LEAST ENERGY NODAL SOLUTION OF A SINGULAR PERTURBED PROBLEM WITH JUMPING NONLINEARITY

E. N. DANCER, SANJIBAN SANTRA, JUNCHENG WEI

ABSTRACT. In this paper we study the asymptotic behavior of the least energy nodal solution of a problem with a jumping nonlinearity.

1. INTRODUCTION

There has been a considerable interest to understand the asymptotic behavior of positive solutions of the elliptic problem

$$\begin{cases} \varepsilon^2 \Delta u - u + f(u) = 0 & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases} \tag{1.1}$$

where $\varepsilon > 0$ is a parameter, f is a superlinear function, Ω is a smooth bounded domain in $\mathbb{R}^N$. Let $F(u) = \int_0^u f(t)dt$. In this paper, we consider the problem

$$\begin{cases} \varepsilon^2 \Delta u - \lambda_1 u^+ + \lambda_2 u^- + f(u) = 0 & \text{in } \Omega \\ u^\pm \neq 0 & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases} \tag{1.2}$$

where $\lambda_1 > 0, \lambda_2 > 0$ with $\lambda_1 \neq \lambda_2$, and $u^\pm = \max\{\pm u, 0\}$. Let $f : \mathbb{R} \to \mathbb{R}$ be a continuously differentiable function satisfying:
(f1) $f(t) = o(t)$ as $t \to 0$;
(f2) $f(t) = O(|t|^p)$ as $t \to +\infty$ for some $p \in (1, \frac{N+2}{N-2})$ if $N \geq 3$ and $p > 1$ if $N = 1, 2$;
(f3) there exists a constant $\theta > 2$ such that $\theta F(t) \leq t f(t)$ where

$$F(t) = \int_0^t f(s)ds;$$

(f4) the function $t \mapsto \frac{f(t)}{|t|}$ is strictly increasing on $\mathbb{R} \setminus \{0\}$.
Problem (1.1) arises in various applications, such as chemotaxis, population genetic, chemical reactor theory. Problem (1.2) arises in the study of population dynamics with jumping nonlinearity [6]. It can also be considered as the limiting problem of the following elliptic system

$$\begin{cases} \varepsilon^2 \Delta u - \lambda_1 u + \mu_1 u^3 + \beta u v^2 = 0 \text{ in } \Omega \\ \varepsilon^2 \Delta v - \lambda_2 v + \mu_2 v^3 + \beta v u^2 = 0 \text{ in } \Omega \\ u, v > 0 \text{ in } \Omega \\ u = v = 0 \text{ on } \partial\Omega \end{cases} \tag{1.3}$$

The system (1.3) arises in the Bose-Einstein condenstates and nonlinear optics. An important phenomena of (1.3) is the so-called *phase separation*. As $\beta \to -\infty$, the components u, v separates and the difference function $u - v$ approaches a solution

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of (1.2) with $f(u) = \mu_1 u_+^3 - \mu_2 u_-^3$. This has been proved for least energy solutions of (**??**) in [2]-[4] and for general bounded solutions in [7] (the case of $N = 2$) and [5] (the case of $N \geq 2$). We refer to [1], [2]-[5], [7], [11], [16], [17] and the references therein.

Existence and concentration of positive solution of this type of problems were extensively studied by Ni-Takagi [13], [14], Ni-Wei [15], del Pino- Felmer [8].

Define

$$I_{\lambda_1}(W) = \frac{1}{2}\int_{\mathbb{R}^N} |\nabla W|^2 + \frac{\lambda_1}{2}\int_{\mathbb{R}^N} W^2 - \int_{\mathbb{R}^N} F(W)$$

and

$$I_{\lambda_2}(W) = \frac{1}{2}\int_{\mathbb{R}^N} |\nabla W|^2 + \frac{\lambda_2}{2}\int_{\mathbb{R}^N} W^2 - \int_{\mathbb{R}^N} F(W).$$

Let W_{λ_1} be a least energy positive solution of

$$\begin{cases} -\Delta u + \lambda_1 u = f(u) & \text{in } \mathbb{R}^N \\ u > 0 & \text{in } \mathbb{R}^N \\ u \in H^1(\mathbb{R}^N) \end{cases} \tag{1.4}$$

and W_{λ_2} be a least positive solution of

$$\begin{cases} -\Delta u + \lambda_2 u = f(u) & \text{in } \mathbb{R}^N \\ u > 0 & \text{in } \mathbb{R}^N \\ u \in H^1(\mathbb{R}^N). \end{cases} \tag{1.5}$$

By Gidas, Ni and Nirenberg [10], it is well known that W_{λ_i} is radially decreasing and decays as

$$W_{\lambda_i}(|x|) \sim e^{-\sqrt{\lambda_i}|x|}|x|^{\frac{1-N}{2}} \text{ as } |x| \to +\infty$$

for $i = 1, 2$. Throughout the course of the paper we will call W_{λ_i}; an entire solution or a ground state.

In this paper, we prove the existence of a least energy nodal solution and show that for ε sufficiently small, the solution has a exactly one positive spike and one negative spike and the spikes concentrate at two distinct points of Ω, in other words they repel each other. Define

$$\varphi(x,y) = \min\left\{\sqrt{\lambda_1}d(x,\partial\Omega), \sqrt{\lambda_2}d(y,\partial\Omega)), \frac{1}{2}\frac{\sqrt{\lambda_1}\sqrt{\lambda_2}}{\sqrt{\lambda_1}+\sqrt{\lambda_2}}|x-y|\right\}.$$

Theorem 1.1. *There exists $\varepsilon_0 > 0$ such that for every $0 < \varepsilon < \varepsilon_0$, the least energy nodal solution $u_\varepsilon \in H_0^1(\Omega)$ of (1.2) having exactly one positive local maximum (hence a global maximum) point P_ε^1 and one negative local minimum (hence a global minimum) point P_ε^2 and*

$$\lim_{\varepsilon\to 0}\varphi(P_\varepsilon^1, P_\varepsilon^2) = \max_{(x,y)\in\overline{\Omega}\times\overline{\Omega}}\varphi(x,y),$$

with $u_\varepsilon(P_\varepsilon^i) \to (-1)^{i-1}W_{\lambda_i}(0)$ and $u_\varepsilon \to 0$ in $C_{loc}^1(\Omega \setminus \{P_\varepsilon^1, P_\varepsilon^2\})$.

Note that for sufficiently small $\varepsilon > 0$, the least energy positive solution to the problem (1.1) has a unique maxima P_ε; u_ε decays exponentially away from P_ε and $d(P_\varepsilon, \partial\Omega) \to \max_{P\in\Omega} d(P,\partial\Omega)$ as $\varepsilon \to 0$, which implies that the solution concentrates at an interior point furthest from the boundary of Ω. This was studied by Ni–Wei [12]. For the least energy nodal solution, the problem was studied by Noussair–Wei [15] when $\lambda_1 = \lambda_2 = 1$ and $f(u) = u^p$. They obtain the same results as in Theorem 1.1. In addition, they prove that $u_\varepsilon(x) = W(\frac{x-P_\varepsilon^1}{\varepsilon}) - W(\frac{x-P_\varepsilon^2}{\varepsilon}) + v_\varepsilon$, where $\|v_\varepsilon\|_{L^\infty(\Omega)} \to 0$ as $\varepsilon \to 0$ and W is the unique solution of the limiting problem. The study of asymptotic behavior involves the uniqueness and non-degeneracy of solution of the limiting problem. Then using the expansion, an asymptotic expansion of the energy is obtained. This approach does not work here since u_+ and u_- are not differentiable. Neither we have uniqueness nor nondegeneracy of the ground state. There is another approach by del Pino and Felmer [8] where they used variational characterizations of positive solutions and symmetrization technique. However their approach works well for positive solutions and does not work for sign-changing solutions. We shall modify the approach of del Pino and Felmer. The problem here is more complicated since the solution is sign-changing and we have to estimate the interaction of the positive and negative components.

