A PRIORI BOUNDS VERSUS MULTIPLE EXISTENCE OF POSITIVE SOLUTIONS FOR A NONLINEAR SCHRÖDINGER SYSTEM

E.N. DANCER, JUNCHENG WEI, AND TOBIAS WETH

ABSTRACT. We study the set of solutions of the nonlinear elliptic system

$$(P) \qquad \begin{cases} -\Delta u + \lambda_1 u = \mu_1 u^3 + \beta v^2 u & \text{in } \Omega, \\ -\Delta v + \lambda_2 v = \mu_2 v^3 + \beta u^2 v & \text{in } \Omega, \\ \ u, v > 0 \quad \text{in } \Omega, \qquad u = v = 0 & \text{on } \partial\Omega, \end{cases}$$

in a smooth bounded domain $\Omega \subset \mathbb{R}^N$, $N \leq 3$ with coupling parameter $\beta \in \mathbb{R}$. This system arises in the study of Bose-Einstein double condensates. We show that the value $\beta = -\sqrt{\mu_1 \mu_2}$ is critical for the existence of a priori bounds for solutions of (P). More precisely, we show that for $\beta > -\sqrt{\mu_1 \mu_2}$, solutions of (P) are a priori bounded. In contrast, when $\lambda_1 = \lambda_2, \mu_1 = \mu_2$, (P) admits an unbounded sequence of solutions if $\beta \leq -\sqrt{\mu_1 \mu_2}$.

1. INTRODUCTION

In this paper we study existence and a priori bounds for solitary wave solutions of the following two-component system of nonlinear Schrödinger equations (also called Gross-Pitaevskii equations):

$$(1.1) \quad \begin{cases} -i\frac{\partial}{\partial t}\Phi_1 = \frac{\bar{h}^2}{2m}\Delta\Phi_1 - V_1(x)\Phi_1 + \mu_1|\Phi_1|^2\Phi_1 + \beta|\Phi_2|^2\Phi_1 & \text{for } \ y \in \Omega, t > 0, \\ -i\frac{\partial}{\partial t}\Phi_2 = \frac{\bar{h}^2}{2m}\Delta\Phi_2 - V_2(x)\Phi_2 + \mu_2|\Phi_2|^2\Phi_2 + \beta|\Phi_1|^2\Phi_2 & \text{for } \ y \in \Omega, t > 0, \\ \Phi_j = \Phi_j(y,t) \in \mathbb{C}\,, \quad j = \ 1, 2, \\ \Phi_j(y,t) = 0 \ \text{ for } \ y \in \partial\Omega, t > 0, j = 1, 2. \end{cases}$$

This system models a binary mixture of Bose-Einstein condensates in two different hyperfine states $|1\rangle$ and $|2\rangle$ (see [6]). Physically, Φ_1 and Φ_2 are the corresponding condensate amplitudes, $\Omega \subseteq \mathbb{R}^N$ is the domain for condensate dwelling, $\bar{h}$ is the Planck constant divided by 2π, m is atom mass, and V_j is the trapping potential for the j-th hyperfine state. Moreover, μ_j and β are the intraspecies and interspecies scattering lengths which determine the interaction of the states.

Throughout the paper, we assume that Ω is a smooth bounded domain and that $\mu_j > 0$, $j = 1, 2$. The latter condition implies that the self-interactions of the single states $|j\rangle$ are *attractive*. The sign of β determines the interaction of state $|1\rangle$ with state $|2\rangle$. When $\beta < 0$, this interaction is *repulsive* (as considered e.g. in [24]). In contrast, when $\beta > 0$, the interaction is *attractive*.

To obtain solitary wave solutions of the form $\Phi_1(x,t) = e^{i\lambda_1 t}u(x)$, $\Phi_2(x,t) = e^{i\lambda_2 t}v(x)$, system (1.1) is reduced to the following elliptic system for u, v:

$$\begin{cases} \frac{\bar{h}^2}{2m}\Delta u - (\lambda_1 + V_1)u + \mu_1 u^3 + \beta uv^2 = 0 \quad \text{in } \ \Omega, \\ \frac{\bar{h}^2}{2m}\Delta v - (\lambda_2 + V_2)v + \mu_2 v^3 + \beta u^2 v = 0 \quad \text{in } \ \Omega, \\ u, v > 0 \ \text{ in } \ \Omega, \ u = v = 0 \ \text{ on } \ \partial\Omega. \end{cases} \tag{1.2}$$

Motivated by recent experimental and theoretical eximinations of double condensates (see [6, 10–12, 24]), system (1.2) has been attracting fastly growing attention. In [16], the existence and asymptotic behavior of least energy solutions is studied in a bounded domain with constant trapping potential, as $\bar{h} \to 0$. In [17], the asymptotic behavior is studied in $\mathbb{R}^N$ under the influence of nonconstant trapping potentials. When $\Omega = \mathbb{R}^N$, least energy and higher energy bound states of (1.2) are investigated in [1, 3, 15, 19, 22, 27].

The purpose of this paper is to analyze the impact of the parameter β (the interspecies scattering length) on a priori bounds and the existence of multiple solutions of (1.2). We first consider a priori estimates for the following more general version of (1.2):

$$\begin{cases} -\Delta u = f(x,u,v) \text{ in } \Omega, \\ -\Delta v = g(x,u,v) \text{ in } \Omega, \\ u, v > 0 \ \text{ in } \ \Omega, \ u = v = 0 \text{ on } \partial\Omega. \end{cases} \tag{1.3}$$

Here f and g are continuous in x and smooth in u and v, and they satisfy the following asymptotic conditions at $+\infty$:

$$f(x,u,v) = f_\infty(u,v) + h_1(x,u,v), \quad g(u,v) = g_\infty(u,v) + h_2(x,u,v) \tag{1.4}$$

where

$$f_\infty(u,v) = \mu_1 u^3 + \beta uv^2, \qquad g_\infty(u,v) = \mu_2 v^3 + \beta u^2 v, \tag{1.5}$$

and

$$\frac{h_i(x,u,v)}{(\max\{u,v\})^3} \to 0 \ \text{ uniformly in } x \in \Omega \text{ for } i = 1,2 \ \text{ as } \max\{u,v\} \to \infty. \tag{1.6}$$

Our first result is the following.

Theorem 1.1. *Assume that (1.4)–(1.6) holds. Then if $N \le 3, \beta > -\sqrt{\mu_1\mu_2}$, there exists a constant $C = C(\beta, \mu_1, \mu_2, \Omega)$ such that for any solution (u,v) of (1.3) we have*

$$\|u\|_{L^\infty(\Omega)}, \ \|v\|_{L^\infty(\Omega)} \le C$$

The proof of Theorem 1.1 relies on Liouville type theorems which we state in Section 2 below. A priori bounds for systems like (1.3) have been studied extensively in recent years, see [4, 5, 21, 29] and the references therein. We point out that in all these papers, it is assumed that the limiting nonlinearity (f_∞, g_∞) is *cooperative* (or *quasimonotone*), i.e,

$$\frac{\partial f_\infty(u,v)}{\partial v} \ge 0, \frac{\partial g_\infty(u,v)}{\partial u} \ge 0. \tag{1.7}$$

For cooperative systems, the maximum principle still works. So one can use various versions of the moving plane method to prove Liouville theorems and a priori estimates. In particular, when $\beta > 0$, Theorem 1.1 follows from results in [5] and [21]. In contrast, our system is *non-cooperative* if $\beta < 0$ and therefore the methods in the above-mentioned papers fail. Our result here seems to be the first in obtaining a priori bounds for *non-cooperative* systems. We may assume that $\beta < 0$ from now on.

In our second result we show that the assumption on β in Theorem 1.1 is optimal. More precisely, we consider the fully symmetric case $\lambda_1 = \lambda_2$, $\mu_1 = \mu_2$ and $V_1 = V_2 \equiv 0$. Then, by a rescaling, (1.2) becomes

$$\begin{cases} -\Delta u + u = u^3 + \beta v^2 u & \text{in } \Omega, \\ -\Delta v + v = v^3 + \beta u^2 v & \text{in } \Omega, \\ \quad u, v > 0 \quad \text{in } \Omega, \quad u = v = 0 & \text{on } \partial\Omega. \end{cases} \tag{1.8}$$

We note that the critical value $-\sqrt{\mu_1\mu_2}$ corresponds to $\beta = -1$ in (1.8). We also point out that (1.8) is invariant under the reflection $(u, v) \to \sigma(u, v) = (v, u)$. This invariance is essential for the following multiplicity result depending on β.

Theorem 1.2. *Let* $N \leq 3$.
(a) If $\beta \leq -1$*, then system (1.8) admits a sequence* $(u_k, v_k)_k$ *of solutions with*

$$\|u_k\|_{L^\infty(\Omega)} + \|v_k\|_{L^\infty(\Omega)} \to \infty.$$

(b) For any positive integer k *there exists a number* $\beta_k > -1$ *such that, for* $\beta < \beta_k$*, system* (1.8) *has at least* k *pairs* $(u, v), (v, u)$ *of solutions.*

We add some comments.

