.12}$$

Observe that, in terms of w, E_1 takes the form

$$E_1 = V_1\left[|\nabla\hat{\varphi} - \varepsilon\widehat{\boldsymbol{A}}_0|^2 - 2i\frac{\nabla w}{w}(\nabla\hat{\varphi} - \varepsilon\widehat{\boldsymbol{A}}_0) - i\varepsilon^2 d_\varepsilon\right].$$

5. Projected linear theory. To solve the problem (3.10), we need first to analyze the possibility to invert the operator $\mathcal{L}^\varepsilon$. It is not expected that this operator to be in general invertible. Indeed, its version L^ε in the $\hat{\phi}$ -variable is a small perturbation of the operator L^0 defined in the above section. When regarded in entire $\mathbb{R}^2$ this operator does have a kernel: functions w_{z_ℓ} (the derivative with respect to z_ℓ such that $z = z_1 + iz_2$) and iw annihilate it. In suitable spaces, these functions are known to span the entire kernel, see [17, 18]. In a suitable "orthogonal" to this kernel, the bilinear form associated to this operator turns out to be uniformly positive definite and hence invertible.

As in [19], we consider the following linear problem, for fixed small $\delta > 0$,

$$\begin{cases} \mathcal{L}^\varepsilon(\psi) = h + c_0\varepsilon^2\chi_{\Omega_\varepsilon\setminus B(\xi',\delta/\varepsilon)} & \text{in } \Omega_\varepsilon, \\ \dfrac{\partial\psi}{\partial\nu} = g & \text{on } \partial\Omega_\varepsilon, \\ \displaystyle\int_{\Omega_\varepsilon\setminus B(\xi',\delta/\varepsilon)} |V_0|^2\psi_1 = 0, \\ \mathrm{Re}\displaystyle\int_{|z|<1} \hat{\phi}(z)\bar{w}_{z_\ell}(z) = 0 \qquad \text{for any } \ell = 1, 2, \end{cases} \tag{5.1}$$

where $h(y) = h_1(y) + ih_2(y)$, $g(y) = g_1(y) + ig_2(y)$ are two complex-valued functions, w_{z_ℓ} is the derivative of w with respect to z_ℓ, $\bar{w}_{z_\ell}$ is the conjugate of w_{z_ℓ} and c_0 is a real constant.

We shall establish a priori estimates for this problem. To this aim we shall conveniently introduce adapted norms. In our proof, in order to get the best upper bound $1/6$ for α, we let

$$\sigma_2 = \frac{1}{6}$$

and σ_1 be any number such that

$$0 < \sigma_1 < \sigma_2 - \alpha.$$

We set $0 < \beta < 1$. Denote $r = |y - \xi'|$ and define

$$\begin{aligned} \|\psi\|_* = \|\hat{\phi}\|_{C^{2,\beta}(|z|<2)} + \|\hat{\phi}\|_{C^{1,\beta}(|z|<3)} \\ + \|\psi_1\|_{L^\infty(r>2)} + \|r\nabla\psi_1\|_{L^\infty(r>2)} \\ + \|r^{1+\sigma_2}\psi_2\|_{L^\infty(r>2)} + \|r^{1+\sigma_2}\nabla\psi_2\|_{L^\infty(r>2)}, \end{aligned} \tag{5.2}$$

$$\|h\|_{**} = \|\hat{h}\|_{C^{0,\beta}(|z|<3)} + \|r^{2+\sigma_1}h_1\|_{L^\infty(r>2)} + \|r^{1+\sigma_2}h_2\|_{L^\infty(r>2)}, \tag{5.3}$$

where $\hat{h}(z) = iw(z)h(z + \xi')$. In addition, we define

$$\begin{aligned} \|g\|_{***} = \varepsilon^{-1}\|g_1\|_{L^\infty(\partial\Omega_\varepsilon)} + \varepsilon^{-2}\|\nabla g_1\|_{L^\infty(\partial\Omega_\varepsilon)} \\ + \varepsilon^{-1-\sigma_2}\|g_2\|_{L^\infty(\partial\Omega_\varepsilon)} + \varepsilon^{-2-\sigma_2}\|\nabla g_2\|_{L^\infty(\partial\Omega_\varepsilon)}. \end{aligned} \tag{5.4}$$

Lemma 2. *Assume that $\xi \in \Lambda$. There exists a constant $C > 0$, dependent on δ but independent of c_0, such that for ε sufficient small, any solution of (5.1) satisfies*

$$\|\psi\|_* \le C\Big[|\log\varepsilon|\|h\|_{**} + \|g\|_{***}\Big].$$

Proof. The proof is similar to that of Lemma 4.1 of [19]. The minor difference is the effect of $\mathbf{A}_0$. For the sake of completeness, we include the proof here.

We argue by contradiction. Let us assume the existence of sequence $\varepsilon_n \to 0$, and the function ψ^n, h^n and g^n which satisfy

$$\begin{cases} \mathcal{L}_n^\varepsilon(\psi^n) = h^n + c_n\varepsilon_n^2\chi_{\Omega_{\varepsilon_n}\setminus B(\xi',\delta/\varepsilon_n)} & \text{in } \Omega_{\varepsilon_n}, \\ \dfrac{\partial\psi^n}{\partial\nu} = g^n & \text{on } \partial\Omega_{\varepsilon_n}, \\ \displaystyle\int_{\Omega_{\varepsilon_n}\setminus B(\xi',\delta/\varepsilon_n)} |V_0|^2\psi_1^n = 0, & \\ \mathrm{Re}\displaystyle\int_{|z|<1} \hat{\phi}^n\bar{w}_{z_\ell} = 0, & \end{cases} \tag{5.5}$$

with

$$\|\psi^n\|_* = 1, \qquad |\log\varepsilon_n|\|h^n\|_{**} + \|g^n\|_{***} \to 0.$$

We observe from (3.11) that the real part of the equation is such that

$$\mathrm{Re}\,\mathcal{L}_n^\varepsilon(\psi^n) = \Delta\psi_1^n + O(\varepsilon_n^3)\nabla\psi_1^n + O(\varepsilon_n^{1-\alpha})\nabla\psi_2^n \qquad \text{in } \Omega_{\varepsilon_n}\setminus B(\xi', \frac{\delta}{\varepsilon_n}).$$

and hence, integrating on $\Omega_{\varepsilon_n}\setminus B(\xi',\delta/\varepsilon_n)$, we get the estimate

$$\begin{aligned} |c_n| &\le C\left|\left(\int_{\partial B(\xi',\frac{\delta}{\varepsilon_n})} - \int_{\partial\Omega_{\varepsilon_n}}\right)\frac{\partial\psi_1^n}{\partial\nu}\right| + C\varepsilon_n^{\sigma_2-\alpha}\|\psi^n\|_* + C\varepsilon^{\sigma_1}\|h^n\|_{**} \\ &\le C\left[\|\psi^n\|_* + \|g^n\|_{***} + \varepsilon^{\sigma_1}\|h^n\|_{**}\right]. \end{aligned}$$

It follows that c_n is bounded. We then assume that $c_n \to c_*$.

Next we will find that actually $c_* = 0$ and that ψ^n approaches 0. Let us set $\tilde{\psi}^n(x) = \psi^n(x/\varepsilon_n)$. It can be directly checked, from the bounds assumed, that given a small number $\delta' > 0$ we have

$$\begin{cases} \Delta\tilde{\psi}_1^n = O(\varepsilon_n^{\sigma_2-\alpha}) + O(\dfrac{\varepsilon_n^{\sigma_1}}{|\log\varepsilon_n|}) + c_n\chi_{\Omega\setminus B(\xi,\delta)} & \text{in } \Omega\setminus B(\xi,\delta'), \\ \dfrac{\partial\tilde{\psi}_1^n}{\partial\nu} = o(1) & \text{on } \partial\Omega. \end{cases}$$

Moreover,

$$\|\tilde{\psi}_1^n\|_\infty \le 1, \qquad \|\nabla\tilde{\psi}_1^n\|_\infty \le C_{\delta'}.$$

Passing to a subsequence, we then get that $\tilde{\psi}_1^n$ converges uniformly over compact subsets of $\Omega\setminus\{\xi\}$ to a function $\tilde{\psi}_1^*$ with $|\tilde{\psi}_1^*| \le 1$ which solves

$$\begin{cases} \Delta\tilde{\psi}_1^* = c_*\chi_{\Omega\setminus B(\xi,\delta)} & \text{in } \Omega\setminus B(\xi,\delta'), \\ \dfrac{\partial\tilde{\psi}_1^*}{\partial\nu} = 0 & \text{on } \partial\Omega. \end{cases}$$

This clearly implies $c_* = 0$ and hence $\tilde{\psi}_1^*$ is a constant. But by passing the third equality in (5.5) to the limit we see that $\tilde{\psi}_1^* = 0$. It follows that $\tilde{\psi}_1^n \to 0$ uniformly and in C^1 sense away from the points ξ. This implies in particular that

$$|\psi_1^n| + \varepsilon^{-1}|\nabla\psi_1^n| \to 0 \qquad \text{in } \Omega_{\varepsilon_n} \setminus B(\xi', \frac{\delta}{2\varepsilon_n}). \tag{5.6}$$

Let us now consider the imaginary part of the equation. From (3.11) we then argue that

$$\begin{cases} -\Delta\psi_2^n + 2|V_0|^2\psi_2^n = o(\varepsilon_n^{1+\sigma_2}) & \text{in } \Omega_{\varepsilon_n} \setminus B(\xi', \frac{\delta}{2\varepsilon_n}), \\ \dfrac{\partial\psi_2^n}{\partial\nu} = o(\varepsilon_n^{1+\sigma_2}) & \text{on } \partial\Omega_{\varepsilon_n}, \end{cases}$$

while in this region $\psi_2^n = O(\varepsilon_n^{1+\sigma_2})$. A suitable use of barriers yields then that

$$|\psi_2^n| + |\nabla\psi_2^n| = o(\varepsilon_n^{1+\sigma_2}) \qquad \text{in } \Omega_{\varepsilon_n} \setminus B(\xi', \frac{\delta}{2\varepsilon_n}). \tag{5.7}$$

Consider now a smooth cut-off function $\check{\eta}$ with $\check{\eta}(s) = 1$ if $s < \frac{1}{2}$, $\check{\eta}(s) = 0$ if $s > 1$, and define

$$\check{\psi}^n(y) = \check{\eta}\left(\frac{\varepsilon_n|y - \xi'|}{\delta}\right)\psi^n(y)..$$

