Double tori solution to an equation of mean curvature and Newtonian potential

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Abstract

Studies of near periodic patterns in many self-organizing physical and biological systems give rise to a nonlocal geometric problem in the entire space involving the mean curvature and the Newtonian potential. One looks for a set in space of the prescribed volume such that on the boundary of the set the sum of the mean curvature of the boundary and the Newtonian potential of the set, multiplied by a parameter, is constant. Despite its simple form, the problem has a rich set of solutions and its corresponding energy functional has a complex landscape. When the parameter is sufficiently large, there exists a solution that consists of two tori: a larger torus and a smaller torus. Due to the axisymmetry, the problem is formulated on a half plane. A variant of the Lyapunov-Schmidt procedure is developed to reduce the problem to minimizing the energy of the set of two exact tori, as an approximate solution, with respect to their radii. A re-parameterization argument shows that the double tori so obtained indeed solves the equation of mean curvature and Newtonian potential. One also obtains the asymptotic formulae for the radii of the tori in terms of the parameter. This double tori set is the first known disconnected solution.

Key words. double tori, axisymmetry, nonlocal geometric problem, mean curvature, Newtonian potential, approximate solution, Lyapunov-Schmidt reduction, re-parameterization.

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1 Introduction

Near periodic patterns arise in many physical and biological systems as orderly outcomes of self-organization principles. Examples include morphological phases in block copolymers, animal coats, and skin pigmentation.

Block copolymers are soft condensed materials that in contrast to crystalline solids are characterized by fluid-like disorder on the molecular scale and a high degree of order on a longer length scale. A diblock copolymer molecule is a linear subchain of A-monomers grafted covalently to another subchain of B-monomers [4, 9]. Because of the repulsion between the unlike monomers, the different type subchains

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tend to segregate, but as they are chemically bonded in chain molecules, segregation of subchains cannot lead to a macroscopic phase separation. Only a local micro-phase separation occurs: micro-domains rich in A-monomers and micro-domains rich in B-monomers emerge as a result. These micro-domains form patterns known as morphological phases. The most common block copolymer morphological phases are the spherical, the cylindrical, and the lamellar phases.

Morphogenesis is a biological process by which an organism develops its shape. Turing postulated the presence of chemical signals and physico-chemical processes such as diffusion, activation, and deactivation in cellular and organismic growth [28]. They control the organized spatial distribution of cells during the embryonic development of an organism. Common in these pattern-forming systems is that a deviation from homogeneity has a strong positive feedback on its further increase. On its own, it would lead to an unlimited increase and spreading. Pattern formation requires in addition a longer ranging confinement of the locally self-enhancing process.

When a pattern in these morphology and morphogenesis problems consists of disconnected components, we may break the study of such a pattern into two separate steps. First a single piece of the pattern is isolated and identified as a solution to a related *profile* problem where one focuses on the interaction of the piece with itself. We call such a piece an ansatz. In the second step we place multiple copies of the ansatz in space and form a set of multiple components, after performing a small modification to each copy. The shape of each component is mostly determined by the ansatz. The interactions between the different components will determine their relative locations and directions in a self-organizing morphology/morphogenesis pattern.

In recent years several morphology/morphogenesis patterns have been constructed this way. See [22] for the multiple disc pattern, modeling the cylindrical morphological phase of block copolymers; see [23] for the multiple ball pattern, modeling the spherical morphological phase; see [11] for the multiple ring pattern, which often appears on animal skins.

This paper is devoted to a profile problem that is geometric and involves the mean curvature and the Newtonian potential. Given two parameters $m > 0$ and $\gamma > 0$ we look for a set E in $\mathbb{R}^n$ and a number λ such that the n-dimensional Lebesgue measure of E is m and on the boundary of E the equation

$$\mathcal{H}(\partial E) + \gamma \mathcal{N}(E) = \lambda \tag{1.1}$$

holds. In (1.1) $\mathcal{H}(\partial E)$ stands for the mean curvature of the boundary of E as an $(n-1)$-dimensional hypersurface or several hypersurfaces, viewed from the side of E, and $\mathcal{N}$ is the Newtonian potential operator. In $\mathbb{R}^3$, the case with which this paper is concerned, the Newtonian potential is given by

$$\mathcal{N}(E)(x) = \int_E \frac{dy}{4\pi|x-y|}. \tag{1.2}$$

The unknown constant λ is the Lagrange multiplier associated with the constraint that the volume of E is m. The problem admits a variational structure. It is the Euler-Lagrange equation of the energy functional

$$\mathcal{J}(E) = \frac{1}{n-1}\mathcal{P}(E) + \frac{\gamma}{2}\int_E \mathcal{N}(E)(x)\,dx, \tag{1.3}$$

defined for the Lebesgue measurable sets whose volume is m. Here $\mathcal{P}(E)$ stands for the perimeter of E, i.e. the $(n-1)$-dimensional area of the boundary of E.

The equation (1.1) is derived as the profile problem for the following geometric problem on a bounded domain. Let D be a bounded domain in $\mathbb{R}^n$, and $a \in (0,1)$ and $\gamma > 0$ be two positive parameters. Find a subset E of D and a number λ such that the n-dimensional Lebesgue measure of E is a times the Lebesgue measure of D and on $\partial_D E$, the part of the boundary of E that is inside D, the equation

$$\mathcal{H}(\partial_D E) + \gamma(-\Delta)^{-1}(\chi_E - a) = \lambda \tag{1.4}$$

holds. If $\partial_D E$ meets the boundary of D, then they meet perpendicularly. Here χ_E is the characteristic function of the set E, i.e. $\chi_E(x) = 1$ if $x \in E$ and $\chi_E(x) = 0$ if $x \in D \backslash E$; the operator $(-\Delta)^{-1}$ is the inverse of $-\Delta$ which comes with the zero Neumann boundary condition. The equation (1.4) also admits a variational structure:

$$\mathcal{J}_D(E) = \frac{1}{n-1}\mathcal{P}_D(E) + \frac{\gamma}{2}\int_D |(-\Delta)^{-1/2}(\chi_E - a)|^2\, dx. \tag{1.5}$$

This functional is defined on subsets E of D whose n-dimensional Lebesgue measure is a times the Lebesgue measure of D. The perimeter of E relative to D, denoted by $\mathcal{P}_D(E)$, is the $(n-1)$-dimensional area of the hypersurface $\partial_D E$, and $(-\Delta)^{-1/2}$ is the positive square root of the operator $(-\Delta)^{-1}$.

The equation (1.4) arises from a number of physical and biological pattern formation problems. It first appeared in [15] as a formal limit of the Ohta-Kawasaki block copolymer theory [16] in the strong segregation regime. The authors noted in [20] that the functional (1.5) is a Γ-limit of the Ohta-Kawasaki density functional under the Γ-convergence theory [6, 14, 13, 12]. This convergence gives a mathematically precise meaning of (1.4) being a limiting problem of the Ohta-Kawasaki theory. The equation (1.4) is also connected to the Gierer-Meinhardt system, an activator-inhibitor type reaction-diffusion system for morphogenesis in cell development [7]. Due to the lack of a variational structure in the Gierer-Meinhardt system, the connection cannot be interpreted as a Γ-convergence property. A formal asymptotic analysis that reduces the Gierer-Menihardt system to (1.4) may be found in [24].

The equation (1.4) is only solved completely in one dimension, where there are a countable number of solutions, each of which is a periodic set and each of which is a local minimizer of $\mathcal{J}$ [20]. In higher dimensions progresses have been made [1, 5, 26], but much remains unknown.

An effective way to study (1.4) in higher dimensions is through solutions to the profile equation (1.1). We explain this approach by an existence result of ours in [23]. The unit ball is a solution to (1.1). This is because that the unit sphere, the boundary of the unit ball, has constant mean curvature, and the Newtonian potential of the unit ball is a radially symmetric function with respect to the center of the ball, so it is also constant on the sphere. The authors showed that on a bounded domain there exists a solution to (1.4) with a number of components. Each component is close to a small ball. All these approximate small balls have almost the same radius. The centers of the balls are determined by minimizing a function that depends only on the domain D. We used the ball, a solution to the profile equation (1.1), as a building block of the multiple ball pattern solution to (1.4). The self-interaction property of a ball to itself is contained in the profile problem (1.1). Only the study of the interactions between distinct balls requires the equation (1.4). Roughly speaking the shape and the size of each component in our self-organizing problem are determined by the profile equation (1.1); the full problem (1.4) is mainly responsible for the locations of the components.

In addition to the unit ball, a less trivial solution is a shell, which is a region bounded by two concentric spheres [19]. It exists when γ is sufficiently large. This shell solution is always unstable. Both the ball and the shell solutions are radially symmetric.

The first non-radially symmetric solution was found by the authors in [25], also for large γ. It has the shape of a toroidal tube, as depicted in Figure 1. Define a function $f = f(\gamma)$ via its inverse

$$\gamma = \frac{2}{f^3 \log \frac{1}{2\pi^2 f^3}}. \tag{1.6}$$

Note that f maps from $(0, \infty)$ to $(0, \infty)$ and

$$\lim_{\gamma \to \infty} f(\gamma) = 0. \tag{1.7}$$

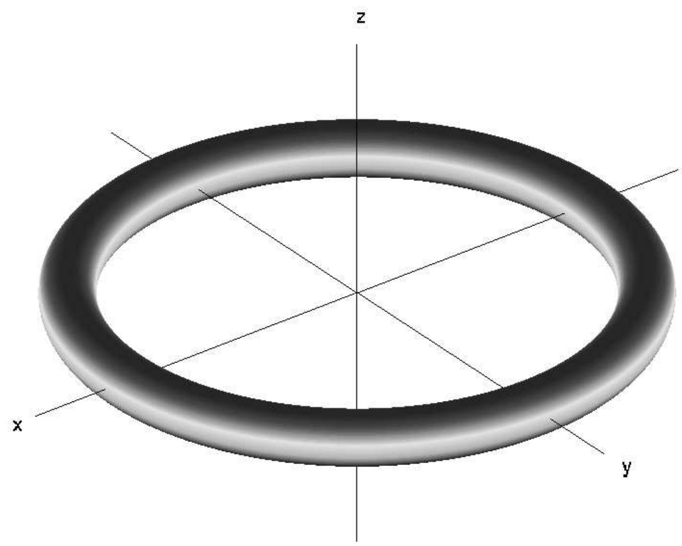

Figure 1: A single torus solution.

Theorem 1.1 ([25]) *When γ is large enough, the equation (1.1) admits a solution of volume equal to 1 which is close to a set enclosed by a torus. The torus is obtained by rotating a circle in the yz-plane about the z-axis. Let p^γ be the distance from the center of the circle to the z-axis and q^γ be the radius of the circle. Then $2\pi^2 p^\gamma (q^\gamma)^2 = 1$ and*

$$\lim_{\gamma\to\infty} \frac{q^\gamma}{f(\gamma)} = 1 \quad \textit{and} \quad \lim_{\gamma\to\infty} 2\pi^2 f^2(\gamma) p^\gamma = 1$$

where the function f is given in (1.6).

Toroidal shaped objects are fascinating and have been observed in many physical systems. Known as the vortex ring in fluid dynamics, it is a region of rotating fluid where the flow pattern takes on a toroidal shape [3]. In a quantum fluid, a vortex ring is formed by a loop of poloidal quantized flow pattern. It was detected in superfluid helium by Rayfield and Reif [18], and more recently in Bose-Einstein condensates by Anderson, *et al*, [2].

In a 2004 Science magazine article, Pochan and his collaborators reported the finding of a morphological phase of toroidal supramolecule assemblies, using a triblock copolymer [17]. They found this phase by combining dilute solution characteristics critical for both bundling of like-charged biopolymers and block copolymer micelle formation. The key to toroid versus classic cylinder micelle formation is the interaction of the negatively charged hydrophilic block of an amphiphilic triblock copolymer with a positively charged divalent organic counterion. This produces a self-attraction of cylindrical micelles that leads to toroid formation, a mechanism akin to the toroidal bundling of semiflexible charged biopolymers such as DNA.

In this paper we go one step further beyond Theorem 1.1 to investigate sets of multiple toroidal components. We prove the existence of a double tori solution, depicted in Figure 2. The solution is close to the union of two regions, each of which is bounded by a torus. Both tori are axisymmetric about the z-axis. In the yz-plane, each torus is represented by a circle. The circle of the inner, smaller torus is centered at $(p_1^\gamma, 0)$ and the circle of the outer, larger torus is centered at $(p_2^\gamma, 0)$ with $p_1^\gamma < p_2^\gamma$. Both circles have approximately the same radius.

The two parameters m and γ in the problem (1.1) can be reduced to one. We can take m to be any convenient number. In this paper, without the loss of generality, we assume that $m = 2$. The case $m \neq 2$

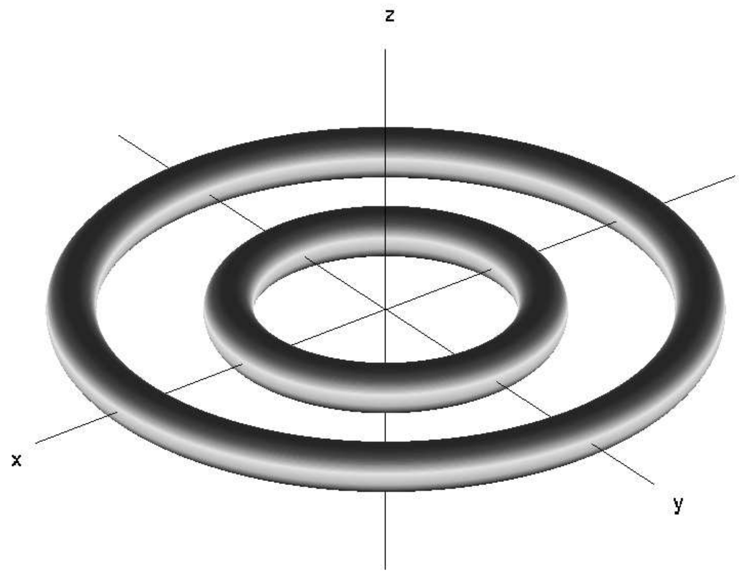

Figure 2: A double tori solution.

can be reduced to the case $m = 2$ by a change of the space variable and a change of γ accordingly. Therefore the problem (1.1) has intrinsically only one parameter, which we take to be γ.