Remark 1.1. (i) For $\beta > -1$, every positive solution of the Dirichlet problem for the scalar equation $-\Delta u + u = u^3$ in Ω gives rise to a *diagonal* solution $\frac{1}{\sqrt{1+\beta}}(u, u)$ of (1.8). In contrast, it will be evident from our construction that the solutions obtained in Theorem 1.2 have different components u, v. Moreover, for $\beta \neq 1$, system (1.8) does not admit nontrivial solutions (u, v) with $u \neq v$ and $u \leq v$ or $v \leq u$ (as is easily seen by multiplying the first equation of (1.8) with v, the second equation with u and integrating). Consequently, all solutions obtained in Theorem 1.2 have intersecting components.
(ii) The proof of Theorem 1.2 relies on a variant of Liusternik-Schnirelman theory on a submanifold $\mathcal{M}$ (depending on β) of the underlying energy space $H_0^1(\Omega) \times H_0^1(\Omega)$. The importance of this manifold is given by the following properties; it contains all solutions of (1.8), it is invariant under the reflection σ, and σ has no fixed points in $\mathcal{M}$ if $\beta \leq -1$.
(iii) The multiplicity statements in Theorem 1.2 carry over to the corresponding problem in the full space $\mathbb{R}^N$ if compactness is restored by restricting to radial functions. More precisely, with essentially the same proof we can show that, for $\beta \leq -1$, system (1.8) admits infinitely many radial bound state solutions if $\Omega = \mathbb{R}^N$, and the number of radial bound states tends to infinity as $\beta \searrow -1$, $\beta > -1$.
(iv) If $\Omega = B_1(0)$ is the unit ball in $\mathbb{R}^N$, a different approach based on a corresponding

parabolic problem shows the existence of radial solutions of (1.8) with a prescribed number of intersections of u and v, see [28].

The paper is organized as follows. In Section 2 we prove the Liouville theorems for the limit system $-\Delta u = f_\infty(u,v)$, $-\Delta v = f_\infty(u,v)$ which are the basis for the a priori estimates asserted in Theorem 1.1. For $N = 1,2$, these Liouville theorems are rather simple consequences of nonexistence results for solutions of the differential inequality $-\Delta w \geq w^3$ obtained in [7, 13, 14]. The case $N = 3$ is essential more involved, since $-\Delta w \geq w^3$ admits solutions if the underlying domain is a half space in $\mathbb{R}^3$, see [14]. We combine a doubling lemma of Poláčik, Quittner and Souplet [20] with a uniform Hopf type estimate on boundary derivatives and a variant of Pohozaev's identity to deal with this case. This procedure is new and might be useful for other non-cooperative elliptic systems. In Section 3 we complete the proof of Theorem 1.1 by a standard blow up argument. Finally, Section 4 contains the proof of Theorem 1.2.

Acknowledgment. The research of the second author is partially supported by an Earmarked Grant from RGC of Hong Kong.

2. Liouville type theorems

As usual, we put $\mathbb{R}^N_+ := \{x \in \mathbb{R}^N \,:\, x_N > 0\}$. In this section we will prove the following Liouville type theorems.

Theorem 2.1. *If* $N \leq 3$, $\beta > -\sqrt{\mu_1\mu_2}$, *and* (u,v) *is a classical solution of the system*

$$\begin{cases} -\Delta u = \mu_1 u^3 + \beta v^2 u & \text{in } \mathbb{R}^N, \\ -\Delta v = \mu_2 v^3 + \beta u^2 v & \text{in } \mathbb{R}^N \end{cases} \tag{2.1}$$

with $u \geq 0$ *and* $v \geq 0$, *then* $(u,v) = (0,0)$.

Theorem 2.2. *Let* $\beta > -\sqrt{\mu_1\mu_2}$.
(i) If $N \leq 2$ *and* (u,v) *is a classical solution of the system*

$$\begin{cases} -\Delta u = \mu_1 u^3 + \beta v^2 u & \text{in } \mathbb{R}^N_+, \\ -\Delta v = \mu_2 v^3 + \beta u^2 v & \text{in } \mathbb{R}^N_+, \\ \;u, v \geq 0 \quad \text{in } \mathbb{R}^N_+, \quad u = v = 0 & \text{on } \partial\mathbb{R}^N_+, \end{cases} \tag{2.2}$$

then $(u,v) = (0,0)$.
(ii) If $N = 3$ *and* (u,v) *is a bounded classical solution of* (2.2), *then* $(u,v) = (0,0)$.

As we shall see below, Theorem 2.1 is a rather immediate corollary of a nonexistence result for supersolutions. For Theorem 2.2, the same is true *only* in case $N \leq 2$. We now recall these nonexistence results. Part (i) of the following theorem is due to Gidas [7], part (ii) due to Laptev [13, 14].

Theorem 2.3. *(i) Suppose that* $0 < q \leq \frac{N}{N-2}$ *if* $N \geq 3$, $0 < q < \infty$ *if* $N = 1, 2$, *and suppose that* $w \in C^2(\mathbb{R}^N)$ *is a nonnegative function satisfying*

$$-\Delta w \geq w^q \qquad \text{in } \mathbb{R}^N.$$

Then $w \equiv 0$.
(ii) Suppose that $0 < q \leq \frac{N+1}{N-1}$ *if* $N \geq 2$, $0 < q < \infty$ *if* $N = 1$, *and suppose that* $w \in C^2(\overline{\mathbb{R}^N_+})$ *is a nonnegative function satisfying*

$$-\Delta w \geq w^q \quad \text{in } \mathbb{R}^N_+, \qquad u = 0 \quad \text{on } \partial\mathbb{R}^N_+.$$

Then $w \equiv 0$.

Proof of Theorem 2.1. If $\beta \geq 0$, then $-\Delta u \geq u^3$ and $-\Delta v \geq v^3$ in $\mathbb{R}^N$, so that $u = v = 0$ by Theorem 2.3(i).
Next we assume that $-\sqrt{\mu_1\mu_2} < \beta < 0$. Then we have the following simple inequality: there exists $\gamma_0 > 0$ such that

$$\mu_1 u^3 + \beta uv^2 + \mu_2 v^3 + \beta u^2 v \geq \gamma_0 (u+v)^3 \text{ for all } u, v \geq 0. \tag{2.3}$$

To see this, we let $t = \frac{v}{u}$ and consider the function

$$t \mapsto \rho(t) := \frac{\mu_1 + \beta t^2 + t(\mu_2 t^2 + \beta)}{(1+t)^3}, \qquad t \geq 0.$$

Then $\rho(0) = \mu_1 > 0$ and $\rho(t) \to \mu_2 > 0$ as $t \to \infty$. Moreover, ρ has no positive zero since there is no solution to $t = \frac{-\beta t^2 - \mu_1}{\mu_2 t^2 + \beta}$ if $\beta > -\sqrt{\mu_1\mu_2}$. Hence $\min_{t \geq 0} \rho(t) > 0$, and from this (2.3) follows.

We now put $w = u + v$. A short computation using (2.3) shows

$$-\Delta w \geq \gamma_0 w^3 \qquad \text{in } \mathbb{R}^N, \tag{2.4}$$

so that $\tilde{w} := \sqrt{\gamma_0} w$ satisfies $-\Delta \tilde{w} \geq \tilde{w}^3$. Since $\tilde{w}$ is nonnegative, we conclude again from Theorem 2.3(i) that $\tilde{w} \equiv 0$. Hence $w \equiv 0$ and therefore $u \equiv 0$ and $v \equiv 0$. □

Part (i) of Theorem 2.2 can be deduced from Theorem 2.3(ii) similarly as in the proof of Theorem 2.1. The case $N = 3$ is much more delicate since the differential inequality $-\Delta w \geq w^3$ admits positive solutions in $\mathbb{R}^3_+$, see [14].
The remainder of this section is devoted to the proof of Theorem 2.2(ii). We first need an a priori singularity and decay estimate for (possibly) singular solutions. The proof of the next Lemma is modeled on an argument of Poláčik, Quittner and Souplet [20].

Lemma 2.1. *There is a constant* $C_1 > 0$ *such that for every solution* (u, v) *of*

$$\begin{cases} -\Delta u = \mu_1 u^3 + \beta v^2 u, \\ -\Delta v = \mu_2 v^3 + \beta u^2 v \end{cases} \quad \text{in } \mathbb{R}^3_+, \qquad u, v \geq 0 \quad \text{in } \mathbb{R}^3_+ \tag{2.5}$$

we have

$$[u + v + |\nabla u|^{\frac{1}{2}} + |\nabla v|^{\frac{1}{2}}](x) \leq C_1 |x_3|^{-1} \qquad \text{for every } x = (x_1, x_2, x_3) \in \mathbb{R}^3_+. \tag{2.6}$$

Proof. Suppose by contradiction that there exists a sequence of solutions $(u_n, v_n)_n$ of (2.5) and a sequence $(x^n)_n \subset \mathbb{R}^3_+$ such that

$$M_n(x^n)|x^n_3| \geq 2n$$

for all n, where the functions $M_n : \mathbb{R}^3_+ \to \mathbb{R}$ are defined by

$$M_n(x) = [u_n + v_n + |\nabla u_n|^{\frac{1}{2}} + |\nabla v_n|^{\frac{1}{2}}](x), \qquad x \in \mathbb{R}^3_+. \tag{2.7}$$

By the Doubling Lemma [20, Lemma 5.1] there exist another sequence $(y^n)_n \subset \mathbb{R}^3_+$ such that

$$M_n(y^n)|y^n_3| \geq 2n \qquad \text{and} \qquad M_n(z) \leq 2M_n(y^n) \quad \text{for } z \in B_{n\lambda_n}(y^n),$$

where $\lambda_n := [M_n(y^n)]^{-1}$. We now define

$$\tilde{u}_n, \tilde{v}_n : B_n(0) \to \mathbb{R}, \qquad \tilde{u}_n(x) = \lambda_n u_n(y^n + \lambda_n x), \quad \tilde{v}_n(x) = \lambda_n v_n(y^n + \lambda_n x).$$