Let us compute the equation satisfied by $\check{\psi}^n$. By (5.6) and (5.7), we observe that, for real and imaginary parts,

$$\nabla_y\check{\eta}\nabla\psi^n = \begin{bmatrix} o(\varepsilon_n^2) \\ o(\varepsilon_n^{2+\sigma_2}) \end{bmatrix}, \qquad \psi^n\Delta_y\check{\eta} = \begin{bmatrix} o(\varepsilon_n^2) \\ o(\varepsilon_n^{3+\sigma_2}) \end{bmatrix},$$

$$2\left(\frac{\nabla V_0}{V_0} - i\varepsilon\widetilde{\boldsymbol{A}}_0^n\right)\nabla_y\check{\eta}\psi^n = \begin{bmatrix} o(\varepsilon_n^{3+\sigma_2-\alpha}) \\ o(\varepsilon_n^{2-\alpha}) \end{bmatrix},$$

$$\check{\eta}\mathcal{L}_n^\varepsilon(\psi^n) = o(1)\begin{bmatrix} \frac{1}{|\log\varepsilon_n|(r+r^{2+\sigma_1})} \\ \frac{1}{|\log\varepsilon_n|(r+r^{1+\sigma_2})} \end{bmatrix}.$$

Thus we get

$$\begin{cases} \mathcal{L}_n^\varepsilon(\check{\psi}^n) = o(1)\begin{bmatrix} \frac{1}{|\log\varepsilon_n|(r+r^{2+\sigma_1})} + \varepsilon_n^2 \\ \frac{1}{|\log\varepsilon_n|(r+r^{1+\sigma_2})} \end{bmatrix} & \text{in } B(\xi', \frac{\delta}{\varepsilon_n}), \\ \check{\psi}^n = 0 & \text{on } \partial B(\xi', \frac{\delta}{\varepsilon_n}). \end{cases} \tag{5.8}$$

Before we proceed with the rest of the proof of Lemma 2, we need to establish the following intermediate result which provides an outer estimate. For notational simplicity we shall omit the subscript n in the quantities involved.

Lemma 3. *There exists positive numbers R_0, C such that for all large n*

$$\begin{aligned} \|\check{\psi}_1\|_{L^\infty(r>R_0)} + \|r\nabla\check{\psi}_1\|_{L^\infty(r>R_0)} + \|r^{1+\sigma_2}\check{\psi}_2\|_{L^\infty(r>R_0)} \\ + \|r^{1+\sigma_2}\nabla\check{\psi}_2\|_{L^\infty(r>R_0)} \le C\Big[\|\check{\phi}\|_{L^\infty(r<2R_0)} + o(1)\Big], \end{aligned} \tag{5.9}$$

where $\check{\phi} = iV_0\check{\psi}$.

Proof. From (4.7) it can be directly checked that the following relations hold for $r > 2$,

$$-\Delta\check{\psi}_1 = O(\frac{1}{r^3})\nabla\check{\psi}_1 + O(\frac{1}{r} + \varepsilon^{1-\alpha})\nabla\check{\psi}_2 + o(\frac{1}{|\log\varepsilon|})\frac{1}{r^{2+\sigma_1}} + o(\varepsilon^2), \tag{5.10}$$

$$-\Delta\check{\psi}_2 + 2|V_0|^2\check{\psi}_2 + O(\frac{1}{r^3})\nabla\check{\psi}_2 = O(\frac{1}{r} + \varepsilon^{1-\alpha})\nabla\check{\psi}_1 + o(\frac{1}{|\log\varepsilon|})\frac{1}{r^{1+\sigma_2}}. \tag{5.11}$$

Let us call p_1, p_2 the respective right hand sides of (5.10) and (5.11). Then we see, provided that $\sigma' < 1 - \sigma_2 - \alpha$,

$$|p_2| \le C\frac{B}{r^{1+\sigma_2}}, \qquad B = \|r^{1-\sigma'}\nabla\check{\psi}_1\|_{L^\infty(r>2)} + o(\frac{1}{|\log\varepsilon|})..$$

The use of a barrier and elliptic estimates then yield

$$|\nabla\check{\psi}_2| + |\check{\psi}_2| \le C\frac{B + \|\check{\psi}_2\|_{L^\infty(r=2)}}{r^{1+\sigma_2}}, \qquad 2 < r < \frac{\delta}{\varepsilon}. \tag{5.12}$$

We now use the above to estimate p_1. Since $\sigma_1 < \sigma_2 - \alpha$, we get

$$|p_1| \le \frac{C}{r^{2+\sigma_1}}\left[\|\nabla\check{\psi}_1\|_{L^\infty(r>2)} + \|r^{1+\sigma_2}\nabla\check{\psi}_2\|_{L^\infty(r>2)} + o(\frac{1}{|\log\varepsilon|})\right] + o(\varepsilon^2),$$

hence by (5.12)

$$|p_1| \le C\frac{B'}{r^{2+\sigma_1}} + o(\varepsilon^2),$$

where

$$B' = \|r^{1-\sigma'}\nabla\check{\psi}_1\|_{L^\infty(r>2)} + \|\check{\psi}_2\|_{L^\infty(r=2)} + o(\frac{1}{|\log\varepsilon|})$$

It is easy to see that a supersolution for (5.10) is given by

$$\omega(z) = \frac{B'}{\sigma_1^2}(1 - \frac{1}{r^{\sigma_1}}) + o(\frac{1}{|\log\varepsilon|})(\delta^2 - r^2\varepsilon^2) + \|\check{\psi}_1\|_{L^\infty(r=2)},$$

and hence

$$\|\check{\psi}_1\|_{L^\infty(r>1)} \le CB' + \|\check{\psi}_1\|_{L^\infty(r=1)}.$$

Next we estimate $\nabla\check{\psi}_1$. Let us define $\tilde{\psi}_1(z) = \check{\psi}_1(\xi' + R(e+z))$ where $|e| = 1$ and $R < \frac{\delta}{\varepsilon}$. Then for $|z| \le \frac{1}{2}$ we have

$$|\Delta\tilde{\psi}_1| \le CB' + o(1).$$

Since we also have $|\tilde{\psi}_1| \le CB'$ in this region, it follows from elliptic estimates that $|\nabla\tilde{\psi}_1(0)| \le CB'$. Since R and e are arbitrary, what we have established is

$$|\check{\psi}_1| + |r\nabla\check{\psi}_1| \le \left[\|r^{1-\sigma'}\nabla\check{\psi}_1\|_{L^\infty(r>2)} + \|\check{\psi}_1\|_{L^\infty(1<r<2)} + o(\frac{1}{|\log\varepsilon|})\right].$$

Now,

$$\|r^{1-\sigma'}\nabla\check{\psi}_1\|_{L^\infty(r>1)} \le R_0^{1-\sigma'}\|\nabla\check{\psi}_1\|_{L^\infty(1<r<R_0)} + \frac{1}{R_0^{\sigma'}}\|r\nabla\check{\psi}_1\|_{L^\infty(r>R_0)},$$

thus fixing R_0 sufficiently large, we obtain

$$|\check{\psi}_1| + |r\nabla\check{\psi}_1| \le C\left[\|\check{\psi}_1\|_{C^1(1<r<R_0)} + o(1)\frac{1}{|\log\varepsilon|}\right] \qquad \text{for } r>2,$$

and also

$$|\check{\psi}_2| + |\nabla\check{\psi}_2| \le \frac{C}{r^{1+\sigma_2}}\left[\|\check{\psi}\|_{C^1(1<r<R_0)} + o(1)\frac{1}{|\log\varepsilon|}\right] \qquad \text{for } r>2.$$

The lemma is proven.

Proof. [Continuation of the proof of Lemma 2] Let us go back to the contradiction argument. Since $\|\psi\|_* = 1$, and the corresponding portion of this norm of ψ goes to zero on the region $r > \frac{\delta'}{\varepsilon}$ for any given $\delta' > 0$, we conclude from the previous lemma that necessarily, for some $C > 0$,

$$\|\check{\phi}\|_{C^2(|z|<R_0)} \ge C,$$

where $\check{\phi}(z) = iw(z)\check{\psi}(\xi' + z)$.

The rest of the proof is similar to the corresponding part of [19, Lemma 4.1]. Namely, we consider the limiting function of $\check{\phi}(z)$, called ϕ_0. ϕ_0 will satisfy $L^0(\phi_0) = 0$ and $\phi_0 = \alpha_0(iw) + \sum_{l=1}^2 \alpha_l \frac{\partial w}{\partial z_l}$. By the orthogonality condition $\int_{\Omega_\varepsilon\setminus B(\xi',\delta/\varepsilon)} |V_0|^2\psi_1 = 0$, $\quad \mathrm{Re}\int_{|z|<1}\hat{\phi}\bar{w}_{z_\ell} = 0$, we then conclude that $\alpha_0 = \alpha_1 = \alpha_2 = 0$ and hence $\phi_0 \equiv 0$, which is a contradiction to (5). The proof is completed.