For $0 < P_1 < P_2$ subject to the constraint $P_1 + P_2 = 2$, define a function

$$(P_1, P_2) \to \sum_{j=1}^{2} \Big(\frac{P_j}{16} + \frac{\pi P_j}{2} G_1(P_j, 0, P_j, 0) \Big) + \pi P_1 G(P_1, 0, P_2, 0). \tag{1.8}$$

In this definition G is a Green's function given in (2.3) and G_1, given in Lemma 2.1, is the the second term in the expansion of G about its singularity. We will show later that the function (1.8) attains its minimum in the interior of its domain. The following is the main result of this paper.

Theorem 1.2 *When γ is sufficiently large, the profile equation (1.1) admits a solution of volume equal to 2 which is a set of two components. Each component is close to a region enclosed by a torus. The two tori are obtained by rotating two circles in the yz-plane about the z-axis. One circle is centered at $(p_1^\gamma, 0)$ of radius q_1^γ, and the second circle is centered at $(p_2^\gamma, 0)$ of radius q_2^γ, where $2\pi^2 \sum_{j=1}^2 p_j^\gamma (q_j^\gamma)^2 = 2$. Moreover*

$$\lim_{\gamma\to\infty} \frac{q_j^\gamma}{f(\gamma)} = 1 \quad \textit{and} \quad \lim_{\gamma\to\infty} 2\pi^2 f^2(\gamma) p_j^\gamma = \Pi_j, \quad j = 1, 2,$$

where the function f is given in (1.6) and (Π_1, Π_2) is a minimum of the function (1.8).

The ball, the shell found in [19], and the single torus found in [25] are all connected sets. The double tori solution discovered here is the first disconnected solution to (1.1).

The proof of Theorem 1.2 starts with a careful study of the Newtonian potential operator (1.2) in the cylindrical coordinates. We obtain a three term asymptotic expansion for the kernel function of this operator, called a Green's function, around its singularity, Lemma 2.1. Next a set of two exact tori is used as an approximate solution and inserted into the left side of the equation (1.1), Lemma 3.2. The result is a quantity, depending on the four radii of the two tori, that differs slightly from a constant. Another important quantity is the energy of this double tori approximate solution, Lemma 3.3, which again depends on the four radii of the tori. Once the approximate solution is well understood, particularly the properties of the linearized operator at the approximate solution, one can use a variant of the Lyapunov-Schmidt procedure developed by the authors for this type of problems to show that an exact solution will deviate only slightly from the approximate solution of exact tori, Lemma 5.3. In the last step we determine the right values of the four radii of the two tori by minimizing the energy of the approximate solution with respect to the radii, Lemma 6.3. Finally we prove in Lemma 6.4 that the double tori obtained this way indeed solves the equation (1.1), using a tricky re-parameterization technique. A few remarks are included in the last section.

2 Green's function

Axisymmetric objects are best described by the cylindrical coordinates, i.e. $x = r\cos\sigma$, $y = r\sin\sigma$, and $z = z$, so that they are independent of the angle variable σ. The variables r and z are in

$$\mathbb{R}^2_+ = \{(r,z) : r > 0,\ z \in \mathbb{R}\}. \tag{2.1}$$

On an axisymmetric set E the operator $\mathcal{N}$ is represented by an integral operator with the kernel being a Green's function. More precisely

$$\mathcal{N}(E)(r,z) = \int_E G(r,z,s,t)\, dsdt \tag{2.2}$$

where the function G is our Green's function defined on $\mathbb{R}^2_+ \times \mathbb{R}^2_+$ with singularity at $(r,z) = (s,t)$. We use the same letter, E here, to denote an axisymmetric set in $\mathbb{R}^3$ and the set in $\mathbb{R}^2_+$ that represents the axisymmetric set. By (1.2) the Green's function G may be written as

$$G(r,z,s,t) = \frac{s}{4\pi}\int_0^{2\pi} \frac{d\sigma}{\sqrt{r^2+s^2-2rs\cos\sigma+(z-t)^2}}, \quad (r,z) \neq (s,t). \tag{2.3}$$

Moreover for each $(s,t) \in \mathbb{R}^2_+$ as a function of (r,z), $G(\cdot,s,t)$ satisfies

$$-\Delta G(\cdot,s,t) - \frac{\partial}{r\partial r}G(\cdot,s,t) = \delta_{(s,t)} \ \text{ in } \mathbb{R}^2_+, \ \frac{\partial}{\partial r}G(0,z,s,t) = 0 \ \ \forall z \in R. \tag{2.4}$$

Lemma 2.1 *For each* $(s,t) \in \mathbb{R}^2_+$

$$\begin{aligned} & G(r,z,s,t) \\ = \ & \frac{1}{2\pi}\log\frac{1}{|(r,z)-(s,t)|} + G_1(r,z,s,t) \\ = \ & \frac{1}{2\pi}\log\frac{1}{|(r,z)-(s,t)|} - \frac{r-s}{4\pi s}\log\frac{1}{|(r,z)-(s,t)|} + G_2(r,z,s,t) \\ = \ & \frac{1}{2\pi}\log\frac{1}{|(r,z)-(s,t)|} - \frac{r-s}{4\pi s}\log\frac{1}{|(r,z)-(s,t)|} + \frac{5(r-s)^2-(z-t)^2}{32\pi s^2}\log\frac{1}{|(r,z)-(s,t)|} \\ & +G_3(r,z,s,t) \end{aligned}$$

where, as functions of (r,z), $G_1(\cdot,s,t) \in C^{\alpha}_{loc}(\mathbb{R}^2_+)$, $G_2(\cdot,s,t) \in C^{1,\alpha}_{loc}(\mathbb{R}^2_+)$, *and* $G_3(\cdot,s,t) \in C^{2,\alpha}_{loc}(\mathbb{R}^2_+)$ *for all* $\alpha \in (0,1)$.

Proof. Since $-\frac{r-s}{4\pi s}\log\frac{1}{|(r,z)-(s,t)|}\in C^{\alpha}_{loc}(\mathbb{R}^2_+)$ and $\frac{5(r-s)^2-(z-t)^2}{32\pi s^2}\log\frac{1}{|(r,z)-(s,t)|}\in C^{1,\alpha}_{loc}(\mathbb{R}^2_+)$, it suffices to show that $G_3(\cdot,s,t)\in C^{2,\alpha}_{loc}(\mathbb{R}^2_+)$. Compute

$$\begin{aligned}
&(-\Delta-\frac{1}{r\partial r})\Big[\frac{1}{2\pi}\log\frac{1}{|(r,z)-(s,t)|}-\frac{r-s}{4\pi s}\log\frac{1}{|(r,z)-(s,t)|}+\frac{5(r-s)^2-(z-t)^2}{32\pi s^2}\log\frac{1}{|(r,z)-(s,t)|}\Big]\\
=\;& -\Delta\Big(\frac{1}{2\pi}\log\frac{1}{|(r,z)-(s,t)|}\Big)-\frac{1}{r\partial r}\Big(\frac{1}{2\pi}\log\frac{1}{|(r,z)-(s,t)|}\Big)\\
&+\Delta\Big(\frac{r-s}{4\pi s}\log\frac{1}{|(r,z)-(s,t)|}\Big)+\frac{1}{r\partial r}\Big(\frac{r-s}{4\pi s}\log\frac{1}{|(r,z)-(s,t)|}\Big)\\
&-\Delta\Big(\frac{5(r-s)^2-(z-t)^2}{32\pi s^2}\log\frac{1}{|(r,z)-(s,t)|}\Big)-\frac{1}{r\partial r}\Big(\frac{5(r-s)^2-(z-t)^2}{32\pi s^2}\log\frac{1}{|(r,z)-(s,t)|}\Big)\\
=\;& \delta_{(s,t)}+\frac{r-s}{2\pi r|(r,z)-(s,t)|^2}\\
&-\frac{r-s}{2\pi s|(r,z)-(s,t)|^2}+\frac{1}{4\pi rs}\log\frac{1}{|(r,z)-(s,t)|}-\frac{(r-s)^2}{4\pi rs|(r,z)-(s,t)|^2}\\
&-\frac{1}{4\pi s^2}\log\frac{1}{|(r,z)-(s,t)|}+\frac{5(r-s)^2-(z-t)^2}{8\pi s^2|(r,z)-(s,t)|^2}\\
&-\frac{5(r-s)}{16\pi rs^2}\log\frac{1}{|(r,z)-(s,t)|}+\frac{5(r-s)^3-(z-t)^2(r-s)}{32\pi rs^2|(r,z)-(s,t)|^2}.
\end{aligned}$$

Note that the above is $\delta_{(s,t)}$ plus a $C^{\alpha}_{loc}(\mathbb{R}^2_+)$ function, since

$$\begin{aligned}
\frac{r-s}{2\pi r|(r,z)-(s,t)|^2}-\frac{r-s}{2\pi s|(r,z)-(s,t)|^2}-\frac{(r-s)^2}{4\pi rs|(r,z)-(s,t)|^2}+\frac{5(r-s)^2-(z-t)^2}{8\pi s^2|(r,z)-(s,t)|^2} &\in C^{0,1}_{loc}(\mathbb{R}^2_+)\\
\frac{1}{4\pi rs}\log\frac{1}{|(r,z)-(s,t)|}-\frac{1}{4\pi s^2}\log\frac{1}{|(r,z)-(s,t)|} &\in C^{\alpha}_{loc}(\mathbb{R}^2_+)\\
-\frac{5(r-s)}{16\pi rs^2}\log\frac{1}{|(r,z)-(s,t)|} &\in C^{\alpha}_{loc}(\mathbb{R}^2_+)\\
\frac{5(r-s)^3-(z-t)^2(r-s)}{32\pi rs^2|(r,z)-(s,t)|^2} &\in C^{0,1}_{loc}(\mathbb{R}^2_+).
\end{aligned}$$

Here $C^{0,1}_{loc}(\mathbb{R}^2_+)$ denotes the space of locally Lipschitz continuous functions on $\mathbb{R}^2_+$. Because of (2.4), we deduce

$$(-\Delta-\frac{1}{r\partial r})G_3(\cdot,s,t)\in C^{\alpha}_{loc}(\mathbb{R}^2_+).$$

The elliptic regularity theory [8] implies that $G_3(\cdot,s,t)\in C^{2,\alpha}_{loc}(\mathbb{R}^2_+)$. □

Some useful properties of G, G_1, G_2 and G_3 are listed below.

Lemma 2.2 *Denote the derivatives of $G=G(r,z,s,t)$ (or G_1, G_2, G_3) with respect to r, z, s, and t by D_1G, D_2G, D_3G and D_4G respectively.*

1. *For $\lambda>0$, $G(\lambda r,\lambda z,\lambda s,\lambda t)=G(r,z,s,t)$.*
2. *For $r\neq s$, $D_2G(r,0,s,0)=0$.*
3. *For $r>0$ and $s>0$, $D_2G_3(r,0,s,0)=0$.*

Proof. Parts 1 and 2 follow directly from (2.3). To prove part 3, we first assume $r \neq s$ and deduce from $D_2G(r,0,s,0) = 0$ in part 2 that

$$\begin{aligned}
D_2G_3(r,0,s,0) &= -D_2\Big|_{z=t=0}\Big[\frac{1}{2\pi}\log\frac{1}{|(r,z)-(s,t)|} - \frac{r-s}{4\pi s}\log\frac{1}{|(r,z)-(s,t)|} \\
&\quad + \frac{5(r-s)^2-(z-t)^2}{32\pi s^2}\log\frac{1}{|(r,z)-(s,t)|}\Big] \\
&= -\Big[-\frac{z-t}{2\pi|(r,z)-(s,t)|^2} + \frac{(r-s)(z-t)}{4\pi s|(r,z)-(s,t)|^2} \\
&\quad -\frac{(z-t)}{16\pi s^2}\log\frac{1}{|(r,z)-(s,t)|} - \frac{5(r-s)^2(z-t)-(z-t)^3}{32\pi s^2|(r,z)-(s,t)|^2}\Big]_{z=t=0} \\
&= 0.
\end{aligned}$$

Since $G_3(\cdot,s,t) \in C^{2,\alpha}_{loc}(\mathbb{R}^2_+)$, $D_2G_3(r,0,s,0) = 0$ remains valid even if $r = s$. □

3 Double tori

Depicted in Figure 2 are two tori in $\mathbb{R}^3$ described by two circles in $\mathbb{R}^2_+$ of (2.1) centered at $(p_j, 0)$ with radii q_j, $j = 1, 2$. The two tori are obtained by rotating the circles about the z-axis. We often use shorthand notations p for (p_1, p_2) and q for (q_1, q_2). To specify the range of p and q we first introduce the scaled variables P_j and Q_j corresponding to p_j and q_j respectively such that

$$p_j = \frac{P_j}{2\pi^2 f^2(\gamma)}, \ q_j = f(\gamma)Q_j, \quad j = 1, 2 \tag{3.1}$$

where $f(\gamma)$ is given in (1.6), and let

$$\Omega = \{(P,Q) = (P_1,P_2,Q_1,Q_2) \in \mathbb{R}^4 : \ 0 < P_1 < P_2, \ Q_1 > 0, \ Q_2 > 0, \ \sum_{j=1}^2 P_jQ_j^2 = 2\}. \tag{3.2}$$

Here we write P for (P_1, P_2) and Q for (Q_1, Q_2). Next we consider a slice of Ω:

$$\widetilde{\Omega}_{II} = \{P \in \mathbb{R}^2 : \ (P_1,P_2,1,1) \in \Omega\} = \{P \in \mathbb{R}^2 : \ 0 < P_1 < P_2, \ P_1 + P_2 = 2\}. \tag{3.3}$$

It will become clear after Lemma 6.1 why the notations Ω and $\widetilde{\Omega}_{II}$ are used. The function (1.8) is defined on $\widetilde{\Omega}_{II}$. We denote this function by $\widetilde{J}_{II}$ now:

$$\widetilde{J}_{II}(P_1,P_2) = \sum_{j=1}^2\Big(\frac{P_j}{16} + \frac{\pi P_j}{2}G_1(P_j,0,P_j,0)\Big) + \pi P_1 G(P_1,0,P_2,0). \tag{3.4}$$

As $P_1 \to 1-$, $P_2 \to 1+$ and consequently $G(P_1,0,P_2,0) \to \infty$ and hence $\widetilde{J}_{II}(P_1,P_2) \to \infty$. On the other hand

$$G_1(P_1,0,P_1,0) = G_1(1,0,1,0) - \frac{1}{2\pi}\log\frac{1}{P_1}$$

implies

$$\frac{d\widetilde{J}_{II}(P_1, 2-P_1)}{dP_1} \to -\infty \ \text{ as } P_1 \to 0.$$

This shows that the global minium value of $\widetilde{J}_{II}$ is achieved in the interior of $\widetilde{\Omega}_{II}$. Denote the set of the global minimum points of $\widetilde{J}_{II}$ by $\widetilde{M}_{II}$:

$$\widetilde{M}_{II} = \{\Pi = (\Pi_1, \Pi_2) \in \widetilde{\Omega}_{II} : \ \widetilde{J}_{II}(\Pi) = \inf_{P \in \widetilde{\Omega}_{II}} \widetilde{J}_{II}(P)\}. \tag{3.5}$$

Now we are ready to specify the range $K \subset \Omega$ of (P, Q):

$$K \text{ is a compact subset of } \Omega \text{ and a neighborhood of the set } \{(\Pi_1, \Pi_2, 1, 1) : \ (\Pi_1, \Pi_2) \in \widetilde{M}_{II}\}. \tag{3.6}$$

More precisely, K is a compact subset of Ω, and the set of the interior points of K, with respect to the relative topology of Ω inherited from the topology of $\mathbb{R}^4$, contains $\{(\Pi_1, \Pi_2, 1, 1) : \ (\Pi_1, \Pi_2) \in \widetilde{M}_{II}\}$ as a subset.