2. Preliminaries

Without loss of generality, we consider $0 < \lambda_1 < \lambda_2$. The associated functional to the problem (1.2) is

$$E_\varepsilon(u) = \int_\Omega \left(\frac{\varepsilon^2}{2}|\nabla u|^2 + \frac{\lambda_1}{2}(u^+)^2 + \frac{\lambda_2}{2}(u^-)^2 - F(u)\right) dx.$$

Note that from (f_2), $E_\varepsilon \in C^{2-0}(H_0^1(\Omega), \mathbb{R})$. Moreover, if $u_\varepsilon \in H_0^1(\Omega)$ is a critical point of E_ε, then $u_\varepsilon \in C^2(\Omega) \cap C(\overline{\Omega})$ and hence u_ε is a classical solution of (1.2). Note that $E_\varepsilon(u) = E_{\varepsilon,\lambda_1}(u) + E_{\varepsilon,\lambda_2}(u)$ where

$$E_{\varepsilon,\lambda_1}(u) = \int_\Omega \left(\frac{\varepsilon^2}{2}|\nabla u^+|^2 + \frac{\lambda_1}{2}(u^+)^2 - F(u^+)\right) dx,$$

$$E_{\varepsilon,\lambda_2}(u) = \int_\Omega \left(\frac{\varepsilon^2}{2}|\nabla u^-|^2 + \frac{\lambda_2}{2}(u^-)^2 - F(u^-)\right) dx.$$

Define the Nehari manifold as

$$\mathcal{N}_\varepsilon = \Big\{u \in H_0^1(\Omega) : u^\pm \not\equiv 0, \varepsilon^2 \int_\Omega |\nabla u^+|^2 + \lambda_1 \int_\Omega (u^+)^2 = \int_\Omega f(u^+)u^+;$$

$$\varepsilon^2 \int_\Omega |\nabla u^-|^2 + \lambda_2 \int_\Omega (u^-)^2 = \int_\Omega f(u^-)u^-\Big\}. \tag{2.1}$$

Define the positive and negative Nehari manifold as

$$\mathcal{N}_\varepsilon^\pm = \{u \in \mathcal{N}_\varepsilon : \pm u \geq 0\} \tag{2.2}$$

respectively. Note that any u belonging to $\mathcal{N}_\varepsilon$ is sign-changing. Moreover, all the sign-changing solutions of (1.2) is contained in $\mathcal{N}_\varepsilon$. Also note that $\mathcal{N}_\varepsilon^+ \cap \mathcal{N}_\varepsilon^- = \emptyset$ and $\mathcal{N}_\varepsilon = \mathcal{N}_\varepsilon^+ + \mathcal{N}_\varepsilon^-$. Let

$$c_\varepsilon = \inf_{u \in \mathcal{N}_\varepsilon} E_\varepsilon(u) \tag{2.3}$$

We will show that there exists $u_\varepsilon \in \mathcal{N}_\varepsilon$ such that $c_\varepsilon = E_\varepsilon(u_\varepsilon)$, that is u_ε is a least energy sign-changing solution. We state some elementary lemma,

Lemma 2.1. *For all $\varepsilon > 0$, $\mathcal{N}_\varepsilon^+$ and $\mathcal{N}_\varepsilon^-$ are closed subsets of $H_0^1(\Omega)$.*

$$0 < c_\varepsilon^+ = \inf_{u \in \mathcal{N}_\varepsilon^+} E_{\varepsilon,\lambda_1}(u) = \inf_{u \in H_0^1(\Omega), u \neq 0} \max_{t \geq 0} E_{\varepsilon,\lambda_1}(tu)$$

and

$$0 < c_\varepsilon^- = \inf_{u\in\mathcal{N}_\varepsilon^-} E_{\varepsilon,\lambda_2}(u) = \inf_{u\in H_0^1(\Omega),u\neq 0}\max_{t\geq 0} E_{\varepsilon,\lambda_2}(tu).$$

Moreover, every minimizer of E_ε *on* $\mathcal{N}_\varepsilon^\pm$ *is positive.*

Proof. This follows trivially by using (f_4) and Sobolev embedding theorem. See [12]. □

Remark 2.2. Similarly if we define

$$\mathcal{H}_\varepsilon = \{u \in H_0^1(\Omega) : \langle E_\varepsilon'(u), u\rangle = 0\}.$$

Then from (f_4) we obtain that for every $u \in H_0^1(\Omega)\setminus\{0\}$, there exists a unique $1 \neq t_1 > 0$ such that $t_1 u \in \mathcal{H}_\varepsilon$ and

$$E_\varepsilon(t_1 u) = \max_{t>0} E_\varepsilon(tu).$$

And using the fact that *supp* $u^+ \cap$ *supp* $u^- = \emptyset$, we obtain that for any $u \in H_0^1(\Omega)\setminus\{0\}$, there exist $0 < t \neq 1$ and $0 < s \neq 1$ such that $tu^+ - su^- \in \mathcal{N}_\varepsilon$ are uniquely defined.

Lemma 2.3. *$\mathcal{N}_\varepsilon$ is a closed subset of $H_0^1(\Omega)$. There exists some $u_\varepsilon \in \mathcal{N}_\varepsilon$ such that c_ε is achieved. Moreover, u_ε is a weak solution and hence a classical nodal solution of (1.2).*

Proof. Let $\varepsilon > 0$ be fixed. Since $c_\varepsilon = \inf_{u\in\mathcal{N}_\varepsilon} E_\varepsilon(u)$, there exists a minimizing sequence $u_{\varepsilon,n} \in \mathcal{N}_\varepsilon$ such that $E_\varepsilon(u_{\varepsilon,n}) \to c_\varepsilon$ as $n \to +\infty$. Note that by $(f3)$, E_ε is coercive on $\mathcal{N}_\varepsilon$, as

$$E_\varepsilon(u_{\varepsilon,n}) \geq (\frac{1}{2} - \frac{1}{\theta})\int_\Omega \left\{\varepsilon^2|\nabla u_{\varepsilon,n}|^2 + \lambda_1(u_{\varepsilon,n}^+)^2 + \lambda_2(u_{\varepsilon,n}^-)^2\right\}$$

and hence $u_{\varepsilon,n}$ is bounded from below on $\mathcal{N}_\varepsilon$. Again by Ekeland's variational principle we have for all $n \in \mathbb{N}$,

$$E_\varepsilon(u_{\varepsilon,n}) \leq c_\varepsilon + \frac{1}{n}$$

$$E_\varepsilon(v) \geq E_\varepsilon(u_{\varepsilon,n}) - \frac{1}{n}\|v_\varepsilon - u_{\varepsilon,n}\|, \ v_\varepsilon \in \mathcal{N}_\varepsilon$$

as before $u_{\varepsilon,n}$ is bounded in $H_0^1(\Omega)$. Let us define $h_{n,\lambda_i}^\pm : \mathbb{R}^3 \to \mathbb{R}$ by

$$\begin{aligned} h_{n,\lambda_1}^+(t,s,l) &= \int_\Omega \Big(\varepsilon^2|\nabla(u_{\varepsilon,n} + t\varphi + su_n^+ + lu_n^-)^+|^2 + \lambda_1((u_{\varepsilon,n} + t\varphi + su_{\varepsilon,n}^+ + lu_{\varepsilon,n}^-)^+)^2 \\ &- f((u_{\varepsilon,n} + t\varphi + su_{\varepsilon,n}^+ + lu_{\varepsilon,n}^-)^+)(u_{\varepsilon,n} + t\varphi + su_{\varepsilon,n}^+ + lu_{\varepsilon,n}^-)^+\Big)dx \end{aligned}$$

$$\begin{aligned} h_{n,\lambda_2}^-(t,s,l) &= \int_\Omega \Big(\varepsilon^2|\nabla(u_{\varepsilon,n} + t\varphi + su_n^+ + lu_n^-)^-|^2 + \lambda_2((u_{\varepsilon,n} + t\varphi + su_{\varepsilon,n}^+ + lu_{\varepsilon,n}^-)^-)^2 \\ &- f((u_{\varepsilon,n} + t\varphi + su_{\varepsilon,n}^+ + lu_{\varepsilon,n}^-)^-)(u_{\varepsilon,n} + t\varphi + su_{\varepsilon,n}^+ + lu_{\varepsilon,n}^-)^-\Big)dx \end{aligned}$$