We next come to the following linear problem

$$\begin{cases} \mathcal{L}^\varepsilon(\psi) = h + c_0\varepsilon^2\chi_{\Omega_\varepsilon\setminus B(\xi',\delta/\varepsilon)} \\ \qquad\qquad + \sum_\ell c_\ell \dfrac{1}{iw(y-\xi')}\chi_{\{r<1/2\}}\dfrac{\partial w}{\partial y_\ell}(y-\xi') \quad \text{in } \Omega_\varepsilon, \\ \dfrac{\partial\psi}{\partial\nu} = g \quad \text{on } \partial\Omega_\varepsilon, \\ \displaystyle\int_{\Omega_\varepsilon\setminus B(\xi',\delta/\varepsilon)} |V_0|^2\psi_1 = 0, \qquad \mathrm{Re}\int_{|z|<1}\hat{\phi}\bar{w}_{z_\ell} = 0, \end{cases} \tag{5.13}$$

Proposition 5.1. *Assume that $d_\varepsilon = O(\varepsilon^{1+\sigma_1-\sigma_2-2\alpha})$. There exists a constant $C > 0$, dependent on δ but independent of c_0, such that for all small ε the following holds: if $|\log\varepsilon|\|h\|_{**} + \|g\|_{***} < \infty$, then there exists a unique solution $\psi = T_\varepsilon(h,g)$ to Problem (5.13). In addition,*

$$\|T_\varepsilon(h,g)\|_* \le C\Big[|\log\varepsilon|\|h\|_{**} + \|g\|_{***}\Big]. \tag{5.14}$$

Proof. Near the point ξ', we recall the definition (4.9) of $\hat{\phi}$ and the deduction below it. Equation (5.13) is then equivalent to

$$L^\varepsilon(\hat{\phi}) = \hat{h} + \sum_\ell c_\ell\chi_{\{r<1/2\}}w_{z_\ell}, \tag{5.15}$$

where $\hat{h}(z) = iw(z)h(\xi' + z)$. Multiplying (5.15) against $\bar{w}_{z_\ell}$, integrating over $B(0, \frac{\delta}{\varepsilon})$ and taking the real part, one gets

$$\mathrm{Re}\int_{B(0,\frac{\delta}{\varepsilon})} L^\varepsilon(\hat{\phi})\bar{w}_{z_\ell} = \mathrm{Re}\int_{B(0,\frac{\delta}{\varepsilon})} \hat{h}\bar{w}_{z_\ell} + c_\ell \,\mathrm{Re}\int_{B(0,\frac{1}{2})} |w_{z_\ell}|^2. \tag{5.16}$$

Integrating by parts, we write

$$\mathrm{Re}\int_{B(0,\frac{\delta}{\varepsilon})} L^\varepsilon(\hat{\phi})\bar{w}_{z_\ell} = \mathrm{Re}\int_{\partial B(0,\frac{\delta}{\varepsilon})} (\frac{\partial\hat{\phi}}{\partial\nu}\bar{w}_{z_\ell} - \hat{\phi}\frac{\partial \bar{w}_{z_\ell}}{\partial\nu}) \\ + \mathrm{Re}\int_{B(0,\frac{\delta}{\varepsilon})} \bar{\hat{\phi}}(L^\varepsilon w_{z_\ell} - L^0 w_{z_\ell}). \tag{5.17}$$

Since $|\nabla w| = O(\varepsilon)$, $|\nabla^2 w| = O(\varepsilon^2)$ and $|\hat{\phi}| \le C|\psi|$, $|\nabla\hat{\phi}| \le C(\varepsilon|\psi| + |\nabla\psi|)$ on $\partial B(0, \frac{\delta}{\varepsilon})$, the boundary integrals can be estimated as

$$\left|\mathrm{Re}\int_{\partial B(0,\frac{\delta}{\varepsilon})} (\frac{\partial\hat{\phi}}{\partial\nu}\bar{w}_{z_\ell} - \hat{\phi}\frac{\partial \bar{w}_{z_\ell}}{\partial\nu})\right| \le C\varepsilon\|\psi\|_*. \tag{5.18}$$

Recall the definition (4.10) of L^ε,

$$(L^\varepsilon - L^0)w_{z_\ell} = 2i\left(\nabla\hat{\varphi} - \varepsilon\widehat{\boldsymbol{A}}_0\right)\nabla w_{z_\ell} \\ - 2i\left(\nabla\hat{\varphi} - \varepsilon\widehat{\boldsymbol{A}}_0\right)\frac{\nabla w}{w}w_{z_\ell} - \frac{\tilde{\eta}(|z|)}{\tilde{\eta} + (1-\tilde{\eta})e^{\frac{\hat{\phi}}{w}}}\frac{E_1}{V_1}w_{z_\ell}.$$

Since $|\nabla w| \le \frac{C}{1+r}$, $|\nabla^2 w| \le \frac{C}{1+r^2}$, it is easy to get

$$\left|\left(\nabla\hat{\varphi} - \varepsilon\widehat{\boldsymbol{A}}_0\right)\nabla w_{z_\ell}\right| \le \frac{C\varepsilon^{1-\alpha}}{1+r^2},$$

and

$$\left|2i\left(\nabla\hat{\varphi} - \varepsilon\widehat{\boldsymbol{A}}_0\right)\frac{\nabla w}{w}w_{z_\ell} + \frac{\tilde{\eta}(|z|)}{\tilde{\eta} + (1-\tilde{\eta})e^{\frac{\hat{\phi}}{w}}}\frac{E_1}{V_1}w_{z_\ell}\right| \le \frac{C\varepsilon^{1-\alpha}}{1+r^2} + \frac{C\varepsilon^{2-2\alpha}}{1+r}.$$

Thus we obtain

$$\left|\mathrm{Re}\int_{B(0,\frac{\delta}{\varepsilon})} \bar{\hat{\phi}}(L^\varepsilon w_{z_\ell} - L^0 w_{z_\ell})\right| = o(|\log\varepsilon|^{-1})\|\psi\|_*. \tag{5.19}$$

Then it holds that by (5.17), (5.18) and (5.19)

$$\left|\mathrm{Re}\int_{B(0,\frac{\delta}{\varepsilon})} L^\varepsilon(\hat{\phi})\bar{w}_{z_\ell}\right| = o(|\log\varepsilon|^{-1})\|\psi\|_*. \tag{5.20}$$

On the other hand, it can be easily checked that

$$\left|\mathrm{Re}\int_{B(0,\frac{\delta}{\varepsilon})} \hat{h}\bar{w}_{z_\ell}\right| \le C\|h\|_{**}.$$

Therefore we conclude that, combining (5.16), (5.20) and the estimate above,

$$|c_\ell| \le o(|\log\varepsilon|^{-1})\|\psi\|_* + C\|h\|_{**}. \tag{5.21}$$

Finally, applying Lemma 2 one gets

$$\|\psi\|_* \le C\Big[|\log\varepsilon|\|h\|_{**} + |\log\varepsilon|\sum_\ell |c_\ell| + \|g\|_{***}\Big],$$

which implies (5.14).

Next we prove the existence. Consider the relations that $\varrho = iV_0\varsigma$ and $\hat\varrho(z) = iw(z)\varsigma(z+\xi')$. Let us define the space

$$\mathcal{H} = \left\{\varrho \in H^1(\Omega_\varepsilon): \int_{\Omega_\varepsilon\setminus B(\xi',\frac{\delta}{\varepsilon})} |V_0|^2\varsigma_1 = 0,\ \mathrm{Re}\int_{|z|<1/2}\hat\varrho\bar w_{z_\ell} = 0 \quad \text{for all } \ell\right\}$$

endowed with the usual inner product $(\varrho,\vartheta) = \int_{\Omega_\varepsilon}\nabla\varrho\nabla\vartheta$. Problem (5.13) can be written via Riesz's representation theorem in the form $\phi + K(\phi) = P$, where K is a linear, compact operator in $\mathcal{H}$, P is determined by h and g. Fredholm alternative then yields the existence and uniqueness assertion.

Remark 5.1. *The previous result implies that the unique solution $\psi = T_\varepsilon(h,g)$ of (5.13) defines a continuous linear map between the corresponding spaces. In addition we can easily know that ψ is continuous with respect to d_ε by the implicit function theorem.*

6. The projected nonlinear problem. This section is devoted to solving Problem (3.10) for a suitable small ψ. Rather than solving this directly, we consider the following intermediate case:

$$\mathcal{L}^\varepsilon(\psi) = R + N(\psi) + c_0\varepsilon^2\chi_{\Omega_\varepsilon\setminus B(\xi',\delta/\varepsilon)} + \sum_{\ell=1}^2 c_\ell \frac{1}{iw(y-\xi')}\chi_{\{r<1/2\}}\frac{\partial w}{\partial y_\ell}(y-\xi') \quad \text{in } \Omega_\varepsilon, \tag{6.1}$$

$$\frac{\partial\psi}{\partial\nu} = F \qquad \text{on } \partial\Omega_\varepsilon, \tag{6.2}$$

$$\int_{\Omega_\varepsilon\setminus B(\xi',\delta/\varepsilon)} |V_0|^2\psi_1 = 0, \qquad \mathrm{Re}\int_{|z|<1}\hat\phi w_{z_\ell} = 0 \quad \forall\ \ell = 1,2. \tag{6.3}$$

Lemma 4. *Assume that $d_\varepsilon = O(\varepsilon^{1+\sigma_1-\sigma_2-2\alpha})$. There is a constant $C>0$ depending only on δ such that for all points $\xi\in\Lambda$ and ε small, Problem (6.1)-(6.3) possesses a unique solution ψ with*

$$\|\psi\|_* \le C\varepsilon^{1-\sigma_2-2\alpha}|\log\varepsilon|.$$

Proof. Using Proposition 5.1, Problem (6.1)-(6.3) is equivalent to the fixed point problem

$$\psi = T_\varepsilon(-R - N(\psi), F) := \mathcal{A}_\varepsilon(\psi).$$

Let

$$\mathcal{S} = \Big\{\psi:\ \|\psi\|_* \le C\varepsilon^{1-\sigma_2-2\alpha}|\log\varepsilon|\Big\}.$$

Regarding the error $R = -\varepsilon^2 d_\varepsilon + R_1 + iR_2$, Lemma 1 yields, for $r > 1$,

$$R_1 = O(\varepsilon^{1-\alpha})\frac{1}{r^3}, \quad R_2 = O(\varepsilon^{1-\alpha})\frac{1}{r} + O(\varepsilon^{2-2\alpha}).$$

Recalling $\widehat{R}$ the error in $\hat{\phi}-$ coordinates (see (4.11)) we also find

$$\|\widehat{R}\|_{C^{0,\beta}(|z|<3)} = O(\varepsilon^{1-\alpha}),$$

and thus we conclude

$$\|R\|_{**} \leq C\varepsilon^{1-\sigma_2-2\alpha}. \tag{6.4}$$

Next we make the following claim:

$$\|N(\psi)\|_{**} \leq C\|\psi\|_*^2. \tag{6.5}$$

In fact, if $r > 2$, $N(\psi)$ reduces to (see (4.6))

$$N(\psi)_1 = -2\nabla\psi_1\nabla\psi_2, \quad N(\psi)_2 = |\nabla\psi_1|^2 - |\nabla\psi_2|^2 + |V_0|^2(e^{-2\psi_2} - 1 + 2\psi_2).$$

The definitions of the $*-$ norm easily yields that in this region

$$|r^{2+\sigma_1}N(\psi)_1| \leq C\frac{\|\psi\|_*^2}{r^{\sigma_2-\sigma_1}}, \qquad |r^{1+\sigma_2}N(\psi)_2| \leq C\frac{\|\psi\|_*^2}{r^{1-\sigma_2}}.$$

On the other hand, if $r < 3$, recall $\widehat{N}(\hat{\phi})$ the operator in the $\hat{\phi}-$ variable, as defined in (4.12). Direct computations obviously show that, from (3.12),

$$|\widehat{N}(\hat{\phi})| \leq C(|\hat{\phi}|^2 + |\nabla\hat{\phi}|^2).$$

Thus we have

$$\|\widehat{N}(\hat{\phi})\|_{C^{0,\beta}(|z|<3)} < C\|\psi\|_*^2,$$

from where the claim (6.5) follows.