From now on we require that (p, q) satisfy the condition

$$(P, Q) \in K, \tag{3.7}$$

where (P, Q) corresponds to (p, q) via (3.1). When γ is sufficiently large, we have

$$0 < p_1 - q_1 < p_1 < p_1 + q_1 < p_2 - q_2 < p_2 < p_2 + q_2 \tag{3.8}$$

for all (p, q) under the condition (3.7). We rotate the two circles centered at $(p_1, 0)$ and $(p_2, 0)$ of radii q_1 and q_2 respectively in $\mathbb{R}^2_+$ about the z-axis, to obtain two tori. The region inside the torus of p_j and q_j is denoted by T_j, $j = 1, 2$. By (3.8) the two sets T_1 and T_2 have no intersection. Set $T = T_1 \cup T_2$.

This double tori set T is used as an approximate solution of (1.1). It should generate a very small error when inserted into the equation (1.1). Between $\mathcal{H}(\partial T)$ and $\mathcal{N}(T)$, the estimation of $\mathcal{N}(T)$ is more involved.

Lemma 3.1 *$\mathcal{N}(T)$ on T_1 is*

$$\begin{aligned}
&\mathcal{N}(T)(p_1 + q_1 \hat{h}_1 \cos\theta_1, q_1 \hat{h}_1 \sin\theta_1) \\
&= \frac{q_1^2}{2} \log \frac{1}{q_1} + q_1^2 \Big[\pi G_3(p_1, 0, p_1, 0) + \frac{1}{4} - \frac{\hat{h}_1^2}{4}\Big] + \pi q_2^2 G(p_1, 0, p_2, 0) \\
&\quad - \frac{q_1^3}{4p_1} \Big(\log \frac{1}{q_1} \Big) \hat{h}_1 \cos\theta_1 + \frac{q_1^3}{p_1} \Big[\pi p_1 D_1 G_3(p_1, 0, p_1, 0) \hat{h}_1 + \frac{\hat{h}_1^3}{16}\Big] \cos\theta_1 + \pi q_1 q_2^2 D_1 G(p_1, 0, p_2, 0) \hat{h}_1 \cos\theta_1 \\
&\quad + O(\frac{|q|^4}{|p|^2} \log \frac{|p|}{|q|}),
\end{aligned}$$

where $\hat{h}_1 \in [0, 1]$; $\mathcal{N}(T)$ on T_2 is

$$\begin{aligned}
&\mathcal{N}(T)(p_2 + q_2 \hat{h}_2 \cos\theta_2, q_2 \hat{h}_2 \sin\theta) \\
&= \frac{q_2^2}{2} \log \frac{1}{q_2} + q_2^2 \Big[\pi G_3(p_2, 0, p_2, 0) + \frac{1}{4} - \frac{\hat{h}_2^2}{4}\Big] + \pi q_1^2 G(p_2, 0, p_1, 0) \\
&\quad - \frac{q_2^3}{4p_2} \Big(\log \frac{1}{q_2} \Big) \hat{h}_2 \cos\theta_2 + \frac{q_2^3}{p_2} \Big[\pi p_2 D_1 G_3(p_2, 0, p_2, 0) \hat{h}_2 + \frac{\hat{h}_2^3}{16}\Big] \cos\theta_2 + \pi q_2 q_1^2 D_1 G(p_2, 0, p_1, 0) \hat{h}_2 \cos\theta_2 \\
&\quad + O(\frac{|q|^4}{|p|^2} \log \frac{|p|}{|q|}),
\end{aligned}$$

where $\hat{h}_2 \in [0, 1]$.

Proof. Since $\mathcal{N}(T) = \mathcal{N}(T_1) + \mathcal{N}(T_2)$, we start with $\mathcal{N}(T_j)$ evaluated at a point in the same T_j. In the Green's function $G(r, z, s, t)$ let

$$(r, z) = (p_j + h_j \cos\theta_j, h_j \sin\theta_j), \quad (s, t) = (p_j + \rho_j \cos\omega_j, \rho_j \sin\omega_j) \tag{3.9}$$

where $h_j, \rho_j \in [0, q_j]$, so both (r, z) and (s, t) are points in the the same T_j. To simplify notations, we temporarily drop the subscript j. Then

$$\begin{aligned}
\mathcal{N}(T_j)(p_j + h_j \cos\theta_j, h_j \sin\theta_j) &= \mathcal{N}(T_j)(p + h\cos\theta, h\sin\theta) \\
&= \int_0^q \int_0^{2\pi} G(p + h\cos\theta, h\sin\theta, p + \rho\cos\omega, \rho\sin\omega)\rho\, d\omega d\rho \\
&= \int_0^q \int_0^{2\pi} \frac{1}{2\pi} \log \frac{1}{\sqrt{h^2 + \rho^2 - 2h\rho\cos(\theta - \omega)}} \Big[1 - \frac{h\cos\theta - \rho\cos\omega}{2(p + \rho\cos\omega)} \\
&\quad + \frac{5(h\cos\theta - \rho\cos\omega)^2 - (h\sin\theta - \rho\sin\omega)^2}{16(p + \rho\cos\omega)^2}\Big]\rho\, d\omega d\rho \\
&\quad + \int_0^q \int_0^{2\pi} G_3(p + h\cos\theta, h\sin\theta, p + \rho\cos\omega, \rho\sin\omega)\rho\, d\omega d\rho \\
&= I + II
\end{aligned} \tag{3.10}$$

where I and II in (3.10) are given by the two integrals above.

Because of the scaling property

$$G_3(\lambda r, \lambda z, \lambda s, \lambda t) = G_3(r, z, s, t) - \frac{1}{2\pi} \log \frac{1}{\lambda} \Big[1 - \frac{r - s}{2s} + \frac{5(r - s)^2 - (z - t)^2}{16 s^2}\Big] \tag{3.11}$$

which follows from Lemma 2.2 part 1 and Lemma 2.1, II becomes

$$\begin{aligned}
II &= -\frac{1}{2\pi} \log \frac{1}{p} \int_0^q \int_0^{2\pi} \Big[1 - \frac{h\cos\theta - \rho\cos\omega}{2(p + \rho\cos\omega)} + \frac{5(h\cos\theta - \rho\cos\omega)^2 - (h\sin\theta - \rho\sin\omega)^2}{16(p + \rho\cos\omega)^2}\Big]\rho\, d\omega d\rho \\
&\quad + \int_0^q \int_0^{2\pi} G_3(1 + \frac{h}{p}\cos\theta, \frac{h}{p}\sin\theta, 1 + \frac{\rho}{p}\cos\omega, \frac{\rho}{p}\sin\omega)\rho\, d\omega d\rho \\
&= II_I + II_{II}
\end{aligned} \tag{3.12}$$

where II_I and II_{II} are given by the two integrals above.

We introduce the scaled variables $\hat{h}$ and $\hat{\rho}$ by

$$h = q\hat{h}, \quad \rho = q\hat{\rho}. \tag{3.13}$$

Then I becomes

$$\begin{aligned}
I &= \frac{q^2}{2\pi} \log \frac{1}{q} \int_0^1 \int_0^{2\pi} \Big[1 - \frac{\hat{h}\cos\theta - \hat{\rho}\cos\omega}{2(p/q + \hat{\rho}\cos\omega)} + \frac{5(\hat{h}\cos\theta - \hat{\rho}\cos\omega)^2 - (\hat{h}\sin\theta - \hat{\rho}\sin\omega)^2}{16(p/q + \hat{\rho}\cos\omega)^2}\Big]\hat{\rho}\, d\omega d\hat{\rho} \\
&\quad + \frac{q^2}{2\pi} \int_0^1 \int_0^{2\pi} \log \frac{1}{\sqrt{\hat{h}^2 + \hat{\rho}^2 - 2\hat{h}\hat{\rho}\cos(\theta - \omega)}} \Big[1 - \frac{\hat{h}\cos\theta - \hat{\rho}\cos\omega}{2(p/q + \hat{\rho}\cos\omega)} \\
&\quad + \frac{5(\hat{h}\cos\theta - \hat{\rho}\cos\omega)^2 - (\hat{h}\sin\theta - \hat{\rho}\sin\omega)^2}{16(p/q + \hat{\rho}\cos\omega)^2}\Big]\hat{\rho}\, d\omega d\hat{\rho} \\
&= I_I + I_{II}
\end{aligned} \tag{3.14}$$

where I_I and I_{II} are the two integrals above.

Combining I_I and II_I we find

$$\begin{aligned}
I_I + II_I &= \frac{q^2}{2\pi}\log\frac{p}{q}\int_0^1\int_0^{2\pi}\Big[1-\frac{\hat{h}\cos\theta-\hat{\rho}\cos\omega}{2(p/q+\hat{\rho}\cos\omega)}+\frac{5(\hat{h}\cos\theta-\hat{\rho}\cos\omega)^2-(\hat{h}\sin\theta-\hat{\rho}\sin\omega)^2}{16(p/q+\hat{\rho}\cos\omega)^2}\Big]\hat{\rho}\,d\omega d\hat{\rho} \\
&= \frac{q^2}{2\pi}\log\frac{p}{q}\Big[A+B+C\Big] \qquad (3.15)
\end{aligned}$$

where

$$\begin{aligned}
A &= \int_0^1\int_0^{2\pi}\hat{\rho}\,d\omega d\hat{\rho} \;=\; \pi \\
B &= \int_0^1\int_0^{2\pi} -\frac{\hat{h}\cos\theta-\hat{\rho}\cos\omega}{2(p/q+\hat{\rho}\cos\omega)}\hat{\rho}\,d\omega d\hat{\rho} \\
&= -\frac{q}{2p}\int_0^1\int_0^{2\pi}\Big(\hat{h}\cos\theta-\hat{\rho}\cos\omega+O(\frac{q}{p})\Big)\hat{\rho}\,d\omega d\hat{\rho} \;=\; -\frac{\pi q}{2p}\hat{h}\cos\theta+O(\frac{q^2}{p^2}) \\
C &= \int_0^1\int_0^{2\pi}\frac{5(\hat{h}\cos\theta-\hat{\rho}\cos\omega)^2-(\hat{h}\sin\theta-\hat{\rho}\sin\omega)^2}{16(p/q+\hat{\rho}\cos\omega)^2}\hat{\rho}\,d\omega d\hat{\rho} \;=\; O(\frac{q^2}{p^2}).
\end{aligned}$$

This shows

$$I_I + II_I = \frac{q^2}{2}\log\frac{p}{q}-\frac{q^3}{4p}\Big(\log\frac{p}{q}\Big)\hat{h}\cos\theta+O(\frac{q^4}{p^2}\log\frac{p}{q}) \qquad (3.16)$$

The term I_{II} must be estimated carefully. We write

$$\begin{aligned}
I_{II} &= \frac{q^2}{2\pi}\int_0^1\int_0^{2\pi}\log\frac{1}{\sqrt{\hat{h}^2+\hat{\rho}^2-2\hat{h}\hat{\rho}\cos(\theta-\omega)}}\Big[1-\frac{\hat{h}\cos\theta-\hat{\rho}\cos\omega}{2(p/q+\hat{\rho}\cos\omega)} \qquad (3.17) \\
&\quad +\frac{5(\hat{h}\cos\theta-\hat{\rho}\cos\omega)^2-(\hat{h}\sin\theta-\hat{\rho}\sin\omega)^2}{16(p/q+\hat{\rho}\cos\omega)^2}\Big]\hat{\rho}\,d\omega d\hat{\rho} \\
&= \frac{q^2}{2\pi}\Big[\widetilde{A}+\widetilde{B}+\widetilde{C}\Big] \qquad (3.18)
\end{aligned}$$

where

$$\begin{aligned}
\widetilde{A} &= \int_0^1\int_0^{2\pi}\log\frac{1}{\sqrt{\hat{h}^2+\hat{\rho}^2-2\hat{h}\hat{\rho}\cos(\theta-\omega)}}\hat{\rho}\,d\omega d\hat{\rho} \\
\widetilde{B} &= -\int_0^1\int_0^{2\pi}\log\frac{1}{\sqrt{\hat{h}^2+\hat{\rho}^2-2\hat{h}\hat{\rho}\cos(\theta-\omega)}}\frac{\hat{h}\cos\theta-\hat{\rho}\cos\omega}{2(p/q+\hat{\rho}\cos\omega)}\hat{\rho}\,d\omega d\hat{\rho} \\
\widetilde{C} &= \int_0^1\int_0^{2\pi}\log\frac{1}{\sqrt{\hat{h}^2+\hat{\rho}^2-2\hat{h}\hat{\rho}\cos(\theta-\omega)}}\frac{5(\hat{h}\cos\theta-\hat{\rho}\cos\omega)^2-(\hat{h}\sin\theta-\hat{\rho}\sin\omega)^2}{16(p/q+\hat{\rho}\cos\omega)^2}\hat{\rho}\,d\omega d\hat{\rho} \;=\; O(\frac{q^2}{p^2}).
\end{aligned}$$

