Traveling waves with multiple and non-convex fronts for a bistable semilinear parabolic equation

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Abstract

We construct new examples of traveling wave solutions to the bistable and balanced semilinear parabolic equation in $\mathbb{R}^{N+1}$, $N \geq 2$. Our first example is that of a traveling wave solution with two non planar fronts that move with the same speed. Our second example is a traveling wave solution with a non convex moving front. To our knowledge no existence results of traveling fronts with these type of geometric characteristics have been previously known. Our approach explores a connection between solutions of the semilinear parabolic PDE and eternal solutions to the mean curvature flow in $\mathbb{R}^{N+1}$. © 2000 Wiley Periodicals, Inc.

Contents

Communications on Pure and Applied Mathematics, Vol. 000, 0001–0065 (2000)

1 Introduction

The problem of finding traveling wave solutions to an autonomous semilinear parabolic PDEs:

$$\Delta v + f(v) = v_t, \quad \in \mathbb{R}^n \times (-\infty, \infty),$$

has been studied extensively since the pioneering work of Kolmogorov, Petrovsky and Piskunow [23] and Fisher [15]. A traveling wave solution propagating in a fixed direction $\mathsf{e} \in \mathbb{R}^n$ with speed c is, by definition, a solution of the form $v(x,t) = u(x - ct\mathsf{e})$. When written in the Galilean frame, the traveling wave problem reduces to the following semilinear elliptic PDE:

$$\Delta u + c\mathsf{e} \cdot \nabla u + f(u) = 0, \quad \text{in } \mathbb{R}^n. \tag{1.1}$$

The most typical scenario is that of a planar front. Taking the ansatz $u(x) = U(x \cdot \mathsf{e})$ reduces (1.1) to the ODE:

$$U'' + cU' + f(U) = 0, \quad \text{in } \mathbb{R}.$$

In this case several examples of existence are well known, most common is the monostable nonlinearity $f(u) = u(1-u)$ (KPP) and the bistable nonlinearity

$$f(u) = (u+a)(1-u^2), \quad a \in (-1,1).$$

In the former case a planar traveling front exists for any $c > 2\sqrt{f'(0)} > 0$ while in the latter case the the nonlinearity determines the speed uniquely

$$c = \frac{\int_{-1}^{1} f(t)}{\int_{-1}^{1} (U')^2}.$$

Note that in the case of *balanced* bistable nonlinearity we have $c = 0$. This means that the traveling wave is a standing wave. These are classical results and we refer the reader for example to [14] for more information. Other related results in the monostable and bistable cases can be found for example in [20], [21] (see also [3], [17], [19], [2] and the references therein).

The case of non-planar fronts is much less understood. Since the subject of this paper is to study the traveling waves with a bistable nonlinearity we will mention some results in this direction. First let us consider f unbalanced i.e. $a \neq 0$. When $n = 2$ a V-shaped traveling wave was found by Ninomiya and Taniguchi and in higher dimension by Hamel, Monneau and Roquejoffre [18]. Let us comment on the $n = 2$ case. Given a traveling wave solution $u(x)$ its traveling front is the nodal set $\{u(x) = 0\}$. It can be proven that the front is asymptotic to two straight lines $y = m|x|$, and that it is convex at ∞ [25]. Moreover, it is shown that the traveling wave solution is stable. These results are generalized to higher dimensions and fronts of more complex geometric structure, which however has the general characteristics of the V-shaped front i.e. the front profiles are asymptotically linear, convex, and as solutions of the parabolic problem the traveling wave solutions are stable, see [18], [26], [27].

Let us now discuss the bistable balanced case. From now on we agree that the direction of propagation will be fixed to coincide with one of the axis. It is convenient to consider the traveling wave problem in $\mathbb{R}^{N+1}$, with x_{N+1} axis as the fixed direction of motion and N corresponding to the dimension of the associated traveling front. We will assume that $N \geq 2$. Thus, if we look for solutions to the parabolic Allen-Cahn equation

$$u_t = \Delta u + u - u^3, \quad \text{in } \mathbb{R}^{N+1} \times (-\infty, \infty), \quad N \geq 2, \tag{1.2}$$

in the following form:

$$u(x,t) = U(x', x_{N+1} - ct), \quad x = (x', x_{N+1}), \tag{1.3}$$

then U will satisfy the *traveling wave* Allen-Cahn equation

$$\Delta U + c\partial_{x_{N+1}} U + U - U^3 = 0. \tag{1.4}$$

In [4], the existence of a traveling wave in the form $U(r, x_{N+1})$, $|x'| = r$, is obtained for any speed $c > 0$. Furthermore, it is shown that asymptotically the $0-$level set of U—denoted here by Γ, is paraboloid-like

$$\lim_{\substack{x_{N+1} \to +\infty \\ (x', x_{N+1}) \in \Gamma}} \frac{r^2}{2x_{N+1}} = \frac{N-1}{c}, \quad \text{if } N \geq 2.$$

In the same paper the case $N = 1$ is treated as well and the traveling front is shown to be asymptotic to a hyperbolic cosine curve. In all cases traveling fronts are connected, convex surfaces.

The objective of this paper is to show that in the bistable balanced case there exist traveling wave solutions whose traveling fronts are non-connected, multicomponent surfaces (Theorem 1.1), and also that there are solutions whose fronts are non-convex (Theorem 1.2). These results are, to our knowledge, the first examples of this type for an autonomous traveling wave problem.

To introduce our results we review some well know facts about the relation between (1.4) and the so called *translating solutions* to the mean curvature flow. These solutions are also called *eternal*, since they exist for all $t \in (-\infty, \infty)$. In general, we say that an evolving in time family of surfaces moves by mean curvature if the following is satisfied:

$$V = \mathbf{H}, \tag{1.5}$$

where $\mathbf{H}$ denotes the mean curvature vector and V the normal velocity of the surface. Translating solutions of this problem are surfaces that do not change shape and are translated by the mean curvature (MC) flow in a fixed direction and with constant velocity. After a rigid motion and rescaling we may assume that a translating solution of the MC flow is represented by a family of surfaces $\{\Sigma + ct\mathrm{e}_{N+1}\}_{t \in \mathbb{R}}$, where Σ is a fixed N dimensional surface in $\mathbb{R}^{N+1}$, and $c \in \mathbb{R}$ is a fixed number. From this we obtain the following equation to determine Σ:

$$H = c\nu_{N+1}, \tag{1.6}$$

where H is the mean curvature and ν is the unit normal vector of the (oriented) surface Σ (recall that $\mathbf{H} = H\nu$). Observe that the family $\{\Sigma + ct\mathrm{e}_{N+1}\}_{t\in\mathbb{R}}$ is a translating solution of the mean curvature flow, which is translated in the direction parallel to the x_{N+1}-axis with the constant speed c.

Let us fix a surface Σ for which (1.6) holds and such that $c = 1$. Let us also define its scaling Σ_ε by

$$y \in \Sigma_\varepsilon \Longleftrightarrow \varepsilon y \in \Sigma. \tag{1.7}$$

Then, denoting the mean curvatures of these surfaces by H_Σ, H_{Σ_ε} respectively we see that if Σ is a translating solution to the mean curvature flow with speed 1 then we have

$$H_{\Sigma_\varepsilon} = \varepsilon \nu_{N+1}, \tag{1.8}$$

which means that the scaled surface moves with the constant speed $c = \varepsilon$. In this paper we will consider ε to be a small parameter, or in other words, we will be interested in translating solutions of the MC flow moving with a small speed.

Several examples of translating solution to the MC equation are known, see for example [1], [5], [24], [28]. Here we will discuss a special eternal solution of the mean curvature flow for which Σ is a graph of a smooth function $F\colon \mathbb{R}^N \to \mathbb{R}$, that is $\Sigma = \{(x', F(x')), x' \in \mathbb{R}^N\}$. In this case (1.6) reduces to

$$\nabla\Big(\frac{\nabla F}{\sqrt{1+|\nabla F|^2}}\Big) = \frac{1}{\sqrt{1+|\nabla F|^2}}. \tag{1.9}$$

It is known from [1] and [5] that there exists a unique rotationally symmetric solution F of (1.9), with the following asymptotic behavior

$$F(r) = \frac{r^2}{2(N-1)} - \log r + 1 + O(r^{-1}), \quad r \gg 1. \tag{1.10}$$

Notice that this asymptotic behavior corresponds (at leading order) to the asymptotic behavior of the nodal set of solutions to (1.4) found in [4]. In what follows we will denote the rotationally symmetric translating solution of the MC flow by Γ and the corresponding scaled surface by Γ_ε. The latter surface is rotationally symmetric, is translating with speed $c = \varepsilon$, and is given as a graph as well:

$$\Gamma_\varepsilon = \{x_{N+1} = \varepsilon^{-1} F(\varepsilon r)\}.$$

The first result in this paper is about existence of a traveling wave solution to (1.4) whose zero level set consists of 2 disjoint components, each of which is asymptotically a paraboloid-like surface in a neighborhood of the rotationally symmetric eternal solution to the mean curvature flow Γ_ε. More precisely we have:

Theorem 1.1. *For each sufficiently small ε, the traveling wave problem (1.4) has a solution u_ε moving with speed $c = \varepsilon$, and with the following properties:*

(1) *The 0-level set of u_ε consists of 2 disjoint, rotationally symmetric and smooth hypersurfaces $\Gamma_\varepsilon^\pm$.*

(2) *The nodal surfaces* $\Gamma_\varepsilon^\pm$ *divide the space into* 3 *disjoint and unbounded components* $\Omega_\varepsilon^\pm$, Ω_ε^0. *Each of the sets* $\Omega_\varepsilon^\pm$ *is a neighborhood, respectively, of* $(0, x_{N+1} = \pm\infty) \in \mathbb{R}^{N+1}$, *and it holds* $u_\varepsilon < 0$ *in* $\Omega_\varepsilon^\pm$. *The set* Ω_ε^0 *contains* Γ_ε *and* $u_\varepsilon > 0$ *in* Ω_ε^0. *Moreover:*

$$\lim_{x_{N+1}\to\pm\infty} u_\varepsilon(x', x_{N+1}) = -1, \quad \forall x' \in \mathbb{R}^N,$$

while at the same time

$$\lim_{\substack{(x', x_{N+1})\to\infty \\ (x', x_{N+1})\in\Gamma_\varepsilon}} u_\varepsilon(x', x_{N+1}) = 1.$$

(3) *For any* $r > 0$ *let* C_r *be the cylinder* $C_r = \{(x', x_{N+1}) \mid |x'| = r\}$. *Let* $\Gamma_\varepsilon^\pm(r) = \Gamma_\varepsilon^\pm \setminus C_r$, *and similarly* $\Gamma_\varepsilon(r) = \Gamma_\varepsilon \setminus C_r$. *Then it holds:*

$$\mathrm{d}\left(\Gamma_\varepsilon^\pm(r), \Gamma_\varepsilon(r)\right) = \mathcal{O}\left(\log\left(\frac{1+\varepsilon^2 r^2}{\varepsilon^2}\right)\right), \text{ as } r \to +\infty, \tag{1.11}$$

where d *is the Hausdorff distance between sets.*

Of course when u_ε is a solution so is $-u_\varepsilon$ so our result provides automatically existence of at least two traveling waves with multiple fronts.

Our construction of a traveling wave solutions of (1.4) with a two-component traveling front gives a more precise information about the moving fronts $\Gamma_\varepsilon^\pm$ and their relation to Γ_ε. In particular it is shown that $\Gamma_\varepsilon^\pm$ are normal graphs over Γ_ε of certain functions $f_\varepsilon^\pm \colon \Gamma_\varepsilon \to \mathbb{R}$, whose asymptotic behavior coincides with the one described in (1.11) above. In section 2.2 we will discuss this in more details and we will introduce, based on formal calculations, a system of nonlinear PDEs on Γ_ε which determines these functions. A schematic view of the situation is included in Figures 1.1 and 1.2.