Note that $h_{n,\lambda_i}^\pm \in C^1$ and $h_{n,\lambda_i}^\pm(0,0,0) = 0$, $\frac{\partial h_{n,\lambda_1}^+}{\partial l}(0,0,0) = 0$, $\frac{\partial h_{n,\lambda_2}^-}{\partial s}(0,0,0) = 0$. As $u_{\varepsilon,n} \in \mathcal{N}_\varepsilon$ and by (f_4) we have

$$\frac{\partial h_{n,\lambda_1}^+}{\partial s}(0,0,0) = \int_\Omega u_{\varepsilon,n}^+\Big(f(u_{\varepsilon,n}^+) - f'(u_{\varepsilon,n}^+)u_{\varepsilon,n}^+\Big)dx < 0,$$

$$\frac{\partial h^-_{n,\lambda_2}}{\partial l}(0,0,0) \quad = \quad \int_\Omega u^-_{\varepsilon,n}\Big(f(u^-_{\varepsilon,n}) - f'(u^-_n)u^-_{\varepsilon,n}\Big)dx < 0.$$

Hence by implicit function theorem, there exist C^1 functions $s_n(t), l_n(t)$ defined in a neighborhood of 0 say $(-\delta_n, +\delta_n)$ such that $s_n(0) = 0$, $l_n(0) = 0$ and

$$h^+_{n,\lambda_1}(t, s_n(t), l_n(t)) = 0, h^-_{n,\lambda_2}(t, s_n(t), l_n(t)) = 0 \ \forall t \in (-\delta_n, +\delta_n)$$

which implies that for $t \in (-\delta_n, +\delta_n)$,

$$v_n = u_{\varepsilon,n} + t\varphi + s_n(t)u^+_{\varepsilon,n} + l_n(t)u^-_{\varepsilon,n} \in \mathcal{N}_\varepsilon \tag{2.4}$$

Moreover, we claim that $s'_n(0)$ and $l'_n(0)$ are bounded. In fact we have

$$\begin{aligned} s'_n(0) &= -\frac{\frac{\partial h^+_{n,\lambda_1}}{\partial t}(0,0,0)}{\frac{\partial h^+_{n,\lambda_1}}{\partial l}(0,0,0)} \\ &= -\frac{\int_\Omega (2\nabla u^+_{\varepsilon,n}\nabla\varphi + 2\lambda_1 u^+_{\varepsilon,n}\varphi - f'(u^+_{\varepsilon,n})u^+_{\varepsilon,n}\varphi - f(u^+_{\varepsilon,n})\varphi)}{\int_\Omega u^+_{\varepsilon,n}(f(u^+_{\varepsilon,n}) - f'(u^+_{\varepsilon,n})u^+_{\varepsilon,n})}. \end{aligned}$$

Using the fact that $u_{\varepsilon,n}$ is bounded in $H^1_0(\Omega)$, the numerator is bounded. We have from Fatou's Lemma

$$\liminf_{n\to+\infty}\int_\Omega u^+_{\varepsilon,n}(f'(u^+_{\varepsilon,n})u^+_{\varepsilon,n} - f(u^+_{\varepsilon,n})) \geq \int_\Omega u^+_\varepsilon(f'(u^+_\varepsilon)u^+_\varepsilon - f(u^+_\varepsilon)) \geq 0.$$

Now we claim that in fact the denominator is bounded away from zero. Also note that by Sobolev inequality we have

$$\left(\varepsilon^2\int_\Omega |\nabla u^+_{\varepsilon,n}|^2 + \lambda_1\int_\Omega (u^+_{\varepsilon,n})^2\right) \geq C. \tag{2.5}$$

Hence we have $\int_\Omega u^+_{\varepsilon,n}(f'(u^+_{\varepsilon,n})u^+_{\varepsilon,n} - f(u^+_{\varepsilon,n})) \geq C$ where $C > 0$ independent of n. Also note that $u_{\varepsilon,n} \in \mathcal{N}_\varepsilon$ this implies that $s'_n(0)$ is bounded. As a result we have

$$\begin{aligned} &E_\varepsilon(u_{\varepsilon,n} + t\varphi + s_n(t)u^+_{\varepsilon,n} + l_n(t)u^-_{\varepsilon,n}) - E_\varepsilon(u_{\varepsilon,n}) \\ \geq\ & -\frac{1}{n}\|t\varphi + s_n(t)u^+_{\varepsilon,n} + l_n(t)u^-_{\varepsilon,n}\| \end{aligned} \tag{2.6}$$

and hence for all $\varphi \in H^1_0(\Omega)$,

$$\begin{aligned} &\left|\frac{d}{dt}E_\varepsilon(u_{\varepsilon,n} + t\varphi + s_n(t)u^+_{\varepsilon,n} + l_n(t)u^-_{\varepsilon,n})\Big|_{t=0}\right| \\ \leq\ & \frac{C}{n}\|\varphi\| + \frac{1}{n}\|s'_n(0)u^+_{\varepsilon,n} + l'_n(0)u^-_{\varepsilon,n}\|. \end{aligned}$$

This implies that

$$\liminf_{n\to+\infty}\frac{d}{dt}E_\varepsilon(u_{\varepsilon,n} + t\varphi + s_n(t)u^+_{\varepsilon,n} + l_n(t)u^-_{\varepsilon,n})\Big|_{t=0} = 0. \tag{2.7}$$

Using the fact (2.7) and $u_{\varepsilon,n} \in \mathcal{N}_\varepsilon$ we have,

$$\begin{aligned} 0 &= \lim_{n\to+\infty}\Big\{\varepsilon^2\int_\Omega \nabla u_{\varepsilon,n}\nabla\varphi + \lambda_1 u_{\varepsilon,n}^+\varphi^+ - \lambda_2 u_{\varepsilon,n}^-\varphi^- - f(u_{\varepsilon,n})\varphi \\ &+ s_n'(0)\Big(\int_\Omega \varepsilon^2|\nabla u_{\varepsilon,n}^+|^2 + \lambda_1(u_{\varepsilon,n}^+)^2 - \int_\Omega f(u_{\varepsilon,n}^+)u_{\varepsilon,n}^+\Big) \\ &+ l_n'(0)\Big(\int_\Omega \varepsilon^2|\nabla u_{\varepsilon,n}^-|^2 + \lambda_2(u_{\varepsilon,n}^-)^2 - \int_\Omega f(u_{\varepsilon,n}^-)u_{\varepsilon,n}^-\Big)\Big\} \\ &= \lim_{n\to+\infty}\Big\{\varepsilon^2\int_\Omega \nabla u_{\varepsilon,n}\nabla\varphi + \lambda_1 u_{\varepsilon,n}^+\varphi^+ - \lambda_2 u_{\varepsilon,n}^-\varphi^- - f(u_{\varepsilon,n})\varphi\Big\} \end{aligned} \tag{2.8}$$

Again as $u_{\varepsilon,n}$ is bounded in $H_0^1(\Omega)$, there exists u_ε such that $u_{\varepsilon,n} \rightharpoonup u_\varepsilon$ in $H_0^1(\Omega)$ as $n \to +\infty$ and hence $u_{\varepsilon,n} \to u_\varepsilon$ in $L^p(\Omega)$ for $p \in (1, \frac{N+2}{N-2})$. As a result, $u_{\varepsilon,n}(x) \to u_\varepsilon(x)$ a. e. as $n \to +\infty$. Note that $u_{\varepsilon,n}^+ \to u_\varepsilon^+$ and $u_{\varepsilon,n}^- \to u_\varepsilon^-$ in $L^p(\Omega)$ and by (2.5) we have $u_\varepsilon^+ \not\equiv 0$. Similarly we have $u_\varepsilon^- \not\equiv 0$ in Ω. Hence u_ε is a weak solution in $\mathcal{N}_\varepsilon$. By standard elliptic regularity, u_ε is a classical nodal solution of (1.2). Now we claim that in fact c_ε is achieved by u_ε.
Since $u_\varepsilon \in \mathcal{N}_\varepsilon$, implies that $c_\varepsilon \le E_\varepsilon(u_\varepsilon)$ and hence we have from (2.8),

$$c_\varepsilon + \frac{1}{n} \ge E_\varepsilon(u_{\varepsilon,n}) = E_\varepsilon(u_\varepsilon) + o(1) + \frac{1}{2}\int_\Omega |\nabla(u_{\varepsilon,n} - u_\varepsilon)|^2 dx$$

which implies that $u_{\varepsilon,n} \to u_\varepsilon$ strongly in $H_0^1(\Omega)$ as $n \to +\infty$ and $E_\varepsilon(u_\varepsilon) = c_\varepsilon$. □

Lemma 2.4. *Let $\omega_{\varepsilon,\lambda_1}$ and $\omega_{\varepsilon,\lambda_2}$ be the least energy solutions of*

$$\begin{cases} -\varepsilon^2\Delta u + \lambda_1 u = f(u) & \text{in } B_r(0) \\ u > 0 & \text{in } B_r(0) \\ u = 0 & \text{on } \partial B_r(0) \end{cases} \tag{2.9}$$

$$\begin{cases} -\varepsilon^2\Delta u + \lambda_2 u = f(u) & \text{in } B_r(0) \\ u > 0 & \text{in } B_r(0) \\ u = 0 & \text{on } \partial B_r(0) \end{cases} \tag{2.10}$$

respectively. Then for sufficiently small $\varepsilon > 0$, we have

$$E_{\varepsilon,\lambda_1}(\omega_{\varepsilon,\lambda_1}) = \varepsilon^N\Big\{I_{\lambda_1}(W_{\lambda_1}) + e^{-\frac{2\sqrt{\lambda_1}r(1+o(1))}{\varepsilon}}\Big\}$$

$$E_{\varepsilon,\lambda_2}(\omega_{\varepsilon,\lambda_2}) = \varepsilon^N\Big\{I_{\lambda_2}(W_{\lambda_2}) + e^{-\frac{2\sqrt{\lambda_2}r(1+o(1))}{\varepsilon}}\Big\}$$

where $o(1) \to 0$ as $\varepsilon \to 0$.