To study $\widetilde{A}$ and $\widetilde{B}$ we need the trigonometric series

$$\log\frac{1}{|1-\beta e^{i\omega}|} = \sum_{n=1}^{\infty}\frac{\beta^n\cos n\omega}{n} \qquad (3.19)$$

for $\beta \in [0,1]$; see [27]. One consequence of (3.19) is that

$$\int_0^{2\pi} \log \frac{1}{\sqrt{\hat{h}^2 + \hat{\rho}^2 - 2\hat{h}\hat{\rho}\cos(\theta - \omega)}}\, d\omega = \begin{cases} -2\pi \log \hat{h} & \text{if} \quad \hat{\rho} < \hat{h} \\ -2\pi \log \hat{\rho} & \text{if} \quad \hat{\rho} \geq \hat{h} \end{cases} \tag{3.20}$$

which implies that

$$\widetilde{A} = -2\pi \int_0^{\hat{h}} (\log \widetilde{h})\hat{\rho}\, d\hat{\rho} - 2\pi \int_{\hat{h}}^1 (\log \widetilde{\rho})\hat{\rho}\, d\hat{\rho} = 2\pi\Big(\frac{1}{4} - \frac{\hat{h}^2}{4}\Big) \tag{3.21}$$

Regarding $\widetilde{B}$ we write

$$\begin{aligned} \widetilde{B} \quad &= \quad -\frac{q}{2p} \int_0^1 \int_0^{2\pi} \log \frac{1}{\sqrt{\hat{h}^2 + \hat{\rho}^2 - 2\hat{h}\hat{\rho}\cos(\theta - \omega)}} \hat{h} \cos\theta \hat{\rho}\, d\omega d\hat{\rho} \\ &\quad\quad + \frac{q}{2p} \int_0^1 \int_0^{2\pi} \log \frac{1}{\sqrt{\hat{h}^2 + \hat{\rho}^2 - 2\hat{h}\hat{\rho}\cos(\theta - \omega)}} \hat{\rho} \cos\omega \hat{\rho}\, d\omega d\hat{\rho} \\ &\quad\quad + O(\frac{q^2}{p^2}) \\ &= \quad \widetilde{B}_1 + \widetilde{B_2} + O(\frac{q^2}{p^2}) \end{aligned} \tag{3.22}$$

where $\widetilde{B}_1$ and $\widetilde{B}_2$ are given by the two integrals above.

Using (3.20) again, we derive

$$\widetilde{B}_1 = -\frac{q}{2p}\Big[\int_0^{\hat{h}} (-2\pi \log \hat{h})\hat{h}\cos\theta \hat{\rho}\, d\hat{\rho} + \int_{\hat{h}}^1 (-2\pi \log \hat{\rho})\hat{h}\cos\theta\hat{\rho}\, d\rho\Big] = \frac{\pi q}{p}\cos\theta\Big(\frac{\hat{h}^3}{4} - \frac{\hat{h}}{4}\Big). \tag{3.23}$$

For $\widetilde{B}_2$ we note that

$$\begin{aligned} \widetilde{B}_2 \quad &= \quad \frac{q}{2p} \int_0^1 \int_0^{2\pi} \log \frac{1}{\sqrt{\hat{h}^2 + \hat{\rho}^2 - 2\hat{h}\hat{\rho}\cos(\theta - \omega)}} \hat{\rho}\cos\omega\hat{\rho}\, d\omega d\hat{\rho} \\ &= \quad \frac{q}{2p} \int_0^1 \int_0^{2\pi} \log \frac{1}{\sqrt{\hat{h}^2 + \hat{\rho}^2 - 2\hat{h}\hat{\rho}\cos(\theta - \omega)}} \hat{\rho}\cos(\theta - \omega)\cos\theta\hat{\rho}\, d\omega d\hat{\rho} \\ &\quad + \frac{q}{2p} \int_0^1 \int_0^{2\pi} \log \frac{1}{\sqrt{\hat{h}^2 + \hat{\rho}^2 - 2\hat{h}\hat{\rho}\cos(\theta - \omega)}} \hat{\rho}\sin(\theta - \omega)\sin\theta\hat{\rho}\, d\omega d\hat{\rho} \\ &= \quad \frac{q}{2p} \int_0^1 \int_0^{2\pi} \log \frac{1}{\sqrt{\hat{h}^2 + \hat{\rho}^2 - 2\hat{h}\hat{\rho}\cos(\theta - \omega)}} \hat{\rho}\cos(\theta - \omega)\cos\theta\hat{\rho}\, d\omega d\hat{\rho}. \end{aligned}$$

To reach the last line, we have used the fact that

$$\frac{q}{2p} \int_0^1 \int_0^{2\pi} \log \frac{1}{\sqrt{\hat{h}^2 + \hat{\rho}^2 - 2\hat{h}\hat{\rho}\cos(\theta - \omega)}} \hat{\rho}\sin(\theta - \omega)\sin\theta\hat{\rho}\, d\omega d\hat{\rho} = 0,$$

which follows from the absence of $\sin n\omega$ terms in (3.19). Another consequence of (3.19) is

$$\begin{aligned}
&\int_0^{2\pi} \log \frac{1}{\sqrt{\hat{h}^2+\hat{\rho}^2-2\hat{h}\hat{\rho}\cos(\theta-\omega)}} \cos(\theta-\omega)\, d\omega \\
&= \int_0^{2\pi} \log \frac{1}{\sqrt{1+\beta^2-2\beta\cos\omega}} \cos\omega\, d\omega \\
&= \int_0^{2\pi} \beta \cos^2\omega\, d\omega \;=\; \pi\beta \quad \text{where} \quad \beta = \begin{cases} \frac{\hat{\rho}}{\hat{h}} & \text{if} \quad \hat{\rho} < \hat{h} \\ \frac{\hat{h}}{\hat{\rho}} & \text{if} \quad \hat{\rho} \geq \hat{h} \end{cases}
\end{aligned} \tag{3.24}$$

Therefore

$$\widetilde{B}_2 = \frac{\pi q}{2p}\Big[\int_0^{\hat{h}} \frac{\hat{\rho}}{\hat{h}}\hat{\rho}^2 \cos\theta\, d\hat{\rho} + \int_{\hat{h}}^1 \frac{\hat{h}}{\hat{\rho}}\hat{\rho}^2\cos\theta\, d\hat{\rho}\Big] = \frac{\pi q}{2p}\cos\theta\Big(\frac{\hat{h}}{2}-\frac{\hat{h}^3}{4}\Big) \tag{3.25}$$

Following (3.23) and (3.25) we deduce

$$\widetilde{B} = \frac{\pi q}{8p}\hat{h}^3\cos\theta + O(\frac{q^2}{p^2}). \tag{3.26}$$

By (3.18), (3.21), and (3.26) we find

$$I_{II} = q^2\Big(\frac{1}{4}-\frac{\hat{h}^2}{4}\Big) + \frac{q^3}{16p}\hat{h}^3\cos\theta + O(\frac{q^4}{p^2}). \tag{3.27}$$

The estimation of II_{II} is straightforward. Because $G_3(\cdot,\cdot,s,t)$ is in $C^{2,\alpha}(\mathbb{R}^2_+)$ and $D_2G_3(1,0,1,0)=0$ according to Lemma 2.2 part 3,

$$\begin{aligned}
II_{II} &= q^2\int_0^1\int_0^{2\pi} G_3(1+\frac{q}{p}\hat{h}\cos\theta, \frac{q}{p}\hat{h}\sin\theta, 1+\frac{q}{p}\hat{\rho}\cos\omega, \frac{q}{p}\hat{\rho}\sin\omega)\hat{\rho}\, d\omega d\hat{\rho} \\
&= q^2\int_0^1\int_0^{2\pi}\Big[G_3(1,0,1,0) + D_1G_3(1,0,1,0)\frac{q}{p}\hat{h}\cos\theta + D_2G_3(1,0,1,0)\frac{q}{p}\hat{h}\sin\theta \\
&\quad + D_3G_3(1,0,1,0)\frac{q}{p}\hat{\rho}\cos\omega + D_4G_3(1,0,1,0)\frac{q}{p}\hat{\rho}\sin\omega + O(\frac{q^2}{p^2})\Big]\hat{\rho}\, d\omega d\hat{\rho} \\
&= q^2\int_0^1\int_0^{2\pi}\Big[G_3(1,0,1,0) + D_1G_3(1,0,1,0)\frac{q}{p}\hat{h}\cos\theta\Big]\hat{\rho}\, d\omega d\hat{\rho} + O(\frac{q^4}{p^2}) \\
&= \pi q^2 G_3(1,0,1,0) + \frac{\pi q^3}{p}D_1G_3(1,0,1,0)\hat{h}\cos\theta + O(\frac{q^4}{p^2}).
\end{aligned} \tag{3.28}$$

We now combine (3.16), (3.27), and (3.28) to conclude that, for $j=1,2$,

$$\begin{aligned}
\mathcal{N}(T_j)(p_j+h_j\cos\theta_j, h_j\sin\theta_j) &= \frac{q_j^2}{2}\log\frac{p_j}{q_j} + q_j^2\Big[\frac{1}{4}-\frac{\hat{h}_j^2}{4}+\pi G_3(1,0,1,0)\Big] \\
&\quad -\frac{q_j^3}{4p_j}\Big(\log\frac{p_j}{q_j}\Big)\hat{h}_j\cos\theta_j + \frac{q_j^3}{p_j}\Big[\frac{\hat{h}_j^3}{16}+\pi D_1G_3(1,0,1,0)\hat{h}_j\Big]\cos\theta_j \\
&\quad +O(\frac{q_j^4}{p_j^2}\log\frac{p_j}{q_j}).
\end{aligned} \tag{3.29}$$

Note that we have restored the subscript j.

Next we consider $\mathcal{N}(T_1)$ on T_2.

$$\mathcal{N}(T_1)(p_2+h_2\cos\theta_2,h_2\sin\theta_2)=\int_0^{q_1}\int_0^{2\pi}G(p_2+h_2\cos\theta_2,h_2\sin\theta_2,p_1+\rho_1\cos\omega_1,\rho_1\sin\omega_1)\rho_1\,d\omega_1 d\rho_1 \tag{3.30}$$

where $h_2\in[0,q_2]$. For this we only need to expand

$$\begin{aligned}&G(p_2+h_2\cos\theta_2,h_2\sin\theta_2,p_1+\rho_1\cos\omega_1,\rho_1\sin\omega_1)\\ =\;& G(p_2,0,p_1,0)+D_1G(p_2,0,p_1,0)h_2\cos\theta_2+D_2G(p_2,0,p_1,0)h_2\sin\theta_2\\ &+D_3G(p_2,0,p_1,0)\rho_1\cos\omega_1+D_4G(p_2,0,p_1,0)\rho_1\sin\omega_1+O(\frac{|q|^2}{|p|^2}).\end{aligned}$$

Since $D_2G(p_2,0,p_1,0)=0$ (Lemma 2.2 part 2), it follows that

$$\mathcal{N}(T_1)(p_2+h_2\cos\theta_2,h_2\sin\theta_2)=\pi q_1^2G(p_2,0,p_1,0)+\pi q_1^2q_2D_1G(p_2,0,p_1,0)\hat{h}_2\cos\theta_2+O(\frac{|q|^4}{|p|^2}). \tag{3.31}$$

Similarly $\mathcal{N}(T_2)$ on T_1 is

$$\mathcal{N}(T_2)(p_1+h_1\cos\theta_1,h_1\sin\theta_1)=\pi q_2^2G(p_1,0,p_2,0)+\pi q_1q_2^2D_1G(p_1,0,p_2,0)\hat{h}_1\cos\theta_1+O(\frac{|q|^4}{|p|^2}). \tag{3.32}$$

By (3.29) and (3.32) we deduce that $\mathcal{N}(T)$ on T_1 is

$$\begin{aligned}&\mathcal{N}(T)(p_1+h_1\cos\theta_1,h_1\sin\theta_1)\\ =\;&\frac{q_1^2}{2}\log\frac{p_1}{q_1}+q_1^2\Big[\frac14-\frac{\hat{h}_1^2}{4}+\pi G_3(1,0,1,0)\Big]+\pi q_2^2G(p_1,0,p_2,0)\\ &-\frac{q_1^3}{4p_1}\Big(\log\frac{p_1}{q_1}\Big)\hat{h}_1\cos\theta_1+\frac{q_1^3}{p_1}\Big[\frac{\hat{h}_1^3}{16}+\pi D_1G_3(1,0,1,0)\hat{h}_1\Big]\cos\theta_1+\pi q_1q_2^2D_1G(p_1,0,p_2,0)\hat{h}_1\cos\theta_1\\ &+O(\frac{|q|^4}{|p|^2}\log\frac{|p|}{|q|}),\end{aligned} \tag{3.33}$$

for $h_1\in[0,q_1]$. And similarly $\mathcal{N}(T)$ on T_2 is

$$\begin{aligned}&\mathcal{N}(T)(p_2+h_2\cos\theta_2,h_2\sin\theta_2)\\ =\;&\frac{q_2^2}{2}\log\frac{p_2}{q_2}+q_2^2\Big[\frac14-\frac{\hat{h}_2^2}{4}+\pi G_3(1,0,1,0)\Big]+\pi q_1^2G(p_2,0,p_1,0)\\ &-\frac{q_2^3}{4p_2}\Big(\log\frac{p_2}{q_2}\Big)\hat{h}_2\cos\theta_2+\frac{q_2^3}{p_2}\Big[\frac{\hat{h}_2^3}{16}+\pi D_1G_3(1,0,1,0)\hat{h}_2\Big]\cos\theta_2+\pi q_2q_1^2D_1G(p_2,0,p_1,0)\hat{h}_2\cos\theta_2\\ &+O(\frac{|q|^4}{|p|^2}\log\frac{|p|}{|q|}),\end{aligned} \tag{3.34}$$

for $h_2\in[0,q_2]$. Since

$$G_3(p_j,0,p_j,0)=G_3(1,0,1,0)-\frac{1}{2\pi}\log\frac{1}{p_j}\quad\text{and}\quad p_jD_1G_3(p_j,0,p_j,0)=D_1G_3(1,0,1,0)+\frac{1}{4\pi}\log\frac{1}{p_j}$$

by (3.11), the lemma follows from (3.33) and (3.34). $\square$

We insert T into the left side of the equation $\mathcal{H}(\partial E)+\gamma\mathcal{N}(E)=\lambda$ and see how much it deviates from a constant.