Then $\tilde{u}_n, \tilde{v}_n$ are nonnegative functions solving

$$\begin{cases} -\Delta \tilde{u}_n = \mu_1(\tilde{u}_n)^3 + \beta(\tilde{v}_n)^2 \tilde{u}_n, \ |x| \leq n, \\ -\Delta \tilde{v}_n = \mu_2(\tilde{v}_n)^3 + \beta(\tilde{u}_n)^2 \tilde{v}_n, \ |x| \leq n. \end{cases} \tag{2.8}$$

Moreover,

$$[\tilde{u}_n + \tilde{v}_n + |\nabla \tilde{u}_n|^{\frac{1}{2}} + |\nabla \tilde{v}_n|^{\frac{1}{2}}](0) = 1 \tag{2.9}$$

and

$$\max_{B_n(0)} [\tilde{u}_n + \tilde{v}_n + |\nabla \tilde{u}_n|^{\frac{1}{2}} + |\nabla \tilde{v}_n|^{\frac{1}{2}}] \leq 2.$$

By standard elliptic estimates, we deduce that a subsequence of $(u_n, v_n)_n$ converges in $C^1_{loc}(\mathbb{R}^N)$ to a solution (u, v) of (2.1) on $\mathbb{R}^N$ which is nonnegative in both components. Since

$$[u + v + |\nabla \tilde{u}|^{\frac{1}{2}} + |\nabla v|^{\frac{1}{2}}](0) = 1$$

by (2.9), (u, v) is a nontrivial solution. This contradicts Theorem 2.1. □

Lemma 2.2. *Let $N = 3$. Then there is a constant $C_2 > 0$ such that, for every solution (u, v) of (2.2) and every $x = (x_1, x_2, x_3) \in \mathbb{R}^3_+$,*

$$w(x) \leq C_2 \sqrt{\partial_{x_3} w(x_1, x_2, 0)}, \tag{2.10}$$

where $w = u + v$.

Proof. Suppose by contradiction that there exists a sequence of solutions $(u_n, v_n)_n$ of (2.2) and a sequence of points $x^n = (x^n_1, x^n_2, x^n_3) \in \mathbb{R}^3_+$, $n \in \mathbb{N}$ such that for $w_n := u_n + v_n$ we have

$$w_n(x^n) > n \sqrt{\partial_{x_3} w_n(x^n_1, x^n_2, 0)} \qquad \text{for all } n.$$

We put $\lambda_n = \frac{1}{w_n(x^n)}$, $y^n := (x^n_1, x^n_2, 0)$, and we consider the rescaled functions

$$\tilde{u}_n, \tilde{v}_n : \mathbb{R}^3_+ \to \mathbb{R}, \qquad \tilde{u}_n(x) = \lambda_n u_n(y^n + \lambda_n x), \quad \tilde{v}_n(x) = \lambda_n v_n(y^n + \lambda_n x)$$

and $\tilde{w}_n = \tilde{u}_n + \tilde{v}_n$. Then $\tilde{u}_n$, $\tilde{v}_n$ solve again the system (2.2), and

$$\sqrt{\partial_{x_3} \tilde{w}_n(0)} = \lambda_n \sqrt{\partial_{x_3} w_n(x^n_1, x^n_2, 0)} \leq \frac{1}{n}. \tag{2.11}$$

Moreover, for $t_n := \lambda_n^{-1} x_n^3$ and $z^n := (0,0,t_n) \in \mathbb{R}^3_+$ we have $\tilde{w}_n(z^n) = 1$, so that $t_n \le C_1$ for all n by Lemma 2.1. We put $\tau = \min\{\frac{1}{2C_1}, \frac{1}{16C_1^2}\}$ and consider

$$B_n := \{x \in \mathbb{R}^3_+ \;:\; |x - z^n| \le \tau t_n^2\}.$$

For $x \in B_n$ we have $|x| \ge t_n - \tau t_n^2 = t_n(1 - \tau t_n) \ge t_n(1 - \tau C_1) \ge \frac{t_n}{2}$ and therefore $|\nabla \tilde{w}_n(x_n)| \le \frac{2C_1^2}{(\frac{t_n}{2})^2} = \frac{8C_1^2}{t_n^2}$ by Lemma 2.1. From this we conclude that

$$\tilde{w}_n(x) \ge \tilde{w}_n(z^n) - \Big(\frac{8C_1^2}{t_n^2}\Big)\tau t_n^2 = 1 - 8C_1^2\tau \ge \frac{1}{2} \qquad \text{for } x \in B_n.$$

We now define the comparison functions

$$g_n : \mathbb{R}^3 \setminus \{\pm z^n\} \to \mathbb{R}, \qquad g_n(x) = \frac{\tau t_n^2}{2}\Big(\frac{1}{|x - z^n|} - \frac{1}{|x + z^n|}\Big)$$

For every n, g_n is a harmonic function which vanishes on $\partial\mathbb{R}^3_+$ and is bounded above by $\frac{1}{2}$ on ∂B_n. On the other hand, $\tilde{w}_n$ satisfies $-\Delta \tilde{w}_n \ge \gamma_0 \tilde{w}_n^3 \ge 0$ in $\mathbb{R}^3_+$ with γ_0 as in (2.3). Moreover, $\tilde{w}_n$ is bounded below by $\frac{1}{2}$ on ∂B_n and vanishes on $\partial\mathbb{R}^3_+$. By the maximum principle, we conclude that $\tilde{w}_n \ge g_n$ in $\mathbb{R}^3_+ \setminus B_n$ and therefore

$$\partial_{x_3}\tilde{w}_n(0) \ge \partial_{x_3} g_n(0) = \frac{\tau t_n^2}{2}\Big(\frac{2}{t_n^2}\Big) = \tau$$

independently of n, contrary to (2.11). The proof is complete. □

Lemma 2.3. *Let $N = 3$, and let (u, v) be a bounded solution of* (2.2)*. Then there is a constant $C_3 > 0$ (possibly depending on u, v) such that, for every $x = (x_1, x_2, x_3) \in \mathbb{R}^3_+$,*

$$|\nabla u(x)| + |\nabla v(x)| \le C_3 \min\{1, |x_3|^{-1}\}\sqrt{\partial_{x_3} w(x_1, x_2, 0)}, \tag{2.12}$$

where $w = u + v$.

Proof. Let $x \in \mathbb{R}^3_+$ be fixed. We distinguish two cases, and we point out that the constants $C_3, C_4, \dots$ chosen below are all independent of x.

Case 1: $|x_3| \ge 1$. Then we consider the rescaled functions $\tilde{u}, \tilde{v}, \tilde{w} : \mathbb{R}^3_+ \to \mathbb{R}$ defined by

$$\tilde{u}(y) = |x_3| u((x_1, x_2, 0) + |x_3| y), \quad \tilde{v}(y) = |x_3| v((x_1, x_2, 0) + |x_3| y) \quad \text{and} \quad \tilde{w}(y) = \tilde{u}(y) + \tilde{v}(y).$$

Since $\tilde{u}$, $\tilde{v}$ solve again the system (2.2), Lemma 2.1 implies that

$$\tilde{w}(y) \le 2C_1 \qquad \text{whenever} \quad |y_3| \ge \frac{1}{2}. \tag{2.13}$$

We put $e_3 = (0,0,1) \in \mathbb{R}^3_+$ and $\Omega_0 = \{y \in \mathbb{R}^3 \;:\; |y - e_3| < \frac{1}{2}\}$. Moreover, we note that

$$-\Delta\tilde{u} = f_1(y)\tilde{u}, \quad \text{and} \quad -\Delta\tilde{v} = f_2(y)\tilde{v} \qquad \text{in } \Omega_0, \tag{2.14}$$

where $f_1 = \mu_1\tilde{u}^2 + \beta\tilde{v}^2$ and $f_2 = \mu_2\tilde{v}^2 + \beta\tilde{u}^2$. By (2.13), we have

$$|f_1|, |f_2| \le C_4 \qquad \text{in } \Omega_0. \tag{2.15}$$

Therefore, using (2.13) together with the standard estimate [9, Theorem 3.9] for solutions of the Poisson equation, we infer that

$$|\nabla \tilde{u}(e_3)| \le C_5\big(\sup_{|y-e_3|\le \frac{1}{4}} \tilde{u}(y) + \sup_{|y-e_3|\le \frac{1}{4}} |f_1(y)\tilde{u}(y)|\big) \le C_6 \sup_{|y-e_3|\le \frac{1}{4}} \tilde{u}(y) \tag{2.16}$$

and similarly

$$|\nabla \tilde{v}(e_3)| \le C_6 \sup_{|y-e_3|\le \frac{1}{4}} \tilde{v}(y). \tag{2.17}$$

Using (2.14), (2.15) and the Harnack inequality (see [9, Theorem 8.20]), we also infer that

$$\sup_{|y-e_3|\le \frac{1}{4}} \tilde{u}(y) \le C_7\, \tilde{u}(e_3), \qquad \sup_{|y-e_3|\le \frac{1}{4}} \tilde{v}(y) \le C_7\, \tilde{v}(e_3) \tag{2.18}$$