Finally, it is obviously from Lemma 1 that

$$\|F\|_{***} \leq C\varepsilon^{2-\sigma_2}.$$

Combining (6.4), (6.5) and the estimate above, since

$$\mathcal{A}_\varepsilon(\psi) \leq C\Big(|\log\varepsilon|\|R\|_{**} + |\log\varepsilon|\|N(\psi)\|_{**} + \|F\|_{***}\Big),$$

we know $\mathcal{A}_\varepsilon$: $\mathcal{S} \longrightarrow \mathcal{S}$.

On the other hand, if ψ^1, $\psi^2 \in \mathcal{S}$ and $r > 2$, it is easy to prove

$$\|N(\psi^1) - N(\psi^2)\|_{**} \leq C\varepsilon^{1-\sigma_2-2\alpha}|\log\varepsilon|\|\psi^1 - \psi^2\|_*.$$

While if $r < 3$, it is also true that

$$\|\widehat{N}(\hat{\phi}^1) - \widehat{N}(\hat{\phi}^2)\|_{C^{0,\beta}(|z|<3)} \leq C\varepsilon^{1-\sigma_2-2\alpha}|\log\varepsilon|\|\hat{\phi}^1 - \hat{\phi}^2\|_{C^2(|z|<3)}.$$

Then we conclude that

$$\|\mathcal{A}_\varepsilon(\psi^1 - \psi^2)\|_{**} \leq C\varepsilon^{1-\sigma_2-2\alpha}|\log\varepsilon|\|\psi^1 - \psi^2\|_*,$$

which tell us $\mathcal{A}_\varepsilon$ is a contraction mapping on $\mathcal{S}$. Hence the existence of a unique solution in $\mathcal{S}$ is proven.

Remark 6.1. *By the implicit function theorem we can show that* $(d_\varepsilon, \xi') \mapsto \psi(d_\varepsilon, \xi')$ *is continuous. Moreover, given* d_ε, $d'_\varepsilon = O(\varepsilon^{1+\sigma_1-\sigma_2-2\alpha})$, *the unique solutions* ψ, ψ' *of Lemma 4 satisfy*

$$\|\psi - \psi'\|_* \le C\varepsilon^{1-\sigma_2-2\alpha}|\log\varepsilon|^2|d_\varepsilon - d'_\varepsilon|. \tag{6.6}$$

Indeed, note that $\psi - \psi' = T_\varepsilon[-(R-R') - (N(\psi) - N'(\psi')), 0]$, *where* R', N' *are the corresponding terms to* d_ε, d'_ε. *Since* $\|R - R'\|_{**} \le C\varepsilon^2|d_\varepsilon - d'_\varepsilon|$ *and*

$$\|N(\psi) - N'(\psi')\|_{**} \le C\varepsilon^{1-\sigma_2-2\alpha}|\log\varepsilon|(|d_\varepsilon - d'_\varepsilon| + \|\psi - \psi'\|_*),$$

using Lemma 2, we get the estimate (6.6).

Proposition 6.1. *There is a unique* $d_\varepsilon = O(\varepsilon^{1-\alpha})$ *such that Problem (6.1)-(6.3) possesses a unique solution* ψ *with* $c_0 = 0$.

Proof. Testing $|V_0|^2$ against (6.1), since $\int_{\Omega_\varepsilon} \frac{1}{iw}\chi\frac{\partial w}{\partial y_\ell}|V_0|^2 = 0$, we have

$$c_0\varepsilon^2 \int_{\Omega_\varepsilon\setminus B(\xi',\delta/\varepsilon)} |V_0|^2 = \mathrm{Re}\int_{\Omega_\varepsilon} |V_0|^2\Big[\mathcal{L}^\varepsilon(\psi) + R + N(\psi)\Big]. \tag{6.7}$$

we will show that there exists a d_ε such that the right hand side is 0. Recall that

$$\begin{aligned}\mathrm{Re}\,\mathcal{L}^\varepsilon(\psi) &= \Delta\psi_1 + 2\,\mathrm{Re}\frac{\nabla V_0}{V_0}\nabla\psi_1 - 2(\mathrm{Im}\frac{\nabla V_0}{V_0} - \varepsilon\widetilde{\boldsymbol{A}}_0)\nabla\psi_2 + \mathrm{Re}\frac{\eta}{\eta + (1-\eta)e^{i\psi}}\frac{E}{V_0}\psi\\ &= \Delta\psi_1 + 2\,\mathrm{Re}\frac{\nabla V_0}{V_0}\nabla\psi_1 - 2(\nabla\tilde\theta + \nabla\tilde\varphi - \varepsilon\widetilde{\boldsymbol{A}}_0)\nabla\psi_2\\ &\qquad + \mathrm{Re}\frac{\eta}{\eta + (1-\eta)e^{i\psi}}\frac{E}{V_0}\psi.\end{aligned} \tag{6.8}$$

Direct computation shows that, on account of $\frac{\partial\psi_1}{\partial\nu} = 0$ on $\partial\Omega_\varepsilon$,

$$\begin{aligned}\int_{\Omega_\varepsilon} |V_0|^2\Delta\psi_1 &= -\int_{\Omega_\varepsilon} 2\,\mathrm{Re}(\overline{V}_0\nabla V_0)\nabla\psi_1 + \int_{\partial\Omega_\varepsilon}|V_0|^2\frac{\partial\psi_1}{\partial\nu}\\ &= -\int_{\Omega_\varepsilon} 2\,\mathrm{Re}(\overline{V}_0\nabla V_0)\nabla\psi_1.\end{aligned}$$

Note that $\nabla S(|y-\xi'|)\nabla\tilde\theta = 0$, $\Delta\tilde\theta = 0$ and on the boundary $\partial_\nu(\tilde\theta + \tilde\varphi) = \varepsilon\widetilde{\boldsymbol{A}}_0\cdot\nu$. We know that

$$\begin{aligned}&\int_{\Omega_\varepsilon} |V_0|^2(\nabla\tilde\theta + \nabla\tilde\varphi - \varepsilon\widetilde{\boldsymbol{A}}_0)\nabla\psi_2\\ =\;& -\int_{\Omega_\varepsilon} 2|V_0|^2\frac{\nabla S(|y-\xi'|)}{S(|y-\xi'|)}(\nabla\tilde\theta + \nabla\tilde\varphi - \varepsilon\widetilde{\boldsymbol{A}}_0)\psi_2\\ &-\int_{\Omega_\varepsilon}|V_0|^2\big[\Delta\tilde\theta + \Delta\tilde\varphi - \varepsilon(\nabla\cdot\widetilde{\boldsymbol{A}}_0)\big]\psi_2 + \int_{\partial\Omega_\varepsilon}|V_0|^2\psi_2\left[\frac{\partial(\tilde\theta + \tilde\varphi)}{\partial\nu} - \varepsilon\widetilde{\boldsymbol{A}}_0\cdot\nu\right]\\ =\;& O(\varepsilon^{1-\alpha})\|\psi\|_*.\end{aligned}$$

We also easily get

$$\left|\int_{\Omega_\varepsilon}\frac{\eta}{\eta + (1-\eta)e^{i\psi}}|V_0|^2\frac{E}{V_0}\psi\right| \le \int_{r<2}|E||V_0\psi| \le C\Big[|d_\varepsilon|\varepsilon^2 + \varepsilon^{1-\alpha}\Big]\|\psi\|_*. \tag{6.9}$$

Combining (6.8) and (6.9), we know that

$$\mathrm{Re}\int_{\Omega_\varepsilon}|V_0|^2\mathcal{L}^\varepsilon(\psi)\le C\Big[|d_\varepsilon|\varepsilon^2+\varepsilon^{1-\alpha}\Big]\|\psi\|_*. \tag{6.10}$$

We directly have, using Lemma 1,

$$\begin{aligned}\mathrm{Re}\int_{\Omega_\varepsilon}|V_0|^2R &= -d_\varepsilon\varepsilon^2\int_{\Omega_\varepsilon}|V_0|^2+\int_{\Omega_\varepsilon}|V_0|^2R_1\\ &= -d_\varepsilon\varepsilon^2\int_{\Omega_\varepsilon}|V_0|^2+O(\varepsilon^{1-\alpha}).\end{aligned}$$

It is easy to check that

$$\mathrm{Re}\int_{\Omega_\varepsilon}|V_0|^2N(\psi)\le C\|\psi\|_*^2.$$

Now since $\varepsilon^2\int_{\Omega_\varepsilon}|V_0|^2>0$, there must exist a

$$d_\varepsilon=O(\varepsilon^{1-\alpha})$$

such that the right hand side of (6.7) is 0, which then gives $c_0=0$.

Suppose we have $d_\varepsilon,\ d_\varepsilon'=O(\varepsilon^{1-\alpha})$ and solutions $\psi,\ \psi'$ such that $c_0=c_0'=0$. From (6.7) and the estimates that follow we obtain

$$|d_\varepsilon-d_\varepsilon'|\le C\varepsilon^{1-\sigma_2-2\alpha}|\log\varepsilon|\|\psi-\psi'\|_*+C\varepsilon^{3-\sigma_2-2\alpha}|\log\varepsilon||d_\varepsilon-d_\varepsilon'|.$$

Using (6.6) we deduce $d_\varepsilon=d_\varepsilon'$. The proof is complete.

7. Proof of Theorem 2.1 and 2.2. By Proposition 6.1, there exists a unique $d_\epsilon=O(\epsilon^{1-\alpha})$ such that

$$\begin{aligned}\mathcal{L}^\varepsilon(\psi)=\ & R+N(\psi)\\ &+\sum_{\ell=1}^{2}c_\ell\frac{1}{iw(y-\xi')}\chi_{\{r<1/2\}}\frac{\partial w}{\partial y_\ell}(y-\xi')\quad\text{in }\Omega_\varepsilon,\end{aligned}\tag{7.1}$$

$$\frac{\partial\psi}{\partial\nu}=\ F\qquad\text{on }\partial\Omega_\varepsilon,\tag{7.2}$$

$$\int_{\Omega_\varepsilon\setminus B(\xi',\delta/\varepsilon)}|V_0|^2\psi_1=0,\qquad \mathrm{Re}\int_{|z|<1}\hat{\phi}w_{z_\ell}=0\quad\forall\ \ell=1,2\tag{7.3}$$

has a unique solution ψ with $\|\psi\|_*\le C\epsilon^{1-\sigma_2-2\alpha}|\log\epsilon|$.