Lemma 3.2 *On ∂T_1*

$$
\begin{aligned}
&\Big(\mathcal{H}(\partial T) + \gamma \mathcal{N}(T)\Big)(p_1 + q_1\cos\theta_1, q_1 \sin\theta_1) \\
&= \frac{1}{2q_1} + \frac{\cos\theta_1}{2p_1} + O(\frac{q_1}{p_1^2}) + \gamma\Big\{\frac{q_1^2}{2}\log\frac{1}{q_1} + \pi q_1^2 G_3(p_1,0,p_1,0) + \pi q_2^2 G(p_1,0,p_2,0) \\
&\quad -\frac{q_1^3}{4p_1}\Big(\log\frac{1}{q_1}\Big)\cos\theta_1 + \frac{q_1^3}{p_1}\Big[\pi p_1 D_1 G_3(p_1,0,p_1,0) + \frac{1}{16}\Big]\cos\theta_1 + \pi q_1 q_2^2 D_1 G(p_1,0,p_2,0)\cos\theta_1 \\
&\quad +O(\frac{|q|^4}{|p|^2}\log\frac{|p|}{|q|})\Big\};
\end{aligned}
$$

on ∂T_2

$$
\begin{aligned}
&\Big(\mathcal{H}(\partial T) + \gamma \mathcal{N}(T)\Big)(p_2 + q_2\cos\theta_2, q_2 \sin\theta_2) \\
&= \frac{1}{2q_2} + \frac{\cos\theta_2}{2p_2} + O(\frac{q_2}{p_2^2}) + \gamma\Big\{\frac{q_2^2}{2}\log\frac{1}{q_2} + \pi q_2^2 G_3(p_2,0,p_2,0) + \pi q_1^2 G(p_2,0,p_1,0) \\
&\quad -\frac{q_2^3}{4p_2}\Big(\log\frac{1}{q_2}\Big)\cos\theta_2 + \frac{q_2^3}{p_2}\Big[\pi p_2 D_1 G_3(p_2,0,p_2,0) + \frac{1}{16}\Big]\cos\theta_2 + \pi q_2 q_1^2 D_1 G(p_2,0,p_1,0)\cos\theta_2 \\
&\quad +O(\frac{|q|^4}{|p|^2}\log\frac{|p|}{|q|})\Big\}.
\end{aligned}
$$

Proof. The mean curvature of the torus ∂T_j at $(r,z) = (p_j + q_j\cos\theta_j, q_j\sin\theta_j)$ is given by

$$
\mathcal{H}(\partial T_j)(p_j + q_j\cos\theta_j, q_j\sin\theta_j) = \frac{1}{2}\Big(\frac{1}{q_j} + \frac{\cos\theta_j}{p_j + q_j\cos\theta_j}\Big) = \frac{1}{2q_j} + \frac{\cos\theta_j}{2p_j} + O(\frac{q_j}{p_j^2}). \tag{3.35}
$$

Lemma 3.2 follows from (3.35) and Lemma 3.1. □

We proceed to estimate $\mathcal{J}(T)$.

Lemma 3.3

$$
\begin{aligned}
\mathcal{J}(T) &= \sum_{j=1}^{2} 2\pi^2 p_j q_j + \frac{\gamma}{2}\Big[\sum_{j=1}^{2}\Big(\pi^2 p_j q_j^4 \log\frac{1}{q_j} + \frac{\pi^2 p_j q_j^4}{4} + 2\pi^3 p_j q_j^4 G_1(p_j,0,p_j,0)\Big) \\
&\quad +4\pi^3 p_1 q_1^2 q_2^2 G(p_1,0,p_2,0)\Big] + O(\frac{\gamma |q|^4}{|p|^2}\log\frac{|p|}{|q|})
\end{aligned}
$$

Proof. The perimeter of T is

$$
\mathcal{P}(T) = \sum_{j=1}^{2} 4\pi^2 p_j q_j, \tag{3.36}
$$

which is the sum of the area of ∂T_1 and the area of ∂T_2.

The nonlocal part of $\mathcal{J}(T)$ is $\frac{\gamma}{2}$ times

$$
2\pi \int_T \mathcal{N}(T)\, r dr dz. \tag{3.37}
$$

Regarding (3.37) we first consider

$$2\pi \int_{T_1} \mathcal{N}(T)\, r dr dz = 2\pi \int_0^{q_1} \int_0^{2\pi} \mathcal{N}(T)(p_1 + h_1\cos\theta_1, h_1 \sin\theta_1)(p_1 + h_1 \cos\theta_1)\, h_1 d\theta_1 dh_1. \tag{3.38}$$

By Lemma 3.1, on T_1

$$\begin{aligned}\mathcal{N}(T)(p_1 + h_1\cos\theta_1, h_1\sin\theta_1) \;=\;& \frac{q_1^2}{2}\log\frac{1}{q_1} + q_1^2\Big[\pi G_3(p_1,0,p_1,0) + \frac{1}{4} - \frac{h_1^2}{4q_1^2}\Big] \\ &+\pi q_2^2 G(p_1,0,p_2,0) \\ &-\frac{q_1^3}{4p_1}\Big(\log\frac{1}{q_1}\Big)\frac{h_1}{q_1}\cos\theta_1 + \frac{q_1^3}{p_1}\Big[\pi p_1 D_1 G_3(p_1,0,p_1,0)\frac{h_1}{q_1} + \frac{h_1^3}{16 q_1^3}\Big]\cos\theta_1 \\ &+\pi q_1 q_2^2 D_1 G(p_1,0,p_2,0)\frac{h_1}{q_1}\cos\theta_1 + O(\frac{|q|^4}{|p|^2}\log\frac{|p|}{|q|}).\end{aligned}$$

Therefore (3.38) becomes

$$\begin{aligned}&2\pi\int_{T_1}\mathcal{N}(T)(r,z)\, r dr dz \\ =\;& 2\pi\int_0^{q_1}\int_0^{2\pi}\Big[\frac{q_1^2}{2}\log\frac{1}{q_1} + q_1^2\Big(\pi G_3(p_1,0,p_1,0) + \frac{1}{4} - \frac{h_1^2}{4q_1^2}\Big)\Big](p_1 + h_1\cos\theta_1)\, h_1 d\theta_1 dh_1 \\ &+2\pi\int_0^{q_1}\int_0^{2\pi}\pi q_2^2 G(p_1,0,p_2,0)(p_1 + h_1\cos\theta_1)\, h_1 d\theta_1 dh_1 \\ &+2\pi\int_0^{q_1}\int_0^{2\pi}\Big[-\frac{q_1^3}{4p_1}\Big(\log\frac{1}{q_1}\Big)\frac{h_1}{q_1}\cos\theta_1 + \frac{q_1^3}{p_1}\Big(\pi p_1 D_1 G_3(p_1,0,p_1,0)\frac{h_1}{q_1} + \frac{h_1^3}{16q_1^3}\Big)\cos\theta_1\Big] \\ &\qquad (p_1 + h_1\cos\theta_1)\, h_1 d\theta_1 dh_1 \\ &+2\pi\int_0^{q_1}\int_0^{2\pi}\pi q_1 q_2^2 D_1 G(p_1,0,p_2,0)\frac{h_1}{q_1}\cos\theta_1(p_1 + h_1\cos\theta_1)\, h_1 dh_1 d\theta_1 + O(\frac{|q|^4}{|p|^2}\log\frac{|p|}{|q|}) \\ =\;& \pi^2 p_1 q_1^4\log\frac{1}{q_1} + 2\pi^3 p_1 q_1^4 G_3(p_1,0,p_1,0) + 2\pi^3 p_1 q_1^2 q_2^2 G(p_1,0,p_2,0) + O(\frac{|q|^4}{|p|^2}\log\frac{|p|}{|q|}) \\ =\;& \pi^2 p_1 q_1^4\log\frac{1}{q_1} + 2\pi^3 p_1 q_1^4 G_1(p_1,0,p_1,0) + 2\pi^3 p_1 q_1^2 q_2^2 G(p_1,0,p_2,0) + O(\frac{|q|^4}{|p|^2}\log\frac{|p|}{|q|}) \qquad (3.39)\end{aligned}$$

where the last line follows from $G_3(p,0,p,0) = G_1(p,0,p,0)$, a consequence of Lemma 2.1.

Similarly

$$\begin{aligned}&2\pi\int_{T_2}\mathcal{N}(T)(r,z)\, r dr dz \\ &\quad = \pi^2 p_2 q_2^4\log\frac{1}{q_2} + 2\pi^3 p_2 q_2^4 G_1(p_2,0,p_2,0) + 2\pi^3 p_2 q_2^2 q_1^2 G(p_2,0,p_1,0) + O(\frac{|q|^4}{|p|^2}\log\frac{|p|}{|q|}). \qquad (3.40)\end{aligned}$$

From (3.36), (3.39), (3.40), and the symmetry $p_1 G(p_1,0,p_2) = p_2 G(p_2,0,p_1,0)$, one finds $\mathcal{J}(T)$. $\square$

4 Perturbation

The solution we will construct is a union of two sets enclosed by two approximate tori. To describe an approximate torus we need a way to perturb the exact double torus T_j. Note that T_j is specified by p_j and

q_j. Let $u_1 = u_1(\theta_1)$ and $u_2 = u_2(\theta_2)$ be two 2π-periodic functions. In the rz-plane $\mathbb{R}^2_+$ define the sets

$$E_j = \bigcup_{\theta_j \in [0,2\pi]} \{(p_j + h_j \cos\theta_j, h_j \sin\theta_j) : \ h_j \in [0, u_j(\theta_j)]\}, \ j = 1,2; \quad E = E_1 \cup E_2. \tag{4.1}$$

One then rotates E about the z-axis and obtains a set in $\mathbb{R}^3$ which is a perturbation of T. This axisymmetric set in $\mathbb{R}^3$ is again denoted by E, if no confusion exists. For this definition to be meaningful we will ensure

$$u_j(\theta_j) > 0, \quad \forall \theta_j \in [0, 2\pi]. \tag{4.2}$$

The perfect tori T_1 and T_2 introduced earlier are described by constant functions:

$$u_j(\theta_j) = q_j, \ \forall \theta_j \in [0, 2\pi]. \tag{4.3}$$

For a perturbed torus E_j the variable u_j generally depends on θ_j.

In terms of u_1 and u_2 the energy of E is given by

$$\mathcal{J}(E) = \sum_{j=1}^{2} \pi \int_0^{2\pi} (p_j + u_j(\theta_j)\cos\theta_j) \sqrt{(u_j'(\theta_j))^2 + u_j^2(\theta_j)}\, d\theta_j + \gamma\pi \int_E \int_E G(r,z,s,t) r\, dsdtdrdz \tag{4.4}$$

where the set E in the double integral is in $\mathbb{R}^2_+$.

The volume of a perturbed torus is given by

$$\begin{aligned} 2\pi \int_{E_j} r dr dz &= 2\pi \int_0^{2\pi} \int_0^{u_j(\theta_j)} (p_j + \rho_j \cos\theta_j) \rho_j d\rho_j d\theta_j \\ &= 2\pi \int_0^{2\pi} \Big[\frac{p_j u_j^2(\theta_j)}{2} + \frac{u_j^3(\theta_j)}{3} \cos\theta_j \Big] d\theta_j. \end{aligned}$$

The volume constraint $|E_1| + |E_2| = 2$ requires that

$$\sum_{j=1}^{2} 2\pi \int_0^{2\pi} \Big[\frac{p_j u_j^2(\theta_j)}{2} + \frac{u_j^3(\theta_j)}{3} \cos\theta_j \Big] d\theta_j = 2. \tag{4.5}$$

The equation (4.5) is a nonlinear constraint on u_1 and u_2. It is often more convenient to work with a different set of variables. Let

$$A(\mu, \alpha, b) = \frac{b\mu^2}{2} + \frac{\mu^3 \cos\alpha}{3}. \tag{4.6}$$

Define $v_j = v_j(\theta_j)$ by

$$v_j(\theta_j) = A(u_j(\theta_j), \theta_j, p_j), \quad j = 1, 2. \tag{4.7}$$

Now both u_j and v_j can be used to described the perturbed torus E_j. Since u_j of a perturbed torus E_j corresponds to q_j of an unperturbed torus T_j and q_j is the radius of a cross section of T_j, we call u_j the radius variable. Since $2\pi \int_0^{2\pi} v_j(\theta_j)\, d\theta_j$ is the volume of E_j, we call v_j the volume variable. In terms of the volume variable the unperturbed torus T_j is described by

$$\psi_j(\theta_j) = \frac{p_j q_j^2}{2} + \frac{q_j^3 \cos\theta_j}{3}, \ \forall \theta_j \in [0, 2\pi]. \tag{4.8}$$

Note that unlike (4.3) the function ψ_j in (4.8) is not constant. The advantage of using v_j is that the constraint (4.5) is simplified to a linear condition

$$\sum_{j=1}^{2} 2\pi \int_0^{2\pi} v_j(\theta_j)\, d\theta_j = 2 \tag{4.9}$$

on the volume variables v_1 and v_2.