Our second result for the traveling wave problem (1.4) has to do with existence of traveling waves whose fronts are non-convex surfaces. In fact in [5] it is proven that in the case of translating solutions of the mean curvature flow in $\mathbb{R}^{N+1}$, $N \geq 2$ there exists a family of rotationally symmetric surfaces Σ_R, $R > 0$, of genus 0 which satisfies:

$$H_{\Sigma_R} = \nu_{R,N+1}.$$

In other words Σ_R is translated by the mean curvature flow in the direction of x_{N+1} axis with speed $c = 1$. Each of these surfaces is formed by taking the union of two graphs of radial functions $W_R^\pm \colon [R, \infty) \to \mathbb{R}$ in $\mathbb{R}^{N+1}$. These functions satisfy the following asymptotic formulas:

$$W_R^\pm(r) = \frac{r^2}{2(N-1)} - \log r + C^\pm + O(r^{-1}), \quad r \gg 1, \tag{1.12}$$

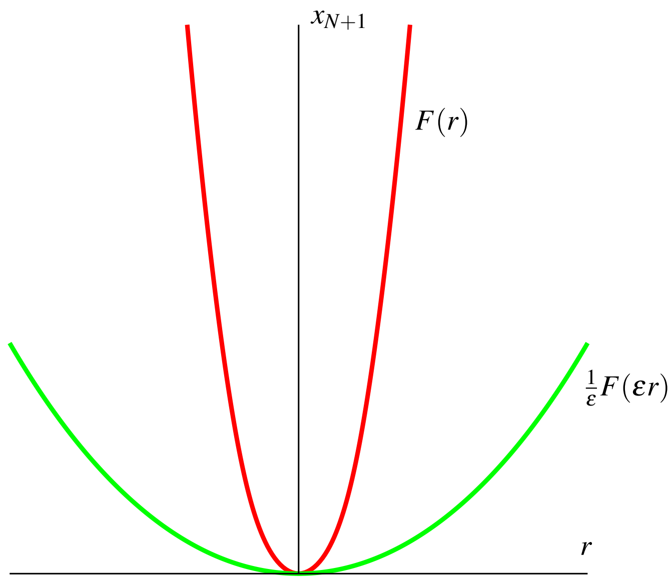

FIGURE 1.1. Schematic view of the surface Γ represented as a graph $x_{N+1} = F(r)$ and moving with speed $c = 1$, and the surface Γ_ε, represented as a graph $x_{N+1} = \frac{1}{\varepsilon}F(\varepsilon r)$ and moving with speed $c = \varepsilon$.

with some constants $C^{\pm}$. The graphs of the functions $W_R^{\pm}$ are called the ends of Γ_R and we will refer to them as the upper end Σ_R^+ and the lower end Σ_R^-, respectively. Comparing (1.12) with (1.10) we see that the ends of each of the surface Σ_R are asymptotically "parallel" to the traveling graph Γ described above. It is easy to see that Σ_R divides the space into two disjoint components, we call them $\Omega_R^{\pm}$, respectively and agree that Ω_R^+ is the component containing the vertical axis x_{N+1} and Ω_R^- is the other one. Sometimes we refer to the surfaces Σ_R as traveling catenoids.

We consider a scaling of Σ_R by a small parameter $\Sigma_{R,\varepsilon} = \frac{1}{\varepsilon}\Sigma_R$. The scaled surfaces move now with speed $c = \varepsilon$. We will denote the ends of the scaled traveling catenoid by $\Sigma_{R,\varepsilon}^{\pm}$. Note that $\Sigma_{\frac{R}{\varepsilon}} \neq \Sigma_{R,\varepsilon}$. Indeed, while both of these surfaces are defined for $r > \frac{R}{\varepsilon}$, $\Sigma_{\frac{R}{\varepsilon}}$ is a traveling catenoid whose speed is $c = 1$, while $\Sigma_{R,\varepsilon}$ is a traveling catenoid whose speed is $c = \varepsilon$. In other words the surfaces Σ_R considered for different R are not simple scalings of one another. See Figure 1.3, which illustrates the situation.

We show the following result:

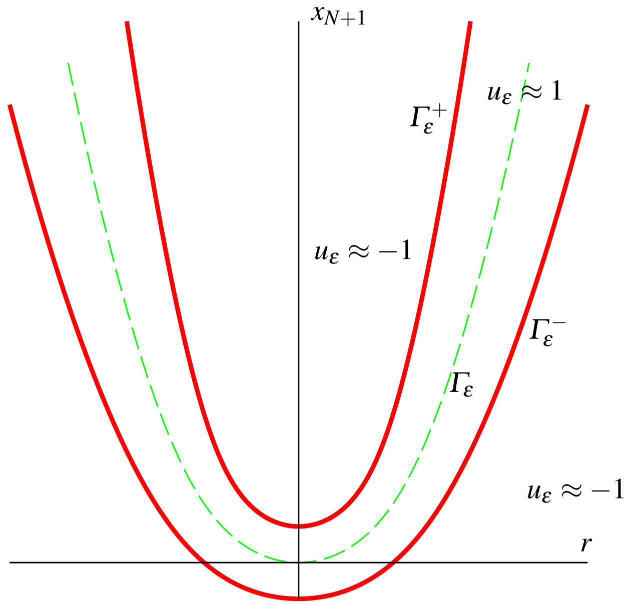

FIGURE 1.2. Illustration of the results of Theorem 1. The surfaces $\Gamma_\varepsilon^{\pm}$ are presented as well as the asymptotic values of the traveling wave solution u_ε.

Theorem 1.2. *For each $R > 0$ and each ε sufficiently small there exists a traveling wave solution u_ε of the problem (1.4) moving with speed $c = \varepsilon$, and such that the following hold:*

(1) *The level set $\tilde{\Sigma}_{R,\varepsilon} = \{u_\varepsilon = 0\}$ is a rotationally symmetric, smooth surface of genus 0.*
(2) *The surface $\tilde{\Sigma}_{R,\varepsilon}$ divides the space into two disjoint components $D^{\pm}_{R,\varepsilon}$ such that $u_\varepsilon > 0$ in $D^+_{R,\varepsilon}$ and $u_\varepsilon < 0$ in $D^-_{R,\varepsilon}$. Moreover, outside of a sufficiently large ball the set $\Omega^-_{R,\varepsilon}$, which is one of the two components into which the traveling catenoid $\Sigma_{R,\varepsilon}$ divides $\mathbb{R}^{N+1}$, is contained in $D^-_{R,\varepsilon}$. We have also:*
$$\lim_{x_{N+1}\to\pm\infty} u_\varepsilon(x', x_{N+1}) = 1, \quad \forall x' \in \mathbb{R}^N.$$
At the same time
$$\lim_{\substack{|(x',x_{N+1})|\to\infty \\ (x',x_{N+1})\in D^-_{R,\varepsilon}\cap\Omega^-_{R,\varepsilon}}} u_\varepsilon(x', x_{N+1}) = -1.$$
(3) *Let $\tilde{\Sigma}^{\pm}_{R,\varepsilon}$ denote the ends of the surface $\tilde{\Sigma}_{R,\varepsilon}$. For each $r > R$ we denote $\tilde{\Sigma}^{\pm}_{R,\varepsilon}(r) = \tilde{\Sigma}^{\pm}_{R,\varepsilon} \setminus C_r$. Correspondingly we introduce the surfaces $\Sigma^{\pm}_{R,\varepsilon}(r) =$*

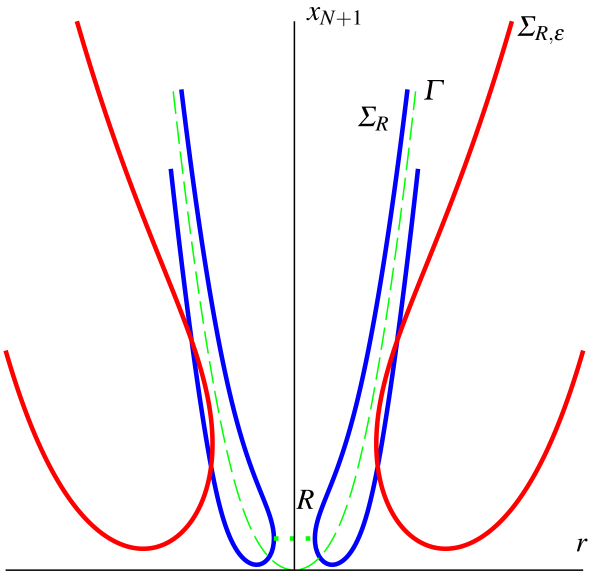

FIGURE 1.3. Schematic view of the traveling catenoid Σ_R and moving with speed $c=1$ and its rescaled version $\Sigma_{R,\varepsilon}$, moving with speed $c=\varepsilon$. The surface Γ is also represented for comparison.

$\Sigma^{\pm}_{R,\varepsilon}\setminus C_r$. *With these notations it holds*

$$\mathrm{d}\,(\tilde\Sigma^{\pm}_{R,\varepsilon}(r),\Sigma^{\pm}_{R,\varepsilon}(r))=\mathscr{O}\big(\log\big(\frac{1+\varepsilon^2r^2}{\varepsilon^2}\big)\big).$$

The existence results in Theorem 1.1 and Theorem 1.2 are rather counterintuitive in view of what happens with the planar fronts. To explain this, let us note that because of the statement (2) in Theorem 1.1 the phase labeled -1 has a tendency to invade the other phase. This is because when we take the limit $u_\varepsilon(x',x_{N+1})$, with x' fixed and $x_{N+1}\to\pm\infty$ then $u_\varepsilon(x',x_{N+1})\to -1$. In the one dimensional situation a solution to the parabolic Allen-Cahn equation with initial data satisfying this condition at ∞ will eventually converge to 01. Thus, if this one dimensional interaction of fronts were the only mechanism present, the nodal hypersurfaces should attract each other and eventually annihilate, and only one phase would remain in the asymptotic limit of infinite time. Based on this a natural statement in higher dimension would then be: if a traveling wave solution of the bistable and balanced problem satisfies $\lim_{x_{N+1}} u(x',x_{N+1})=-1$ then $u\equiv -1$.

This turns out to be false because of the mediating effect of the geometry of the front. Indeed, we see that in the situation described by the theorems one stable

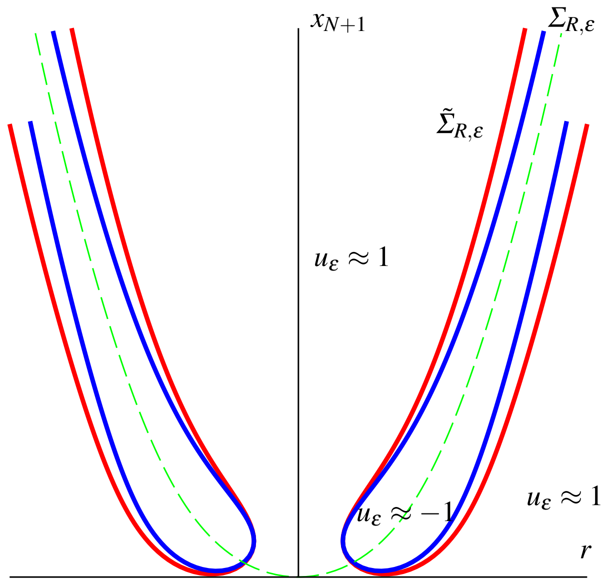

FIGURE 1.4. Illustration of the results of Theorem 2. The surface $\tilde{\Sigma}_{R,\varepsilon}$ is presented as well as the asymptotic values of the traveling wave solution u_ε For comparison we include also the surface $\Sigma_{\varepsilon,R}$.

phase, say -1, is "surrounded" by the other phase, say $+1$, which is also stable thanks to the fact that the nonlinearity is bistable. The nonlinearity being balanced as well, the two phases move with equal speed, and their initial configuration is translated with constant speed and is preserved for all times. As a result we have an eternal solutions to the parabolic Allen-Cahn equation. The main effort in this paper is to give a quantitive form of this by deriving and solving, a system of PDEs, called the Jacobi-Toda system, for the moving fronts.

Before we close this section, we make several important remarks as well as open questions.

Remark 1.3. The results of Theorems 1.1 and 1.2 hold for general *balanced* non-linearity

$$\Delta U + c\partial_{x_{N+1}} U + f(U) = 0 \tag{1.13}$$

where $f(U) = F^{'}(U)$ and $F \in \mathscr{C}^4(\mathbb{R})$ has two equal wells ± 1 with $F(-1) = F(1) = 0$ and $f^{'}(\pm 1) < 0$. The proofs are similar but the notation and details of some computations become quite cumbersome. For this reason we chose here to work with the cubic, balanced nonlinearity $f(u) = u(1-u^2)$.

On the other hand it is also possible to construct solutions with multiple traveling fronts when the nonlinearity is unbalanced (see [13]).

Remark 1.4. In the statement of Theorem 1.1 we have assumed that $N \geq 2$. When $N = 1$ the traveling wave solution to the mean curvature front is the well known *grim reaper* and its properties are quite different. In particular the ends of the grim reaper become parallel at ∞ and as a result to find multiple front traveling waves for the traveling wave problem (1.1) one would have to take into account strong interactions of the ends of the grim reaper. This situation resembles somewhat the one of Theorem 1.2 but the problem seems quite technical and is beyond the scope of this paper. In this context it is worth mentioning that according to a result of Gui [16] traveling wave solutions (1.1) with one front must be even symmetric. An interesting question would be then whether multiple front traveling wave solutions with no even symmetry exist.

Remark 1.5. In the proof of Theorem 1.1, for brevity we have only dealt with the case of $k = 2$ front traveling wave. The techniques can be extended to multiple front traveling wave ($k > 2$) but the technical details render the proofs bit longer. The main issue which is the solution of the Jacobi-Toda system can handled similarly as in [12].

This paper is organized as follows. First, we explain on the formal level the result in Theorem 1.1 introducing in the process the Jacobi-Toda system for a traveling solution to the mean curvature flow. Next, we solve this system for Γ_ε. This is in fact the core of our paper. Then we use the infinite dimensional Lyapunov-Schmidt reduction procedure to show the existence for (1.1). Finally, we prove Theorem 1.2.

2 The Jacobi-Toda system and multi component traveling fronts

The discussion in this section is mostly formal however we think that it is useful in order to understand the role played by the Jacobi-Toda system in the existence of traveling wave with multiple components. We chose to work in the setting that is more general than the one of Theorem 1.1 to emphasize the universality of this system. The notations and many calculations presented here will be used throughout the paper.