In this section we will choose a suitable ξ_ε to make $c_\ell=0$ in (7.1)-(7.2), which completes the proofs of Theorems 2.1-2.2.

In the following, we calculate the expansions of c_ℓ.

Testing (6.1) against $iw\frac{\partial\bar{w}}{\partial y_\ell}(y-\xi')$ and integrating over Ω_ε, we have

$$c_\ell\int_{r<1/2}\left|\frac{\partial w}{\partial y_\ell}(y-\xi')\right|^2=\mathrm{Re}\int_{\Omega_\varepsilon}\left[\mathcal{L}^\varepsilon(\psi)+R+N(\psi)\right]iw\frac{\partial\bar{w}}{\partial y_\ell}(y-\xi')\ \mathrm{d}y.$$

Noting that $iw\mathcal{L}^\varepsilon(\psi) = L^\varepsilon(\hat\phi)$, we have

$$\begin{aligned}
&\mathrm{Re}\int_{\Omega_\varepsilon} iw\mathcal{L}^\varepsilon(\psi)\frac{\partial\bar w}{\partial y_\ell}(y-\xi') = \mathrm{Re}\int_{\Omega_\varepsilon} L^\varepsilon(\hat\phi)\frac{\partial\bar w}{\partial y_\ell}(y-\xi')\\
=\ &\mathrm{Re}\int_{\Omega_\varepsilon} L^0(\hat\phi)\frac{\partial\bar w}{\partial y_\ell}(y-\xi')\\
&+\mathrm{Re}\int_{\Omega_\varepsilon}\left[2i(\nabla\tilde\varphi-\varepsilon\widetilde{\boldsymbol{A}}_0)\nabla\hat\phi - 2i(\nabla\tilde\varphi-\varepsilon\widetilde{\boldsymbol{A}}_0)\frac{\nabla w}{w}\hat\phi - \frac{\eta}{\eta+(1-\eta)e^{\hat\phi/w}}\frac{E}{V_0}\hat\phi\right]\frac{\partial\bar w}{\partial y_\ell}\\
=\ &\mathrm{Re}\int_{\Omega_\varepsilon} L^0(\hat\phi)\frac{\partial\bar w}{\partial y_\ell}(y-\xi')\ \mathrm{d}y + O(\varepsilon^{2-\sigma_2-3\alpha})|\log\varepsilon|^2.
\end{aligned}$$

It is direct to check that, since $L^0(\frac{\partial w}{\partial y_\ell}) = 0$,

$$\begin{aligned}
\left|\mathrm{Re}\int_{\Omega_\varepsilon} L^0(\hat\phi)\frac{\partial\bar w}{\partial y_\ell}(y-\xi')\right| &\le \left|\mathrm{Re}\int_{\partial\Omega_\varepsilon}\frac{\partial\hat\phi}{\partial\nu}\frac{\partial\bar w}{\partial y_\ell} - \frac{\partial}{\partial\nu}(\frac{\partial\bar w}{\partial y_\ell})\hat\phi\right| + \left|\mathrm{Re}\int_{\Omega_\varepsilon} L^0(\frac{\partial w}{\partial y_\ell})\bar{\hat\phi}\right|\\
&\le\ C\varepsilon^{2-\sigma_2-2\alpha}|\log\varepsilon|.
\end{aligned}$$

Thus,

$$\mathrm{Re}\int_{\Omega_\varepsilon} iw\mathcal{L}^\varepsilon(\psi)\frac{\partial\bar w}{\partial y_\ell}(y-\xi') = O(\varepsilon^{2-\sigma_2-3\alpha})|\log\varepsilon|^2. \tag{7.4}$$

Recall that

$$\begin{aligned}
iwR = iw(y-\xi')\Bigg\{&\varepsilon^2 d_\varepsilon + 2\frac{\nabla S(|y-\xi'|)}{S(|y-\xi'|)}(\nabla\tilde\varphi-\varepsilon\widetilde{\boldsymbol{A}}_0)\\
&+2i\nabla\tilde\theta(\nabla\tilde\varphi-\varepsilon\widetilde{\boldsymbol{A}}_0) + i|\nabla\tilde\varphi-\varepsilon\widetilde{\boldsymbol{A}}_0|^2\Bigg\}.
\end{aligned}$$

It is easy to check that, since $\|\boldsymbol{A}_0\| = O(\varepsilon^{-\alpha})$,

$$\begin{aligned}
&\mathrm{Re}\int_{\Omega_\varepsilon} iwR\frac{\partial\bar w}{\partial y_\ell}(y-\xi')\\
&= 2\int_{\Omega_\varepsilon}\left(\partial_\ell\tilde\theta\frac{\nabla S(|y-\xi'|)}{S(|y-\xi'|)} - \frac{\partial_\ell S(|y-\xi'|)}{S(|y-\xi'|)}\nabla\tilde\theta\right)|w(y-\xi')|^2\widetilde{\boldsymbol{\Psi}}_0^\varepsilon\ \mathrm{d}y + O(\varepsilon^{2-2\alpha})\\
&= 2\widetilde{\boldsymbol{\Psi}}^\varepsilon(\xi')\int_{\Omega_\varepsilon}\left(\partial_\ell\tilde\theta\frac{\nabla S(|y-\xi'|)}{S(|y-\xi'|)} - \frac{\partial_\ell S(|y-\xi'|)}{S(|y-\xi'|)}\nabla\tilde\theta\right)|w(y-\xi')|^2\mathrm{d}y + O(\varepsilon^{2-2\alpha}),
\end{aligned} \tag{7.5}$$

where $\widetilde{\boldsymbol{\Psi}}_0^\varepsilon(y,\xi') = \nabla\tilde\varphi - \varepsilon\widetilde{\boldsymbol{A}}_0 = \varepsilon\boldsymbol{\Psi}_0^\varepsilon(x,\xi)$ and $\widetilde{\boldsymbol{\Psi}}^\varepsilon(\xi') = \widetilde{\boldsymbol{\Psi}}_0^\varepsilon(\xi',\xi')$. We can also easily get that

$$\left|\mathrm{Re}\int_{\Omega_\varepsilon} iwN(\psi)\frac{\partial\bar w}{\partial y_\ell}(y-\xi')\right| \le C\varepsilon^{2-2\sigma_2-4\alpha}|\log\varepsilon|^2. \tag{7.6}$$

Combining (7.4), (7.5) and (7.6), we get that

$$
\begin{aligned}
&c_\ell \int_{r<1/2} \left| \frac{\partial w}{\partial y_\ell}(y-\xi') \right|^2 \\
&= 2\widetilde{\mathbf{\Psi}}^\varepsilon(\xi') \int_{\Omega_\varepsilon} \left(\partial_\ell \tilde{\theta} \frac{\nabla S(|y-\xi'|)}{S(|y-\xi'|)} - \frac{\partial_\ell S(|y-\xi'|)}{S(|y-\xi'|)} \nabla\tilde{\theta} \right) |w(y-\xi')|^2 \mathrm{d}y \\
&\qquad\qquad + O(\varepsilon^{2-2\sigma_2-4\alpha}|\log\varepsilon|^2).
\end{aligned} \tag{7.7}
$$

Direct computation shows that

$$
\begin{aligned}
&\left(\partial_{y_1} \tilde{\theta} \frac{\nabla S(|y-\xi'|)}{S(|y-\xi'|)} - \frac{\partial_{y_1} S(|y-\xi'|)}{S(|y-\xi'|)} \nabla\tilde{\theta} \right) |w|^2 = \left(0, -\frac{S(|y-\xi'|)S'(|y-\xi'|)}{|y-\xi'|} \right), \\
&\left(\partial_{y_2} \tilde{\theta} \frac{\nabla S(|y-\xi'|)}{S(|y-\xi'|)} - \frac{\partial_{y_2} S(|y-\xi'|)}{S(|y-\xi'|)} \nabla\tilde{\theta} \right) |w|^2 = \left(-\frac{S(|y-\xi'|)S'(|y-\xi'|)}{|y-\xi'|}, 0 \right).
\end{aligned} \tag{7.8}
$$

Proof. [Proof of Theorem 2.1] Now $\boldsymbol{A}_0$ and Φ_a are independent of ε, so we can make $\alpha = 0$ in all the previous estimates and $\mathbf{\Psi}^\varepsilon$ is replaced by $\mathbf{\Psi}$. Since $\xi_0 \in \Omega$ is a stable zero point of $\mathbf{\Psi}$ and $\int_{\Omega_\varepsilon} \frac{S(|y-\xi'|)S'(|y-\xi'|)}{|y-\xi'|} \to C > 0$, there exists a ξ_ε such that the right side hand of (7.7) equals 0, which implies $c_\ell = 0$. Moreover $\xi_\varepsilon \to \xi_0$. Proposition 6.1 already tells us that there exists a small $d_\varepsilon \to 0$ such that $c_0 = 0$. Thus we completely solved the problem (3.10) and therefore Theorem 2.1 follows.

Proof. [Proof of Theorem 2.2] Since ξ_0^ε is a uniformly non-degenerate inner zero point of $\mathbf{\Psi}^\varepsilon$ and $\int_{\Omega_\varepsilon} \frac{S(|y-\xi'|)S'(|y-\xi'|)}{|y-\xi'|} \to C > 0$, there exists a ξ_ε such that the right side hand of (7.7) equals 0, which implies $c_\ell = 0$. Recalling $\sigma_2 = \frac{1}{6}$ and from (7.7), we also easily get $|\xi_\varepsilon - \xi_0^\varepsilon| = O(\varepsilon^{\frac{2}{3}-4\alpha})$. Recall Proposition 6.1 again, Theorem 2.2 follows directly.

8. Sketch proof of Theorems 2.3 and 2.4. Since the proof is similar to Theorem 2.2, we will give a sketch in this section. Our main idea, different from the proof of Theorem 2.1-2.2, is to use the symmetry of the unit disk and invariance of the equation to make $d_\epsilon = 0$.