Combining (2.16), (2.17), (2.18) and Lemma 2.2, we find that

$$|\nabla \tilde{u}(e_3)| + |\nabla \tilde{v}(e_3)| \le C_8\, \tilde{w}(e_3) \le C_9 \sqrt{\partial_{x_3} \tilde{w}(0)}.$$

We conclude that

$$\begin{aligned} |\nabla u(x)| + |\nabla v(x)| &= |x_3|^{-2}\Big(|\nabla \tilde{u}(e_3)| + |\nabla \tilde{v}(e_3)|\Big) \le C_9\, |x_3|^{-2}\sqrt{\partial_{x_3}\tilde{w}(0)} \\ &= C_9\, |x_3|^{-1}\sqrt{\partial_{x_3} w(x_1,x_2,0)}. \end{aligned} \tag{2.19}$$

Case 2: $|x_3| \le 1$. We note that u and v solve the linear equations

$$-\Delta u = g_1(y)\tilde{u}, \quad \text{and} \quad -\Delta v = g_2(y)\tilde{v} \qquad \text{in } \mathbb{R}^3_+, \tag{2.20}$$

where $g_1 = \mu_1 u^2 + \beta v^2$ and $g_2 = \mu_2 v^2 + \beta u^2$. Since u and v are bounded by assumption, the functions g_1, g_2 are also bounded in $\mathbb{R}^3_+$. Since u, v are classical solutions satisfying Dirichlet boundary conditions on $\partial\mathbb{R}^3_+$, standard estimates up to the boundary for solutions of the Poisson equation (see e.g. [9, Theorem 4.16]) yield that

$$|\nabla u(x)| \le C_{10}\big(\sup_{|y-x|\le \frac{1}{2}} u(y) + \sup_{|y-x|\le \frac{1}{2}} |g_i(y)u(y)|\big) \le C_{11} \sup_{|y-x|\le \frac{1}{2}} u(y). \tag{2.21}$$

and

$$|\nabla v(x)| \le C_{11} \sup_{|y-x|\le \frac{1}{2}} v(y). \tag{2.22}$$

Moreover, applying the Harnack inequality up to the boundary of Berestycki, Caffarelli and Nirenberg [2, Theorem 1.3] to (2.20), it follows that

$$\sup_{|y-x|\le \frac{1}{2}} u(y) \le C_{12}\, u(x_1,x_2,1) \qquad \text{and} \qquad \sup_{|y-x|\le \frac{1}{2}} v(y) \le C_{12}\, v(x_1,x_2,1). \tag{2.23}$$

Combining (2.21), (2.22), (2.23) and Lemma 2.2, we find that

$$|\nabla u(x)| + |\nabla v(x)| \le C_{13}\sqrt{\partial_{x_3} w(x_1,x_2,0)}. \tag{2.24}$$

Combining (2.19) and (2.24), we conclude that

$$|\nabla u(x)| + |\nabla v(x)| \le (C_9 + C_{13}) \min\{1, |x_3|^{-1}\} \sqrt{\partial_{x_3} w(x_1,x_2,0)} \qquad \text{for all } x \in \mathbb{R}^3_+.$$

Now the claim follows with $C_3 := C_9 + C_{13}$. $\square$

The following lemma is related to a Pohozaev type identity.

Lemma 2.4. *Let $N = 3$. For $r > 0$ consider the set $Z_r = \{(x', t) : x' \in \mathbb{R}^2, |x'| \leq r, t > 0\} \subset \mathbb{R}^3_+$, whose boundary consists of the two parts $C_r = \{(x', t) : x' \in \mathbb{R}^2, |x'| = r, t > 0\}$ and $D_r = \{(x', 0) : x' \in \mathbb{R}^2, |x'| \leq r\}$. Let ν denote the outer unit normal vector field on C_r. Then*

$$\int_{D_r} [(\partial_{x_3} u)^2 + (\partial_{x_3} v)^2] \, d\mu_2 = 2 \int_{C_r} [(\partial_\nu u)(\partial_{x_3} u) + (\partial_\nu v)(\partial_{x_3} v) \, d\mu_2 \tag{2.25}$$

for every $r > 0$ and every solution (u, v) of (2.2).

Here and in the following, μ_k denotes the k-dimensional Hausdorff measure.

Proof. We use the fact that (2.2) is a gradient system, i.e. it can be written as $-\Delta u = \partial_u F(u, v)$, $-\Delta v = \partial_v F(u, v)$ with

$$F : \mathbb{R}^2 \to \mathbb{R}, \qquad F(u, v) = \frac{\mu_1}{4} u^4 + \frac{\mu_2}{4} v^4 + \frac{\beta}{2} u^2 v^2.$$

Hence for every $x' \in \mathbb{R}^2$ and $s > 0$ we have

$$\begin{aligned} -\int_0^s [\Delta u \, \partial_{x_3} u + \Delta v \, \partial_{x_3} v](x', t) \, dt &= \int_0^s [\partial_u F(u, v) \partial_{x_3} u + \partial_v F(u, v) \partial_{x_3} v](x', t) \, dt \\ = \int_0^s \frac{d}{dt} F(u(x', t), v(x', t)) \, dt &= F(u, v)(x', s) - F(u, v)(x', 0) = F(u, v)(x', s). \end{aligned}$$

Since also

$$\Delta u \partial_{x_3} u + \Delta v \partial_{x_3} v = [\Delta_{x'} u \, \partial_{x_3} u + \Delta_{x'} v \, \partial_{x_3} v] + \frac{1}{2} \partial_{x_3} [(\partial_{x_3} u)^2 + (\partial_{x_3} v)^2],$$

we obtain

$$\begin{aligned} &[(\partial_{x_3} u)^2 + (\partial_{x_3} v)^2](x', 0) = \\ &[(\partial_{x_3} u)^2 + (\partial_{x_3} v)^2](x', s) + 2F(u, v)(x', s) + 2 \int_0^s [\Delta_{x'} u \, \partial_{x_3} u + \Delta_{x'} v \, \partial_{x_3} v](x', t) \, dt. \end{aligned}$$

Letting $s \to \infty$ and using the decay estimate of Lemma 2.1, we get

$$[(\partial_{x_3} u)^2 + (\partial_{x_3} v)^2](x', 0) = 2 \int_0^\infty [\Delta_{x'} u \, \partial_{x_3} u + \Delta_{x'} v \, \partial_{x_3} v](x', t) \, dt.$$

Hence for $r > 0$ we have by using Green's formula in each of the slices $D_r \times \{s\}$, $s > 0$,

$$\begin{aligned} \int_{D_r} (\partial_{x_3} u)^2 + (\partial_{x_3} v)^2 \, d\mu_2 &= 2 \int_{Z_r} [\Delta_{x'} u \, \partial_{x_3} u + \Delta_{x'} v \, \partial_{x_3} v] \, d\mu_3 \\ &= 2 \int_{C_r} [\partial_\nu u \partial_{x_3} u + \partial_\nu v \partial_{x_3} v] \, d\mu_2 - 2 \int_{Z_r} [\nabla_{x'} u \nabla_{x'} (\partial_{x_3} u) + \nabla_{x'} v \nabla_{x'} (\partial_{x_3} v)] \, d\mu_3, \end{aligned}$$

where

$$\begin{aligned}\int_{Z_r}[\nabla_{x'}u\nabla_{x'}(\partial_{x_3}u)+\nabla_{x'}v\nabla_{x'}(\partial_{x_3}v)]\,d\mu_3&=\frac{1}{2}\int_{|x'|\le r}\int_0^\infty\frac{d}{dt}\big(|\nabla_{x'}u|^2+|\nabla_{x'}v|^2\big)(x',t)\,dt\,dx'\\&=-\frac{1}{2}\int_{D_r}\big(|\nabla_{x'}u|^2+|\nabla_{x'}v|^2\big)d\mu_2=0,\end{aligned}$$

since $\nabla_{x'}u=\nabla_{x'}v=0$ on $\partial\mathbb{R}^3_+$. We conclude that

$$\int_{D_r}[(\partial_{x_3}u)^2+(\partial_{x_3}v)^2]\,d\mu_2=2\int_{C_r}[\partial_\nu u\partial_{x_3}u+\partial_\nu v\partial_{x_3}v]\,d\mu_2,$$

as claimed. □

Proof of Theorem 2.2(ii) (completed).
Let $N=3$, and suppose by contradiction that (u,v) is a nontrivial bounded solution of (2.2). Put $h(r)=\int_{D_r}[(\partial_{x_3}u)^2+(\partial_{x_3}v)^2]\,d\mu_2$ and $S_r=\{(x',0)\,:\,x'\in\mathbb{R}^2,\ |x'|=r\}$. Then (2.25) and Lemma 2.3 imply

$$\begin{aligned}h(r)&=2\int_{C_r}[\partial_\nu u\partial_{x_3}u+\partial_\nu v\partial_{x_3}v]\,d\mu_2\le 2\int_{C_r}[|\nabla u|^2+|\nabla v|^2]\,d\mu_2\\&\le 2C_3^2\int_{|x'|=r}\partial_{x_3}w(x',0)\Big(\int_0^\infty\min\{1,t^{-2}\}\,dt\Big)\,dx'\\&\le C_{14}\int_{S_r}\partial_{x_3}(u+v)\,d\mu_1\ \le C_{15}[\mu_1(S_r)]^{\frac{1}{2}}\Big(\int_{S_r}[(\partial_{x_3}u)^2+(\partial_{x_3}v)^2]\,d\mu_1\Big)^{\frac{1}{2}}\\&\le C_{16}\sqrt{r\,h'(r)}.\end{aligned}$$