Proof. For the proof see [8]. □

Let $\Lambda = \{x \in \Omega : \sqrt{\lambda_1}|x - P_1| = \sqrt{\lambda_2}|x - P_2|\}$.

Lemma 2.5. *We have for $\varepsilon > 0$ sufficiently small*

$$c_\varepsilon \le \varepsilon^N\Big\{I_{\lambda_1}(W_{\lambda_1}) + I_{\lambda_2}(W_{\lambda_2}) + e^{-\frac{2\varphi(P_1,P_2)}{\varepsilon}} + o(e^{-\frac{2\varphi(P_1,P_2)}{\varepsilon}})\Big\}. \tag{2.11}$$

Proof. Let v_ε be a positive solution of

$$\begin{cases} -\varepsilon^2\Delta u + \lambda_1 u = f(u) & \text{in } B_{r_1}(P_1) \\ u > 0 & \text{in } B_{r_1}(P_1) \\ u = 0 & \text{on } B_{r_1}(P_1) \end{cases} \tag{2.12}$$

where $r_1 = \min\{d(P_1, \partial\Omega), d(P_1, \Lambda)\}$. Let w_ε be a positive solution of

$$\begin{cases} -\varepsilon^2\Delta u + \lambda_2 u = f(u) & \text{in } B_{r_2}(P_2) \\ u > 0 & \text{in } B_{r_2}(P_2) \\ u = 0 & \text{on } B_{r_2}(P_2) \end{cases} \tag{2.13}$$

where $r_2 = \min\{d(P_2, \partial\Omega), d(P_2, \Lambda)\}$. Note that supp $v_\varepsilon \cap$supp $w_\varepsilon = \emptyset$ and $v_\varepsilon \in \mathcal{N}_\varepsilon^+$ and $w_\varepsilon \in \mathcal{N}_\varepsilon^-$. Then we have $v_\varepsilon - w_\varepsilon \in \mathcal{N}_\varepsilon$ and hence we have from (2.12) and (2.13),

$$\begin{aligned} c_\varepsilon &\le E_\varepsilon(v_\varepsilon - w_\varepsilon) \\ &\le E_{\varepsilon,\lambda_1}(v_\varepsilon) + E_{\varepsilon,\lambda_2}(w_\varepsilon) \\ &\le \varepsilon^N \Big\{ I_{\lambda_1}(W_{\lambda_1}) + e^{-\frac{2r_1}{\varepsilon}} + I_{\lambda_2}(W_{\lambda_2}) + e^{-\frac{2r_2}{\varepsilon}} \\ &+ o(e^{-\frac{2r_1}{\varepsilon}}) + o(e^{-\frac{2r_2}{\varepsilon}}) \Big\}. \end{aligned}$$

Hence we have,

$$\begin{aligned} c_\varepsilon &\le \varepsilon^N \Big\{ I_{\lambda_1}(W_{\lambda_1}) + e^{-\frac{2\min\{r_1,r_2\}}{\varepsilon}} + I_{\lambda_2}(W_{\lambda_2}) \\ &+ o(e^{-\frac{2\min\{r_1,r_2\}}{\varepsilon}}) \Big\} \\ &\le \varepsilon^N \Big\{ I_{\lambda_1}(W_{\lambda_1}) + I_{\lambda_2}(W_{\lambda_2}) + e^{-\frac{2\varphi(P_1,P_2)}{\varepsilon}} \\ &+ o(e^{-\frac{2\varphi(P_1,P_2)}{\varepsilon}}) \Big\}. \end{aligned} \tag{2.14}$$

□

Corollary 2.6. *We also have* $c_\varepsilon \ge \varepsilon^N \Big\{ I_{\lambda_1}(W_{\lambda_1}) + I_{\lambda_2}(W_{\lambda_2}) + o(1) \Big\}$.

Proof.

$$c_\varepsilon = \inf_{u\in\mathcal{N}_\varepsilon} \{E_{\varepsilon,\lambda_1}(u) + E_{\varepsilon,\lambda_2}(u)\} \ge \inf_{u\in\mathcal{N}_\varepsilon^+} E_{\varepsilon,\lambda_1}(u) + \inf_{u\in\mathcal{N}_\varepsilon^-} E_{\varepsilon,\lambda_2}(u)$$

this implies the result. □

Lemma 2.7. *As* $\varepsilon \to 0$,

$$\frac{d(P_\varepsilon^1, \partial\Omega)}{\varepsilon} \to +\infty, \frac{d(P_\varepsilon^2, \partial\Omega)}{\varepsilon} \to +\infty, \frac{|P_\varepsilon^1 - P_\varepsilon^2|}{\varepsilon} \to +\infty.$$

Proof. As $\varepsilon^2\Delta u_\varepsilon(P_\varepsilon^1)\le 0$ it implies that $f(u_\varepsilon(P_\varepsilon^1))\ge\lambda_1 u_\varepsilon(P_\varepsilon^1)$ which implies that $Cu_\varepsilon^{p-1}(P_\varepsilon^1)\ge\lambda_1$, hence there exists a positive constant β such that $u_\varepsilon(P_\varepsilon^1)\ge\beta$ and similarly we obtain that $u_\varepsilon(P_\varepsilon^2)\le-\beta$. Also by Lemma 2.5,

$$\varepsilon^2\int_\Omega|\nabla u_\varepsilon|^2+\lambda_1\int_\Omega(u_\varepsilon^+)^2+\lambda_2\int_\Omega(u_\varepsilon^-)^2\le C\varepsilon^N$$

and hence by Moser iteration we obtain $\|u_\varepsilon\|_{L^\infty(\Omega)}\le C$.
Suppose that $\lim\limits_{\varepsilon\to0}\dfrac{d(P_\varepsilon^1,\partial\Omega)}{\varepsilon}\le C$. By scaling $v_\varepsilon(x)=u_\varepsilon(\varepsilon x+P_\varepsilon^1)$, then (1.2) reduces to,

$$\begin{cases}\Delta v_\varepsilon-\lambda_1 v_\varepsilon+\lambda_2 v_\varepsilon^-+f(v_\varepsilon)=0 & \text{in } \Omega_\varepsilon\\ v_\varepsilon^\pm\neq 0 & \text{in } \Omega_\varepsilon\\ v_\varepsilon=0 & \text{on } \partial\Omega_\varepsilon\end{cases}\tag{2.15}$$

where $\Omega_\varepsilon=\frac{x-P_\varepsilon^1}{\varepsilon}$. Note that from (2.5), $\|v_\varepsilon\|_{H_0^1(\Omega_\varepsilon)}\le C$; there exists $W\in H^1(\mathbb{R}^N)$ we have $v_\varepsilon\rightharpoonup W$ in $H^1(\mathbb{R}^N)$ and by Sobolev embedding theorem we have $v_\varepsilon\to W$ in $L^p_{loc}(\mathbb{R}^N)$. Hence $v_\varepsilon\to W$ point-wise almost everywhere in Ω. Also by Schauder estimates, it follows that there exists $C>0$ such that $\|v_\varepsilon\|_{C^{2,\beta}_{loc}(\mathbb{R}^N)}\le C$ for some $0<\beta\le1$. Hence by the Ascoli-Arzela's theorem there exists $W\neq0$ such that