Observe that (2.2) is invariant under the transformation

$$
u(x_1, -x_2) = \overline{u(x_1, x_2)}. \tag{8.1}
$$

Thus we may consider ξ in the set

$$
\Lambda_D = \Big\{ \xi = (\xi_1, \xi_2) \in D : \ \xi_2 = 0 \text{ and } \operatorname{dist}(\xi, \partial D) > \delta \Big\}.
$$

The first approximate solution for degree $+1$ is same as before

$$
U_0(x) = S\left(\frac{|x-\xi|}{\varepsilon} \right) e^{i[\theta(x) + \varphi(x)]},
$$

where φ satisfies (3.4). The degree -1 case is also given as stated in Remark 3.1. It is easy to see that, for $\xi \in \Lambda_D$,

$$
\theta(x_1, -x_2) = -\theta(x_1, x_2), \qquad \varphi(x_1, -x_2) = -\varphi(x_1, x_2).
$$

So we have

$$U_0(x_1,-x_2)=\overline{U_0(x_1,x_2)}.$$

Note that $u(x)$ is a solution of (2.2) if and only if $v(y)=u(\varepsilon y)$ satisfies

$$\begin{cases}(i\nabla+\varepsilon\widetilde{\boldsymbol{A}}_0)^2 v+(|v|^2-1)v+i\widetilde{\Phi}_a v=0 & \text{in } D_\varepsilon,\\ (i\nabla+\varepsilon\widetilde{\boldsymbol{A}}_0)v\cdot\nu=0 & \text{on } \partial D_\varepsilon.\end{cases}$$

We will look for a solution v with the form

$$v=\eta(V_0+iV_0\psi)+(1-\eta)V_0e^{i\psi}$$

where V_0, η are defined as before. For the symmetry (8.1), we impose now that $\psi(y)$ is such that

$$\psi(y_1,-y_2)=-\overline{\psi(y_1,y_2)}, \tag{8.2}$$

and ψ satisfy

$$\begin{cases}\mathcal{L}^\varepsilon(\psi)=R+N(\psi) & \text{in } D_\varepsilon,\\ \dfrac{\partial\psi}{\partial\nu}=F & \text{on } \partial D_\varepsilon,\end{cases}$$

where $\mathcal{L}^\varepsilon(\psi)$, R, $N(\psi)$ and F are defined as in Section 3. Just note that now

$$E(y)=(i\nabla+\varepsilon\widetilde{\boldsymbol{A}}_0)^2V_0+(|V_0|^2-1)V_0+i\varepsilon^2\widetilde{\Phi}_aV_0,$$

and

$$R(y_1,-y_2)=-\overline{R(y_1,y_2)},\qquad N(\psi)(y_1,-y_2)=-\overline{N(\psi)(y_1,y_2)}.$$

There is only a slight change in the following proof. First consider

$$\begin{cases}\mathcal{L}^\varepsilon(\psi)=h & \text{in } D_\varepsilon,\\ \dfrac{\partial\psi}{\partial\nu}=g & \text{on } \partial D_\varepsilon,\\ \operatorname{Re}\displaystyle\int_{|z|<1}\hat{\phi}(z)\bar{w}_{z_\ell}(z)=0.\end{cases} \tag{8.3}$$

Of course, we should assume $h(y_1,-y_2)=-\overline{h(y_1,y_2)}$, $g(y_1,-y_2)=-\overline{g(y_1,y_2)}$. By (8.2), we have $\psi_1(y_1,0)=0$ so that the constant c_0 in (5.1) is not needed. By the same reduction process, the same result as Lemma 2 also holds for any solution of (8.3). Second, it is then easy to know that Proposition 5.1 holds for the problem

$$\begin{cases}\mathcal{L}^\varepsilon(\psi)=h+\displaystyle\sum_\ell c_\ell\frac{1}{iw(y-\xi')}\chi_{\{r<1/2\}}\frac{\partial w}{\partial y_\ell}(y-\xi') & \text{in } D_\varepsilon,\\ \dfrac{\partial\psi}{\partial\nu}=g & \text{on } \partial D_\varepsilon,\\ \operatorname{Re}\displaystyle\int_{|z|<1}\hat{\phi}(z)\bar{w}_{z_\ell}(z)=0.\end{cases}$$

The process for the corresponding nonlinear problem discussed in later sections is similar and even simpler due to the disappearance of d_ε. We only remark here that we have $c_2 = 0$ automatically in Section 7 because of the symmetry. Note that $\mathbf{\Psi}^\varepsilon(\xi) = \mathbf{\Psi}^\varepsilon(\xi_1, 0) = (\Psi_1^\varepsilon(\xi_1), \Psi_2^\varepsilon(\xi_1))$ and denote $\xi_0^\varepsilon = (\xi_{01}^\varepsilon, 0)$. Considering (7.8), we only need to assume that ξ_{01}^ε is a uniformly non-degenerate inner zero point of Ψ_2^ε (or for ε-independent case, $\xi_0 \in \Omega$ is a stable zero point of Ψ_2). Then there is a $\xi_\varepsilon = (\xi_{\varepsilon 1}, 0)$ such that $c_1 = 0$.

Proof. [Continuation Proof of Theorem 2.3] From Theorem 2.1, we should prove the existence of a stable zero point of Ψ_2. Decompose the solution of (3.4) to $\varphi = \varphi_1 + \varphi_2$, which satisfy the following equations

$$\begin{cases} \Delta\varphi_1 - Jx_2 = 0 & \text{in } D, \\ \dfrac{\partial \varphi_1}{\partial \nu} = 0 & \text{on } \partial D, \\ \displaystyle\int_D \varphi_1 = 0 \end{cases} \tag{8.4}$$

and, for degree $+1$,

$$\begin{cases} \Delta\varphi_2 = 0 & \text{in } D, \\ \dfrac{\partial \varphi_2}{\partial \nu} = -\dfrac{(x-\xi)^\perp \cdot \nu}{|x-\xi|^2} & \text{on } \partial D, \\ \displaystyle\int_D \varphi_2 = 0, \end{cases} \tag{8.5}$$

while for degree -1,

$$\begin{cases} \Delta\varphi_2 = 0 & \text{in } D, \\ \dfrac{\partial \varphi_2}{\partial \nu} = \dfrac{(x-\xi)^\perp \cdot \nu}{|x-\xi|^2} & \text{on } \partial D, \\ \displaystyle\int_D \varphi_2 = 0. \end{cases} \tag{8.6}$$

By the method of Fourier series, we obtain the unique solution of (8.4)

$$\varphi_1(x) = \frac{J}{8} \sin\theta_0 (r_0^3 - 3r_0) \tag{8.7}$$

where $(r_0(x), \theta_0(x))$ is the usual polar coordinates centered at 0.

As for (8.5), note that there exists a harmonic conjugate of φ_2, namely a harmonic function $\varphi_2^\perp$ satisfying

$$\partial_{x_1}\varphi_2^\perp = -\partial_{x_2}\varphi_2, \qquad \partial_{x_2}\varphi_2^\perp = \partial_{x_1}\varphi_2.$$

It is easy to check that on ∂D, denote ρ as the unit tangent of ∂D,

$$\partial_\rho \left(\varphi_2^\perp + \log\frac{1}{|x-\xi|} \right) = 0.$$

Since the harmonic conjugate is defined up to an additive constant, we impose

$$\varphi_2^\perp + \log\frac{1}{|x-\xi|} = 0 \qquad \text{on } \partial D.$$

Let $G(x,\xi)$ be the Green's function on D under the Dirichlet boundary condition, i. e.

$$\begin{cases} -\Delta_x G(x,\xi) = 2\pi\delta_{x=\xi} & x \in D, \\ G(x,\xi) = 0 & x \in \partial D. \end{cases}$$

It is known that the Green's function can be written as

$$G(x,\xi) = \log\frac{1}{|x-\xi|} + \log(|\xi||x-\bar{\xi}|)$$

where $\bar{\xi} = \frac{\xi}{|\xi|^2}$. Therefore, by the uniqueness of $\varphi_2^{\perp}$, we get

$$\varphi_2^{\perp}(x) = \log(|\xi||x-\bar{\xi}|) \qquad \text{in } D.$$

So

$$\partial_{x_1}\varphi_2 = \frac{(x-\bar{\xi})_2}{|x-\bar{\xi}|^2}, \qquad \partial_{x_2}\varphi_2 = -\frac{(x-\bar{\xi})_1}{|x-\bar{\xi}|^2},$$

which implies, for degree $+1$,

$$\partial_{x_1}\varphi_2(\xi) = -\frac{\xi_2}{1-|\xi|^2}, \qquad \partial_{x_2}\varphi_2(\xi) = \frac{\xi_1}{1-|\xi|^2}. \tag{8.8}$$

Obviously for degree -1, we have

$$\partial_{x_1}\varphi_2(\xi) = \frac{\xi_2}{1-|\xi|^2}, \qquad \partial_{x_2}\varphi_2(\xi) = -\frac{\xi_1}{1-|\xi|^2}. \tag{8.9}$$

By Remark 2.1, we shall find a $\xi_0 = (\xi_{01}, 0) \in \Lambda_D$ such that ξ_{01} is a non-degenerate zero point of Ψ_2. For degree $+1$, we get

$$\mathbf{\Psi}(\xi) = \begin{bmatrix} \dfrac{J}{4}|\xi|^2\sin\theta_0(\xi)\cos\theta_0(\xi) - \dfrac{|\xi|\sin\theta_0(\xi)}{1-|\xi|^2} \\ -\dfrac{3}{8}J + \dfrac{3}{8}J|\xi|^2 - \dfrac{1}{4}J|\xi|^2\cos^2\theta_0(\xi) + \dfrac{|\xi|\cos\theta_0(\xi)}{1-|\xi|^2} \end{bmatrix}.$$

Note that

$$\Psi_1((\xi_{01},0)) = 0\,, \qquad \Psi_2((\xi_{01},0)) = -\frac{3}{8}J + \frac{1}{8}J\xi_{01}^2 + \frac{\xi_{01}}{1-\xi_{01}^2}.$$

It is easy to find a unique zero point $\xi_{01} = 0$ if $J = 0$, $\xi_{01} \approx 3J/8 \neq 0$ for small $J \neq 0$ in the considered region, which can also be checked using the software *Mathematica* accurately. It is also easy to know that $\Psi_2'(\xi_{01}) \neq 0$ at this ξ_{01}, which implies that it is stable. Therefore the vortex lies at $\xi_\varepsilon = (\xi_{\varepsilon 1}, 0) \to (\xi_{01}, 0)$. Similarly for degree -1,

$$\mathbf{\Psi}(\xi) = \begin{bmatrix} \dfrac{J}{4}|\xi|^2\sin\theta_0(\xi)\cos\theta_0(\xi) + \dfrac{|\xi|\sin\theta_0(\xi)}{1-|\xi|^2} \\ -\dfrac{3}{8}J + \dfrac{3}{8}J|\xi|^2 - \dfrac{1}{4}J|\xi|^2\cos^2\theta_0(\xi) - \dfrac{|\xi|\cos\theta_0(\xi)}{1-|\xi|^2} \end{bmatrix}.$$

There also exists a unique non-degenerate zero point of Ψ_2 near $-3J/8$. The proof is concluded.