It follows that $\frac{1}{(C_{16})^2r}-\frac{h'(r)}{h^2(r)}\le 0$, which implies that $g(r):=\frac{\ln r}{(C_{16})^2}+\frac{1}{h(r)}$ is nonincreasing in $r>0$. However, $g(r)\to\infty$ as $r\to\infty$, which yields a contradiction. The proof is finished. □

3. A PRIORI BOUNDS IN THE CASE $\beta>-\sqrt{\mu_1\mu_2}$

In this section we complete the proof of Theorem 1.1, and we fix $\beta>-\sqrt{\mu_1\mu_2}$. We proceed by contradiction, assuming that there is a sequence of solutions (u_n,v_n) to (1.3) with

$$\max_{x\in\Omega}u_n(x)+\max_{x\in\Omega}v_n(x)\to+\infty\qquad\text{as } n\to\infty. \tag{3.1}$$

We follow a blow up procedure introduced by Gidas and Spruck [8] for scalar equations which has already been generalized to elliptic systems, see e.g. [5] and [4]. Since the method is standard, we only sketch the argument. Without loss of generality, we may assume that

$$M_n:=\max_{x\in\Omega}u_n(x)\ge\max_{x\in\Omega}v_n(x). \tag{3.2}$$

Let $x_n \in \Omega$ satisfy $u_n(x_n) = M_n$. Now we perform a rescaling, setting $\Omega_n = \{y \in \mathbb{R}^N : x_n + \frac{y}{\sqrt{M_n}} \in \Omega\}$ and defining functions $U_n, V_n : \Omega_n \to \mathbb{R}$ by

$$U_n(y) = \frac{u_n(x_n + \frac{y}{\sqrt{M_n}})}{M_n}, \qquad V_n(y) = \frac{v_n(x_n + \frac{y}{\sqrt{M_n}})}{M_n} \qquad \text{for } y \in \Omega_n. \tag{3.3}$$

Then

$$1 := \max_{y \in \Omega_n} U_n(y) \geq \max_{y \in \Omega_n} V_n(y), \tag{3.4}$$

and (U_n, V_n) solves the rescaled problem

$$\begin{cases} -\Delta U_n = \mu_1 U_n^3 + \beta U_n V_n^2 + \dfrac{h_1(x_n + \frac{y}{\sqrt{M_n}},\, u_n, v_n)}{M_n^3} & \text{in } \Omega_n, \\ -\Delta V_n = \mu_1 V_n^3 + \beta V_n U_n^2 + \dfrac{h_2(x_n + \frac{y}{\sqrt{M_n}},\, u_n, v_n)}{M_n^3} & \text{in } \Omega_n, \\ \quad U_n = V_n = 0 & \text{on } \partial\Omega_n. \end{cases} \tag{3.5}$$

Using (1.6), we see that

$$\sup_{x \in \Omega} \frac{h_i(x, u_n, v_n)}{M_n^3} \to 0 \qquad \text{as } n \to \infty. \tag{3.6}$$

Passing to a subsequence if necessary, we consider two cases.

Case 1: $\sqrt{M_n} d(x_n, \partial\Omega) \to +\infty$. In this case, Ω_n approaches $\mathbb{R}^N$ in the sense that any compact subset of $\mathbb{R}^N$ is contained in $\bigcap\limits_{m \geq n} \Omega_m$ for n large enough. Using elliptic regularity theory as in [4, 5], we may assume that $(U_n, V_n) \to (U_0, V_0)$ uniformly on compact subsets of $\mathbb{R}^N$ where (U_0, V_0) is a solution of

$$-\Delta U_0 = \mu_1 U_0^3 + \beta U_0 V_0^2, \qquad -\Delta V_0 = \mu_2 V_0^3 + \beta U_0^2 V_0 \qquad \text{in } \mathbb{R}^N \tag{3.7}$$

with $0 \leq U_0(y) \leq 1$, $0 \leq V_0(y) \leq 1$ for $y \in \mathbb{R}^N$ and $U_0(0) = 1$. This is impossible by Theorem 2.1.

Case 2: $d_n := \sqrt{M_n} d(x_n, \partial\Omega) \to d_0 \geq 0$. In this case we consider Ω_n, U_n and V_n as before and let $y_n \in \partial\Omega_n$ be a point where

$$|y_n| = \operatorname{dist}(0, \partial\Omega_n) = d_n.$$

Rotating Ω_n suitably, we may assume that $y_n = t_n e_N$, where $e_N = (0, \ldots, 0, 1)$ is the n-th coordinate vector and $t_n = -d_n \to -d_0$ as $n \to \infty$. In this case, Ω_n approaches the half space $H := \{x \in \mathbb{R}^N : x_N > -d_0\}$ in the sense that $\Omega_n \cap B_R(0) \to H \cap B_R(0)$ for every $R > 0$ with respect to the Hausdorff distance. As in [4, 5, 8] we may now pass to a subsequence such that $(U_n, V_n) \to (U_0, V_0)$ uniformly on compact subsets of $\mathbb{R}^N_+$, where now (U_0, V_0) is a solution of the following limiting problem on H.

$$\begin{cases} -\Delta U_0 = \mu_1 U_0^3 + \beta U_0 V_0^2 & \text{in } H, \\ -\Delta V_0 = \mu_2 V_0^3 + \beta U_0^2 V_0 & \text{in } H, \\ \quad U_0 = V_0 = 0 & \text{on } \partial H. \end{cases} \tag{3.8}$$

Moreover, $0 \le U_0(y) \le 1$, $0 \le V_0(y) \le 1$ for $y \in \mathbb{R}^N$ and $U_0(0) = 1$ (a posteriori this implies that $d_0 > 0$). This is impossible by Theorem 2.2.
Since in both cases we have come to a contradition, the proof of Theorem 1.1 is complete.

4. MULTIPLE POSITIVE SOLUTIONS IN THE SYMMETRIC CASE

In this section we prove Theorem 1.2. Throughout this section, we assume that $\lambda_1 = \lambda_2 = 1, \mu_1 = \mu_2 = 1$ and $\beta < 0$. We put $\mathcal{H} = H_0^1(\Omega) \times H_0^1(\Omega)$, and we consider the energy functional $E \in C^2(\mathcal{H}, \mathbb{R})$ defined by

$$E(u,v) = \frac{1}{2}(\|u\|^2 + \|v\|^2) - \frac{1}{4}\int_\Omega (|u^+|^4 + |v^+|^4)\,dx - \frac{\beta}{2}\int_\Omega u^2v^2\,dx.$$

Here and in the following, $u^+ = \max\{u, 0\}$, $u^- = -\min\{u, 0\}$ and $\|u\|^2 = \int_\Omega(|\nabla u|^2 + |u|^2)\,dx$ for $u \in H_0^1(\Omega)$. Moreover, for a function $u \in L^s(\Omega)$, we denote by $|u|_s$ the usual L^s-norm of u. We are interested in *nontrivial* critical points (u,v) of E. These are critical points with $u \neq 0$ *and* $v \neq 0$, as opposed to semitrivial critical points which are of the form $(u,0)$ or $(0,v)$.

Lemma 4.1. *Every nontrivial critical point $(u,v) \in \mathcal{H}$ of E is a classical solution of (1.8).*

Proof. A critical point $(u,v) \in \mathcal{H}$ is a weak solution of the system

$$\begin{cases} -\Delta u + (1 - \beta v^2)u = (u^+)^3 & \text{in } \Omega, \\ -\Delta v + (1 - \beta u^2)v = (v^+)^3 & \text{in } \Omega, \\ u = v = 0 & \text{on } \partial\Omega. \end{cases} \tag{4.1}$$

Multiplying these equations with u^- resp. v^- and integrating, we get

$$\int_\Omega |\nabla u^-|^2 + \int_\Omega (1 - \beta v^2)|u^-|^2 = 0 = \int_\Omega |\nabla v^-|^2 + \int_\Omega (1 - \beta u^2)|v^-|^2.$$

Since $\beta < 0$, we conclude that $u, v \ge 0$, and therefore (u,v) is a weak solution of the original system in (1.8). By standard elliptic regularity, (u,v) is a classical solution. If $u \not\equiv 0$ and $v \not\equiv 0$, we conclude that $u, v > 0$ in Ω by the strong maximum principle. □

Next we put

$$\mathcal{M} = \left\{ (u,v) \in \mathcal{H},\ u, v \neq 0 \;\middle|\; \begin{array}{l} \|u\|^2 - \beta \int_\Omega u^2v^2 = \int_\Omega |u^+|^4, \\ \|v\|^2 - \beta \int_\Omega u^2v^2 = \int_\Omega |v^+|^4. \end{array} \right\}$$
$$= \{(u,v) \in \mathcal{H},\ u, v \neq 0 \mid \partial_u E(u,v)u = 0,\ \partial_v E(u,v)v = 0\}$$

Clearly, all nontrivial critical points (u,v) of E are contained in $\mathcal{M}$.

Lemma 4.2. *(i) $\mathcal{M}$ is a C^2-submanifold of H of codimension two.*
(ii) If (u,v) is a critical point of the restriction $E_\mathcal{M}$ of E to $\mathcal{M}$, then (u,v) is a nontrivial critical point of E.
(iii) $E(u,v) = \frac{1}{4}(\|u\|^2 + \|v\|^2)$ for $(u,v) \in \mathcal{M}$.
(iv) $E_\mathcal{M} : \mathcal{M} \to \mathbb{R}$ satisfies the Palais-Smale condition.