$$\|v_\varepsilon-W\|_{C^2_{loc}(\mathbb{R}^N)}\to0 \text{ as } \varepsilon\to0$$

where W is a positive solution satisfying

$$\begin{cases}\Delta W-\lambda_1 W+f(W)=0 & \text{in } \mathbb{R}^N_+\\ W(0)\ge\beta, W\in H^1 & \\ W=0 & \text{on } \partial\mathbb{R}^N_+\end{cases}\tag{2.16}$$

where $\mathbb{R}^N_+=\{y:y_n>-a\}$. Then by result in [9] we obtain $W\equiv0$, a contradiction. Similarly $\lim\limits_{\varepsilon\to0}\dfrac{d(P_\varepsilon^2,\partial\Omega)}{\varepsilon}=+\infty$. Now we prove that $\lim\limits_{\varepsilon\to0}\dfrac{|P_\varepsilon^1-P_\varepsilon^2|}{\varepsilon}=+\infty$.
By applying the Schauder estimates we obtain a $C>0$ such that $\|\varepsilon Du_\varepsilon\|_{L^\infty}\le C$. If possible let $\lim\limits_{\varepsilon\to0}\dfrac{|P_\varepsilon^1-P_\varepsilon^2|}{\varepsilon}=\delta<+\infty$. Then it easily follows that $u_\varepsilon(P_\varepsilon^1)\ge\beta$ and $u_\varepsilon(P_\varepsilon^2)\le-\beta$ which implies that $u_\varepsilon(P_\varepsilon^1)-u_\varepsilon(P_\varepsilon^2)\ge2\beta$. Then

$$2\beta\le|u_\varepsilon(P_\varepsilon^1)-u_\varepsilon(P_\varepsilon^2)|\le\varepsilon\|Du_\varepsilon\|_\infty\frac{|P_\varepsilon^1-P_\varepsilon^2|}{\varepsilon}.$$

Suppose $P_\varepsilon=\frac{P_\varepsilon^1-P_\varepsilon^2}{\varepsilon}$. Then along a subsequence $|P_\varepsilon|\to\delta\in(0,+\infty)$. Define $v_\varepsilon=u_\varepsilon(\varepsilon y+P_\varepsilon^1)$. Then $v_\varepsilon\to W$ in $C^2_{loc}(\mathbb{R}^N)$ and W satisfies

$$\begin{cases}-\Delta W+\lambda_1W^+-\lambda_2W^-=f(W) & \text{in } \mathbb{R}^N\\ W(0)\ge\beta,\ W(P)\le-\beta & \\ W\in H^1(\mathbb{R}^N) & \end{cases}\tag{2.17}$$

where $P=\lim_{\varepsilon\to0}\frac{P_\varepsilon^1-P_\varepsilon^2}{\varepsilon}$ which implies that W is a nodal solution of (2.17) and hence a critical point of the functional

$$I_\infty(u)=\int_{\mathbb{R}^N}\left(\frac12|\nabla u|^2+\frac{\lambda_1}{2}(u^+)^2+\frac{\lambda_2}{2}(u^-)^2-F(u)\right)dx$$

and in particular we have $\langle I'_\infty(W), W^\pm\rangle = 0$ and $W \in \mathcal{N}_\infty$ where

$$\mathcal{N}_\infty = \Big\{u \in H^1(\mathbb{R}^N) : u^\pm \not\equiv 0, \int_{\mathbb{R}^N} |\nabla u^+|^2 + \lambda_1 \int_{\mathbb{R}^N} (u^+)^2 = \int_{\mathbb{R}^N} f(u^+)u^+;$$
$$\int_{\mathbb{R}^N} |\nabla u^-|^2 + \lambda_2 \int_{\mathbb{R}^N} (u^-)^2 = \int_{\mathbb{R}^N} f(u^-)u^-\Big\}.$$

But by (2.1) we know that $\varepsilon^N(I_{\lambda_1}(W_{\lambda_1}) + I_{\lambda_2}(W_{\lambda_2}) + o(1)) \geq \varepsilon^N(I_\infty(W^+) + I_\infty(W^-) + o(1))$. This implies

$$I_\infty(W^+) + I_\infty(W^-) \leq I_{\lambda_1}(W_{\lambda_1}) + I_{\lambda_2}(W_{\lambda_2}) = c_{\lambda_1} + c_{\lambda_2}$$

where c_{λ_i} is a mountain pass critical value with respect to the functional I_{λ_i}, i.e.

$$c_{\lambda_i} = \inf_{u\in H^1(\mathbb{R}^N), u\not\equiv 0, \int_{\mathbb{R}^N}|\nabla u|^2 + \lambda_i \int_{\mathbb{R}^N} u^2 = \int_{\mathbb{R}^N} f(u)u} I_{\lambda_i}(u). \tag{2.18}$$

Also it easily follows that $I_\infty(W^+) = I_{\lambda_1}(W^+) \geq c_{\lambda_1}$, $I_\infty(W^-) = I_{\lambda_2}(W^-) \geq c_{\lambda_2}$. Since any minimizer is a non-negative solution of weak solution, we have $c_{\lambda_1} = I_{\lambda_1}(W^+), c_{\lambda_2} = I_{\lambda_2}(W^-)$. Thus $W^+ = W_{\lambda_1}(x-R)$ and $W^- = W_{\lambda_2}(x-S)$ for some R, S in $\mathbb{R}^N$. So $W^+(P) = W_{\lambda_1}(P-R) > 0$. However $W(P) \leq -\beta, W^+(P) = 0$, hence we have a contradiction. □

Lemma 2.8. *For sufficiently small $\varepsilon > 0$, u_ε has exactly one positive local maximum and one negative local minimum.*

Proof. Note that from Lemma 2.5, we obtain that $c_\varepsilon \leq \varepsilon^N(I_{\lambda_1}(W_{\lambda_1}) + I_{\lambda_2}(W_{\lambda_2}) + o(1))$. Suppose it has two positive local maxima as P_ε and Q_ε and negative local minimum R_ε. Then it follows similarly as in Lemma 2.7 one can show that $\frac{|P_\varepsilon - Q_\varepsilon|}{\varepsilon} \to +\infty$, $\frac{|Q_\varepsilon - R_\varepsilon|}{\varepsilon} \to +\infty$ and $\frac{|P_\varepsilon - R_\varepsilon|}{\varepsilon} \to +\infty$ as $\varepsilon \to 0$. Also note that $\frac{1}{2}f(u_\varepsilon)u_\varepsilon - F(u_\varepsilon) \geq 0$ by assumption,

$$\begin{aligned}
c_\varepsilon &= E_\varepsilon(u_\varepsilon) = \int_\Omega (\frac{1}{2}f(u_\varepsilon)u_\varepsilon - F(u_\varepsilon))dx \\
&\geq \int_{B_{\varepsilon R}(P_\varepsilon)} (\frac{1}{2}f(u_\varepsilon)u_\varepsilon - F(u_\varepsilon)) + \int_{B_{\varepsilon R}(Q_\varepsilon)} (\frac{1}{2}f(u_\varepsilon)u_\varepsilon - F(u_\varepsilon)) + \\
&+ \int_{B_{\varepsilon R}(R_\varepsilon)} (\frac{1}{2}f(u_\varepsilon)u_\varepsilon - F(u_\varepsilon)) \\
&\geq \varepsilon^N\Big(2I_{\lambda_1}(W_{\lambda_1}) + I_{\lambda_2}(W_{\lambda_2}) + o(1)\Big) \qquad (2.19)
\end{aligned}$$

a contradiction to Lemma 2.5. Hence u_ε has exactly one positive maximum and one negative minimum. □

Now let us define

$$d_\varepsilon = \min\Big\{\sqrt{\lambda_1}d(P_\varepsilon^1, \partial\Omega), \sqrt{\lambda_2}d(P_\varepsilon^2, \partial\Omega), \frac{\sqrt{\lambda_1}\sqrt{\lambda_2}}{\sqrt{\lambda_1}+\sqrt{\lambda_2}}|P_\varepsilon^1 - P_\varepsilon^2|\Big\}.$$

Then by the above lemma $\frac{d_\varepsilon}{\varepsilon} \to +\infty$ as $\varepsilon \to 0$. Now let us re-scale the problem by $\overline{\varepsilon} = \frac{\varepsilon}{d_\varepsilon}$ and $\overline{x} = d_\varepsilon \overline{x}$. Then we have

$$\Delta u - \lambda_1 u^+ + \lambda_2 u^- + f(u) = 0 \text{ in } \overline{\Omega}_{d_\varepsilon} = \frac{\Omega}{d_\varepsilon}. \tag{2.20}$$

In this re-scaling

$$\overline{d}_\varepsilon = \min\left\{\sqrt{\lambda_1} d(\overline{P_\varepsilon}^1, \partial\Omega_{d_\varepsilon}), \sqrt{\lambda_2} d(\overline{P_\varepsilon}^2, \partial\Omega_{d_\varepsilon}), \frac{\sqrt{\lambda_1}\sqrt{\lambda_2}}{\sqrt{\lambda_1}+\sqrt{\lambda_2}}|\overline{P_\varepsilon}^1 - \overline{P_\varepsilon}^2|\right\} = 1.$$

Without loss of generality we can consider $\overline{d}_\varepsilon \geq 1$.