2.1 Geometric background

Let us consider a parametrized, regular, N dimensional surface $\Sigma(t)$ for which (1.6) is satisfied. We will consider its parametrization over a family of open sets $\mathscr{U}_\alpha \subset \mathbb{R}^N$, $\alpha \in \mathscr{A}$, and associated smooth maps $q_\alpha \colon \mathscr{U}_\alpha \to \mathbb{R}^{N+1}$ such that their images cover Σ, namely $\bigcup_{\alpha\in\mathscr{A}} q_\alpha(\mathscr{U}_\alpha) = \Sigma$. Furthermore we fix an orientation on Σ and by ν we will denote the vector field of the unit normal vectors. Let us consider a tubular neighborhood D_δ of Σ given by:

$$D_\delta = \{|\mathrm{dist}\,(\Sigma, x)| < \delta\} \subset \mathbb{R}^{N+1},$$

where dist denotes the signed distance. All our calculations below have local character and for this reason we will fix a pair $(q_\alpha, \mathscr{U}_\alpha)$ and, for simplicity of notation, drop the subscript α. For each sufficiently small δ the map

$$(s,z) \longmapsto X \in D_\delta \cap q(\mathscr{U}), \quad \text{where } X(s,z) = q(s) + z\nu(s), \quad s = (s_1, \ldots, s_N) \in \mathscr{U},$$

is a diffemorphism onto $D_\delta \cap q(\mathscr{U})$ (we will consistently abuse the notation writing $\nu(s)$ instead of $\nu(q(s))$). In the sequel we will work with the scaled version of Σ, namely Σ_ε, and we will denote its parametrization and the unit normal by q_ε, ν_ε, respectively. It is easy to see that the following relations hold:

$$q_\varepsilon(s) = \varepsilon^{-1} q(\varepsilon s), \quad \nu_\varepsilon(s) = \nu(\varepsilon s), \quad s \in \varepsilon^{-1}\mathscr{U},$$

and that similar scaling formulas can be derived for other functions defined on Σ_ε. We also have local coordinates in $D_{\delta/\varepsilon}$ which we will still denote by (s,z) and the map X_ε defined by:

$$X_\varepsilon(s,z) = q_\varepsilon(s) + z\nu_\varepsilon(s).$$

It is convenient to introduce the following notation for functions $f\colon D_{\delta/\varepsilon} \to \mathbb{R}$:

$$(X_\varepsilon^* f)(s,z) = (f \circ X_\varepsilon)(s,z).$$

The function $X_\varepsilon^* f\colon X_\varepsilon^{-1}(D_{\delta/\varepsilon}) \to \mathbb{R}$ can be interpreted as the pull back of f via parametrization X_ε. In a similar way we define the pull back of a map $f\colon D_{\delta/\varepsilon} \to \mathbb{R}^d$, $d \geq 1$ via X_ε. By $(X^* f)$ we denote the pull back of $f\colon D_\delta \to \mathbb{R}^d$, via X.

We will now derive formulas expressing Δ and $\partial_{x_{N+1}}$ in $D_{\delta/\varepsilon}$, in terms of $(s,z) \in \varepsilon^{-1}\mathscr{U}$. We define for each $z \in (-\delta/\varepsilon, \delta/\varepsilon)$

$$\Sigma_{\varepsilon,z} = \{x \in D_{\delta/\varepsilon} \mid \operatorname{dist}(\Sigma_\varepsilon, x) = z\}.$$

In other words $\Sigma_{\varepsilon,z}$ is the surface obtained from Σ_ε by translation in the direction of the normal by z. Then the well known formula gives:

$$\Delta = \Delta_{\Sigma_{\varepsilon,z}} + \partial_z^2 - H_{\Sigma_{\varepsilon,z}} \partial_z, \tag{2.1}$$

where $H_{\Sigma_{\varepsilon,z}}$ denotes the mean curvature of $\Sigma_{\varepsilon,z}$. We need to expand these operators in terms of the variable z. By g_{Σ_ε} and $g_{\Sigma_{\varepsilon,z}}$, respectively, we will denote the metric on Σ_ε, $\Sigma_{\varepsilon,z}$ (induced from $\mathbb{R}^{N+1}$). In terms of $s \in \varepsilon^{-1}\mathscr{U}$ we get the following expressions:

$$g_{\Sigma_{\varepsilon,z},ij} = g_{\Sigma_\varepsilon,ij} + \varepsilon z a_{\varepsilon,ij} + \varepsilon^2 z^2 b_{\varepsilon,ij}, \tag{2.2}$$

where

$$\begin{aligned} g_{\Sigma_\varepsilon,ij} &= (\partial_j q \cdot \partial_i q)(\varepsilon s), \quad a_{\varepsilon,ij}(s) = (\partial_j q \cdot \partial_i \nu)(\varepsilon s) + (\partial_i q \cdot \partial_j \nu)(\varepsilon s), \\ b_{\varepsilon,ij}(s) &= (\partial_i \nu \cdot \partial_j \nu)(\varepsilon s). \end{aligned} \tag{2.3}$$

Then, for the matrix $g^{-1}_{\Sigma_{\varepsilon,z}} = (g^{ij}_{\Sigma_{\varepsilon,z}})_{i,j=1,\ldots,N}$ we get, provided that $|\varepsilon z|$ is sufficiently small:

$$g^{-1}_{\Sigma_{\varepsilon,z}} = g^{-1}_{\Sigma_\varepsilon} + \varepsilon z A_\varepsilon + \varepsilon^2 z^2 B_\varepsilon, \tag{2.4}$$

where

$$A_\varepsilon = A(\varepsilon s), \quad B_\varepsilon = B(\varepsilon s, \varepsilon z),$$

and $A\colon \mathscr{U} \to \mathbb{R}^N \times \mathbb{R}^N$, $B\colon \mathscr{U} \times (-\delta, \delta) \to \mathbb{R}^N \times \mathbb{R}^N$ are smooth matrix functions. The expression for the Laplace-Beltrami operator on Σ_ε in local coordinates is:

$$\begin{aligned}\Delta_{\Sigma_\varepsilon} &= \frac{1}{\sqrt{\det(g_{\Sigma_\varepsilon})}}\partial_j(\sqrt{\det(g_{\Sigma_\varepsilon})}g^{ij}_{\Sigma_\varepsilon}\partial_i)\\ &= g^{ij}_{\Sigma_\varepsilon}\partial_{ij} + \frac{1}{\sqrt{\det(g_{\Sigma_\varepsilon})}}\partial_j(\sqrt{\det(g_{\Sigma_\varepsilon})}g^{ij}_{\Sigma_\varepsilon})\partial_i\\ &= g^{ij}_{\Sigma_\varepsilon}\partial_{ij} - g^{k\ell}\Gamma^i_{\Sigma_\varepsilon,k\ell}\partial_i,\end{aligned}$$

where $\Gamma^i_{\Sigma_\varepsilon,k\ell}$ are the Christoffell symbols. A similar formula holds for $\Delta_{\Sigma_{\varepsilon,z}}$. Using this we can write:

$$\Delta_{\Sigma_{\varepsilon,z}} = \Delta_{\Sigma_\varepsilon} + \mathbb{A}_{\varepsilon,ij}\partial_{ij} + \mathbb{B}_{\varepsilon,i}\partial_i.$$

where

$$\begin{aligned}\mathbb{A}_{\varepsilon,ij} &= g^{ij}_{\Sigma_{\varepsilon,z}} - g^{ij}_{\Sigma_\varepsilon},\\ \mathbb{B}_{\varepsilon,i} &= g^{k\ell}_{\Sigma_{\varepsilon,z}}[\Gamma^i_{\Sigma_{\varepsilon,z},k\ell} - \Gamma^i_{\Sigma_\varepsilon,k\ell}] + \Gamma^i_{\Sigma_\varepsilon,k\ell}[g^{k\ell}_{\Sigma_{\varepsilon,z}} - g^{k\ell}_{\Sigma_\varepsilon}].\end{aligned}$$

Expressions in local coordinates for $\mathbb{A}_{\varepsilon,ij}$, $\mathbb{B}_{\varepsilon,i}$ can be further derived using the above expansions, however their exact form is not crucial here. The point is that, formally, these functions are small in terms of $|\varepsilon z|$. Finally, for future reference, we notice that if $f_\varepsilon \in \mathscr{C}^2(\Sigma_\varepsilon)$ is identified with $f \in \mathscr{C}^2(\Sigma)$ through the formula $(X^*_\varepsilon f_\varepsilon)(s) = (X^* f)(\varepsilon s)$, then

$$(X^*_\varepsilon \Delta_{\Sigma_\varepsilon} f_\varepsilon)(s) = \varepsilon^2 (X^* \Delta_\Sigma f)(\varepsilon s). \tag{2.5}$$

Next, we will expand the mean curvature $H_{\Sigma_{\varepsilon,z}}$. To this end we will denote by $\mathbb{k}_{\varepsilon,j}$, $j = 1, \dots, N$ the principal curvatures of Σ_ε. Then we have

$$\begin{aligned}H_{\Sigma_{\varepsilon,z}} &= \sum_{j=1}^N \frac{\mathbb{k}_{\varepsilon,j}}{1 - z\mathbb{k}_{\varepsilon,j}}\\ &= \sum_{j=1}^N \mathbb{k}_{\varepsilon,j} + z\sum_{j=1}^N \mathbb{k}^2_{\varepsilon,j} + z^2 R_{\Sigma_\varepsilon},\\ &= H_{\Sigma_\varepsilon} + z|A_{\Sigma_\varepsilon}|^2 + z^2 R_{\Sigma_\varepsilon},\end{aligned} \tag{2.6}$$

where

$$R_{\Sigma_\varepsilon} = \sum_{j=1}^N \mathbb{k}^3_{\varepsilon,j} + z\sum_{j=1}^N \mathbb{k}^4_{\varepsilon,j} + \dots,$$

and $|A_{\Sigma_\varepsilon}|$ is the norm of the second fundamental form on Σ_ε. Denoting by $\mathbb{k}_j$, $j = 1,\dots,N$ the principal curvatures of Σ it is straightforward to see that $(X_\varepsilon^*\mathbb{k}_{\varepsilon,j})(s) = \varepsilon(X^*\mathbb{k}_j)(\varepsilon s)$, hence

$$(X_\varepsilon^*|A_{\Sigma_\varepsilon}|^2)(s) = \varepsilon^2(X^*|A_\Sigma|^2)(\varepsilon s). \tag{2.7}$$

To compute the expression for $\partial_{x_{N+1}} \equiv \partial_{N+1}$ in local coordinates of $D_{\delta/\varepsilon}$ we observe that for any function f_ε in $D_{\delta/\varepsilon}$ we have

$$\partial_{N+1} f_\varepsilon = \nabla f_\varepsilon \cdot \nabla(\pi_{\varepsilon,N+1}),$$

where $\pi_{\varepsilon,j}\colon D_{\delta/\varepsilon} \to \mathbb{R}$ denotes the projection on the jth coordinate. Furthermore we have the following formula for the gradient (interpreted as a vector field defined on $D_{\delta/\varepsilon}$):

$$\nabla f_\varepsilon = \nabla_{\Sigma_{\varepsilon,z}} f_\varepsilon + \partial_z f_\varepsilon \partial_z, \tag{2.8}$$

where $\nabla_{\Sigma_{\varepsilon,z}}$ denotes the gradient vector field on the hypersurface $\Sigma_{\varepsilon,z}$.

The formula for the gradient in the local coordinates $(s,z) \in \varepsilon^{-1}\mathscr{U} \times (-\delta/\varepsilon, \delta/\varepsilon)$ is given by

$$(X_\varepsilon^*\nabla f_\varepsilon) = \partial_j(X_\varepsilon^* f_\varepsilon) g^{ij}_{\Sigma_{\varepsilon,z}} \partial_i + \partial_z(X_\varepsilon^* f_\varepsilon)\partial_z,$$

hence:

$$\begin{aligned} X_\varepsilon^*(\partial_{N+1} f_\varepsilon) &= (X_\varepsilon^*\nabla_{\Sigma_{\varepsilon,z}} f_\varepsilon)\cdot(X_\varepsilon^*\nabla_{\Sigma_{\varepsilon,z}}\pi_{\varepsilon,N+1}) + X_\varepsilon^*(\partial_{\nu_\varepsilon} f_\varepsilon) X_\varepsilon^*(\partial_{\nu_\varepsilon}\pi_{\varepsilon,N+1}) \\ &= g^{ij}_{\Sigma_{\varepsilon,z}} \partial_j(X_\varepsilon^* f)\partial_i(X_\varepsilon^*\pi_{\varepsilon,N+1}) + \partial_z(X_\varepsilon^* f_\varepsilon)\partial_z(X_\varepsilon^*\pi_{\varepsilon,N+1}). \end{aligned}$$

Observe that $X_\varepsilon^*\pi_{\varepsilon,N+1} = q_{\varepsilon,N+1} + z\nu_{\varepsilon,N+1}$, hence, using (2.4) and neglecting those terms that carry a factor of εz in front, we get the following asymptotic formula, valid whenever $|\varepsilon z|$ is small:

$$X_\varepsilon^*(\partial_{N+1} f_\varepsilon) \approx g^{ij}_{\Sigma_\varepsilon} \partial_j(X_\varepsilon^* f)\partial_i(q_{\varepsilon,N+1}) + \partial_z(X_\varepsilon^* f_\varepsilon)\nu_{\varepsilon,N+1}. \tag{2.9}$$

Here and below we denote $f \approx g$ when $f - g$ is a lower order term.