Proof. (i) The Sobolev embedding $H^1_0(\Omega) \hookrightarrow L^4(\Omega)$ implies that for $(u,v) \in \mathcal{M}$ we have

$$C\|u\|^4 \ge |u|_4^4 \ge \|u\|^2 \qquad \text{and} \qquad C\|v\|_4^4 \ge |v|_4^4 \ge \|v\|^2 \tag{4.2}$$

with a constant $C > 0$, hence

$$\|u\|, \|v\| \ge C^{-1/2} \qquad \text{for all } (u,v) \in \mathcal{M}. \tag{4.3}$$

Moreover, $\mathcal{M} = \{(u,v) \in \mathcal{H} \;:\; u,v \neq 0,\ F(u,v) = (0,0)\}$, where $F \in C^2(\mathcal{H}, \mathbb{R}^2)$ is given by

$$F(u,v) = \begin{pmatrix} F_1(u,v) \\ F_2(u,v) \end{pmatrix} = \begin{pmatrix} \|u\|^2 - \beta \int_\Omega u^2v^2 - \int_\Omega |u^+|^4 \\ \|v\|^2 - \beta \int_\Omega u^2v^2 - \int_\Omega |v^+|^4 \end{pmatrix}. \tag{4.4}$$

Note that for $(u,v) \in \mathcal{M}$ we have

$$\partial_u F_1(u,v)u = 2\|u\|^2 - 2\beta \int_\Omega u^2v^2 - 4\int_\Omega |u^+|^4 = -2\int_\Omega |u^+|^4 \neq 0$$

and

$$\partial_v F_2(u,v)v = 2\|v\|^2 - 2\beta \int_\Omega u^2v^2 - 4\int_\Omega |v^+|^4 = -2\int_\Omega |v^+|^4 \neq 0,$$

whereas $\partial_v F_1(u,v)v = -2\int_\Omega u^2v^2 = \partial_u F_2(u,v)u$. Consequently,

$$T_{u,v} := \left(\begin{array}{cc} \partial_u F_1(u,v)u & \partial_v F_1(u,v)v \\ \partial_u F_2(u,v)u & \partial_v F_2(u,v)v \end{array} \right) = \left(\begin{array}{cc} -2\int_\Omega |u^+|^4 & -2\beta \int_\Omega u^2v^2 \\ -2\beta \int_\Omega u^2v^2 & -2\int_\Omega |v^+|^4 \end{array} \right) \quad \in \mathbb{R}^{2\times 2}.$$

Since $(u,v) \in \mathcal{M}$, we have $\int_\Omega |u^+|^4 > -\beta \int_\Omega u^2v^2 \ge 0$ and $\int_\Omega |v^+|^4 > -\beta \int_\Omega u^2v^2 \ge 0$, which implies that $T_{u,v}$ is negative definite. Hence the vectors $F'(u,v)(u,0)$ and $F'(u,v)(0,v)$ are linearly independent in $\mathbb{R}^2$, so that $F'(u,v) : \mathcal{H} \to \mathbb{R}^2$ is onto. We therefore conclude that $\mathcal{M}$ is a C^2-submanifold of H of codimension two.
(ii) If $(u,v) \in \mathcal{M}$ is a critical point of $E|_\mathcal{M}$, then there are Lagrangian multipliers $\lambda_1, \lambda_2 \in \mathbb{R}$ such that

$$\lambda_1 F_1'(u,v) + \lambda_2 F_2'(u,v) = E'(u,v) \qquad \text{in } \mathcal{H}^*. \tag{4.5}$$

Applying this to $(u,0)$ and $(0,v)$, respectively, gives

$$T_{u,v}\begin{pmatrix} \lambda_1 \\ \lambda_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \qquad \text{with } T_{u,v} \text{ as above.}$$

Since $T_{u,v}$ is negative definite, $\lambda_1 = \lambda_2 = 0$, so that $E'(u,v) = 0$ by (4.5).
(iii) For $(u,v) \in \mathcal{M}$ we have

$$\begin{aligned} E(u,v) &= \frac{1}{2}(\|u\|^2 + \|v\|^2) - \frac{1}{4}\int_\Omega (|u^+|^4 + |v^+|^4)\,dx - \frac{\beta}{2}\int_\Omega u^2v^2\,dx \\ &= \frac{1}{2}(\|u\|^2 + \|v\|^2) - \frac{1}{4}\left(\|u\|^2 + \|v\|^2 - 2\beta\int_\Omega u^2v^2\right) - \frac{\beta}{2}\int_\Omega u^2v^2\,dx \\ &= \frac{1}{4}(\|u\|^2 + \|v\|^2). \end{aligned}$$

(iv) Let $(u_k,v_k)_k \subset \mathcal{M}$ be a Palais Smale sequence for $E_{\mathcal{M}}$. Then $(u_k,v_k)_k$ is bounded in $\mathcal{H}$ by (iii). Passing to a subsequence, we may assume that $(u_k,v_k) \rightharpoonup (u,v) \in \mathcal{H}$ and $u_k \to u$, $v_k \to v$ in $L^4(\Omega)$. We note that

$$u^+ \neq 0 \qquad \text{and} \qquad v^+ \neq 0. \tag{4.6}$$

Indeed, suppose by contradiction that $u^+ = 0$. Then

$$\lim_{k\to\infty} |u_k^+|_4 \to 0 \qquad \text{and} \qquad \limsup_{k\to\infty} \beta \textstyle\int_\Omega u_k^2 v_k^2 \le 0,$$

so that $\|u_k\| \to 0$ since $u_k \in \mathcal{M}$. This contradicts (4.3). Similarly we exclude that $v^+ = 0$.

Next we note that

$$o(1) = E'_{\mathcal{M}}(u_k,v_k) = E'(u_k,v_k) - \lambda_1^k F_1'(u_k,v_k) + \lambda_2^k F_2'(u_k,v_k) \qquad \text{as } k \to \infty \tag{4.7}$$

for appropriate sequences $(\lambda_1^k)_k, (\lambda_2^k)_k \subset \mathbb{R}$, where F_1, F_2 are defined in (4.4). Since the sequence $(u_k,v_k)_k$ is bounded in $\mathcal{H}$, we find that

$$\begin{aligned} o(1) &= \begin{pmatrix} E'(u_k,v_k)(u_k,0) - [\lambda_1^k F_1'(u_k,v_k) + \lambda_2^k F_2'(u_k,v_k)](u_k,0) \\ E'(u_k,v_k)(0,v_k) - [\lambda_1^k F_1'(u_k,v_k) + \lambda_2^k F_2'(u_k,v_k)](0,v_k) \end{pmatrix} \\ &= -\begin{pmatrix} [\lambda_1^k F_1'(u_k,v_k) + \lambda_2^k F_2'(u_k,v_k)](u_k,0) \\ [\lambda_1^k F_1'(u_k,v_k) + \lambda_2^k F_2'(u_k,v_k)](0,v_k) \end{pmatrix} = -T_{u_k,v_k} \begin{pmatrix} \lambda_1^k \\ \lambda_2^k \end{pmatrix} \\ &= (-T_{u,v} + o(1)) \begin{pmatrix} \lambda_1^k \\ \lambda_2^k \end{pmatrix} \end{aligned} \tag{4.8}$$

Since $(u_k,v_k) \in \mathcal{M}$ for every k, the weak convergence implies that

$$\|u\|^2 - \beta \textstyle\int_\Omega u^2v^2 \le \int_\Omega |u^+|^4 \qquad \text{and} \qquad \|v\|^2 - \beta \int_\Omega u^2v^2 \le \int_\Omega |v^+|^4.$$

So as in the proof of (i) it follows that $T_{u,v}$ is negative definite, and therefore $\lambda_1^k, \lambda_2^k \to 0$ by (4.8). Since $F_1'(u_k,v_k)$ and $F_2'(u_k,v_k)$ remain bounded in $\mathcal{H}^*$ as $k \to \infty$, we now infer from (4.7) that $E'(u_k,v_k) \to 0$. It is then standard to deduce that (u,v) is a weak solution of

$$\begin{cases} -\Delta u + u = (u^+)^3 + \beta v^2 u & \text{in } \Omega, \\ -\Delta v + v = (v^+)^3 + \beta u^2 v & \text{in } \Omega, \\ \qquad u = v = 0 & \text{on } \partial\Omega. \end{cases} \tag{4.9}$$

Multiplying the first equation by u and integrating by parts we get

$$\|u\|^2 = |u^+|_4^4 + \beta \textstyle\int_\Omega v^2u^2 = \lim\limits_{k\to\infty} (|u_k^+|_4^4 + \beta \int_\Omega v_k^2 u_k^2) = \lim\limits_{k\to\infty} \|u_k\|^2$$

since $(u_k,v_k) \in \mathcal{M}$. This implies that $u_k \to u$ strongly in $H_0^1(\Omega)$. Similarly we find that $v_k \to v$ strongly in $H_0^1(\Omega)$, so that $(u_k,v_k) \to (u,v)$ strongly in $\mathcal{H}$. □

To prove the existence of multiple critical points of E, we consider the sets $\mathcal{M}^c := \{(u,v) \in \mathcal{M} \,:\, E(u,v) \le c\}$ and

$$\begin{aligned} K_c &:= \{(u,v) \in \mathcal{M} \,:\, E(u,v) = c,\ E'(u,v) = 0\} \\ &= \{(u,v) \in \mathcal{M} \,:\, E_{\mathcal{M}}(u,v) = c,\ E'_{\mathcal{M}}(u,v) = 0\}. \end{aligned}$$

for every $c \in \mathbb{R}$, and we note that the functional E and $\mathcal{M}, \mathcal{M}^c$ and K_c are invariant with respect to the involution

$$\sigma : \mathcal{H} \to \mathcal{H}, \qquad (u,v) \mapsto \sigma(u,v) = (v,u).$$

We put

$$c(\beta) := \inf\{E(u,v) \,:\, (u,v) \in \mathcal{M} \text{ is a fixed point of } \sigma\}.$$

Note that, in contrast to the notation introduced up to now, we stress the dependence of $c(\beta)$ on the parameter β in view of the following simple but crucial fact.