Let us also denote the inverse of A where $\nu = A(\mu, \alpha, b)$, as a function of μ with α and b held fixed, by $\mu = B(\nu, \alpha, b)$ such that

$$u_j(\theta_j) = B(v_j(\theta_j), \theta_j, p_j). \tag{4.10}$$

Let us write the half of the area of the perturbed torus E_j as

$$\pi \int_0^{2\pi} L_j(v_j'(\theta_j), v_j(\theta_j), \theta_j)\, d\theta_j \tag{4.11}$$

where the Lagrangian L_j is given by

$$L_j(\dot{\nu}, \nu, \alpha) = (p_j + \mu \cos\alpha)\sqrt{\dot{\mu}^2 + \mu^2}. \tag{4.12}$$

In (4.12) $(\dot{\nu}, \nu, \alpha)$ and $(\dot{\mu}, \mu, \alpha)$ transfer according to the following rules.

$$\begin{pmatrix} \dot{\mu} \\ \mu \\ \alpha \end{pmatrix} \to \begin{pmatrix} \dot{\nu} \\ \nu \\ \alpha \end{pmatrix} = \begin{pmatrix} \dot{\mu} A_\mu(\mu, \alpha, p_j) + A_\alpha(\mu, \alpha, p_j) \\ A(\mu, \alpha, p_j) \\ \alpha \end{pmatrix}, \tag{4.13}$$

$$\begin{pmatrix} \dot{\nu} \\ \nu \\ \alpha \end{pmatrix} \to \begin{pmatrix} \dot{\mu} \\ \mu \\ \alpha \end{pmatrix} = \begin{pmatrix} \dot{\nu} B_\nu(\nu, \alpha, p_j) + B_\alpha(\nu, \alpha, p_j) \\ B(\nu, \alpha, p_j) \\ \alpha \end{pmatrix}. \tag{4.14}$$

We use the shorthand notation v for v_1 and v_2, and write $\mathcal{J}(v)$ for the energy of the perturbed double tori E described by the volume variables v_1 and v_2:

$$\mathcal{J}(v) = \sum_{j=1}^{2} \int_0^{2\pi} L_j(v_j'(\theta_j), v_j(\theta_j), \theta_j)\, d\theta_j + \gamma\pi \int_{E_v}\int_{E_v} G(r, z, s, t) r\, ds dt dr dz. \tag{4.15}$$

Here the set in $\mathbb{R}^2_+$ described by v_1 and v_2 is denoted by E_v.

The variation of $\mathcal{J}$ in the direction of ϕ is given by

$$\frac{d\mathcal{J}(v + \epsilon\phi)}{d\epsilon}\Big|_{\epsilon=0} = \sum_{j=1}^{2} 2\pi \int_0^{2\pi} \mathcal{H}_j(v_j)(\theta_j)\phi_j(\theta_j)\, d\theta_j + \gamma \sum_{j=1}^{2} 2\pi \int_0^{2\pi} \mathcal{N}_j(v)(\theta_j)\phi_j(\theta_j)\, d\theta_j. \tag{4.16}$$

In (4.16) the mean curvature $\mathcal{H}_j$ is now an operator on the volume variable v_j and the Newtonian potential $\mathcal{N}_j$ is an operator on both v_1 and v_2. Taking the constraint (4.9) into consideration, we deduce that at a critical point of $\mathcal{J}$ the system of equations

$$\mathcal{H}_j(v_j)(\theta_j) + \gamma\mathcal{N}_j(v)(\theta_j) = \lambda, \quad j = 1, 2, \tag{4.17}$$

holds. Let us introduce the operator $\mathcal{S} = (\mathcal{S}_1, \mathcal{S}_2)$ so that

$$\mathcal{S}_j(v) = \mathcal{H}_j(v_j) + \gamma\mathcal{N}_j(v) - \frac{1}{2}\sum_{j=1}^{2}\left[\overline{\mathcal{H}_j(v_j)} + \gamma\overline{\mathcal{N}_j(v)}\right], \quad j = 1, 2. \tag{4.18}$$

Note that a bar over a function denotes its average over $[0, 2\pi]$, i.e.

$$\overline{\mathcal{H}_j(v_j)} = \frac{1}{2\pi}\int_0^{2\pi} \mathcal{H}_j(v_j)(\theta_j)\, d\theta_j. \tag{4.19}$$

Then (4.17) is equivalent to

$$\mathcal{S}(v) = 0. \tag{4.20}$$

When specifying the domain and the target space of $\mathcal{S}$, we make use of the mirror symmetry of the sought after solution with respect to the horizontal xy-plane in addition to the axisymmetry with respect to the z-axis. Set

$$\mathcal{X} = \{(v_1, v_2) :\ v_j \in H^2(S^1), v_j > 0,\ v_j(\theta_j) = v_j(2\pi - \theta_j),\ (j = 1, 2),\ 2\pi \sum_{j=1}^{2} \int_0^{2\pi} v_j(\theta)\, d\theta_j = 2\}. \tag{4.21}$$

Note that in this definition v_1 is a function of θ_1 and v_2 is a function of θ_2, where θ_1 and θ_2 are different variables. The target space is

$$\mathcal{Y} = \{(y_1, y_2) :\ y_j \in L^2(S^1),\ y_j(\theta_j) = y_j(2\pi - \theta_j),\ (j = 1, 2),\ 2\pi \sum_{j=1}^{2} \int_0^{2\pi} y_j(\theta)\, d\theta_j = 0\} \tag{4.22}$$

where y_1 is a function of θ_1 and y_2 is a function of θ_2. The conditions $v_j(\theta_j) = v_j(2\pi - \theta_j)$ and $y_j(\theta_j) = y_j(2\pi - \theta_j)$ in (4.21) and (4.22) impose a mirror symmetry with respect to the xy-plane. This symmetry will greatly simplify the proof of Theorem 1.2. See the appendix for more discussion on the mirror symmetry.

The norm of $\mathcal{Y}$, denoted by $\|y\|_{L^2}$, is given by

$$\|y\|_{L^2} = (\|y_1\|^2_{L^2(S^1)} + \|y_2\|^2_{L^2(S^2)})^{1/2}, \text{ where } y = (y_1, y_2).$$

Here we have identified the interval $[0, 2\pi]$ with the unit circle S^1. Similarly the norm in $\mathcal{X}$ is denoted by $\|v\|_{H^2}$:

$$\|v\|_{H^2} = (\|v_1\|^2_{H^2(S^1)} + \|v_2\|^2_{H^2(S^1)})^{1/2}.$$

A 2π-periodic function is regarded as a function on S^1. Here $H^2(S^1)$ is the Sobolev space of twice weakly differentiable L^2 functions with L^2-integrable first and second order derivatives on the circle S^1.

5 Reduction

We fix the parameters p_i and q_i satisfying the condition (3.7) in this section. The exact double tori T of the radii p_i and q_i is described by the volume variable $\psi = (\psi_1, \psi_2)$. The linearized operator of $\mathcal{S}$ at ψ is the operator

$$\mathcal{S}'(\psi) :\ \mathcal{X}' \to \mathcal{Y}. \tag{5.1}$$

Here $\mathcal{X}'$ is the domain of $\mathcal{S}'(\psi)$ given by

$$\mathcal{X}' = \{(\phi_1, \phi_2) :\ \phi_j \in H^2(S^1),\ \phi_j(\theta_j) = \phi_j(2\pi - \theta_j),\ (j = 1, 2),\ 2\pi \sum_{j=1}^{2} \int_0^{2\pi} \phi_j(\theta_j)\, d\theta_j = 0\}. \tag{5.2}$$

A subspace of $\mathcal{X}'$ plays an important role. Let

$$\mathcal{X}'_* = \{(\phi_1, \phi_2) \in \mathcal{X}' :\ \int_0^{2\pi} \phi_j(\theta_j) \cos\theta_j\, d\theta_j = \int_0^{2\pi} \phi_j(\theta_j)\, d\theta_j = 0,\ j = 1, 2\}. \tag{5.3}$$

Similarly define a subspace of $\mathcal{Y}$:

$$\mathcal{Y}_* = \{(y_1, y_2) \in \mathcal{Y} :\ \int_0^{2\pi} y_j(\theta_j) \cos\theta_j\, d\theta_j = \int_0^{2\pi} y_j(\theta_j)\, d\theta_j = 0,\quad j = 1, 2\}. \tag{5.4}$$

The orthogonal projection from $\mathcal{Y}$ to $\mathcal{Y}_*$ is denoted by Π so that $\mathcal{X}'_* = \Pi\mathcal{X}'$ and $\mathcal{Y}_* = \Pi\mathcal{Y}$.

Lemma 5.1 *There exists $C > 0$ independent of p, q and γ such that*

$$\|\phi\|_{H^2} \le Cp^{-1/2} \|\Pi \mathcal{S}'(\psi)\phi\|_{L^2}$$

for all $\phi \in \mathcal{X}'_*$. Moreover the operator $\Pi\mathcal{S}'(\psi)$ is one-to-one and onto from $\mathcal{X}'_*$ to $\mathcal{Y}_*$.

Proof. The proof is similar to that of [25, Lemma 4.1]. The key here is to identify $\Pi\mathcal{S}'(\psi)$ as the sum of a dominant operator $\mathcal{L}$ and a negligible operator $\widetilde{\mathcal{L}}$. On $\mathcal{X}'_*$ the dominant part $\mathcal{L} = (\mathcal{L}_1, \mathcal{L}_2)$ is given by

$$\mathcal{L}_1\phi = -\sqrt{2}p_1^{1/2}(\phi_1'' + \phi_1), \quad \mathcal{L}_2\phi = -\sqrt{2}p_2^{1/2}(\phi_2'' + \phi_2). \tag{5.5}$$

Two families of eigenvalues exist: $\sqrt{2}p_1^{1/2}(n^2-1)$ $(n = 2, 3, 4, ...)$ corresponding to eigenvectors $(\cos n\theta_1, 0)$, and $\sqrt{2}p_2^{1/2}(n^2 - 1)$ $(n = 2, 3, 4, ...)$ corresponding to $(0, \cos n\theta_2)$. The statement in the lemma holds true if $\Pi\mathcal{S}'(\psi)$ were $\mathcal{L}$. However since $\Pi\mathcal{S}(\psi) = \mathcal{L} + \widetilde{\mathcal{L}}$ and $\widetilde{\mathcal{L}}$ is small compared to $\mathcal{L}$, the lemma is also true for $\Pi\mathcal{S}(\psi)$. $\square$

The second Fréchet derivative of $\mathcal{S}$ is estimated in the next lemma. We omit the proof since it is similar to that of [22, Lemma 3.2].

Lemma 5.2 *There exists $C > 0$ independent of p, q and γ such that*

$$\|\mathcal{S}''(v)(\phi_1, \phi_2)\|_{L^2} \le Cp^{1/2}\|\phi_1\|_{H^2}\|\phi_2\|_{H^2}$$

for all v near ψ, in the sense $\|v - \psi\|_{H^2} < \frac{1}{2}\|\psi\|_{H^2}$, and all $\phi_1, \phi_2 \in \mathcal{X}'$.

The nonlinear operator $\mathcal{S}$ maps from a neighborhood of ψ in $\mathcal{X}$ to $\mathcal{Y}$. An element φ exists in $\mathcal{X}'_*$ such that

$$\mathcal{S}_1(\psi + \varphi)(\theta_1) = A_1 + B_1 \cos\theta_1, \quad \mathcal{S}_2(\psi + \varphi)(\theta_2) = A_2 + B_2 \cos\theta_2 \tag{5.6}$$

for some $A_1, A_2, B_1, B_2 \in \mathbb{R}$. The equations in (5.6) may be written as

$$\Pi\mathcal{S}(\psi + \varphi) = 0. \tag{5.7}$$

The last equation may be solved by a contraction mapping argument with the help of Lemmas 5.1 and 5.2. The proof is essentially the same as that of [25, Lemma 5.1], hence we omit.

Lemma 5.3 *There exists $\varphi \in \mathcal{X}'_*$ such that φ solves (5.7) and $\|\varphi\|_{H^2} \le Cf^6(\gamma)$ where C is a sufficiently large constant independent of γ.*

We remark that (5.7) is solved for each given (p, q) satisfying (3.7). The parameters p and q are held fixed in this section. In the next section we will vary p and q.

6 Minimization

We prove Theorem 1.2 in this section. From now on we emphasize dependences on p and q now and denote the exact double tori T by $T(p, q)$, which is describe by the volume variable $\psi(\cdot, p, q)$. By Lemma 5.3 there exists $\varphi(\cdot, p, q) \in \mathcal{X}'_*$ such that $\Pi\mathcal{S}(\psi(\cdot, p, q) + \varphi(\cdot, p, q)) = 0$, i.e. (5.6) holds. In this section we find particular p and q denoted by p^γ and q^γ such that $\mathcal{S}(\psi(\cdot, p^\gamma, q^\gamma) + \varphi(\cdot, p^\gamma, q^\gamma)) = 0$.

Let us denote the set specified by the volume variable $\psi(\cdot, p, q) + \varphi(\cdot, p, q)$ by $E(p, q)$.

Lemma 6.1 $\mathcal{J}(E(p, q)) = \mathcal{J}(T(p, q)) + O(f^{11}(\gamma))$.

The proof of this lemma, which we omit, is similar to that of [25, Lemma 6.1]. The quantity $\mathcal{J}(E(p,q))$ may also be expressed in terms of the scaled variables P_j and Q_j defined in (3.1).

Lemma 6.2

$$\begin{aligned}\mathcal{J}(E(p,q)) &= \frac{1}{f(\gamma)}\sum_{j=1}^{2}\Big[P_jQ_j+\frac{P_jQ_j^4}{2}\Big]+\gamma f^2(\gamma)\Big[\sum_{j=1}^{2}\Big(\frac{P_jQ_j^4}{4}\log\frac{1}{Q_j}+\frac{P_jQ_j^4}{16}+\frac{\pi P_jQ_j^4}{2}G_1(P_j,0,P_j,0)\Big)\\ &\quad +\pi P_1Q_1^2Q_2^2G(P_1,0,P_2,0)\Big]+O(f^5(\gamma)).\end{aligned}$$

Proof. Use Lemmas 3.3 and 6.1 and the formulas

$$G(\lambda r,\lambda z,\lambda s,\lambda t)=G(r,z,s,t),\quad G_1(\lambda r,\lambda z,\lambda s,\lambda t)=G_1(r,z,s,t)-\frac{1}{2\pi}\log\frac{1}{\lambda}.\quad\Box \tag{6.1}$$

Lemma 6.2 gives the first two orders of the expansion of $\mathcal{J}(E(p,q))$ with respect to γ. Recall Ω from (3.2), and introduce two functions J_I and J_{II} defined on Ω:

$$J_I(P,Q) = \sum_{j=1}^{2}\Big(P_jQ_j+\frac{P_jQ_j^4}{2}\Big), \tag{6.2}$$

$$J_{II}(P,Q) = \sum_{j=1}^{2}\Big(\frac{P_jQ_j^4}{4}\log\frac{1}{Q_j}+\frac{P_jQ_j^4}{16}+\frac{\pi P_jQ_j^4}{2}G_1(P_j,0,P_j,0)\Big)+\pi P_1Q_1^2Q_2^2G(P_1,0,P_2,0). \tag{6.3}$$

Denote the set of the minimum points of J_I in Ω by M_I:

$$M_I=\{(P,Q)\in\Omega:\ J_I(P,Q)=\inf_{(P',Q')\in\Omega}J_I(P',Q')\}. \tag{6.4}$$

Since

$$P_jQ_j+\frac{P_jQ_j^4}{2}=P_jQ_j^2\Big(\frac{1}{Q_j}+\frac{Q_j^2}{2}\Big)\geq P_jQ_j^2\Big(\frac{3}{2}\Big)$$

and the equality is achieved if $Q_j=1$, the constraint $\sum_{j=1}^2 P_jQ_j^2=2$ implies that

$$M_I=\{(P,Q)\in\Omega:\ Q=(1,1)\}. \tag{6.5}$$

On the set M_I, $J_I(P,(1,1))=3$. Recall

$$\widetilde{J}_{II}(P)=J_{II}(P,(1,1))=\sum_{j=1}^{2}\Big(\frac{P_j}{16}+\frac{\pi P_j}{2}G_1(P_j,0,P_j,0)\Big)+\pi P_1G(P_1,0,P_2,0)$$

from (1.8) and (3.4) for $P\in\widetilde{\Omega}_{II}$, and also $\widetilde{M}_{II}$ from (3.5), the set of the minimum points of $\widetilde{J}_{II}$.