To find the scaling formula for this expression we observe that if $f_\varepsilon \in \mathscr{C}^2(D_{\delta/\varepsilon})$ and $f \in \mathscr{C}^2(D_\delta)$ are related through the formula $(X_\varepsilon^* f_\varepsilon)(s,z) = (X^* f)(\varepsilon s, \varepsilon z)$ then

$$X_\varepsilon^*(\nabla_{\Sigma_\varepsilon} f_\varepsilon) = \varepsilon X^*(\nabla_\Sigma f),$$

and in particular, since we have:

$$(X_\varepsilon^*\pi_{\varepsilon,N+1})(s,z) = \varepsilon^{-1}(X^*\pi_{N+1})(\varepsilon s, \varepsilon z), \quad \nu_{\varepsilon,N+1}(s) = \nu_{N+1}(\varepsilon s),$$

therefore

$$\begin{aligned} X_\varepsilon^*(\partial_{N+1} f_\varepsilon)(s,z) &\approx \varepsilon X^*(\partial_{N+1} f)(\varepsilon s, \varepsilon z) \\ &= \varepsilon\big[\big(X^*(\nabla_\Sigma f\cdot\nabla_\Sigma \pi_{N+1})\big) + \big(X^*(\partial_{\nu_{N+1}} f)(X^*\partial_{\nu_{N+1}}\pi_{N+1})\big)\big](\varepsilon s, \varepsilon z) \\ &= \varepsilon\big[g^{ij}_\Sigma(\partial_j X^* f)(\partial_i q_{N+1}) + \partial_z(X^* f)\nu_{N+1}\big](\varepsilon s, \varepsilon z). \end{aligned} \tag{2.10}$$

2.2 A model for multi component traveling waves

In this section we will describe an approximate form of the multiple traveling wave solution to the equation (1.4), where $c = \varepsilon$ is considered to be a small parameter. This approximate solution models the multiple traveling waves in the sense that the true solution to (1.4) with $c = \varepsilon$ is its small perturbation, as $\varepsilon \to 0$. In general it is reasonable to assume that each component of the multiple traveling wave is a normal graph over an eternal, translating solution of the MC flow, represented by the hypersurface Σ_ε. Moreover, the profile of each component of the traveling front should locally resemble one dimensional solution of (1.4) with $\varepsilon = 0$. Given these observations we will proceed now with more precise definitions.

Let H be the unique odd and monotonically increasing heteroclinic solution of (1.4) in one dimension:

$$H'' + H(1 - H^2) = 0, \quad \text{in } \mathbb{R}.$$

For future reference let us recall that $H(t) = \tanh\left(\frac{t}{\sqrt{2}}\right)$

Furthermore, let $f_j \colon \Sigma \to \mathbb{R}$, $j = 1, \dots, k$, $k > 1$ be smooth functions such that $f_j < f_{j+1}$. We also set for convenience $f_0 = -\infty$ and $f_{k+1} = \infty$. In our formal considerations we do not restrict k, however, to keep the paper at a reasonable length the rigorous construction is carried on for $k = 2$ only (see Remark 1.5).

We now define the approximate solution u_ε, through its expression in local coordinates $(q, \mathscr{U})$, by:

$$(X_\varepsilon^* u_\varepsilon)(s,z) = \sum_{j=1}^{k} (-1)^{j+1} H\big(z - (X^* f_j)(\varepsilon s)\big) + \frac{1}{2}\big(1 - (-1)^{k+1}\big), \tag{2.11}$$
$$\text{where } s \in \varepsilon^{-1}\mathscr{U}, z \in (-\delta/\varepsilon, \delta/\varepsilon).$$

Later on we will have to be more specific about the way the approximate solution is defined outside of $D_{\delta/\varepsilon}$ (which is in fact a nontrivial matter) but for our formal considerations it suffices to know u_ε in $D_{\delta/\varepsilon}$. In the sequel we will denote $f_j(\varepsilon s) = f_{\varepsilon,j}(s)$, so that $f_{\varepsilon,j} \colon \Sigma_\varepsilon \to \mathbb{R}$ and that the following relation holds $(X_\varepsilon^* f_{\varepsilon,j})(s) = (X^* f_j)(\varepsilon s)$, $s \in \varepsilon^{-1}\mathscr{U}$.

In order to solve (1.4) we will further introduce a new unknown function ϕ, and look for a solution in the form $u = u_\varepsilon + \phi$. Substituting into (1.4) with $c = \varepsilon$ we get

$$\Delta u + \varepsilon \partial_{x_{N+1}} u + f(u) = S(u_\varepsilon) + \mathbb{L}(\phi) + N(\phi), \quad f(u) = u(1 - u^2),$$

where

$$\begin{aligned} S(u_\varepsilon) &= \Delta u_\varepsilon + \varepsilon \partial_{x_{N+1}} u_\varepsilon + f(u_\varepsilon), \\ \mathbb{L}(\phi) &= \Delta \phi + \varepsilon \partial_{x_{N+1}} \phi + f'(u_\varepsilon)\phi, \\ N(\phi) &= f(u_\varepsilon + \phi) - f(u_\varepsilon) - f'(u_\varepsilon)\phi. \end{aligned}$$

Then, roughly speaking, (1.4) is reduced to finding ϕ and $f_{\varepsilon,j}$, $j=1,\dots,k$ such that

$$\mathbb{L}(\phi)+S(u_\varepsilon)+N(\phi)=0. \tag{2.12}$$

As we will see later on this problem requires further modification and in particular to solve it we will analyze in details invertibility properties of the linear operator $\mathbb{L}$. Let us notice one important fact in this context. If by $H'_{\varepsilon,j}$ we denote

$$(X_\varepsilon^* H'_{\varepsilon,j})(s,z)=H'\big(z-(X^* f_{\varepsilon,j})(s)\big),$$

then

$$\mathbb{L}(H'_{\varepsilon,j})=o(1),\quad \varepsilon\to 0.$$

Thus the inverse of the linear operator $\mathbb{L}$ is not expected to be uniformly bounded as $\varepsilon\to 0$, since the function $H'_{\varepsilon,j}$ is in the approximate kernel of $\mathbb{L}$. On the other hand to solve (2.12) for ϕ we would like to use a fixed point argument for the operator

$$\phi\longmapsto -\mathbb{L}^{-1}(S(u_\varepsilon)+N(\phi)),$$

and this clearly requires that $\|\mathbb{L}^{-1}\|$ be bounded independently on ε. A standard way to deal with this difficulty is to employ the method of infinite dimensional Lyapunov-Schmidt reduction. The idea is simple: for any function $\psi\colon D_{\delta/\varepsilon}\to\mathbb{R}$ we define a projection operator Π_ε by

$$(X_\varepsilon^*\Pi_\varepsilon\psi)=(X_\varepsilon^* H'_{\varepsilon,j})(s,z)\frac{\int_{-\delta/\varepsilon}^{\delta/\varepsilon}[(X_\varepsilon^*\psi)(X_\varepsilon^* H'_{\varepsilon,j})](s,z)\,dz}{\int_{-\delta/\varepsilon}^{\delta/\varepsilon}(X_\varepsilon^* H'_{\varepsilon,j})^2(s,z)\,dz}.$$

Next we decompose $\phi=\phi^{\parallel}+\phi^{\perp}$ where

$$(X_\varepsilon^*\phi^{\parallel})=(X_\varepsilon^*\Pi_\varepsilon\phi).$$

Then problem (2.12) reduces to:

$$\Pi_\varepsilon[\mathbb{L}(\phi)+S(u_\varepsilon)+N(\phi)]=0, \tag{2.13}$$

$$(\mathrm{Id}-\Pi_\varepsilon)[\mathbb{L}(\phi)+S(u_\varepsilon)+N(\phi)]=0. \tag{2.14}$$

Neglecting formally terms involving $N(\phi)$ and $\mathbb{L}(\phi)$ in (2.13), which should be of lower order, this condition reads:

$$\int_{-\delta/\varepsilon}^{\delta/\varepsilon}\big(X_\varepsilon^*[S(u_\varepsilon)H'_{\varepsilon,j}]\big)(s,z)\,dz=0,\quad j=1,\dots,k,\quad \forall s\in\varepsilon^{-1}\mathscr{U}. \tag{2.15}$$

Recall here that we work with a fixed pair $(q,\mathscr{U})$ belonging to the parametrization $(q_\alpha,\mathscr{U}_\alpha)_{\alpha\in\mathscr{A}}$ of Σ, but of course this condition needs to be satisfied for all $\mathscr{U}_\alpha$.

We will now use equations (2.15) to derive the Jacobi-Toda system on Σ_ε. We will write for each fixed j:

$$\begin{aligned}
&\int_{-\delta/\varepsilon}^{\delta/\varepsilon} \big(X_\varepsilon^*(S(u_\varepsilon)H'_{\varepsilon,j})\big)(s,z)\,dz \\
&\approx \int_{-\delta/\varepsilon}^{\delta/\varepsilon} X_\varepsilon^*\big((\Delta H_{\varepsilon,j}+\varepsilon\partial_{x_{N+1}}H_{\varepsilon,j}+f(H_{\varepsilon,j}))H'_{\varepsilon,j})\big)(s,z)\,dz \\
&\quad + \int_{-\delta/\varepsilon}^{\delta/\varepsilon} X_\varepsilon^*\big([f(\sum_{i=0}^{2}H_{\varepsilon,j+i-1})-\sum_{i=0}^{2}f(H_{\varepsilon,j+i-1})]H'_{\varepsilon,j}\big)(s,z)\,dz.
\end{aligned} \tag{2.16}$$

Observe that above we took into account only terms representing the interactions of the jth wave with its "immediate" neighbors. The remaining terms represent interactions of the jth wave with those waves whose distances to the jth wave are large enough to render their interactions negligible.

Now we will consider the integrand of the first of the integrals on the right hand side of (2.16). Using the expressions for Δ, $\partial_{x_{N+1}}$, and neglecting small terms (as in the previous section), we get:

$$\begin{aligned}
X_\varepsilon^* S(H_{\varepsilon,j}) &\approx \partial_{zz}X_\varepsilon^* H_{\varepsilon,j} + X_\varepsilon^* f(H_{\varepsilon,j}) \\
&\quad + X_\varepsilon^*(\varepsilon\nu_{N+1}-H_{\Sigma_\varepsilon})\partial_z X_\varepsilon^* H_{\varepsilon,j} \\
&\quad + X_\varepsilon^*\big[\big(\Delta_{\Sigma_\varepsilon}-z|A_{\Sigma_\varepsilon}|^2\partial_z\big)H_{\varepsilon,j}+\varepsilon\nabla_{\Sigma_\varepsilon}H_{\varepsilon,j}\cdot\nabla_{\Sigma_\varepsilon}(\pi_{\varepsilon,N+1})\big].
\end{aligned}$$

Consecutive terms above are organized in such a way that the first term is simply 0 by definition of $H_{\varepsilon,j}$, the second term is also 0 since Σ_ε is an eternal solution of the mean curvature flow translating with speed $c=\varepsilon$, and the third is of order $\mathcal{O}(\varepsilon^2)$. In this term we will separate those parts whose projection Π_ε onto $H'_{\varepsilon,j}$ is nonzero from the rest:

$$\begin{aligned}
X_\varepsilon^* S(H_{\varepsilon,j}) &\approx X_\varepsilon^*\big[\big(-\Delta_{\Sigma_\varepsilon}f_{\varepsilon,j}-|A_{\Sigma_\varepsilon}|^2 f_{\varepsilon,j}-\varepsilon\nabla_{\Sigma_\varepsilon}f_{\varepsilon,j}\cdot\nabla_{\Sigma_\varepsilon}(\pi_{\varepsilon,N+1})\big)H'_{\varepsilon,j}\big] \\
&\quad + X_\varepsilon^*(|\nabla_{\Sigma_\varepsilon}f_{\varepsilon,j}|^2 H''_{\varepsilon,j}) - (z-X_\varepsilon^* f_{\varepsilon,j})X_\varepsilon^*(|A_{\Sigma_\varepsilon}|^2 H'_{\varepsilon,j}).
\end{aligned} \tag{2.17}$$

Taking this formula into account it is not hard to show that

$$\begin{aligned}
&\int_{-\delta/\varepsilon}^{\delta/\varepsilon} X_\varepsilon^* S(H_{\varepsilon,j})H'_{\varepsilon,j} \\
&\approx -c_0 X_\varepsilon^*\big(\Delta_{\Sigma_\varepsilon}f_{\varepsilon,j}+|A_{\Sigma_\varepsilon}|^2 f_{\varepsilon,j}+\varepsilon\nabla_{\Sigma_\varepsilon}f_{\varepsilon,j}\cdot\nabla_{\Sigma_\varepsilon}(\pi_{\varepsilon,N+1})\big)(s)\big) \\
&\quad = -\varepsilon^2 c_0 X^*\big(\Delta_\Sigma f_j+|A_\Sigma|^2 f_j+\nabla_\Sigma f_j\cdot\nabla_\Sigma(\pi_{N+1})\big)(\varepsilon s),
\end{aligned} \tag{2.18}$$

where $c_0=\int_{\mathbb{R}}(H')^2$.