Lemma 4.3. *$c(\beta) = \infty$ for $\beta \le -1$, and* $\lim\limits_{\substack{\beta \to -1 \\ \beta > -1}} c(\beta) = \infty$.

Proof. It follows immediately from the definition of $\mathcal{M}$ that σ has no fixed fixed points in $\mathcal{M}$ for $\beta \le -1$, hence $c(\beta) = \infty$. If $-1 < \beta < 0$ and $(u,u) \in M$ for some $u \in H_0^1(\Omega)$, then

$$\|u\|^2 = |u^+|_4^4 + \beta |u|_4^4 \le (1+\beta)|u|_4^4 \le C(1+\beta)\|u\|^4,$$

where the constant C is given independently of β by the Sobolev embedding $H_0^1(\Omega) \hookrightarrow L^4(\Omega)$ as in (4.2). We conclude that $\|u\|^2 \ge \frac{1}{C(1+\beta)}$ and therefore $E(u,u) \ge \frac{1}{2C(1+\beta)}$ by Lemma 4.2(iii). Since $\frac{1}{2C(1+\beta)} \to \infty$ as $\beta \to -1$, the claim follows. □

Using the Palais-Smale condition for the functional $E_{\mathcal{M}} : \mathcal{M} \to \mathbb{R}$ and the fact that $\mathcal{M}$ is a $C^{1,1}$-manifold, we obtain the following equivariant deformation lemma. Since the proof is standard, we omit it.

Proposition 4.1. *Let $c \in \mathbb{R}$, and let $N \subset \mathcal{M}$ be a relative open σ-invariant neighborhood of K_c. Then there exists $\varepsilon > 0$ and a C^1-deformation $\eta : [0,1] \times \mathcal{M}^{c+\varepsilon} \setminus N \to \mathcal{M}^{c+\varepsilon}$ such that, for all $(u,v) \in \mathcal{M}^{c+\varepsilon} \setminus N$ and $s \in [0,1]$,*

$$\eta(0,(u,v)) = (u,v), \quad \eta(1,(u,v)) \in \mathcal{M}^{c-\varepsilon} \quad \text{and} \quad \sigma[\eta(s,(u,v))] = \eta(s,\sigma(u,v))).$$

For any closed σ-invariant subset $A \subset M$ we now define the genus $\gamma(A)$ as the smallest $n \in \mathbb{N} \cup \{0\}$ such that there exists a continuous map $h : A \to \mathbb{R}^n \setminus \{0\}$ with $h(\sigma(u,v)) = -h(u,v)$ for all $(u,v) \in A$. As usual, we set $\gamma(A) = \infty$ if no such map h exists. In particular, $\gamma(A) = \infty$ if A contains a fixed point of σ. By definition we have $\gamma(\varnothing) = 0$. We list some properties of γ.

Lemma 4.4. *Let $A, B \subset \mathcal{M}$ be closed and σ-invariant.*

(i) *If $A \subset B$, then $\gamma(A) \le \gamma(B)$.*
(ii) $\gamma(A \cup B) \le \gamma(A) + \gamma(B)$.
(iii) *If $h : A \to \mathcal{M}$ is continuous and σ-equivariant, then $\gamma(A) \le \gamma(\overline{h(A)})$.*

If A does not contain fixed points of σ, then:

(iv) *if $\gamma(A) > 1$, then A is an infinite set;*

(v) *if A is compact, then $\gamma(A) < \infty$, and there exists a relatively open σ-invariant neighborhood N of A in $\mathcal{M}$ such that $\gamma(A) = \gamma(\overline{N})$.*

Finally,

(vi) *if S is the boundary of a bounded symmetric neighborhood of zero in a k-dimensional normed vector space und $\psi : S \to \mathcal{M}$ is a continuous map satisfying $\psi(-u) = \sigma(\psi(u))$, then $\gamma(\psi(S)) \geq k$.*

Note that in (vii) the set $\psi(S)$ is closed since S is compact.

Proof. Properties (i) and (iii) are immediate consequences of the definition of γ. Properties (ii) and (v) can be proved precisely as in the case of the Krasnoselski genus, see e.g. [25, Proposition 5.4].
To prove (iv), we note that a finite σ-invariant subset $A \subset \mathcal{M}$ without fixed points can be written as

$$A = \{(u_1, v_1), \dots, (u_n, v_n), \sigma(u_1, v_1), \dots, \sigma(u_n, v_n)\},$$

where the $(u_i, v_i), \sigma(u_i, v_i) \in \mathcal{M}$, $i = 1, \dots, n$ are pairwise different. Therefore a continuous map $h : A \to \mathbb{R} \setminus \{0\}$ is defined by

$$h(u_i, v_i) = -1 \quad \text{and} \quad h(\sigma(u_i, v_i)) = 1 \qquad \text{for } i = 1, \dots, n,$$

showing that $\gamma(A) = 1$.
Property (vi) is proved by contradiction, assuming that there exists a continuous map $h : \psi(S) \to \mathbb{R}^{k-1} \setminus \{0\}$ with $h(\sigma(u, v)) = -h(u, v)$. Then $h \circ \psi : S \to \mathbb{R}^{k-1} \setminus \{0\}$ is an odd and continuous map, which contradicts the Borsuk-Ulam Theorem (see e.g. [26, Theorem D.17.]). □

Proposition 4.2. *For every $c < c(\beta)$ we have $\gamma(K_c) < \infty$, and there exists $\varepsilon > 0$ such that*

$$\gamma(\mathcal{M}^{c+\varepsilon}) \leq \gamma(\mathcal{M}^{c-\varepsilon}) + \gamma(K_c) \tag{4.10}$$

Proof. Since $E_{\mathcal{M}}$ satisfies the Palais Smale condition, the set K_c is compact, and it does not contain fixed points of σ by definition of $c(\beta)$. Hence $\gamma(K_c) < \infty$ by Lemma 4.4(v), and there exists a relative open σ-invariant neighborhood $N \subset \mathcal{M}$ of K_c in $\mathcal{M}$ with $\gamma(\overline{N}) = \gamma(K_c)$. Let $\varepsilon > 0$ and $\eta : [0, 1] \times \mathcal{M}^{c+\varepsilon} \setminus N \to \mathcal{M}^{c+\varepsilon}$ be chosen as in the statement of Proposition 4.1. Put $\eta_1 := \eta(1, \cdot) : \mathcal{M}^{c+\varepsilon} \setminus N \to \mathcal{M}^{c-\varepsilon}$. Since η_1 is σ-equivariant, Lemma 4.4(iii) implies that $\gamma(\mathcal{M}^{c+\varepsilon} \setminus N) \leq \gamma(\mathcal{M}^{c-\varepsilon})$ and therefore

$$\gamma(\mathcal{M}^{c+\varepsilon}) \leq \gamma(\mathcal{M}^{c+\varepsilon} \setminus N) + \gamma(\overline{N}) \leq \gamma(\mathcal{M}^{c-\varepsilon}) + \gamma(K_c),$$

as claimed. □

The nondecreasing sequence of Lyusternik-Schnirelman type levels associated to the genus γ is defined by

$$c_k := \inf\{c \in \mathbb{R} \,:\, \gamma(\mathcal{M}^c) \geq k\}, \qquad k \in \mathbb{N}.$$

We note the following.

Proposition 4.3. *(i) For every k, $c_k < \infty$ is bounded independently of $\beta < 0$.*
(ii) $c_k \to \bar{c}$ as $k \to \infty$, where $c(\beta) \le \bar{c} \le \infty$.
(iii) If $c := c_k = c_{k+1} = \cdots = c_l < c(\beta)$ for some $l \ge k$, then $\gamma(K_c) \ge l - k + 1$.
(iv) If $c_k < c(\beta)$, then $K_{c_k} \neq \varnothing$, and $\mathcal{M}^{c_k}$ contains at least k pairs $(u,v),(v,u)$ of critical points of E.