We view $\mathcal{J}(E(p,q))$ as a function of $(P,Q)\in K$, where K is given in (3.6), through the set $E(p,q)$ and the scaling (3.1). The next lemma shows that $\mathcal{J}(E(p,q))$ is minimized at an interior point of K.

Lemma 6.3 *Let (P^γ,Q^γ) be a minimum of $\mathcal{J}(E(p,q))$ in K. Suppose that $(P^\gamma,Q^\gamma)\to(P^\infty,Q^\infty)$ as $\gamma\to\infty$, possibly along a subsequence. Then $P^\infty\in\widetilde{M}_{II}$ and $Q^\infty=(1,1)$. Hence (P^γ,Q^γ) is an interior point of K if γ is sufficiently large.*

Proof. We write $\mathcal{J}(P,Q)$ for $\mathcal{J}(E(p,q))$ in this proof. Lemma 6.2 shows that for $(P,Q)\in K$,

$$\mathcal{J}(P,Q)=\frac{1}{f(\gamma)}J_I(P,Q)+\gamma f^2(\gamma)J_{II}(P,Q)+O(f^5(\gamma)). \tag{6.6}$$

Then as $\gamma\to\infty$,

$$f(\gamma)\mathcal{J}(P^\gamma,Q^\gamma)\to J_I(P^\infty,Q^\infty) \quad \text{and} \quad f(\gamma)\mathcal{J}(P^\infty,(1,1))\to J_I(P^\infty,(1,1)),$$

possibly along a subsequence. Since $\mathcal{J}(P^\gamma,Q^\gamma)\le\mathcal{J}(P^\infty,(1,1))$, we deduce $J_I(P^\infty,Q^\infty)\le J_I(P^\infty,(1,1))$. Therefore $(P^\infty,Q^\infty)\in M_I$, i.e. $Q^\infty=(1,1)$.

Note that the minimum value of J_I is 3. Next we compare

$$\begin{aligned}
&\liminf_{\gamma\to\infty}\frac{1}{\gamma f^3(\gamma)}\Big(f(\gamma)\mathcal{J}(P^\gamma,Q^\gamma)-3\Big)\\
=&\liminf_{\gamma\to\infty}\Big[\frac{1}{\gamma f^3(\gamma)}\Big(J_I(P^\gamma,Q^\gamma)-3\Big)+J_{II}(P^\gamma,Q^\gamma)+O(\frac{f^3(\gamma)}{\gamma})\Big]\\
=&\liminf_{\gamma\to\infty}\frac{1}{\gamma f^3(\gamma)}\Big(J_I(P^\gamma,Q^\gamma)-3\Big)+J_{II}(P^\infty,Q^\infty)\\
=&\liminf_{\gamma\to\infty}\frac{1}{\gamma f^3(\gamma)}\Big(J_I(P^\gamma,Q^\gamma)-3\Big)+\widetilde{J}_{II}(P^\infty)
\end{aligned}$$

with

$$\begin{aligned}
&\lim_{\gamma\to\infty}\frac{1}{\gamma f^3(\gamma)}\Big(f(\gamma)\mathcal{J}(\Pi,(1,1))-3\Big)\\
=&\lim_{\gamma\to\infty}\Big[\frac{1}{\gamma f^3(\gamma)}\Big(J_I(\Pi,(1,1))-3\Big)+J_{II}(\Pi,(1,1))+O(\frac{f^3(\gamma)}{\gamma})\Big]\\
=&J_{II}(\Pi,(1,1)) \;=\; \widetilde{J}_{II}(\Pi)
\end{aligned}$$

where Π is any point in $\widetilde{M}_{II}$. Because $\mathcal{J}(P^\gamma,Q^\gamma)\le\mathcal{J}(\Pi,(1,1))$,

$$\liminf_{\gamma\to\infty}\frac{1}{\gamma f^3(\gamma)}\Big(J_I(P^\gamma,Q^\gamma)-3\Big)+\widetilde{J}_{II}(P^\infty)\le\widetilde{J}_{II}(\Pi).$$

Since 3 is the minimum value of J_I,

$$\liminf_{\gamma\to\infty}\frac{1}{\gamma f^3(\gamma)}\Big(J_I(P^\gamma,Q^\gamma)-3\Big)\ge 0,$$

and consequently $\widetilde{J}_{II}(P^\infty)\le\widetilde{J}_{II}(\Pi)$. Since Π is a minimum of $\widetilde{J}_{II}$, P^∞ is also a minimum. $\square$

Let p^γ and q^γ be the radii corresponding to P^γ and Q^γ through (3.1). Before completing the proof of Theorem 1.2, the last thing we need to show is that $E(p^\gamma,q^\gamma)$ solves (1.1).

Lemma 6.4 *The set described by $\psi(\cdot,p^\gamma,q^\gamma)+\varphi(\cdot,p^\gamma,q^\gamma)$ satisfies the equation (1.1), i.e. $\mathcal{S}(\psi(\cdot,p^\gamma,q^\gamma)+\varphi(\cdot,p^\gamma,q^\gamma))=0$.*

Proof. Since $\Pi\mathcal{S}(\psi+\varphi)=0$, there exist A_k and B_k in $\mathbb{R}$ ($k=1,2$) such that

$$\mathcal{S}_k(\psi(\cdot,p,q)+\varphi(\cdot,p,q))=A_k+B_k\cos\theta_k. \tag{6.7}$$

The constants A_k and B_k depend on (p,q). When (p,q) is (p^γ, q^γ) these constants are denoted by A_k^γ and B_k^γ respectively. We will prove the lemma by showing that

$$A_1^\gamma = A_2^\gamma = B_1^\gamma = B_2^\gamma = 0. \tag{6.8}$$

Introduce a new variable $m = (m_1, m_2)$ where $m_j = 2\pi^2 p_j q_j^2$ $(j = 1,2)$ is the volume of T_j, and treat $\mathcal{J}(E(p,q))$ as a function of m and p instead of p and q. The original constraint on p and q now becomes a constraint on m: $m_1 + m_2 = 2$. We write $\mathcal{J}(m,p)$ for $\mathcal{J}(E(p,q))$, and differentiate $\mathcal{J}(m,p)$ with respect to m_j. Since $\mathcal{J}(m,p)$ depends on m only through $\psi + \varphi$,

$$\begin{aligned}
\frac{\partial \mathcal{J}(m,p)}{\partial m_k} &= \sum_{j=1}^{2} 2\pi \int_0^{2\pi} \mathcal{H}_j(\psi_j + \varphi_j) \frac{\partial(\psi_j + \varphi_j)}{\partial m_k}\, d\theta_j + \gamma \sum_{j=1}^{2} 2\pi \int_0^{2\pi} \mathcal{N}_j(\psi + \varphi) \frac{\partial(\psi_j + \varphi_j)}{\partial m_k}\, d\theta_j \\
&= \sum_{j=1}^{2} 2\pi \int_0^{2\pi} \Big(\mathcal{S}_j(\psi + \varphi) - \lambda(\psi + \varphi)\Big) \frac{\partial(\psi_j + \varphi_j)}{\partial m_k}\, d\theta_j
\end{aligned}$$

where we have written

$$\lambda(\psi + \varphi) = \frac{1}{2} \sum_{j=1}^{2} \Big[\overline{\mathcal{H}_j(\psi_j + \varphi_j)} + \gamma \overline{\mathcal{N}_j(\psi + \varphi)}\Big]$$

for short. Since $\int_0^{2\pi} \psi_j(\theta_j)\, d\theta_j = \pi p_j q_j^2 = \frac{m_j}{2\pi}$, $\int_0^{2\pi} \frac{\partial \psi_j}{\partial m_k}\, d\theta_j = \frac{\partial}{\partial m_k} \int_0^{2\pi} \psi_j(\theta_j)\, d\theta_j = \frac{1}{2\pi}$ if $k = j$ and $= 0$ if $k \neq j$. Also by the definition of $\mathcal{X}'_*$ where φ belongs, $\int_0^{2\pi} \frac{\partial \varphi_j}{\partial m_k}\, d\theta_j = \frac{\partial}{\partial m_k} \int_0^{2\pi} \varphi_j(\theta_j)\, d\theta_j = \frac{\partial 0}{\partial m_k} = 0$. Consequently

$$\begin{aligned}
\frac{\partial \mathcal{J}(m,p)}{\partial m_k} &= \sum_{j=1}^{2} 2\pi \int_0^{2\pi} \Big(\mathcal{S}_j(\psi + \varphi) - \lambda(\psi + \phi)\Big) \frac{\partial(\psi_j + \varphi_j)}{\partial m_k}\, d\theta_j \\
&= \sum_{j=1}^{2} 2\pi \int_0^{2\pi} \mathcal{S}_j(\psi + \varphi) \frac{\partial(\psi_j + \varphi_j)}{\partial m_k}\, d\theta_j - \lambda(\psi + \varphi) \\
&= \sum_{j=1}^{2} 2\pi \int_0^{2\pi} (A_j + B_j \cos\theta_j) \frac{\partial(\psi_j + \varphi_j)}{\partial m_k}\, d\theta_j - \lambda(\psi + \varphi) \\
&= \sum_{j=1}^{2} 2\pi \int_0^{2\pi} B_j \cos\theta_j \frac{\partial(\psi_j + \varphi_j)}{\partial m_k}\, d\theta_j + A_k - \lambda(\psi + \varphi).
\end{aligned}$$

Note that $\varphi \in \mathcal{X}'_*$ implies that $\varphi_j \perp \cos\theta_j$. Hence

$$\frac{\partial \mathcal{J}(m,p)}{\partial m_k} = \sum_{j=1}^{2} 2\pi \int_0^{2\pi} B_j \cos\theta_j \frac{\partial \psi_j}{\partial m_k}\, d\theta_j + A_k - \lambda(\psi + \varphi).$$

Because $\psi_j(\theta_j) = \frac{p_j q_j^2}{2} + \frac{q_j^3 \cos\theta_j}{3} = \frac{m_j}{2\pi} + \frac{1}{3}(\frac{1}{2\pi^2 p_j})^{3/2} m_j^{3/2} \cos\theta_j$,

$$\frac{\partial \mathcal{J}(m,p)}{\partial m_k} = \frac{m_k^{1/2}}{2^{3/2}\pi p_k^{3/2}} B_k + A_k - \lambda(\psi + \varphi).$$

Let (m^γ, p^γ) be a minimum of $\mathcal{J}(m,p)$ corresponding to (p^γ, q^γ). Under the constraint $m_1 + m_2 = 2$, there exists a Lagrange multiplier $\eta \in \mathbb{R}$ such that

$$\frac{(m_k^\gamma)^{1/2}}{2^{3/2}\pi (p_k^\gamma)^{3/2}} B_k^\gamma + A_k^\gamma - \lambda = \eta, \quad k = 1,2. \tag{6.9}$$

We have denoted $\lambda(\psi+\varphi)$ at (m^γ, p^γ) simply by λ.

Next we compute $\frac{\partial \mathcal{J}(m,p)}{\partial p_k}$. This is a more complicated story. The quantity $\mathcal{J}(m,p)$ depends on p in two ways. First it depends on $\psi+\varphi$ which in turn depends on p. Second since $(p_j, 0)$ is the center of a coordinate system from which $\mathcal{J}$ is defined, the functional $\mathcal{J}$ depends directly on p_j as parameters. The crucial step here is to use the coordinate system centered at the fixed point p_j^γ to re-parameterize the sets originally described by the coordinate system centered at nearby p_j, an idea first used by Ren and Wei in [22, 21]. Under the new coordinate system $\mathcal{J}(m,p)$ will depend on p indirectly, only through the variables that describe the sets. This re-parameterization operation can be done when p_j is close to p_j^γ. Let us call the polar coordinate systems centered at p_j and p_j^γ the p_j-coordinate system and the p_j^γ-coordinate system respectively. We need to make transformations between the two.