Proof. [Continuation Proof of Theorem 2.4] From Theorem 2.2, it is sufficient to prove the existence of a uniformly non-degenerate inner zero point of Ψ_2. Decompose the solution of (3.4) to $\varphi = \varphi_1 + \varphi_2$. φ_1 satisfies

$$\begin{cases} \Delta\varphi_1 - (\nabla \cdot \boldsymbol{A}_0) - Jx_2 = 0 & \text{in } D, \\ \dfrac{\partial \varphi_1}{\partial \nu} = \boldsymbol{A}_0 \cdot \nu & \text{on } \partial D, \\ \displaystyle\int_D \varphi_1 = 0, & \end{cases} \tag{8.10}$$

and φ_2 is just defined as in (8.5) and (8.6).

Case of $\boldsymbol{A}_0 = \frac{H}{2}(-x_2, x_1)$

We obtain the unique solution of (8.10)

$$\varphi_1(x) = \frac{J}{8}\sin\theta_0(r_0^3 - 3r_0).$$

For degree +1, recalling (8.8), we know that

$$\boldsymbol{\Psi}^\varepsilon(\xi) = \begin{bmatrix} \dfrac{J}{4}|\xi|^2\sin\theta_0(\xi)\cos\theta_0(\xi) - \dfrac{|\xi|\sin\theta_0(\xi)}{1-|\xi|^2} + \dfrac{H}{2}|\xi|\sin\theta_0(\xi) \\ -\dfrac{3}{8}J + \dfrac{3}{8}J|\xi|^2 - \dfrac{1}{4}J|\xi|^2\cos^2\theta_0(\xi) + \dfrac{|\xi|\cos\theta_0(\xi)}{1-|\xi|^2} - \dfrac{H}{2}|\xi|\cos\theta_0(\xi) \end{bmatrix}.$$

Thus

$$\Psi_1^\varepsilon((\xi_{01}, 0)) = 0\,, \quad \Psi_2^\varepsilon((\xi_{01}, 0)) = -\frac{3}{8}J + \frac{1}{8}J\xi_{01}^2 + \frac{\xi_{01}}{1-\xi_{01}^2} - \frac{H}{2}\xi_{01},$$

and then for large H, there exists a unique zero point $\xi_{01}^\varepsilon = 0$ if $J = 0$,

$$\xi_{01}^\varepsilon \approx -\frac{3J}{4(H-2)} \tag{8.11}$$

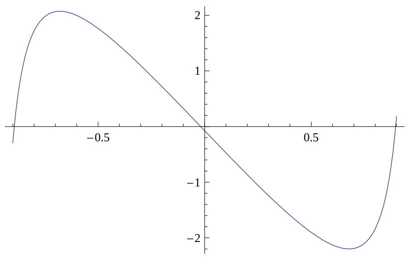
if $J \neq 0$. (Software *Mathematica* can be used to get the accurate solution.) It can be easily checked that

$$\frac{\mathrm{d}\Psi_2^\varepsilon}{\mathrm{d}\xi_{01}} = \frac{2\xi_{01}^2}{(1-\xi_{01}^2)^2} + \frac{J\xi_{01}}{4} - \frac{H}{2} + \frac{1}{1-\xi_{01}^2}.$$

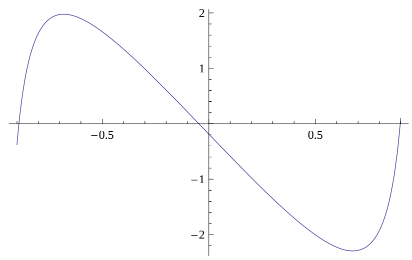
Thus at ξ_{01}^ε, $|\frac{\mathrm{d}\Psi_2^\varepsilon}{\mathrm{d}\xi_{01}}| \geq C$ where C is independent of ε. So ξ_{01}^ε is a uniformly non-degenerate inner zero point. Furthermore we already know that the vortex position $\xi_\varepsilon = (\xi_{\varepsilon 1}, 0)$ should satisfy $|\xi_{\varepsilon 1} - \xi_{01}^\varepsilon| = o(|\log\varepsilon|^{-1})$ by Theorem 2.2, therefore if $J \neq 0$, considering $H \sim \frac{|\log\varepsilon|}{2}$, we conclude that the vortex must have a nonzero shift on x_1 axis from origin for any fixed small $\varepsilon \neq 0$. See some examples in Fig. 8.1.

For degree -1,

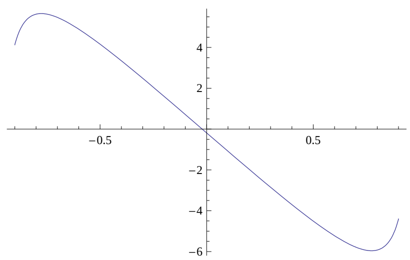
$$\boldsymbol{\Psi}^\varepsilon(\xi) = \begin{bmatrix} \dfrac{J}{4}|\xi|^2\sin\theta_0(\xi)\cos\theta_0(\xi) + \dfrac{|\xi|\sin\theta_0(\xi)}{1-|\xi|^2} + \dfrac{H}{2}|\xi|\sin\theta_0(\xi) \\ -\dfrac{3}{8}J + \dfrac{3}{8}J|\xi|^2 - \dfrac{1}{4}J|\xi|^2\cos^2\theta_0(\xi) - \dfrac{|\xi|\cos\theta_0(\xi)}{1-|\xi|^2} - \dfrac{H}{2}|\xi|\cos\theta_0(\xi) \end{bmatrix}.$$



(a) For $H = 10$, $J = 0.2$, $\xi_{01}^{\varepsilon} \approx -0.0187$



(b) For $H = 10$, $J = 0.5$, $\xi_{01}^{\varepsilon} \approx -0.0469$



(c) For $H = 20$, $J = 0.5$, $\xi_{01}^{\varepsilon} \approx -0.0208$

FIG. 8.1. *The figure of* $\Psi_2^{\varepsilon}(\xi_{01})$ *for the solution with degree* $+1$ *when* $\boldsymbol{A}_0 = \frac{H}{2}(-x_2, x_1)$

Similar argument as degree $+1$ shows that for large H, there exists a unique uniformly non-degenerate inner zero point $\xi_{01}^{\varepsilon} = 0$ if $J = 0$,

$$\xi_{01}^{\varepsilon} \approx -\frac{3J}{4(H+2)} \tag{8.12}$$

if $J \neq 0$. The remaining properties are due to the same reason. See some examples in Fig. 8.2.

Case of $\boldsymbol{A}_0 = \left(-\frac{H}{2}x_2, \frac{H}{2}x_1 - \frac{J}{2}x_1^2\right)$

The unique solution of (8.10) is

$$\varphi_1(x) = \frac{J}{6} r_0 \sin\theta_0 (r_0^2 \sin^2\theta_0 - 3).$$

For degree $+1$,

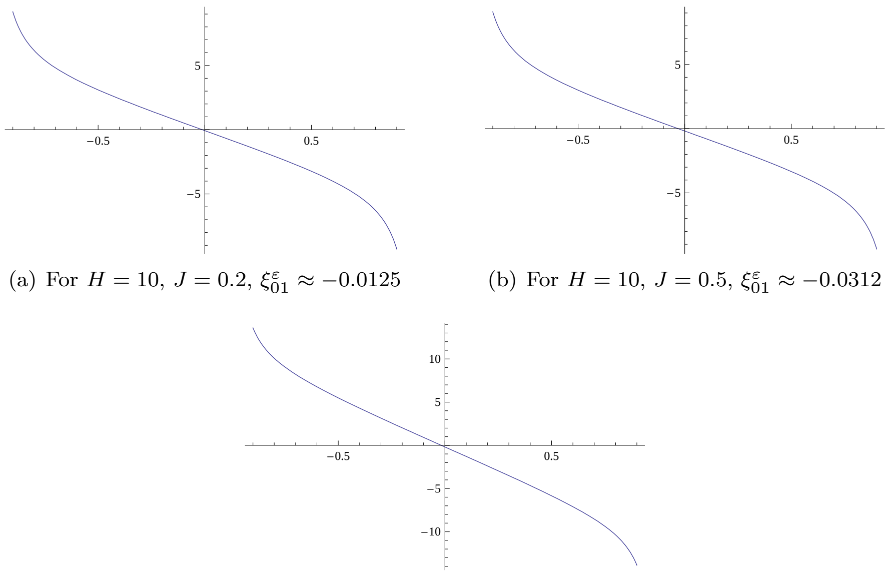
$$\boldsymbol{\Psi}^{\varepsilon}(\xi) = \begin{bmatrix} -\dfrac{|\xi|\sin\theta_0(\xi)}{1-|\xi|^2} + \dfrac{H}{2}|\xi|\sin\theta_0(\xi) \\ \dfrac{J}{2}|\xi|^2 - \dfrac{J}{2} + \dfrac{|\xi|\cos\theta_0(\xi)}{1-|\xi|^2} - \dfrac{H}{2}|\xi|\cos\theta_0(\xi) \end{bmatrix}.$$