Lemma 2.9. *For any $0 < \delta' < 1$, there exists a constant $C > 0$ independent of δ such that*

$$u_\varepsilon^+ \leq Ce^{-\frac{\sqrt{\lambda_1}(1-\delta')|x-P_\varepsilon^1|}{\varepsilon}} \text{ and } u_\varepsilon^- \leq Ce^{-\frac{\sqrt{\lambda_2}(1-\delta')|x-P_\varepsilon^2|}{\varepsilon}} \ \forall x \in \Omega.$$

Proof. Let $v_\varepsilon^i(y) = u_\varepsilon(\varepsilon y + P_\varepsilon^i)$. Then $v_\varepsilon^1 \to W_{\lambda_1}$ in $C^2_{loc}(\mathbb{R}^N)$. Also we have $W_{\lambda_1}(r) \leq Ce^{-\sqrt{\lambda_1}r}$ for all r. Let $R = \ln\frac{C}{\zeta}$ such that $\zeta = Ce^{-R}$. Then there exist an $\varepsilon_0 > 0$ such that $v_\varepsilon^+(y) \leq W_{\lambda_1}(y) + \zeta \leq 2\zeta$. Let us consider the domain $\Omega^1 = \Omega \setminus B_{\varepsilon R}(P_\varepsilon^1)$ where $R > 0$ is large. Hence we can choose a $\zeta > 0$, independent of ε such that $v_\varepsilon^+ \leq C$ on $\partial B_R(0)$. This implies that $u_\varepsilon^+ \leq 2\zeta$ on $\partial B_{\varepsilon R}(P_\varepsilon^1)$. For any $0 < \delta' < 1$, choose ζ in such a way that

$$\frac{f(u_\varepsilon)}{\lambda_1 u_\varepsilon^+} < \delta',$$

consider the equation with $u_\varepsilon > 0$

$$-\varepsilon^2\Delta u_\varepsilon + \lambda_1 u_\varepsilon = \frac{f(u_\varepsilon)}{u_\varepsilon}u_\varepsilon \text{ in } \Omega^1.$$

Then we obtain,

$$(2.21) \qquad \begin{cases} -\varepsilon^2\Delta u_\varepsilon + (1-\delta')\lambda_1 u_\varepsilon \leq 0 & \text{in } \Omega^1 \\ u_\varepsilon > 0 & \text{in } \Omega^1 \\ u_\varepsilon \leq 2\zeta & \text{in } \partial B_{\varepsilon R}(P_\varepsilon^1) \\ u_\varepsilon = 0 & \text{on } \partial\Omega. \end{cases}$$

Using a comparison argument we obtain $u_\varepsilon^+ \leq Ce^{-\frac{\sqrt{\lambda_1}(1-\delta')|x-P_\varepsilon^1|}{\varepsilon}}$. We obtain the other estimate similarly. □

3. Lower bound of the energy expansion

In order to obtain the greatest lower bound of the energy E_ε we consider three cases.

Case 1 Suppose that

$$\frac{d_\varepsilon}{\sqrt{\lambda_1}d(P_\varepsilon^1, \partial\Omega)} \to 1 \text{ as } \varepsilon \to 0.$$

Note that

$$c_\varepsilon \geq \inf_{u\in\mathcal{N}_\varepsilon^+} E_{\varepsilon,\lambda_1}(u) + \inf_{u\in\mathcal{N}_\varepsilon^-} E_{\varepsilon,\lambda_2}(u).$$

We use del Pino-Felmer's symmetrization technique in [8] to conclude that

$$E_{\varepsilon,\lambda_1}(u_\varepsilon^+) \geq \varepsilon^N\left\{I_{\lambda_1}(W_{\lambda_1}) + \frac{1}{2}e^{-2\frac{\sqrt{\lambda_1}(d(P_\varepsilon^1,\partial\Omega)+o(1))}{\varepsilon}}\right\}.$$

We also deduce that

$$E_{\varepsilon,\lambda_2}(u_\varepsilon^-) \geq \varepsilon^N\left\{I_{\lambda_2}(W_{\lambda_2}) + \frac{1}{2}e^{-2\frac{(d_\varepsilon+o(1))}{\varepsilon}}\right\}$$

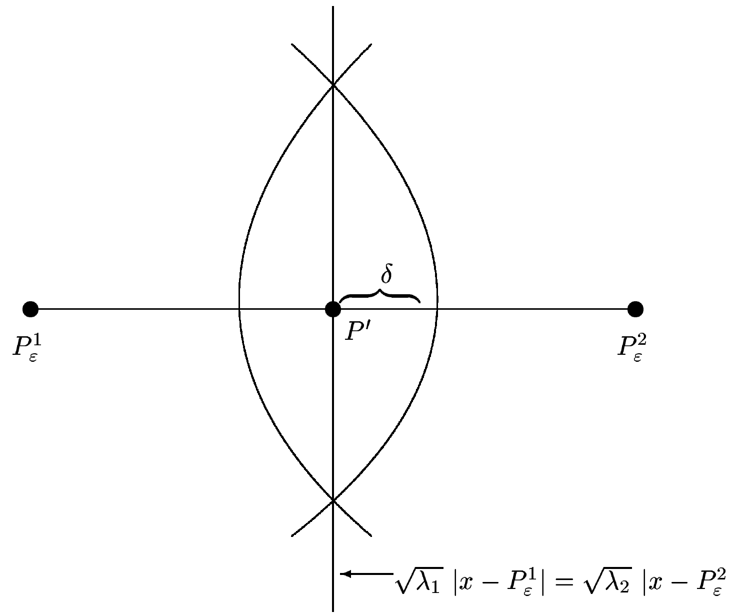

FIGURE 1. The region of intersection

and as $d_\varepsilon = \sqrt{\lambda_1}d(P_\varepsilon^1, \partial\Omega) + o(1)$, we have

$$c_\varepsilon \geq \varepsilon^N \left(I_{\lambda_1}(W_{\lambda_1}) + I_{\lambda_2}(W_{\lambda_2}) + e^{-\frac{2(d_\varepsilon + o(1))}{\varepsilon}} \right). \tag{3.1}$$

Case 2 Suppose that

$$\frac{d_\varepsilon}{\sqrt{\lambda_2}d(P_\varepsilon^2, \partial\Omega)} \to 1 \text{ as } \varepsilon \to 0.$$

Then we argue as in Case 1.
Case 3 Suppose that

$$d_\varepsilon = \frac{\sqrt{\lambda_1}\sqrt{\lambda_2}}{\sqrt{\lambda_1}+\sqrt{\lambda_2}}|P_\varepsilon^1 - P_\varepsilon^2|$$

and $d_\varepsilon \geq (1+5\delta)\sqrt{\lambda_1}d(P_\varepsilon^1, \partial\Omega)$, $d_\varepsilon \geq (1+5\delta)\sqrt{\lambda_2}d(P_\varepsilon^2, \partial\Omega)$. Then we define $|P' - P_\varepsilon^1| = \frac{\sqrt{\lambda_2}}{\sqrt{\lambda_1}+\sqrt{\lambda_2}}|P_\varepsilon^1 - P_\varepsilon^2| = d_{\varepsilon,1}$. Then we have

$$|P' - P_\varepsilon^2| = \frac{\sqrt{\lambda_1}}{\sqrt{\lambda_1}+\sqrt{\lambda_2}}|P_\varepsilon^1 - P_\varepsilon^2| = d_{\varepsilon,2}.$$