The rest of the proof requires very precise notations for functions and their derivatives. Let us write $v_j = V_j(\theta_j, p)$ for the function $\psi_j + \varphi_j$, emphasizing its dependence on p. Here V_j is a volume variable describing a component of the set E_v under the p_j-coordinate system. Denote the corresponding radius variable by $u_j = U_j(\theta_j, p)$, which is also under the p_j-coordinates. Note that functions are now denoted by capital letters and their dependent variables are denoted by the corresponding lower case letters. The same set under the p_j^γ-coordinates is described either by a radius variable $\widetilde{u}_j = \widetilde{U}_j(\eta_j, p)$, or a volume variable $\widetilde{v}_j = \widetilde{V}_j(\eta_j, p)$. Functions denoted by letters with a tilde are under the p_j^γ-coordinates, while the functions denoted by letters without a tilde are under the p_j-coordinates. The two sets of variables are connected through the following transformation rule

$$p_j + U_j(\theta_j, p)\cos\theta_j = p_j^\gamma + \widetilde{u}_j \cos\eta_j \tag{6.10}$$

$$U_j(\theta_j, p)\sin\theta_j = \widetilde{u}_j \sin\eta_j \tag{6.11}$$

These two equations implicitly define two functions X_j and Y_j such that $\widetilde{u}_j = X_j(\theta_j, p)$ and $\eta_j = Y_j(\theta_j, p)$. Note that p^γ is fixed and not considered as a variable. We also need the inverse of Y_j, with respect to θ_j, which we denote by $\theta_j = Z_j(\eta_j, p)$ such that $\eta_j = Y_j(Z_j(\eta_j, p), p)$. Now we are ready to define $\widetilde{u}_j = \widetilde{U}_j(\eta_j, p)$:

$$\widetilde{u}_j = \widetilde{U}_j(\eta_j, p) = X_j(Z_j(\eta_j, p), p). \tag{6.12}$$

Under the p_j^γ-coordinates, $\mathcal{J}(m,p)$ depends on p only through $\widetilde{V}$. Standard variational calculations show that

$$\frac{\partial \mathcal{J}(m,p)}{\partial p_k} = \sum_{j=1}^{2} 2\pi \int_0^{2\pi} \left(\mathcal{S}_j(\widetilde{V}) - \lambda(\widetilde{V})\right)\frac{\partial \widetilde{V}_j}{\partial p_k}\, d\eta_j = \sum_{j=1}^{2} 2\pi \int_0^{2\pi} \mathcal{S}_j(\widetilde{V})\frac{\partial \widetilde{V}_j}{\partial p_k}\, d\eta_j. \tag{6.13}$$

The last equality of (6.13) follows from the fact

$$\int_0^{2\pi} \frac{\partial \widetilde{V}_j}{\partial p_k}\, d\eta_j = \frac{\partial}{\partial p_k}\int_0^{2\pi} \widetilde{V}_j\, d\eta_j = \frac{\partial m_j}{2\pi \partial p_k} = 0. \tag{6.14}$$

Our main task now is to find $\frac{\partial \widetilde{V}_j}{\partial p_k}$ at $(m,p) = (m^\gamma, p^\gamma)$. For clarity we sometimes denote differentiation by the D-notation. For example we write

$$\frac{\partial \widetilde{V}_j(\eta_j, p)}{\partial p_k} = D_{k+1}\widetilde{V}(\eta_j, p). \tag{6.15}$$

Here p_k is the $(k+1)$-th variable of $\widetilde{V}_j$ and D_{k+1} denotes differentiation with respect to this variable.

Since

$$\widetilde{V}_j = \frac{p_j^\gamma \widetilde{U}_j^2}{2} + \frac{\widetilde{U}_j^3\cos\eta_j}{3},$$

$$\frac{\partial \widetilde{V}_j(\eta_j,p)}{\partial p_k} = (p_j^\gamma \widetilde{U}_j(\eta_j,p) + \widetilde{U}_j^2(\eta_j,p)\cos\eta_j)\frac{\partial \widetilde{U}_j(\eta_j,p)}{\partial p_k}. \tag{6.16}$$

We start with $\frac{\partial \widetilde{U}_j(\eta_j,p)}{\partial p_k}$ in (6.16). By (6.12)

$$\frac{\partial \widetilde{U}_j(\eta_j,p)}{\partial p_k} = \frac{\partial X_j(Z_j(\eta_j,p),p)}{\partial p_k} = D_1X_j(Z_j(\eta_j,p),p)D_{k+1}Z_j(\eta_j,p) + D_{k+1}X_j(Z_j(\eta_j,p),p). \tag{6.17}$$

By implicit differentiation

$$D_{k+1}Z_j(\eta_j,p) = -\frac{D_{k+1}Y_j(\theta_j,p)}{D_1Y_j(\theta_j,p)}. \tag{6.18}$$

The derivatives of X_j and Y_j are calculated from the transformations (6.10) and (6.11). Let

$$\begin{bmatrix} \Gamma_I(\theta_j,\widetilde{u}_j,\eta_j,p) \\ \Gamma_{II}(\theta_j,\widetilde{u}_j,\eta_j,p) \end{bmatrix} := \begin{bmatrix} p_j + U_j(\theta_j,p)\cos\theta_j - p_j^\gamma - \widetilde{u}_j\cos\eta_j \\ U_j(\theta_j,p)\sin\theta_j - \widetilde{u}_j\sin\eta_j \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}. \tag{6.19}$$

In this definition p_j^γ are constants, not variables. Then implicit differentiation of (6.19) yields

$$\begin{aligned} \begin{pmatrix} D_1X_j & D_{k+1}X_j \\ D_1Y_j & D_{k+1}Y_j \end{pmatrix} &= -\begin{pmatrix} D_2\Gamma_I & D_3\Gamma_I \\ D_2\Gamma_{II} & D_3\Gamma_{II} \end{pmatrix}^{-1} \begin{pmatrix} D_1\Gamma_I & D_{k+3}\Gamma_I \\ D_1\Gamma_{II} & D_{k+3}\Gamma_{II} \end{pmatrix} \\ &= -\begin{pmatrix} -\cos\eta_j & \widetilde{u}_j\sin\eta_j \\ -\sin\eta_j & -\widetilde{u}_j\cos\eta_j \end{pmatrix}^{-1} \begin{pmatrix} \frac{\partial U_j}{\partial\theta_j}\cos\theta_j - U_j\sin\theta_j & \delta_{jk} + \frac{\partial U_j}{\partial p_k}\cos\theta_j \\ \frac{\partial U_j}{\partial\theta_j}\sin\theta_j + U_j\cos\theta_j & \frac{\partial U_j}{\partial p_k}\sin\theta_j \end{pmatrix} \\ &= -\begin{pmatrix} -\cos\eta_j & -\sin\eta_j \\ \frac{\sin\eta_j}{\widetilde{u}_j} & -\frac{\cos\eta_j}{\widetilde{u}_j} \end{pmatrix} \begin{pmatrix} \frac{\partial U_j}{\partial\theta_j}\cos\theta_j - U_j\sin\theta_j & \delta_{jk} + \frac{\partial U_j}{\partial p_k}\cos\theta_j \\ \frac{\partial U_j}{\partial\theta_j}\sin\theta_j + U_j\cos\theta_j & \frac{\partial U_j}{\partial p_k}\sin\theta_j \end{pmatrix}. \end{aligned}$$

Here $\delta_{jk} = 1$ if $j = k$ and $= 0$ if $j \neq k$.

Since our goal is to evaluate $\frac{\partial \widetilde{V}}{\partial p_k}$ at $(m,p) = (m^\gamma, p^\gamma)$, where $\theta = \eta$ and $\widetilde{U}(\theta,p^\gamma) = U(\eta,p^\gamma)$, we deduce

$$\begin{pmatrix} D_1X_j & D_{k+1}X_j \\ D_1Y_j & D_{k+1}Y_j \end{pmatrix}\Big|_{(m,p)=(m^\gamma,p^\gamma)} = \begin{pmatrix} \frac{\partial U_j(\theta_j,p^\gamma)}{\partial\theta_j} & \delta_{jk}\cos\theta_j + \frac{\partial U_j(\theta_j,p^\gamma)}{\partial p_k} \\ 1 & -\delta_{jk}\frac{\sin\theta_j}{U_j(\theta_j,p^\gamma)} \end{pmatrix}. \tag{6.20}$$

By (6.18) and (6.20), (6.17) becomes

$$\frac{\partial \widetilde{U}_j}{\partial p_k}\Big|_{(m,p)=(m^\gamma,p^\gamma)} = \delta_{jk}\frac{D_1U_j(\theta_j,p^\gamma)\sin\theta_j}{U_j(\theta_j,p^\gamma)} + \delta_{jk}\cos\theta_j + D_{k+1}U_j(\theta_j,p^\gamma). \tag{6.21}$$

Returning to the volume variable, we deduce, from (6.16),

$$\frac{\partial \widetilde{V}_j}{\partial p_k}\Big|_{(m,p)=(m^\gamma,p^\gamma)} = (p_j^\gamma U_j + U_j^2\cos\theta_j)\Big[\delta_{jk}\frac{D_1U_j\sin\theta_j}{U_j} + \delta_{jk}\cos\theta_j + D_{k+1}U_j\Big]. \tag{6.22}$$

We single out the leading order term in (6.22) to obtain

$$\frac{\partial \widetilde{V}_j}{\partial p_k}\Big|_{(m,p)=(m^\gamma,p^\gamma)} = p_j^\gamma q_j^\gamma(\delta_{jk}\cos\theta_j + o(1)). \tag{6.23}$$

To reach (6.23) note that $U_j(\theta_j,p^\gamma)$ is a small perturbation of the constant q_j^γ by Lemma 5.3, so $\frac{D_1U_j}{U_j}$ is small. The fact that $D_{k+1}U_j$ is small can be proved in a way similar to the proof of [22, Lemma 4.2] or that of [21, Lemma 7.3].

Consider (6.13) at $(m, p) = (m^\gamma, p^\gamma)$. On the one hand, since (m^γ, p^γ) minimizes $\mathcal{J}$,

$$\frac{\partial \mathcal{J}(m^\gamma, p^\gamma)}{\partial p_k} = 0.$$

On the other hand, since $\mathcal{S}_j(\widetilde{V}(\cdot, p^\gamma)) = A_j^\gamma + B_j^\gamma \cos\theta_j$, by (6.13) and (6.14)

$$\frac{\partial \mathcal{J}(m^\gamma, p^\gamma)}{\partial p_k} = \sum_{j=1}^{2} 2\pi \int_0^{2\pi} B_j^\gamma \cos\theta_j \frac{\partial \widetilde{V}_j(\theta_j, p^\gamma)}{\partial p_k}\, d\theta_j.$$

We obtain the key equations

$$\sum_{j=1}^{2} B_j^\gamma \int_0^{2\pi} \cos\theta_j \frac{\partial \widetilde{V}_j(\theta_j, p^\gamma)}{\partial p_k}\, d\eta_j = 0, \quad k = 1, 2. \tag{6.24}$$

Combining (6.23) and (6.24) we find

$$\sum_{j=1}^{2} p_j^\gamma q_j^\gamma (\delta_{jk}\pi + o(1)) B_j^\gamma = 0, \quad k = 1, 2. \tag{6.25}$$

Since (6.25) is a non-singular linear homogeneous system for B_j^γ when γ is sufficiently large,

$$B_1^\gamma = B_2^\gamma = 0. \tag{6.26}$$

Finally with (6.26) we go back to (6.9) to deduce that

$$A_k^\gamma - \lambda = \eta, \;\; k = 1, 2,$$

i.e. $A_k^\gamma = \lambda + \eta$ is independent of k. Therefore $\mathcal{S}_k(V(\cdot, p^\gamma)) = \lambda + \eta$, $k = 1, 2$. Since $\mathcal{S}(V(\cdot, p^\gamma)) \in \mathcal{Y}$ where each member satisfies the condition $\sum_{k=1}^{2} \int_0^{2\pi} y_k(\theta_k)\, d\theta_k = 0$ according to (4.22), we must have $\lambda + \eta = 0$, i.e.

$$A_1^\gamma = A_2^\gamma = 0. \tag{6.27}$$

This proves (6.8). □

7 Discussion

Stability. It is not yet known whether the single torus solution found in [25] is stable, i.e. whether it is a local minimizer of $\mathcal{J}$ among all sets of the unit volume. However among axisymmetric sets of the unit volume, the single torus solution is in a sense stable. The double tori solution constructed in this paper is different. Even within the class of axisymmetric sets, the double tori solution is still unstable. Let us consider a configuration of two perfect tori, both axisymmetric about the z axis. In the (r, z)-plane $\mathbb{R}^2_+$ suppose one is represented by a circle centered at (p_1, z_1) of radius q_1 and the other by a circle centered at (p_2, z_2) of radius q_2. Denote the union of the two regions bounded by the tori by $T(p, q, z)$. Similar to Lemma 3.3 one finds

$$\begin{aligned}\mathcal{J}(T(p, q, z)) \;\; = \;\; & \sum_{j=1}^{2} 2\pi^2 p_j q_j + \frac{\gamma}{2}\Big[\sum_{j=1}^{2}\Big(\pi^2 p_j q_j^4 \log\frac{1}{q_j} + \frac{\pi^2 p_j q_j^4}{4} + 2\pi^3 p_j q_j^4 G_1(p_j, z_j, p_j, z_j)\Big) \\ & + 4\pi^3 p_1 q_1^2 q_2^2 G(p_1, z_1, p_2, z_2)\Big] + O\Big(\frac{\gamma |q|^4}{|p|^2}\Big|\log\frac{|q|}{|p|}\Big|\Big).\end{aligned} \tag{7.1}$$

In (7.1) $G_1(p_j, z_j, p_j, z_j)$ is independent of z_j, but $G(p_1, z_1, p_2, z_2)$ is decreasing in $|z_1 - z_2|$ which can be seen from (2.3). If we take the double tori solution and move one torus up and one torus down in the z-direction, the energy becomes smaller. This implies that even in the class of axisymmetric sets, the double tori solution cannot be a local minimizer of $\mathcal{J}$.

Mirror symmetry. We have imposed the mirror symmetry about the xy-plane on all sets in this paper; namely in (4.21) we required $v_j(\theta_j) = v_j(2\pi - \theta_j)$ for all $\theta_j \in S^1$. This mirror symmetry prevents sets from moving in the z-direction, in contrast to the argument above. Consequently we were able to find the double tori solution by minimizing $\mathcal{J}(E(p, q))$ with respect to (p, q). This procedure actually implies that the double tori solution is stable in the class of axisymmetric and mirror symmetric sets. If we do not impose mirror symmetry and work with sets only with axisymmetry, then we cannot find the double tori solution simply by minimization. We will have to (1) make an approximate solution $T(p, q, z)$ with two tori of radii q_j centered at (p_j, z_j) in $\mathbb{R}^2_+$, (2) improve $T(p, q, z)$ to $E(p, q, z)$ that solves the equations

$$\mathcal{S}_j(v) = A_j + B_j \cos\theta_j + C_j \sin\theta_j, \quad j = 1, 2, \tag{7.2}$$

and (3) employ a mini-max type argument to find a saddle point $(p^\gamma, q^\gamma, z^\gamma)$ at which the constants A_j, B_j and C_j all vanish.

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