Thus

$$\Psi_1^{\varepsilon}((\xi_{01},0)) = 0\,, \qquad \Psi_2^{\varepsilon}((\xi_{01},0)) = \frac{J}{2}\xi_{01}^2 - \frac{J}{2} + \frac{\xi_{01}}{1-\xi_{01}^2} - \frac{H}{2}\xi_{01},$$

and then for large H, there exists a unique zero point $\xi_{01}^{\varepsilon} = 0$ if $J = 0$, $\xi_{01}^{\varepsilon} \approx -\frac{J}{H-2}$ if $J \neq 0$ small. (In fact, when both J and H are large and $J \sim H$, we have a zero

$$\xi_{01} \sim \frac{2H - \sqrt{4H^2 + 3J^2}}{J}.) \tag{8.13}$$



(a) For $H = 10,\ J = 0.2,\ \xi_{01}^{\varepsilon} \approx -0.0125$

(b) For $H = 10,\ J = 0.5,\ \xi_{01}^{\varepsilon} \approx -0.0312$

(c) For $H = 20,\ J = 0.5,\ \xi_{01}^{\varepsilon} \approx -0.0170$

FIG. 8.2. *The figure of* $\Psi_2^{\varepsilon}(\xi_{01})$ *for the solution with degree* -1 *when* $\boldsymbol{A}_0 = \frac{H}{2}(-x_2, x_1)$

We easily check that

$$\frac{\mathrm{d}\Psi_2^{\varepsilon}}{\mathrm{d}\xi_{01}} = \frac{2\xi_{01}^2}{(1-\xi_{01}^2)^2} + J\xi_{01} - \frac{H}{2} + \frac{1}{1-\xi_{01}^2},$$

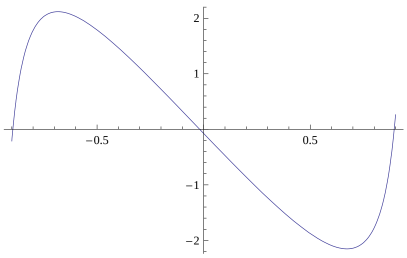
thus at ξ_{01}^{ε}, $|\frac{\mathrm{d}\Psi_2^{\varepsilon}}{\mathrm{d}\xi_{01}}| \geq C$ where C is independent of ε. So ξ_{01}^{ε} is a uniformly non-degenerate inner zero point. Furthermore we already know that the vortex position $\xi_{\varepsilon} = (\xi_{\varepsilon 1}, 0)$ should satisfy $|\xi_{\varepsilon 1} - \xi_{01}^{\varepsilon}| = o(|\log\varepsilon|^{-1})$ by Theorem 2.2, therefore if $J \neq 0$, considering $H \sim \frac{|\log\varepsilon|}{2}$, we conclude that the vortex must have a nonzero shift on x_1 axis from origin for any fixed small $\varepsilon \neq 0$. See examples in Fig. 8.3.

For degree -1,

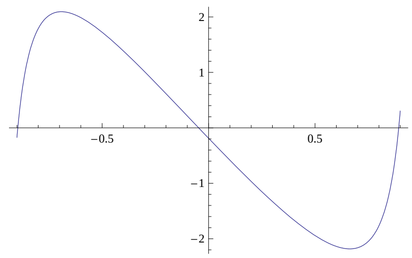
$$\boldsymbol{\Psi}^{\varepsilon}(\xi) = \begin{bmatrix} \dfrac{|\xi|\sin\theta_0(\xi)}{1-|\xi|^2} + \dfrac{H}{2}|\xi|\sin\theta_0(\xi) \\ \dfrac{J}{2}|\xi|^2 - \dfrac{J}{2} - \dfrac{|\xi|\cos\theta_0(\xi)}{1-|\xi|^2} - \dfrac{H}{2}|\xi|\cos\theta_0(\xi) \end{bmatrix}.$$

For large H, there exists a unique zero point $\xi_{01}^{\varepsilon} = 0$ if $J = 0$, $\xi_{01}^{\varepsilon} \approx -\frac{J}{H+2}$ if $J \neq 0$ small. The remaining properties can also be checked as in the case of degree $+1$. See some examples in Fig. 8.4. The proof is complete.

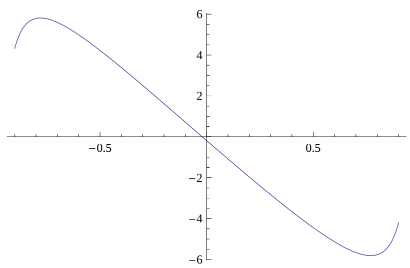
9. Conclusions and open questions. As a simplified Ginzburg-Landau model, the HKHF model retains much of the features of the original Ginzburg-Landau model of superconductivity and is useful in the study of vortex interactions in the presence of both an applied magnetic field and an applied current. Given the applied magnetic field close to the lower critical field at which there are single vortex ground state solutions being the ground state solutions, numerical simulations suggest that for small applied current, solutions with stationary vortex locations still exist while larger applied current can generate periodic vortex motion as depicted in Fig 9.1 [11].



(a) For $H = 10,\ J = 0.2,\ \xi_{01}^{\varepsilon} \approx -0.0250$

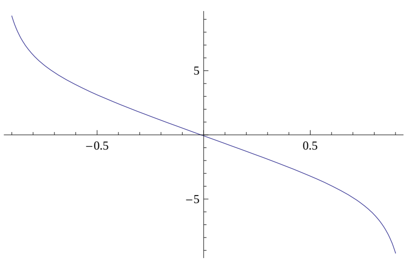


(b) For $H = 10,\ J = 0.5,\ \xi_{01}^{\varepsilon} \approx -0.0623$

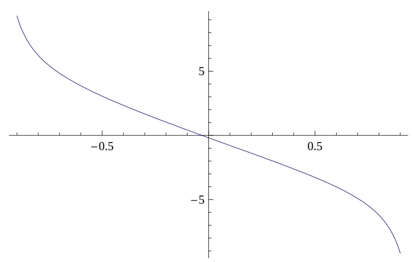


(c) For $H = 20,\ J = 0.5,\ \xi_{01}^{\varepsilon} \approx -0.0278$

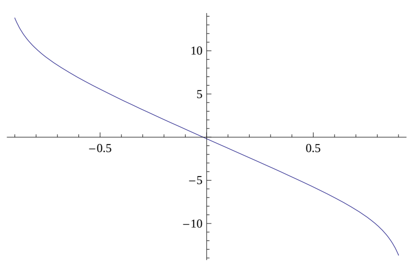
FIG. 8.3. *The figure of* $\Psi_2^{\varepsilon}(\xi_{01})$ *for the solution with degree* $+1$ *when* $\boldsymbol{A}_0 = \left(-\frac{H}{2}x_2, \frac{H}{2}x_1 - \frac{J}{2}x_1^2\right)$



(a) For $H = 10,\ J = 0.2,\ \xi_{01}^{\varepsilon} \approx -0.0167$



(b) For $H = 10,\ J = 0.5,\ \xi_{01}^{\varepsilon} \approx -0.0416$



(c) For $H = 20,\ J = 0.5,\ \xi_{01}^{\varepsilon} \approx -0.0227$

FIG. 8.4. *The figure of* $\Psi_2^{\varepsilon}(\xi_{01})$ *for the solution with degree* $+1$ *when* $\boldsymbol{A}_0 = \left(-\frac{H}{2}x_2, \frac{H}{2}x_1 - \frac{J}{2}x_1^2\right)$

The change of stationary vortex solution to the time periodic vortex motion suggests the existence of the critical applied current as conjectured in [10, 11]. As a first step towards rigorously proving such a conjecture, based on the idea of [19], we demonstrated in this paper the existence of stationary single vortex solutions of the HKHF model for a range of applied current. This provides partial justification to the

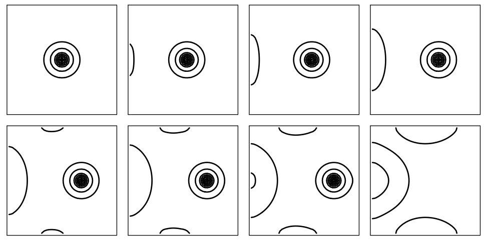


FIG. 9.1. *Motion of a single vortex in the presence of an applied current: contour plots of $|\psi_0(x,t)|$ are shown at different values of time t (from left to right then top to bottom).*

existence of the critical current in the context of the HKHF model. While we used a more general technique for non-variational problems, the present theory only verified the existence of the vortex solutions but makes no implication on their stability. We note in particular the freedom in choosing the signs for the vortex in the constructed solutions and it is obviously those having signs opposite to the applied magnetic field that would be energetically less favorable. Such stability analysis require a closer examination of the zeros of the functions $\boldsymbol{\Psi}(\xi)$ and $\boldsymbol{\Psi}^{\varepsilon}(\xi)$ which we leave for future studies. Moreover, our results are still limited to very small currents, in comparison with the applied magnetic field so that the shift of vortex positions remains a small perturbation. It remains to investigate the situation of a larger shift when the current increases, and to show that for large enough applied current, periodic in time solutions with vortices moving across the spatial domain can exist.

Concerning the existence of the critical current as characterized above, such a theory provides only a partial picture for the solution of the HKHF or the original Ginzburg-Landau model. In the existing literature, there have been lots of studies of simpler diagrams, such as those for time-independent Ginzburg-Landau models in the absence of the applied current (see for example [1]), and the more recent study of a one-dimensional time-dependent model with an applied voltage but in the absence of the applied magnetic field [22]. Yet, in [11], it has been suggested a much richer bifurcation diagram can be studied with both the applied magnetic field and the applied electric current as parameters. In this sense, much more analytical works are needed. Furthermore, we have not introduced the variety of pinning mechanisms discussed in the literature into our discussion. The result of the existence of the stationary vortex solution in our setting, despite the effect of the applied current, is due to the geometric construction and the barrier imposed by the applied magnetic field. Similar studies can be made in the future to consider the effect of various pinning mechanisms and the balance of pinning forces and the Lorentz force generated by the applied current. We stress again that such studies can still be carried out using the variants of the HKHF model such as the following equation

$$a(x)\left[\frac{\partial\psi_0}{\partial t}+i\Phi_a\psi_0\right]+(i\nabla+\boldsymbol{A}_0)^T a(x)M(x)(i\nabla+\boldsymbol{A}_0)\psi_0+\frac{a(x)}{\epsilon^2}(|\psi_0|^2-f(x))\psi_0=0\,,$$

with the scalar functions $a=a(x)$, $f=f(x)$ and tensor functions $M=M(x)$. This is a generalization of (1.1) for which we have $a=f\equiv 1$, and $M=I$, but it can also model various aspects of the pinning effect. For example, with $a\equiv 1$ and $M=I$, we can use a variable function f to model the normal inclusions. Alternatively, we can use $f\equiv 1$, $M=I$ and a non-uniform $a=a(x)$ to model inhomogeneities in film thickness. These generalizations certainly open up more questions to be rigorously studied in the future.

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