We consider balls $B_{d_{\varepsilon,1}+\delta}(P_\varepsilon^1)$ and $B_{d_{\varepsilon,2}+\delta_2}(P_\varepsilon^2)$, where $0 < \delta << d_{\varepsilon,1}$ is small and $\delta_2 \sim \frac{\sqrt{\lambda_2}}{\sqrt{\lambda_1}}\delta$ is defined by

$$(d_{\varepsilon,1}+\delta)^2 - d_{\varepsilon,1}^2 = (d_{\varepsilon,2}+\delta_2)^2 - d_{\varepsilon,2}^2. \tag{3.2}$$

Define the intersection $\Gamma_\varepsilon = B_{d_{\varepsilon,1}+\delta}(P^1_\varepsilon) \cap B_{d_{\varepsilon,2}+\delta}(P^2_\varepsilon)$. Then the total volume of $\Gamma_\varepsilon \approx \delta O(\delta^{\frac{N-1}{2}})$. Since $\Gamma_\varepsilon = (\Gamma_\varepsilon \cap \{u_\varepsilon \geq 0\}) \cup (\Gamma_\varepsilon \cap \{u_\varepsilon \leq 0\})$, we either have $|\Gamma_\varepsilon \cap \{u_\varepsilon \geq 0\}| \leq \frac{1}{2}|\Gamma_\varepsilon|$ or $|\Gamma_\varepsilon \cap \{u_\varepsilon \leq 0\}| \leq \frac{1}{2}|\Gamma_\varepsilon|$.

Without loss of generality, let

$$|\Gamma_\varepsilon \cap \{u_\varepsilon \geq 0\}| \leq \frac{1}{2}|\Gamma_\varepsilon|$$

Thus

$$|B_{d_{\varepsilon,1}+\delta}(P^1_\varepsilon) \cap \{u_\varepsilon > 0\}| \leq |B_{d_{\varepsilon,1}+\delta}(P^1_\varepsilon)| - \frac{1}{2}|\Gamma_\varepsilon| = |B_{r_\varepsilon}(0)|$$

where $r_\varepsilon = (d_{1,\varepsilon} + \delta)(1-\eta)$ for some $0 < \eta < 1$, where $\eta \sim \delta\sqrt{\delta}$. Let us define a smooth function

$$\chi(x) = \begin{cases} 1 & \text{if } |x - P^1_\varepsilon| \leq (d_{\varepsilon,1} + \delta)(1-\eta) \\ 0 & \text{if } |x - P^1_\varepsilon| \geq (d_{\varepsilon,1} + \delta) \end{cases} \tag{3.3}$$

and $0 \leq \chi \leq 1$ and $|\nabla\chi| \leq \frac{C}{(d_{\varepsilon,1}+\delta)\eta}$. Then support of $u^+_\varepsilon\chi^2$ is contained in $B_{d_{\varepsilon,1}+\delta}(P^1_\varepsilon)$. Multiplying (1.2) by $u^+_\varepsilon\chi^2$ we obtain

$$\int_\Omega \varepsilon^2\nabla u_\varepsilon \nabla(u^+_\varepsilon\chi^2) + \lambda_1(u^+_\varepsilon)^2\chi^2 = \int_\Omega f(u_\varepsilon)u^+_\varepsilon\chi^2 \tag{3.4}$$

Now let us compute

$$\begin{aligned}
\int_\Omega \varepsilon^2\nabla u_\varepsilon \nabla(u^+_\varepsilon\chi^2) &= \int_\Omega \varepsilon^2\nabla u^+_\varepsilon \nabla(u^+_\varepsilon\chi^2) \\
&= \int_\Omega \varepsilon^2\nabla u^+_\varepsilon\Big\{\chi\nabla(u^+_\varepsilon\chi) + u^+_\varepsilon\chi\nabla\chi\Big\} \\
&= \int_\Omega \varepsilon^2\Big\{(\nabla(u^+_\varepsilon\chi) - u^+_\varepsilon\nabla\chi)\nabla(u^+_\varepsilon\chi) + u^+_\varepsilon\chi\nabla\chi\nabla u^+_\varepsilon\Big\} \\
&= \int_\Omega \varepsilon^2\Big\{|\nabla(u^+_\varepsilon\chi)|^2 - u^+_\varepsilon\nabla\chi\nabla(u^+_\varepsilon\chi) + u^+_\varepsilon\chi\nabla\chi\nabla u^+_\varepsilon\Big\} \\
&= \int_\Omega \varepsilon^2\Big\{|\nabla(u^+_\varepsilon\chi)|^2 - u^+_\varepsilon\chi\nabla\chi\nabla u^+_\varepsilon - (u^+_\varepsilon)^2|\nabla\chi|^2 + u^+_\varepsilon\chi\nabla\chi\nabla u^+_\varepsilon\Big\} \\
&= \varepsilon^2\int_\Omega |\nabla(u^+_\varepsilon\chi)|^2 - \varepsilon^2\int_\Omega (u^+_\varepsilon)^2|\nabla\chi|^2 \qquad (3.5)
\end{aligned}$$

where

$$\varepsilon^2\int_\Omega (u^+_\varepsilon)^2|\nabla\chi|^2 \leq C\varepsilon^N e^{-\sqrt{\lambda_1}\frac{2(1-\frac{\eta}{2})(d_{\varepsilon,1}+\delta)}{\varepsilon}}. \tag{3.6}$$

On the other hand

$$\begin{aligned}
\int_\Omega f(u_\varepsilon)u^+_\varepsilon\chi^2 &= \int_\Omega f(u^+_\varepsilon\chi)u^+_\varepsilon\chi + \int_\Omega \{f(u^+_\varepsilon\chi) - f(u_\varepsilon)\chi\}u^+_\varepsilon\chi \\
&= \int_\Omega f(u^+_\varepsilon\chi)u^+_\varepsilon\chi + O(\varepsilon^N e^{-\frac{(p+1)\sqrt{\lambda_1}(d_{\varepsilon,1}+\delta)(1-\frac{\eta}{2})}{\varepsilon}}). \qquad (3.7)
\end{aligned}$$

Note that in order to derive (3.6) and (3.7), we use the fact (f_2), Lemma 2.9 and (3.3)

$$u^+_\varepsilon \leq Ce^{-\frac{\sqrt{\lambda_1}(1-\delta')|x-P^1_\varepsilon|}{\varepsilon}}, \ \delta' = \frac{\eta}{2(1-\eta)},$$

$$\int_\Omega \{f(u_\varepsilon^+\chi)u_\varepsilon^+\chi - f(u_\varepsilon)\}u_\varepsilon^+\chi \le Ce^{-\frac{\sqrt{\lambda_1}(p+1)(d_{\varepsilon,1}+\delta)(1-\delta')}{\varepsilon}} \le Ce^{-\frac{\sqrt{\lambda_1}(p+1)(d_{\varepsilon,1}+\delta)(1-\frac{\eta}{2})}{\varepsilon}}.$$

Hence combining (3.4), (3.5) and (3.7) we have

$$\begin{aligned} &\varepsilon^2\int_\Omega |\nabla(u_\varepsilon^+\chi)|^2 + \lambda_1\int_\Omega (u_\varepsilon^+\chi)^2 \\ (3.8)\qquad &= \int_\Omega f(u_\varepsilon^+\chi)u_\varepsilon^+\chi + O\Big(\varepsilon^N e^{-\frac{2\sqrt{\lambda_1}(d_{\varepsilon,1}+\delta)(1-\frac{\eta}{2})}{\varepsilon}}\Big). \end{aligned}$$

Let $v_\varepsilon = t_\varepsilon u_\varepsilon^+\chi$ where t_ε is such that

$$\varepsilon^2\int_\Omega |\nabla v_\varepsilon|^2 + \lambda_1\int_\Omega v_\varepsilon^2 = \int_\Omega f(v_\varepsilon)v_\varepsilon$$

Now we claim that

$$t_\varepsilon = 1 + O\Big(e^{-\frac{2\sqrt{\lambda_1}(1-\frac{\eta}{2})(d_{\varepsilon,1}+\delta)}{\varepsilon}}\Big).$$

Define $\tilde\sigma : [0,+\infty)\times[0,\beta^\star) \to \mathbb{R}$ such that

$$\tilde\sigma(t,\beta) = \int_\Omega f(tu_\varepsilon^+\chi)u_\varepsilon^+\chi - \int_\Omega f(u_\varepsilon^+\chi)u_\varepsilon^+\chi - \beta\int_\Omega f'(u_\varepsilon^+\chi)(u_\varepsilon^+\chi)^2$$

for some $\beta^\star > 0$. Then $\tilde\sigma \in C^1$. Note that $\tilde\sigma(1,0) = 0$ and

$$\tilde\sigma_t(1,0) = \int_\Omega f'(u_\varepsilon^+\chi)(u_\varepsilon^+\chi)^2 \neq 0.$$