Proof. (i) Let $W \subset H_0^1(\Omega)$ be a k-dimensional subspace consisting of functions $u \in H_0^1(\Omega)$ with $\int_\Omega u = 0$, and let $S := \{u \in W \,:\, \|u\| = 1\}$. Then $u^+ \neq 0$ and $u^- \neq 0$ for every $u \in S$. We therefore may consider the map

$$\psi : S \to \mathcal{M}, \qquad \psi(u) = \left(\left(\frac{\|u^+\|^2}{|u^+|_4^4}\right)^{1/2} u^+, \left(\frac{\|u^-\|^2}{|u^-|_4^4}\right)^{1/2} u^-\right).$$

Clearly ψ is continuous, and $\psi(-u) = \sigma(\psi(u))$ for every $u \in S$. Hence $\gamma(\psi(S)) \ge k$ by Lemma 4.4(vi) and therefore $c_k \le \sup_{u \in S} E(\psi(u)) < \infty$. By definition of ψ and Lemma 4.2(iii), the value of $\sup_{u\in S} E(\psi(u))$ does not depend on β. Hence the claim follows.
(ii) Suppose by contradiction that $c_k \to \bar{c} < c(\beta)$ as $k \to \infty$. Choosing $\varepsilon > 0$ as in Proposition 4.2 for $c = \bar{c}$, we find that $\bar{c} - \varepsilon < c_k$ for k large, hence $\gamma(\mathcal{M}^{\bar{c}-\varepsilon})$ is finite. By Proposition 4.2 we therefore conclude that $\gamma(\mathcal{M}^{\bar{c}+\varepsilon}) \le \gamma(\mathcal{M}^{\bar{c}-\varepsilon}) + \gamma(K_{\bar{c}}) < \infty$, which contradicts the fact that $\bar{c} \ge c_k$ for all k.
(iii) By assumption and the definition of the Lyusternik-Schnirelman values we have $\gamma(\mathcal{M}^{c-\varepsilon}) \le k - 1$ and $\gamma(\mathcal{M}^{c+\varepsilon}) \ge l$ for every $\varepsilon > 0$, hence $\gamma(K_c) \ge l - k + 1$ by Proposition 4.2.
(iv) If $c_k < c(\beta)$, then (iii) implies that $\gamma(K_{c_k}) \ge 1$, hence K_{c_k} is a nonempty σ-invariant set. If $c_1 < c_2 < \cdots < c_k$, we conclude that $\mathcal{M}^{c_k}$ contains at least k pairs of critical points of E. On the other hand, if $c_i = c_j$ for some $i < k$ and $j > i$, then $\gamma(\mathcal{K}_{c_i}) > 1$ by (iii), and therefore $\mathcal{K}_{c_i}$ is an infinite set by Lemma 4.4(iv). Hence in this case $\mathcal{M}^{c_k}$ contains infinitely many pairs of critical points of E. ☐

We now complete the

Proof of Theorem 1.2. (a) Choosing $(u_k, v_k) \in K_{c_k}$ for every k, we get a sequence of nontrivial critical points of E with $E(u_k, v_k) \to \infty$, hence $\|u_k\|^2 + \|v_k\|^2 \to \infty$ by Lemma 4.2(iii). Since

$$|\Omega|^4(|u_k|_\infty^4 + |v_k|_\infty^4) \ge |u_k|_4^4 + |v_k|_4^4 \ge \|u_k\|^2 + \|v_k\|^2,$$

we conclude that $|u_k|_\infty + |v_k|_\infty \to \infty$ as $k \to \infty$.
(b) Let k be a given positive integer. By Lemma 4.3 and Proposition 4.3(i), there exists $\beta_k > -1$ such that for $\beta < \beta_k$ we have $c_k < c(\beta)$. Hence E has at least k pairs of nontrivial critical points by Proposition 4.3(iv), and therefore (1.8) admits at least k pairs (u,v), (v,u) of solutions. ☐

References

[1] A. Ambrosetti and E. Colorado, Bound and ground states of coupled nonlinear Schrodinger equations, *C.R. Math. Acad. Sci. Paris* 342 (2006), 453-458.

[2] H. Berestycki, L.A. Caffarelli, L. Nirenberg, Inequalities for second-order elliptic equations with applications to unbounded domains. I *Duke Math. J.* 81 (1996), 467–494.

[3] T. Bartsch, Z.-Q. Wang and J.C. Wei, Bound states for a coupled Schrödinger system, preprint.

[4] D.G. de Figueiredo, B. Sirakov, Liouville type theorems, monotonicity results and a priori bounds for positive solutions of elliptic systems, *Math. Ann.* 333 (2005), 231-260.

[5] D.G. de Figueiredo, J.F. Yang, A priori bounds for positive solutions of a non-variational elliptic system, *Comm. Partial Differential Equations* 26 (2001), 2305-2321.

[6] B. D. Esry, C. H. Greene, J. P. Burke Jr., J. L. Bohn, Hartree-Fock theory for double condensates, *Phys. Rev. Lett.* 78 (1997), 3594-3597.

[7] B. Gidas. Symmetry properties and isolated singularities of positive solutions of nonlinear elliptic equations. In *Nonlinear partial differential equations in engineering and applied science (Proc. Conf., Univ. Rhode Island, Kingston, R.I., 1979)*, volume 54 of *Lecture Notes in Pure and Appl. Math.*, pages 255–273. Dekker, New York, 1980.

[8] B. Gidas, J. Spruck, A priori bounds for positive solutions of nonlinear elliptic equations, *Comm. Part. Diff. Eq.* 6 (1981), 883–901.

[9] D. Gilbarg, N.S. Trudinger, Elliptic partial differential equations of second order. Springer-Verlag, Berlin, Reprint of the 1998 ed., 2001.

[10] S. Gupta, Z. Hadzibabic, M.W. Zwierlein, C.A. Stan, K. Dieckmann, C.H. Schunck, E.G.M. van Kempen, B.J. Verhaar, W. Ketterle, Radio-Frequency Spectroscopy of Ultracold Fermions, *Science* 300 (2003), 1723-1726.

[11] D.S. Hall, R. Matthews, J.R. Ensher, C.E. Wiemann, E.A. Cornell, Dynamics of component separation in a binary mixture of Bose-Einstein condensates, *Phys. Rev. Lett.* 81 (1998), 1539-1542.

[12] C.J. Myatt, E.A. Burt, R.W. Ghrist, E.A. Cornell, C.E. Wieman Production of two overlapping Bose-Einstein condensates by sympathetic cooling, *Phys. Rev. Lett.* 78 (1997), 586-589.

[13] G. G. Laptev. Absence of global positive solutions of systems of semilinear elliptic inequalities in cones. *Izv. Ross. Akad. Nauk Ser. Mat.* 64 (2000), 107–124.

[14] G. G. Laptev, On the nonexistence of solutions of elliptic differential inequalities in conic domains. *Mat. Zametki* 71 (2002), 855–866.

[15] T.-C. Lin and J.-C. Wei, Ground state of N coupled Nonlinear Schrödinger Equations in R^n, $n \leq 3$, *Communications in Mathematical Physics* 255 (2005), 629-653.

[16] T.-C. Lin and J.-C. Wei, Spikes in two coupled nonlinear Schrodinger equations, *Ann. de l'Institut H. Poincaré Analyse Nonlinéare* **22** (4) (2005), 403-439.

[17] T. C. Lin and J. Wei, Spikes in two-component systems of Nonlinear Schrödinger equations with trapping potentials, *J. Diff. Eqns.* 2006, to appear.

[18] E. Mitidieri, S. Pohozaev, A priori estimates and the absence of solutions of nonlinear partial differential equations and inequalities, *Tr. Mat. Inst. Steklova* 234, 1-384 (2001)

[19] L.A. Maia, E. Montefusco and B. Pellacci, Positive solutions for a weakly coupled nonlinear Schrödinger system, preprint.

[20] P. Poláčik, P. Quittner, and P. Souplet. Singularity and decay estimates in superlinear problems via Liouville-type theorems. preprint.

[21] W. Reichel, H. Zou, Non-existence results for semilinear cooperative elliptic systems via moving spheres, *J. Differential Equations* 161 (2000), 219–243.

[22] B. Sirakov, Least energy solitary waves for a system of nonlinear Schrodinger equations, *Comm. Math. Physics*, to appear.

[23] M.A.S. Souto, A priori estimates and and existence of positive solutions of nonlinear cooperative elliptic systems. *Diff. Int. Eq.* 8 (1995), 1245-1258.

[24] E. Timmermans, Phase separation of Bose-Einstein condensates, *Phys. Rev. Lett.* 81 (1998), 5718-5721.

[25] M. Struwe. *Variational methods*, volume 34 of *Ergebnisse der Mathematik und ihrer Grenzgebiete (3) [Results in Mathematics and Related Areas (3)]*. Springer-Verlag, Berlin, second edition, 1996. Applications to nonlinear partial differential equations and Hamiltonian systems.
[26] M. Willem. *Minimax theorems.* Progress in Nonlinear Differential Equations and their Applications, 24. Birkhäuser Boston Inc., Boston, MA, 1996.
[27] J.C. Wei, T. Weth, Nonradial symmetric bound states for a system of coupled Schrödinger equations, *Rend. Lincei Mat. Appl.*, to appear.
[28] J.C. Wei, T. Weth, Radial solutions and phase separation in a system of two coupled Schrödinger equations, *Arch. Rational Mech. Anal.*, to appear.
[29] H. Zou, A priori estimates for a semilinear elliptic system without variational structure and their application, *Math. Ann.* 323 (2002), 713–735.

School of Mathematics and Statistics, University of Sydney, Australia
E-mail address: `normd@maths.usyd.edu.au`

Department of Mathematics, Chinese University of Hong Kong, Shatin, Hong Kong
E-mail address: `wei@math.cuhk.edu.hk`

Mathematische Institut, Universität Giessen, Arndtstr.2, 35392 Giessen, Germany
E-mail address: `tobias.weth@math.uni-giessen.de`