Similarly we will separate the integrand in the second integral in (2.16) into the parts whose projection onto $H'_{\varepsilon,j}$ is nontrivial, and the rest. Here we use the fact that from $H(t)=\tanh(\frac{t}{\sqrt{2}})$ we get $1-H^2=\sqrt{2}H'$. After some elementary manipulations we find

(2.19)
$$f\Big(\sum_{i=0}^{2} H_{\varepsilon,j+i-1}\Big) - \sum_{i=0}^{2} f(H_{\varepsilon,j+i-1}) \approx 3\sqrt{2}H'_{\varepsilon,j}(H_{\varepsilon,j-1}-1) + 3\sqrt{2}H'_{\varepsilon,j}(H_{\varepsilon,j+1}+1),$$

where the terms that we have neglected turn out to have small contributions when projected onto $H'_{\varepsilon,j}$. To compute the projection Π_ε let us recall that

$$H(t)-1 \approx -2e^{-\sqrt{2}t}, \quad t\to\infty, \quad H(t)+1 \approx 2e^{\sqrt{2}t}, \quad t\to-\infty.$$

Then, we obtain the following as the leading order term in the second integral in (2.16):

(2.20)
$$\begin{aligned} 3\sqrt{2}\int_{-\delta/\varepsilon}^{\delta/\varepsilon} X_\varepsilon^*\left[(H'_{\varepsilon,j})^2(H_{\varepsilon,j-1}-1)\right](s,z)\,dz + 3\sqrt{2}\int_{-\delta/\varepsilon}^{\delta/\varepsilon} X_\varepsilon^*\left[H'_{\varepsilon,j}(H_{\varepsilon,j+1}+1)\right](s,z)\,dz \\ \approx 6\sqrt{2}c_1 X_\varepsilon^*(e^{\sqrt{2}(f_{\varepsilon,j-1}-f_{\varepsilon,j})} - e^{\sqrt{2}(f_{\varepsilon,j}-f_{\varepsilon,j+1})})(s) \\ = 6\sqrt{2}c_1 X^*(e^{\sqrt{2}(f_{j-1}-f_{\varepsilon,j})} - e^{\sqrt{2}(f_j-f_{j+1})})(\varepsilon s) \end{aligned}$$

where

$$c_1 = \int_{-\infty}^{\infty} \big(H'(t)\big)^2 e^{\sqrt{2}t}\,dt.$$

Denoting

$$\alpha_0 = \frac{c_0}{6\sqrt{2}c_1} = \frac{1}{6\sqrt{2}}\frac{\int_{\mathbb{R}}(H')^2}{\int_{\mathbb{R}}\big(H'(t)\big)^2 e^{\sqrt{2}t}} = \frac{\sqrt{2}}{24},$$

we find that to leading order (2.15) is equivalent to:

$$\begin{aligned} \alpha_0\big(\Delta_{\Sigma_\varepsilon} f_{\varepsilon,j} + |A_{\Sigma_\varepsilon}|^2 f_{\varepsilon,j} + \nabla_{\Sigma_\varepsilon} f_{\varepsilon,j}\cdot\nabla_\Sigma(\pi_{\varepsilon,N+1})\big) \\ - e^{\sqrt{2}(f_{\varepsilon,j-1}-f_{\varepsilon,j})} + e^{\sqrt{2}(f_{\varepsilon,j}-f_{\varepsilon,j+1})} = 0. \end{aligned} \tag{2.21}$$

This system of k equations will be called the *Jacobi-Toda system on* Σ_ε. Let us recall that we have set $f_{\varepsilon,0} = -\infty$ and $f_{\varepsilon,k+1} = \infty$ to close the system. Let us also observe that by scaling back to Σ we get the following singular perturbation problem:

$$\begin{aligned} \alpha_0\varepsilon^2\big(\Delta_\Sigma f_j + |A_{\Sigma_\varepsilon}|^2 f_j + \nabla_\Sigma f_j\cdot\nabla_\Sigma(\pi_{N+1})\big) \\ - e^{\alpha_1(f_{j-1}-f_j)} + e^{\alpha_1(f_j-f_{j+1})} = 0. \end{aligned} \tag{2.22}$$

Solutions of (2.21) and (2.22) are related through the formula $f_{\varepsilon,j}(\cdot) = f_j(\varepsilon\cdot)$. We should mention here that a similar system appears in the context of foliation by interfaces [12] and [10].

3 An existence result for the Jacobi-Toda system

3.1 Rotationally symmetric eternal solutions

The formal calculations of the previous section show that to prove Theorem 1.1 we need to find a suitbale approximation of the components of the traveling front and this in turn requires solving the Jacobi-Toda system (2.22). This will be done in several steps in this section. We begin by writing the Jacobi-Toda system for a special solution of (1.6). Assuming that the surface Σ is given as a graph $\Sigma = \{x_{N+1} = F(x'), x' \in \mathbb{R}^N\}$, and that $c = 1$, we obtain that (1.6) is equivalent to:

$$\nabla\Big(\frac{\nabla F}{\sqrt{1+|\nabla F|^2}}\Big) = \frac{1}{\sqrt{1+|\nabla F|^2}}. \tag{3.1}$$

We will further assume that $F(x') = F(|x'|)$ i.e Σ is rotationally symmetric. Denoting $|x'| = r$ we get:

$$\frac{F_{rr}}{1+F_r^2} + (N-1)\frac{F_r}{r} = 1. \tag{3.2}$$

The following result is proven in [1] in the case $N = 2$ and in general in [5]:

Proposition 3.1. *There exists an entire, rotationally symmetric, and strictly convex, graphical eternal solution to the mean curvature flow. This solution satisfies (3.2) and consequently it is translating with speed $c = 1$. Additionally the following asymptotic expansion as $r \to \infty$ is valid:*

$$F(r) = \frac{r^2}{2(N-1)} - \log r + 1 + \mathcal{O}(r^{-1}). \tag{3.3}$$

In the sequel by Γ we will denote the surface corresponding to the rotationally symmetric eternal solution described in Proposition 3.1.

The Jacobi-Toda system (2.22) for Γ becomes:

$$\varepsilon^2\alpha_0(\Delta_\Gamma f_j + |A_\Gamma|^2 f_j + \nabla_\Gamma f_j \cdot \nabla_\Gamma F) - e^{\sqrt{2}(f_{j-1}-f_j)} + e^{\sqrt{2}(f_j - f_{j+1})} = 0.$$

Our theory of solvability of the Jacobi-Toda system will be valid for functions of the radial variable r only and so we need to express the Jacobi-Toda system on Γ in terms of the radial variable r. For what follows it will be convenient to denote:

$$L[v] = \Delta_\Gamma v + |A_\Gamma|^2 v + \nabla_\Gamma v \cdot \nabla_\Gamma F. \tag{3.4}$$

Now, we will find the expression of this operator when restricted to functions $v = v(r)$ i.e. functions depending on the radial variable only. The Laplace-Betrami operator for a surface $x_{N+1} = F(r)$ acting on $v = v(r)$ is

$$\begin{aligned}\Delta_\Gamma v &= \frac{1}{r^{N-1}\sqrt{1+F_r^2}}\frac{\partial}{\partial r}\Big(\frac{r^{N-1}}{\sqrt{1+F_r^2}}\frac{\partial}{\partial r}\Big)v \\ &= \frac{v_{rr}}{1+F_r^2} + \Big(\frac{N-1}{r} - \frac{F_r}{1+F_r^2}\Big)v_r.\end{aligned} \tag{3.5}$$

The principal curvatures are given by

$$\mathbb{k}_1 = \ldots = \mathbb{k}_{N-1} = \frac{F_r}{r\sqrt{1+F_r^2}}, \quad \mathbb{k}_N = \frac{F_{rr}}{(1+F_r^2)^{3/2}}, \tag{3.6}$$

hence

$$|A_\Gamma|^2 = \sum_{j=1}^N \mathbb{k}_j^2 = \frac{(N-1)F_r^2}{r^2(1+F_r^2)} + \frac{F_{rr}^2}{(1+F_r^2)^3}. \tag{3.7}$$

Finally we have:

$$\nabla_\Gamma v \cdot \nabla_\Gamma F = \frac{v_r F_r}{1+F_r^2},$$

hence we find the following expression for the operator L acting on radial functions (we denote this operator by L_r):

$$L_r[v] = \frac{v_{rr}}{1+F_r^2} + \frac{(N-1)v_r}{r} + \Big(\frac{(N-1)F_r^2}{r^2(1+F_r^2)} + \frac{F_{rr}^2}{(1+F_r^2)^3}\Big)v. \tag{3.8}$$

3.2 Weighted Hölder norms on Γ

We will now proceed to define some weighted norms that we will use in the sequel. First we recall that in general, for a function h given on a manifold Σ we have, in some local coordinates:

$$\nabla_\Sigma h = g^{ij}\partial_j h \partial_i, \quad (D^2_\Sigma h)_{ij} = g^{ij}\partial_{ij}h - g^{ij}\Gamma^k_{ij}\partial_k h.$$

We refer to the vector ∇_Σ as the gradient and to matrix D^2_Σ as the second derivative matrix of h.

Now, in the case at hand we can use the fact that the surface Γ is rotationally symmetric to find ∇_Γ and D^2_Γ. In particular, when $h = h(r)$ i.e. we are dealing with a radial function then we have the following relations:

$$\begin{aligned}
|\nabla_\Gamma h(r)| &\le \frac{C|\partial_r h(r)|}{\sqrt{1+|F_r(r)|^2}}, \\
|\partial_r h(r)| &\le C\sqrt{1+|F_r(r)|^2}|\nabla_\Gamma h(r)|, \\
|D^2_\Gamma h(r)| &\le \frac{C(|\partial_r^2 h(r)| + r^{-1}|\partial_r h(r)|}{1+|F_r(r)|^2}, \\
|\partial_r^2 h(r)| &\le C(1+|F_r(r)|^2)(|D^2_\Gamma h(r)| + |\nabla_\Gamma h(r)|)
\end{aligned}$$

We define the following weighted norms for $\mathscr{C}^{2,\mu}$ functions on Γ:

$$\begin{aligned}
\|h\|_{\mathscr{C}^{0,\mu}_\beta(\Gamma)} &= \sup_{y\in\Gamma}(1+|F_r(|y'|)|^2)^\beta \|h\|_{\mathscr{C}^{0,\mu}(B(y,1)\cap\Gamma)}, \quad y = (y', y_{N+1}), \\
\|h\|_{\mathscr{C}^{2,\mu}_\beta(\Gamma)} &= \|h\|_{\mathscr{C}^{0,\mu}_\beta(\Gamma)} + \|\nabla_\Gamma h\|_{\mathscr{C}^{0,\mu}_\beta(\Gamma)} + \|D^2_\Gamma h\|_{\mathscr{C}^{0,\mu}_\beta(\Gamma)}.
\end{aligned}$$

3.3 A non-homogeneous Jacobi-Toda system

We observe that so far we were considering the Jacobi-Toda system with the right hand side equal to 0. However, as we will we see later on, we have to deal with a more general, non-homogeneous Jacobi-Toda system. This is because in our formal considerations we neglected some terms, that are of lower order but need to be eventually taken into account. Also in this case we assume $k = 2$ and thus we get the following problem:

$$\begin{aligned} \varepsilon^2\alpha_0 L[f_1] + e^{\sqrt{2}(f_1-f_2)} &= \varepsilon^2 h_1, \\ \varepsilon^2\alpha_0 L[f_2] - e^{\sqrt{2}(f_1-f_2)} &= \varepsilon^2 h_2. \end{aligned} \tag{3.9}$$

where $f_j\colon \Gamma \to \mathbb{R}$, $h_j\colon \Gamma \to \mathbb{R}$. To solve the above problem we will assume that f_j, h_j are radial functions. In the remaining part of this section we will consider the problem of the existence of solutions to (3.9) under some assumptions about the decay in r and smallness in ε for the right hand side. In general we will assume that

$$\|h_j\|_{\mathscr{C}^{0,\mu}_\beta(\Gamma)} \leq C\varepsilon^\tau, \quad \tau > 0, \quad \beta > 1. \tag{3.10}$$

Let us explain briefly why a non-homogeneous problem (3.9) with this type of right hand side appears in our considerations. Going back to the formal calculations in section 2.2 we notice that in (2.17) we expanded the mean curvature according to (2.6), and we neglected the error term R_{Σ_ε}. In the case considered here i.e. $\Sigma = \Gamma$, this term is small in terms of ε, and it decays like $\mathscr{O}((1+r^2)^{-\frac{3}{2}})$ when $r \to \infty$.

We have the following:

Proposition 3.2. *Consider the Jacobi-Toda system (3.9) where h_j, $j = 1,2$ are radial functions satisfying (3.10). There exists a solution of this problem such that, the functions u, v defined by*

$$u = \sqrt{2}(f_2 - f_1), \quad v = \sqrt{2}(f_1 + f_2),$$

satisfy

(3.11)

$$\begin{aligned} u(r) &= \log\frac{2\sqrt{2}}{\varepsilon^2\alpha_0|A_\Gamma(r)|^2} + \mathscr{O}(\log\log\frac{1}{\varepsilon^2|A_\Gamma(r)|^2}), \quad \textit{as } \varepsilon r \to 0^+, \textit{ or } \varepsilon r \gg 1, \\ |v(r)| &\leq C\varepsilon^\tau(1+r^2)^{-\frac{1}{2}}\log(2+r^2), \end{aligned}$$

where $|A_\Gamma(r)|$ is the norm of the second fundamental form on Γ.