Hence by implicit function theorem, there exists a C^1 function $\beta \mapsto t(\beta)$ such that $\tilde\sigma(t(\beta),\beta) = 0$, for small β and $t(0) = 1$. Letting $t_\varepsilon = 1+\beta$, we have from (3.8)

$$\beta \sim \frac{\varepsilon^2\int_\Omega |\nabla u_\varepsilon^+\chi|^2 + \lambda_1\int_\Omega (u_\varepsilon^+\chi)^2 - \int_\Omega f(u_\varepsilon^+\chi)u_\varepsilon^+\chi}{\varepsilon^2\int_\Omega |\nabla u_\varepsilon^+\chi|^2 + \lambda_1\int_\Omega (u_\varepsilon^+\chi)^2 - \int_\Omega f'(u_\varepsilon^+)(u_\varepsilon^+\chi)^2}.$$

Hence

$$\beta \sim \frac{O\Big(\varepsilon^N e^{-\frac{2\sqrt{\lambda_1}(d_{\varepsilon,1}+\delta)(1-\frac{\eta}{2})}{\varepsilon}}\Big)}{\int_\Omega f(u_\varepsilon^+\chi)u_\varepsilon^+\chi - \int_\Omega f'(u_\varepsilon^+\chi)(u_\varepsilon^+\chi)^2}$$

which implies $\beta = O(e^{-\frac{2\sqrt{\lambda_1}(1-\frac{\eta}{2})(d_{\varepsilon,1}+\delta)}{\varepsilon}})$. Then we obtain,

$$\frac{\varepsilon^2}{2}\int_{B_{d_{1,\varepsilon}+\delta}(P_\varepsilon^1)} |\nabla v_\varepsilon|^2 = \frac{\varepsilon^2}{2}\int_{B_{d_{1,\varepsilon}+\delta}(P_\varepsilon^1)} |\nabla(u_\varepsilon^+\chi)|^2 + \varepsilon^2\beta\int_{B_{d_{1,\varepsilon}+\delta}(P_\varepsilon^1)} |\nabla u_\varepsilon^+\chi|^2 + O(\beta^2\varepsilon^N),$$

$$\frac{\lambda_1}{2}\int_{B_{d_{1,\varepsilon}+\delta}(P_\varepsilon^1)} v_\varepsilon^2 = \frac{\lambda_1}{2}\int_{B_{d_{1,\varepsilon}+\delta}(P_\varepsilon^1)} (u_\varepsilon^+\chi)^2 + \lambda_1\beta\int_{B_{d_{1,\varepsilon}+\delta}(P_\varepsilon^1)} (u_\varepsilon^+\chi)^2 + O(\beta^2\varepsilon^N),$$

and

$$\int_{B_{d_{1,\varepsilon}+\delta}(P_\varepsilon^1)} F(v_\varepsilon) = \int_{B_{d_{1,\varepsilon}+\delta}(P_\varepsilon^1)} F(u_\varepsilon^+\chi) + \beta\int_{B_{d_{1,\varepsilon}+\delta}(P_\varepsilon^1)} f(u_\varepsilon^+\chi)u_\varepsilon^+\chi + O(\beta^2\varepsilon^N)$$

and

$$\varepsilon^2\int_{B_{d_{1,\varepsilon}+\delta}(P_\varepsilon^1)} |\nabla u_\varepsilon^+\chi|^2 + \lambda_1\int_{B_{d_{1,\varepsilon}+\delta}(P_\varepsilon^1)} (u_\varepsilon^+\chi)^2 - \int_{B_{d_{1,\varepsilon}+\delta}(P_\varepsilon^1)} f(u_\varepsilon^+\chi)u_\varepsilon^+\chi = O(\beta\varepsilon^N)$$

Using the above facts we have,

$$\begin{aligned}
&\frac{\varepsilon^2}{2}\int_{B_{d_{1,\varepsilon}+\delta}(P^1_\varepsilon)}|\nabla v_\varepsilon|^2+\frac{\lambda_1}{2}\int_{B_{d_{1,\varepsilon}+\delta}(P^1_\varepsilon)}v_\varepsilon^2-\int_{B_{d_{1,\varepsilon}+\delta}(P^1_\varepsilon)}F(v_\varepsilon)\\
=\;&\frac{\varepsilon^2}{2}\int_{B_{d_{1,\varepsilon}+\delta}(P^1_\varepsilon)}|\nabla u^+_\varepsilon\chi|^2+\frac{\lambda_1}{2}\int_{B_{d_{1,\varepsilon}+\delta}(P^1_\varepsilon)}(u^+_\varepsilon\chi)^2-\int_{B_{d_{1,\varepsilon}+\delta}(P^1_\varepsilon)}F(u^+_\varepsilon\chi)\\
+\;&O(\varepsilon^N|t_\varepsilon-1|^2)\\
=\;&\int_{B_{d_{1,\varepsilon}+\delta}(P^1_\varepsilon)}\left(\frac{1}{2}f(u^+_\varepsilon\chi)u^+_\varepsilon\chi-F(u^+_\varepsilon\chi)\right)+O(\varepsilon^N|t_\varepsilon-1|^2)\\
=\;&\int_\Omega\left(\frac{1}{2}f(u^+_\varepsilon)u^+_\varepsilon-F(u^+_\varepsilon)\right)+O\left(\varepsilon^N|t_\varepsilon-1|^2+e^{-\frac{\sqrt{\lambda_1}(p+1)(1-\frac{\eta}{2})(d_{\varepsilon,1}+\delta)}{\varepsilon}}\right)\\
(3.9)=\;&E_{\varepsilon,\lambda_1}(u^+_\varepsilon)+\varepsilon^N O\left(e^{-\frac{\sqrt{\lambda_1}(2+\sigma)(d_{\varepsilon,1}+\delta)}{\varepsilon}}\right)
\end{aligned}$$

for some $\sigma\in(0,\min(1,p-1))$. Thus we have

$$\begin{aligned}
E_{\varepsilon,\lambda_1}(u^+_\varepsilon)\;&\geq\;\inf_{\mathcal{N}^+_\varepsilon}E_{\varepsilon,\lambda_1,B_{d_\varepsilon+\delta}(P^1_\varepsilon)}(v)-C\varepsilon^N e^{-\frac{\sqrt{\lambda_1}(2+\sigma)(d_{\varepsilon,1}+\delta)}{\varepsilon}}\\
&\geq\;\varepsilon^N\left\{I_{\lambda_1}(W_{\lambda_1})+e^{-\frac{2\sqrt{\lambda_1}(1-\frac{\eta}{2})(d_{\varepsilon,1}+\delta)}{\varepsilon}}\right\}-C\varepsilon^N e^{-\frac{\sqrt{\lambda_1}(2+\sigma)(d_{\varepsilon,1}+\delta)}{\varepsilon}}\\
&\geq\;\varepsilon^N\left\{I_{\lambda_1}(W_{\lambda_1})+\frac{1}{2}e^{-\frac{2\sqrt{\lambda_1}(1-\frac{\eta}{2})(d_{\varepsilon,1}+\delta)}{\varepsilon}}\right\}\\
&\geq\;\varepsilon^N\left\{I_{\lambda_1}(W_{\lambda_1})+\frac{1}{2}e^{-\frac{2(1-\frac{\eta}{2})(d_\varepsilon+\delta)}{\varepsilon}}\right\}.
\end{aligned}$$

Similarly we obtain the estimate for $E_{\varepsilon,\lambda_2}(u^-_\varepsilon)$. This proves the result.

Proof of Theorem 1.1. This follows from Lemma 2.5 and Section 3. □

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E.N. Dancer, School of Mathematics and Statistics, The University of Sydney NSW 2006, Australia.

E-mail address: `normd@maths.usyd.edu.au`

S. Santra, Department of Mathematics, The Chinese University of Hong Kong, Shatin, Hong Kong.

E-mail address: `ssantra@math.cuhk.edu.hk`

J. Wei, Department of Mathematics, The Chinese University of Hong Kong, Shatin, Hong Kong.

E-mail address: `wei@math.cuhk.edu.hk`