To describe the strategy let us denote

$$h = \frac{\sqrt{2}}{\alpha_0}(h_2 - h_1), \quad \text{and } g = \frac{\sqrt{2}}{\alpha_0}(h_2 + h_1).$$

Then we get the following decoupled system:

$$L[u] - \frac{2\sqrt{2}}{\varepsilon^2\alpha_0}e^{-u} = h \tag{3.12}$$

$$L[v] = g \tag{3.13}$$

Let us discuss briefly the second of the above equations. The key observation is that the operator L has a decaying, positive element in its kernel

$$\phi_0(r) = \frac{1}{\sqrt{1+F_r^2}} \sim \frac{1}{r}, \quad r \gg 1, \tag{3.14}$$

from which we can solve (3.13) by a standard ODE method.

The solvability theory for the nonlinear equation (3.12) is where the real difficulty lies. Our approach will be to first use an approximation scheme to find a suitable asymptotic approximation of the solution of (3.12), after which we will be in a position to use a fixed point argument to solve the non-homogeneous problem, with the right hand side satisfying (3.10).

The following sections are devoted to the proof of Proposition 3.2.

3.4 Solvability theory for the operator L

We begin by proving the claim that we have made in (3.14), namely that ϕ_0 is in the kernel of L. Note that since Γ is an eternal graph solution to the MC flow then so is $\Gamma + \tau \mathrm{e}_{N+1}$, namely the graph of $x_{N+1} = F(x')$ translated by τ in the direction of the x_{N+1}-axis. This results in an invariance of the nonlinear operator on the left hand side of (3.2), which we will take advantage of in the proof of the following:

Lemma 3.3. *The function $\phi_0 = \frac{1}{\sqrt{1+F_r^2}}$ satisfies $L[\phi_0] = 0$ i.e. it is a positive, decaying element in* Ker$L$.

Proof. Let us consider the nonlinear operator

$$\mathscr{H}(\Phi) = \frac{\Phi_{rr}}{1+\Phi_r^2} + (N-1)\frac{\Phi_r}{r}.$$

Taking variations of this operator of the form $\Phi_\sigma = F + \sigma\phi$, $\phi = \phi(r)$ we get:

$$\frac{d}{d\sigma}\mathscr{H}(\Phi_\sigma)\,|_{\sigma=0} \equiv \mathscr{H}'[\phi] = \frac{\phi_{rr}}{1+F_r^2} - \frac{2F_{rr}F_r\phi_r}{(1+F_r^2)^2} + \frac{(N-1)\phi_r}{r}.$$

In particular we have $\mathscr{H}'[1] \equiv 0$. In addition the following relation is not hard to prove, again assuming that $\phi = \phi(r)$:

$$L[\phi] = \mathscr{H}'[\phi\sqrt{1+F_r^2}].$$

From this the assertion of the lemma follows immediately. □

The existence result for (3.13) follows from the following.

Lemma 3.4. *Let g be a $\mathscr{C}^{0,\mu}(\Gamma)$ radial function such that*

$$\|g\|_{\mathscr{C}^{0,\mu}_{\beta}(\mathbb{R}_+)} < \infty, \quad \beta \geq 1.$$

There exists a unique, bounded solution to

$$L[v] = g, \tag{3.15}$$

such that:

$$\|v\|_{\mathscr{C}^{2,\mu}_{\beta-1}(\Gamma)} \leq C\|g\|_{\mathscr{C}^{0,\mu}_{\beta}(\Gamma)}. \tag{3.16}$$

Proof. Since the function g in (3.15) is radial we can use ODE methods to solve the equation. Given $\phi_0 > 0$ as in Lemma 3.3, which is a radial solution of $L[\phi] = 0$ we find the second linearly independent solution ϕ_1 of $(1+|F_r(r)|^2)L_r[\phi] = 0$ (recall that L_r is the radial form of L) by the reduction of order formula:

$$\phi_1(r) = \phi_0(r)\int_r^\infty (1+|F_r(\rho)|^2)\exp[-A(\rho)]\,d\rho,$$
$$A(\rho) = \int_1^\rho \frac{(N-1)(1+|F_r(\eta)|^2)}{\eta}\,d\eta.$$

From this we readily get that

$$\phi_1(r) \sim \begin{cases} \log r, & N=2, \\ r^{2-N}, & N>2, \end{cases} \qquad r \ll 1, \qquad \phi_1(r) \sim re^{-r^2}, \quad r \gg 1.$$

Denoting by $W(r) = W(1)\exp[-A(r)]$ the Wronskian, and letting $\tilde{g}(r) = (1+|F_r(r)|^2)g(r)$ we write:

$$v(r) = -\phi_0(r)\int_0^r \frac{\phi_1(\rho)\tilde{g}(\rho)}{W(\rho)}\,d\rho + \phi_1(r)\int_0^r \frac{\phi_0(\rho)\tilde{g}(\rho)}{W(\rho)}\,d\rho.$$

The assertion of the Lemma follows from a straightforward argument, using the asymptotic formulas for the functions $\phi_0(r)$ and $\phi_1(r)$. □

3.5 Solving for u: the approximate solution

Our goal in this and the following section is to solve the problem (3.12). Of course once this is done the Proposition 3.2 will be proven. We begin by finding an approximate solution of (3.12) assuming that $h \equiv 0$, which is equivalent to solving:

$$\mathscr{S}_\delta[u] = 0, \tag{3.17}$$

where

$$\mathscr{S}_\delta[u] \equiv L[u] - \delta^{-2}e^{-u}, \quad \delta = \frac{\varepsilon\sqrt{\alpha_0}}{2^{3/4}}, \tag{3.18}$$

and L is the linear operator defined in (3.4). For the purpose of finding a suitable approximate solution we will consider a sequence of approximations $\mathrm{v}_k = v_0 + v_1 + \cdots + v_k$. Once an accurate enough approximation is found the nonlinear problem (3.12) can be reduced to a fixed point problem. This step involves inverting the linear operator obtained by linearization of the nonlinear operator $\mathscr{S}_\delta$ around the approximate solution and will be dealt with in the next section.

The nonlinear operator $\mathscr{S}_\delta$ can be written explicitly (using the notation of section 3.1):

$$\mathscr{S}_\delta[v] = \Delta_\Gamma v + \nabla_\Gamma v \cdot \nabla_\Gamma F + |A_\Gamma|^2 v - \delta^{-2} e^{-v}.$$

We will now describe the construction of an approximate solution of (3.17) . The leading order term of this approximation is found by solving for v_0 the following equation:

$$|A_\Gamma|^2 v_0 = \frac{1}{\delta^2} e^{-v_0} \Longrightarrow v_0 e^{v_0} = \frac{1}{\delta^2 |A_\Gamma|^2}. \tag{3.19}$$

For brevity we denote $b(r) = |A_\Gamma(y)|^2$, $y = (y', y_{N+1})$, $r = |y'|$. Now, equation (3.19) implies that

$$v_0(r) = \log \frac{1}{\delta^2 b(r)} - \log\log \frac{1}{\delta^2 b(r)} + \mathscr{O}(\log\log|\log \delta^2 b(r)|). \tag{3.20}$$

This asymptotic formula is valid when $\delta \ll 1$. This follows from the fact that $b(r) = 1 + \mathscr{O}(r^2)$, $r \to 0$ and on the other hand $b(r) = \frac{N-1}{r^2} + \mathscr{O}(r^{-4})$, $r \gg 1$.

Let us also observe the following relations:

$$v_0' = -\frac{b'}{b}\frac{v_0}{1+v_0}, \quad v_0'' = -\left(\frac{b'}{b}\right)' \frac{v_0}{1+v_0} - \left(\frac{b'}{b}\right)^2 \frac{v_0}{(1+v_0)^3}, \tag{3.21}$$

from which the asymptotic behavior of the derivatives of v_0 of any order can be readily deduced. In particular we observe that

$$|v_0^{(j)}| \le \frac{C}{(r+1)^j}, \quad j = 1, 2, \ldots. \tag{3.22}$$

Accepting v_0 as the leading order approximation, and assuming that the next approximate solution is of the form $\mathrm{v}_1 = v_0 + v_1$, we are left with the following problem:

$$|A_\Gamma|^2 v_1 - \frac{1}{\delta^2}(e^{-v_0 - v_1} - e^{-v_0}) = -\left[\Delta_\Gamma v_0 + \nabla_\Gamma v_0 \cdot \nabla_\Gamma F\right] \equiv \rho_0. \tag{3.23}$$

This is a nonlinear equation with the right hand side that satisfies

$$|\rho_0(y)| \le \frac{C}{(1+r)^2}, \quad r = |y'|. \tag{3.24}$$

This follows from the fact that v_0 is a smooth function on Γ and (3.22). Using this we can find a smooth solution of the equation (3.23) which satisfies:

$$|v_1^{(j)}(y)| \le \frac{C}{\log\left(\frac{2+r^2}{\delta^2}\right)} \frac{1}{(1+r)^j}, \quad j = 0, 1, \ldots. \tag{3.25}$$

The next terms in the approximate solutions will be determined inductively. It is important to keep in mind that the approximations we want to construct must be decaying functions of both $\frac{1}{\log \delta^2}$ and r. Given $\mathrm{v}_{k-1} = v_0 + v_1 + \cdots + v_{k-1}$, for which we already know (suitably adapted) relations (3.24)–(3.25) we determine v_k by solving:

(3.26)
$$|A_\Gamma|^2 v_k - \frac{1}{\delta^2}\left(e^{-v_0-v_1-\cdots-v_k} - e^{-v_0-v_1-\cdots-v_{k-1}}\right) = -\left[\Delta_\Gamma v_{k-1} + \nabla_\Gamma v_{k-1}\cdot\nabla_\Gamma F\right] \equiv \rho_{k-1}.$$

Solving this equation gives $\mathrm{v}_k = v_0 + v_1 + \cdots + v_k$, where:

$$|v_k^{(j)}(y)| \le \frac{C}{\left(\log\frac{2+r^2}{\delta^2}\right)^k}\frac{1}{(1+r)^{k+j-1}}, \quad j = 0,1,\dots, \tag{3.27}$$

and

$$|\rho_k(y)| = \left|\Delta_\Gamma v_k + \nabla_\Gamma v_k\cdot\nabla_\Gamma F\right| \le \frac{C}{\left(\log\frac{2+r^2}{\delta^2}\right)^k}\frac{1}{(1+r)^{k+1}}. \tag{3.28}$$

Thus we have proven:

Lemma 3.5. *For each $k > 1$ there exists a function v_k such that*

$$\mathscr{S}_\delta[\mathrm{v}_k] \le \frac{C}{\left(\log\frac{2+r^2}{\delta^2}\right)^k}\frac{1}{(1+r)^{k+1}}.$$

Another parametrization of Γ

The next step in the proof of Proposition 3.2 it to linearize the operator $\mathscr{S}_\delta$ around v_k and find a solution of $\mathscr{S}_\delta[u] = g$ in the form $u = \mathrm{v}_k + h$ using ODE methods.

To have a convenient form of the linear operator $\mathscr{S}'_\delta[\mathrm{v}_k]$ we define another parametrization of Γ, which is obtained by taking the arc length along the curve $(r, F(r))$. Thus we define:

$$s = \int_0^r \sqrt{1+F_r^2}\,d\rho. \tag{3.29}$$

Of course the function $r \mapsto s(r)$ is invertible and its inverse is $s \mapsto r(s)$. We also note the following relations:

$$\begin{aligned} c|\partial_s h| &\le |\nabla_\Gamma h| \le C|\partial_s h| \\ c(|\partial_s^2 h| + s^{-1}|\partial_s h|) &\le |D_\Gamma^2 h| \le C(|\partial_s^2 h| + s^{-1}|\partial_s h|). \end{aligned} \tag{3.30}$$

Using the asymptotic formula (3.3) for F we get that

$$s \sim r, \quad r \ll 1, \ \text{and}, \qquad s = \frac{r^2}{2(N-1)} + \mathscr{O}(\log r), \quad r \gg 1. \tag{3.31}$$

By a straightforward computation we obtain the following expression for the operator L but now in terms of the arc-lenght variable s:

$$L_s[v] = v_{ss} + a(s)v_s + b(s)v, \tag{3.32}$$

where

$$a(s) = \frac{F_r(r(s)) + \frac{N-1}{r(s)}}{\sqrt{1+F_r^2(r(s))}}, \quad b(s) = |A_\Gamma(r(s))|^2. \tag{3.33}$$

Note that

$$\begin{aligned} a(s) &= \frac{N-1}{s}\big(1+\mathscr{O}(s^2)\big), \quad s \ll 1, \qquad a(s) = 1+\mathscr{O}(s^{-1}), \quad s \gg 1, \\ b(s) &= \frac{N-1}{r^2(s)} + \mathscr{O}(r^{-4}) = \frac{1}{2s} + \mathscr{O}(s^{-2}\log s), \quad s \gg 1, \end{aligned} \tag{3.34}$$

and that in general $a(s)$, $b(s) > 0$ since Γ is convex and $F_r(0) = 0$. We also have $b(0) = 1$ and $b'(0) = 0$. Another important fact is that

$$b''(0) = -\frac{N^2+4N+2}{N^4(N+2)} < 0, \quad N = 2, \dots. \tag{3.35}$$

This last identity follows by a direct computation. Setting $\mathfrak{b}_N = \frac{N^2+4N+2}{2N^4(N+2)}$ we have

$$b(s) = 1 - \mathfrak{b}_N s^2 + \mathscr{O}(s^4), \quad s \to 0. \tag{3.36}$$

Definition of the linearized operator $\mathfrak{L}_\delta$

From the above considerations we see that linearization of $\mathscr{S}_\delta$ around the approximate solution v_k expressed in terms of r is the following operator

$$\mathfrak{L}_\delta[h] = \frac{h_{rr}}{1+F_r^2} + \frac{N-1}{r}h_r + p_\delta(r)h, \quad p_\delta(r) = b(r)(1+v_0 e^{-\mathrm{v}_k + v_0}). \tag{3.37}$$

We will often use the approximate solution v_k expressed in terms of the arc length variable s, which we will denote by $\mathrm{u}_k(s) = \mathrm{v}_k(r(s))$. We will also set $u_j(s) = v_j(r(s))$, $j = 0, 1, \dots$. We let $b(s) = b(r(s))$.

Later on we will consider the linearized operator in the space of functions which decay both in s and $\log(\frac{s}{\delta^2})$ as s increases. We will see that for our purposes we need to determine v_k (or u_k) for k sufficiently large.

With some abuse of notation we will denote by the same symbol $\mathfrak{L}_\delta$ the linearized operator expressed in terms of the arc length variable s:

$$\mathfrak{L}_\delta[h] = h_{ss} + a(s)h_s + p_\delta(s)h, \quad p_\delta(s) = b(s)\big(1+u_0(s)e^{-(u_1(s)+\dots+u_k(s))}\big). \tag{3.38}$$

Our goal is to find a right inverse of $\mathfrak{L}_\delta$. The idea is very simple. Since (3.38) is an ODE an inverse can always be written using the variation of parameters formula. To control the norm of $\mathfrak{L}_\delta^{-1}$ we need to understand the behavior of a fundamental set. This is complicated by the fact that the operator, on the one hand depends on δ, and on the other hand its properies change as s varies from 0 to ∞.

In fact we observe that from (3.19)–(3.20) and (3.26)–(3.27) it follows that

$$p_\delta(s) \sim \log\frac{1}{\delta^2}, \tag{3.39}$$

when $s \le \bar{s}$ with some $\bar{s} > 0$ fixed, independent on δ, while when $s \gg 1$ we have

$$p_\delta(s) \sim \frac{\log \frac{s}{\delta^2}}{s}. \tag{3.40}$$

This can be summarized:

$$p_\delta(s) \sim \frac{1}{2+s} \log\big(\frac{2+s}{\delta^2}\big),$$

for all s and $\delta \ll 1$. At the same time $a(s) \sim \frac{1}{s}$ $s \ll 1$ and $a(s) \sim 1$, $s \gg 1$. In particular we will need to study carefully $\mathfrak{L}_\delta$ in these ranges of s.

3.6 An inverse of $\mathfrak{L}_\delta$

In this section, we solve the following problem

$$\mathfrak{L}_\delta[h] = g(s). \tag{3.41}$$

Clearly, solving this problem is the key to implement a fixed point argument needed to solve (3.12). The point is to construct a right inverse of $\mathfrak{L}_\delta$ which is bounded in suitable Hölder weighted norms. Let us define these norms first:

$$\begin{aligned} \|g\|_{\mathscr{C}^{0,\mu}_{\beta,\nu}(\mathbb{R}_+)} &= \sup_{s>1} \big\{(2+s)^\beta \big(\log \frac{2+s}{\delta^2}\big)^\nu \|g\|_{\mathscr{C}^{0,\mu}((s-1,s+1))}\big\}, \\ \|g\|_{\mathscr{C}^{\ell,\mu}_{\beta,\nu}(\mathbb{R}_+)} &:= \sum_{j=0}^{\ell} \|g^{(j)}\|_{\mathscr{C}^{0,\mu}_{\beta,\nu}(\mathbb{R}_+)}. \end{aligned} \tag{3.42}$$

Because of the relations (3.30) these norms are easily translated into the norms of g as a function (of the radial variable) on Γ.

More precisely we will show:

Lemma 3.6. *Suppose that $\beta > 0$, $\nu > 0$. Then there exists a constant $C > 0$ and a solution h to (3.41) such that*

$$\|h\|_{\mathscr{C}^{0,\mu}_{\beta,\nu}(\mathbb{R}_+)} + \|h'\|_{\mathscr{C}^{0,\mu}_{\beta+1,\nu}(\mathbb{R}_+)} + \|h''\|_{\mathscr{C}^{0,\mu}_{\beta+1,\nu}(\mathbb{R}_+)} \le C(\log \frac{1}{\delta^2})^{4+2\beta} \|g\|_{\mathscr{C}^{0,\mu}_{\beta+1,\nu+1}(\mathbb{R}_+)}. \tag{3.43}$$

In the rest of this section we prove this important lemma.

To begin with we make the following transformation:

$$\hat{h} = \exp\left(\frac{1}{2}\int_1^s a(\tau)\, d\tau\right) h. \tag{3.44}$$

Then, when $s \to 0$, $\hat{h} \sim s^{(N-1)/2} h$ and when $s \to +\infty$, $\hat{h} \sim e^{s/2} h$, by (3.34). Equation (3.41) is transformed to

$$\hat{h}'' + (p_\delta(s) - \hat{a}(s))\hat{h}(s) = \hat{g}, \tag{3.45}$$

where

$$\hat{a} = \frac{1}{2}a' + \frac{1}{4}a^2, \quad \hat{g} = \exp\left(\frac{1}{2}\int_1^s a(\tau)\, d\tau\right) g.$$

In what follows we will mainly work with the transformed equation (3.45). The idea of the proof of the lemma follows the same lines as the construction of the approximate solutions. The situation now is more complicated since we have to consider a second order ODE.

Let us denote

$$\hat{\mathfrak{L}}_\delta[h] = h'' + \hat{p}_\delta h, \quad \hat{p}_\delta = p_\delta - \hat{a}.$$

When we consider the opertator $\hat{\mathfrak{L}}_\delta$ for functions defined in the interval $I_1 = (0, s_1)$, for some $s_1 > 0$ independent on δ then we refer to this problem as the inner problem. We speak of the outer problem when we take $I_{s_\delta} = (s_\delta, \infty)$, $s_\delta \gg s_1 > 0$ as the domain of the functions involved.

First, we will describe the way we chose s_1 and s_δ. For $s \to 0$, we have, by (3.34)–(3.36)

$$\begin{aligned} p_\delta(s) &= \big(1 - \mathfrak{b}_N s^2 + \mathscr{O}(s^4)\big)\big(\log\frac{1}{\delta^2} + 1 + \mathscr{O}(s^2)\big), \\ \hat{a}(s) &= s^{-2}\big[\frac{(N-2)^2}{4} - \frac{1}{4}\big] + \mathscr{O}(1). \end{aligned} \tag{3.46}$$

As a consequence there exist an $M > 0$ and $s_1 > \frac{M}{\sqrt{\log \frac{1}{\delta^2}}} > 0$, which is independent of δ, such that

$$\hat{p}_\delta(s) = p_\delta(s) - \hat{a}(s) > 0, \quad \frac{M}{\sqrt{\log\frac{1}{\delta^2}}} \le s \le s_1. \tag{3.47}$$

When $s \to \infty$ we have by (3.34) that p_δ satisfies (3.40) and

$$\hat{a}(s) = \frac{1}{4} + \mathscr{O}(s^{-1}), \tag{3.48}$$

with similar formulas for the derivatives. From this we can find the asymptotic behavior of $\hat{p}_\delta(s)$ for s large, and infer the existence of $s_2 \ge s_1$, again independent of δ, such that for $s > s_2$ it holds:

$$\hat{p}'_\delta(s) \le 0. \tag{3.49}$$

Observe that s_1 and s_2 in general do not coincide and we need to solve an intermediate problem to glue the inner solution and the solution for s between s_1 and s_2. Finally, we will assume that δ is chosen sufficiently small, so that

$$\hat{p}_\delta(s) > 0, \quad s_1 < s < s_2. \tag{3.50}$$

This can be achieved since, when s is bounded away from 0 and ∞ independently on δ, we have $\hat{p}_\delta(s) \sim p_\delta(s) \sim b(s)\log\frac{1}{\delta^2}$. For future references we observe that from (3.49) and (3.50) it follows that the exists a unique s_δ such that $\hat{p}_\delta(s_\delta) = 0$ and

$$\hat{p}_\delta(s) > 0, \quad s_1 \le s < s_\delta, \quad \hat{p}_\delta(s) < 0, \quad s > s_\delta. \tag{3.51}$$

Actually, from (3.34) it follows that there exist constants $M_1 < M_2$ such that

$$s_\delta \in (M_1 \log \frac{1}{\delta^2}, M_2 \log \frac{1}{\delta^2}). \tag{3.52}$$

One more observation we make is that on any interval $I = (0, s^*)$, with $s^* < C \log \frac{1}{\delta^2}$, the norms $\|\cdot\|_{\mathscr{C}^{\ell,\mu}_{\beta,\nu}(I)}$ and $\|\cdot\|_{\mathscr{C}^{\ell,\mu}_{\beta,0}(I)}$ are equivalent in the following sense:

$$\|g\|_{\mathscr{C}^{\ell,\mu}_{\beta,\nu}(I)} \le C\big(\log \frac{1}{\delta^2}\big)^\nu \|g\|_{\mathscr{C}^{\ell,\mu}_{\beta,0}(I)} \le C\|g\|_{\mathscr{C}^{\ell,\mu}_{\beta,\nu}(I)}.$$

We agree that $\|\cdot\|_{\mathscr{C}^{\ell,\mu}_{\beta,0}(I)} = \|\cdot\|_{\mathscr{C}^{\ell,\mu}_{\beta}(I)}$. We will use this equivalence of norms when we consider the operator $\mathfrak{L}_\delta$ on the interval $(0, s_\delta)$.

The inner problem for the operator $\mathfrak{L}_\delta$

In this section we will consider the following problem:

$$\begin{aligned} \mathfrak{L}_\delta[h_i] &= g, \quad \text{in } I_1 = (0, s_1) \\ h_i(0) &= 0, \quad h_i'(0) = 0. \end{aligned} \tag{3.53}$$

Our goal is to show that there exists a unique solution h_i to (3.53) such that

$$\|h_i\|_{\mathscr{C}^{2,\mu}_{\beta}(I_1)} \le C \log \frac{1}{\delta^2} \|g\|_{\mathscr{C}^{0,\mu}_{\beta+1}(I_1)}. \tag{3.54}$$

We will work with the transformed operator $\mathfrak{L}_\delta$ so that (3.53) becomes:

$$\begin{aligned} \hat{\mathfrak{L}}_\delta[\hat{h}_i] &= \hat{g}, \quad \text{in } I_1 = (0, s_1), \\ \hat{h}_i(0) &= 0, \quad \hat{h}_i'(0) = 0. \end{aligned} \tag{3.55}$$

For convenience we will denote $\lambda = \sqrt{1 + \log \frac{1}{\delta^2}}$. Taking into account the asymptotic behavior of $b(s)$ and $\hat{a}(s)$ when $s \to 0$ we see that the operator $\hat{\mathfrak{L}}_\delta$ can be written in the form:

$$\hat{\mathfrak{L}}_\delta[\hat{h}] = \hat{h}'' + \big[\lambda^2 - s^{-2}\big(\frac{(N-2)^2}{4} - \frac{1}{4}\big)\big]\big(1 + \mathscr{O}(s^2)\big)\hat{h}.$$

It is convenient to make further change of variables setting:

$$\hat{h}_i(s) = \tilde{h}_i(\lambda s), \quad \hat{g}(s) = \tilde{g}(\lambda s), \quad \hat{p}_\delta(s) = \lambda^{-2}\tilde{p}(\lambda s) \quad \text{etc.}$$

Then, denoting by $\tilde{\mathfrak{L}}_\delta$ the re-scaled operator we have:

$$\tilde{\mathfrak{L}}_\delta[\tilde{h}] = \tilde{h}'' + \big[1 - s^{-2}\big(\frac{(N-2)^2}{4} - \frac{1}{4}\big)\big]\big(1 + \mathscr{O}(\lambda^{-2}s^2)\big)\tilde{h},$$

and (3.55) becomes

$$\tilde{\mathfrak{L}}_\delta[\tilde{h}_i] = \lambda^{-2}\tilde{g}, \quad \text{in } I_\lambda = (0, \lambda s_1).$$

Formally $\tilde{\mathfrak{L}}_\delta[\tilde{h}] = 0$ resembles the modified Bessel equation and the operator $\tilde{\mathfrak{L}}_\delta$ should have an element of the kernel $\tilde{h}_{i,1}$ such that

$$\tilde{h}_{i,1}(s) \sim s^{\frac{1}{2}} J_{\frac{N-2}{2}}(s), \tag{3.56}$$

where $J_{\frac{N-2}{2}}(s)$ is the Bessel function. The second linearly independent element in the kernel is such that

$$\tilde{h}_{i,2}(s) \sim s^{\frac{1}{2}} J_{\frac{-N+2}{2}}(s), \tag{3.57}$$

when $\frac{N-2}{2}$ is not an integer and

$$\tilde{h}_{i,2}(s) \sim s^{\frac{1}{2}} Y_{\frac{N-2}{2}}(s),$$

when $\frac{N-2}{2}$ is an integer, where $Y_{\frac{N-2}{2}}$ is the modified Bessel function of the second kind [6].

We choose the solution to (3.55) given by

$$\tilde{h}_i(s) = -\lambda^{-2}\tilde{h}_{i,1}(s)\int_0^s \tilde{h}_{i,2}(\tau)\tilde{g}(\tau)\,d\tau + \lambda^{-2}\tilde{h}_{i,2}(s)\int_0^s \tilde{h}_{i,1}(\tau)\tilde{g}(\tau)\,d\tau. \tag{3.58}$$

Note that $\tilde{h}_i(0) = 0$, $\tilde{h}_i'(0) = 0$ since, after the change of variables, we have $\tilde{g}(s) = \mathcal{O}(s^{\frac{N-1}{2}})$.

To make use of the above formula and to estimate $\tilde{h}_i$ we need some information about the functions $\tilde{h}_{i,j}$, $j = 1,2$. We recall that the Bessel functions oscillate and the same is expected for $\tilde{h}_{i,j}$. We observe first, that passing to the limit over compacts we can justify the asymptotic statements (3.56)–(3.57), and show the uniform convergence of $\tilde{h}_{i,j}$ to the corresponding solutions of the Bessel equation as $\lambda \to \infty$. In particular it follows that for each $K > 0$ and each sufficiently large λ the function $\tilde{h}_{i,1}$ is uniformly bounded on the interval $(0,K)$, and for each small $\tau > 0$ the function $\tilde{h}_{i,2}$ is uniformly bounded over the interval (τ, K). Furthermore, taking K sufficiently large, we may assume that

$$\tilde{p}(s) = \left[1 - s^{-2}\left(\frac{(N-2)^2}{4} - \frac{1}{4}\right)\right]\left(1 + \mathcal{O}(\lambda^{-2}s^2)\right) > 0, \quad s \in (K, \lambda s_1).$$

In fact we even have

$$c_1 \le \tilde{p}(s) \le c_2, \quad s \in (K, \lambda s_1),$$

with some constants $c_1, c_2 > 0$. Now we will make an important observation: let $\tilde{h}$ be a solution of $\tilde{\mathfrak{L}}_\delta[\tilde{h}] = 0$ in $(K, \lambda s_1)$ and consider the following expressions:

$$Q_1(\tilde{h}) \equiv [\tilde{h}'(s)]^2 + \tilde{p}(s)[\tilde{h}(s)]^2, \quad Q_2(\tilde{h}) = \frac{[\tilde{h}'(s)]^2}{\tilde{p}(s)} + [\tilde{h}(s)]^2.$$

It is easy to see that

$$\frac{d}{ds}Q_1(\tilde{h}) = \tilde{p}'(\tilde{h})^2, \quad \frac{d}{ds}Q_2(\tilde{h}) = -\frac{\tilde{p}'}{\tilde{p}}(\tilde{h}')^2. \tag{3.59}$$

Let now $K \le \xi_1 < \xi_2 < \lambda s_1$ be two points such that $\tilde h'(\xi_j) = 0$. Then from (3.59) and the bound on $\tilde p$ it follows that there exist constants C_1, C_2 such that

$$C_2[\tilde h(\xi_1)]^2 \ge [\tilde h(\xi_2)]^2 \ge C_1[\tilde h(\xi_1)]^2, \tag{3.60}$$

as long as $\tilde p'$ does not change sign in the interval (ξ_1, ξ_2) (recall that $\tilde p > 0$ in $(K, \lambda s_1)$).

We claim that from this, and the uniform bound for the functions $\tilde h_{i,j}$ for $s < K$, which we have already proven, it follows that these functions are actually bounded uniformly for $s \ge K$ as well. To prove this we observe that from (3.46) it follows that

$$\tilde p(s) = \big\{[1 - \mathfrak{b}_N\lambda^{-2}s^2 + \mathscr{O}(\lambda^{-4}s^4)] - s^{-2}\big[\frac{(N-2)^2}{4} - \frac{1}{4}\big]\big\}\big(1 + \mathscr{O}(\lambda^{-2}s^2)\big),$$

hence when $N = 2,3$ we have $\tilde p'(s) < 0$ for all $s \in (0, \lambda s_1)$ while when $N > 3$ there exists a unique $s_\lambda < C\sqrt{\lambda}$ such that

$$\tilde p'(s) > 0, \quad s \in (0, s_\lambda), \quad \tilde p'(s) < 0, \quad s \in (s_\lambda, \lambda s_1).$$

Therefore when $N = 2,3$ the uniform bound on $\tilde h_{i,j}$ follows immediately from (3.60). When $N > 3$ we need to consider the growth of $\tilde h_{i,j}$ between $\zeta_1 < s_\lambda < \zeta_2$ where ζ_ℓ are zeros of $\tilde h_{i,j}$. Observe that since $\tilde p(s)$ is bounded uniformly for $0 < s < \lambda s_1$, therefore using the relations (3.59), but now considering those points ζ at which $\tilde h_{i,j}(\zeta) = 0$ we get, as long as $\zeta < s_\lambda$, that $[\tilde h'_{i,j}(\zeta)]^2$ is bounded uniformly in λ. Then, for each $s \in (\zeta_1, s_\lambda)$, we get

$$\frac{d}{ds}Q_2(\tilde h_{i,j})(s) \le 0 \Longrightarrow C[\tilde h'_{i,j}(\zeta_1)]^2 \ge [\tilde h'_{i,j}(s)]^2 + \tilde p(s)[\tilde h_{i,j}(s)]^2,$$

and in particular $[\tilde h'_{i,j}(s_\lambda)]^2 + [\tilde h_{i,j}(s_\lambda)]^2$ is bounded. A similar argument, but using $Q_1(\tilde h_{i,j})(s)$ for $s \in (s_\lambda, \zeta_2)$, gives that $[\tilde h'_{i,j}(s)]^2 + [\tilde h_{i,j}(s)]^2$ is bounded as well. Now (3.60) applies in $(\zeta_2, \lambda s_1)$ and the claim follows.

The asymptotic formulas (3.56)–(3.57) for s small, and the uniform bound on $\tilde h_{i,j}$, together with the variation of parameters formula (3.58), give the following bound:

$$\|s^{\frac{1-N}{2}}\tilde h_i\|_{\mathscr{C}^0(0,K)} \le \frac{C}{\lambda^2}\|s^{2+\frac{1-N}{2}}\tilde g\|_{\mathscr{C}^0(0,K)}. \tag{3.61}$$

On the other hand, the uniform bounds on $\tilde h_{i,j}$ yield:

$$\|s^{\frac{1-N}{2}}\tilde h_i\|_{\mathscr{C}^0(K,\lambda s_1)} \le \frac{C}{\lambda^2}\|s^{1+\frac{1-N}{2}}\tilde g\|_{\mathscr{C}^0(K,\lambda s_1)}. \tag{3.62}$$

Scaling back this estimates we get for the solution of the inner problem the following estimate

$$\|h_i\|_{\mathscr{C}^{0,\mu}(I_1)} \le C\|g\|_{\mathscr{C}^{0,\mu}(I_1)}.$$

Using then equation (3.53) we can write:

$$h_{ss} + a(s)h_s = g - p_\delta(s)h,$$

and since $p_\delta(s) \sim \log\frac{1}{\delta^2}$ on I_1:

$$\|h_i\|_{\mathscr{C}^{2,\mu}(I_1)} \leq C\log\frac{1}{\delta^2}\|g\|_{\mathscr{C}^{0,\mu}(I_1)},$$

from where we get (3.54), using the fact on the interval $I_1 = (0,s_1)$, with s_1 bounded independently on δ, the weight in the definition of $\mathscr{C}^{0,\mu}_\beta$ norm is bounded by a constant.

Continuation of the solution from $s = s_1$ to $s = s_2$

Let $s_1 < s_2$ be as defined above (see (3.47)–(3.50)). We will solve now,

$$\begin{aligned} &\hat{\mathfrak{L}}_\delta[\hat{h}_n] = \hat{g}, \quad \text{in } I_2 = (s_1,s_2) \\ &\hat{h}_n(s_1) = \hat{h}_i(s_1) \quad \hat{h}_n{}'(s_1) = \hat{h}_i'(s_1). \end{aligned} \tag{3.63}$$

Let us recall that in the interval considered here we have $\hat{p}_\delta(s) > 0$, $\hat{p}_\delta(s) \sim b(s)\log\frac{1}{\delta^2}$, and s_2 is a point such that $p_\delta'(s) \leq 0$, for $s > s_2$.

The solution of (3.63) can be written using the variation of parameters formula

$$\begin{aligned} \hat{h}_n(s) = \hat{h}_{n,1}(s)\hat{h}_i(s_1) + \hat{h}_{n,2}(s)\hat{h}_i'(s_1) - \hat{h}_{n,1}(s)\int_{s_1}^{s}\hat{h}_{2,n}(\tau)\hat{g}(\tau)\,d\tau \\ + \hat{h}_{n,2}(s)\int_{s_1}^{s}\hat{h}_{1,n}(\tau)\hat{g}(\tau)\,d\tau, \end{aligned} \tag{3.64}$$

where the $\hat{h}_{n,j}$ form a fundamental set of the ODE (3.63) with

$$\hat{h}_{n,1}(s_1) = 1 = \hat{h}_{n,2}'(s_1), \quad \hat{h}_{n,1}'(s_1) = 0 = \hat{h}_{n,2}(s_1).$$

Using the fact that, by the choice of s_1, s_2 and δ in (3.47)–(3.50), $\hat{p}_\delta(s) > c > 0$ in I_2, we can employ the identities (3.59) to obtain a uniform bound on $[\hat{h}_{n,j}(s)]^2$ and $[\hat{h}_{n,j}'(s)]^2$ in I_2.

Then from the estimate on $\hat{h}_j(s_1)$ and $\hat{h}_i'(s_1)$ and (3.64) we get, after changing back to the original functions h_n and g

$$\|h_n\|_{\mathscr{C}^{0,\mu}(I_2)} \leq C\log\frac{1}{\delta^2}\|g\|_{\mathscr{C}^{0,\mu}(I_1\cup I_2)}, \tag{3.65}$$

hence we get, again using the equation:

$$\|h_n\|_{\mathscr{C}^{2,\mu}(I_2)} \leq C\big(\log\frac{1}{\delta^2}\big)^2\|g\|_{\mathscr{C}^{0,\mu}(I_1\cup I_2)}, \tag{3.66}$$

and since s_2 is bounded:

$$\|h_n\|_{\mathscr{C}^{2,\mu}_\beta(I_2)} \leq C\big(\log\frac{1}{\delta^2}\big)^2\|g\|_{\mathscr{C}^{0,\mu}_{\beta+1}(I_1\cup I_2)}. \tag{3.67}$$

Continuation of the solution from $s = s_2$ to $s = s_\delta$

Next we will solve,

$$\begin{aligned} &\hat{\mathfrak{L}}_\delta[\hat{h}_m] = \hat{g}, \quad \text{in } I_3 = (s_2, s_\delta), \\ &\hat{h}_m(s_2) = \hat{h}_n(s_2) \quad \hat{h}_m{}'(s_2) = \hat{h}_n'(s_2), \end{aligned} \tag{3.68}$$

where s_δ is defined in (3.51). Notice that in I_3 we have $\hat{p}_\delta'(s) < 0$ however $\hat{p}_\delta(s)$ is not bounded away from 0 since by definition of s_δ, $\hat{p}_\delta(s_\delta) = 0$. But we can still use the quadratic form $Q_1(h)$ in (3.59) to find a uniform bound on $[\hat{h}_{m,j}'(s)]^2$, where the $\hat{h}_{m,j}$ are elements of a fundamental set. From this we find:

$$|\hat{h}_{m,j}(s)| \leq C(1 + (s - s_2)), \quad s \in I_3. \tag{3.69}$$

Then, the variation of parameters formula gives:

$$\begin{aligned} \hat{h}_m(s) = \hat{h}_{m,1}(s)\hat{h}_n(s_2) + \hat{h}_{m,2}(s)\hat{h}_n'(s_2) - \hat{h}_{m,1}(s)\int_{s_2}^s \hat{h}_{2,m}(\tau)\hat{g}(\tau)\,d\tau \\ + \hat{h}_{m,2}(s)\int_{s_2}^s \hat{h}_{1,m}(\tau)\hat{g}(\tau)\,d\tau. \end{aligned} \tag{3.70}$$

Multiplying this identity by $\exp\{-\frac{1}{2}\int_1^s a(\tau)\,d\tau\}$ and using (3.69) we infer that the function

$$h_m(s) = \hat{h}_m(s)\exp\{-\frac{1}{2}\int_1^s a(\tau)\,d\tau\},$$

satisfies

$$\|h_m\|_{\mathscr{C}^{0,\mu}(I_3)} \leq C\big(\log\frac{1}{\delta^2}\big)(|h_n(s_2)| + |h_n{}'(s_2)|) + C\big(\log\frac{1}{\delta^2}\big)^2\|g\|_{\mathscr{C}^{0,\mu}_{\beta+1}(I_3)}.$$

Taking into account (3.66) we find:

$$\|h_m\|_{\mathscr{C}^{0,\mu}(I_3)} \leq C\big(\log\frac{1}{\delta^2}\big)^3\|g\|_{\mathscr{C}^{0,\mu}_{\beta+1}(I_1\cup I_2\cup I_3)}, \tag{3.71}$$

and then using the equation $\mathfrak{L}_\delta[h_m] = g$ in I_3:

$$\|h_m\|_{\mathscr{C}^{2,\mu}(I_3)} \leq C\big(\log\frac{1}{\delta^2}\big)^4\|g\|_{\mathscr{C}^{0,\mu}_{\beta+1}(I_1\cup I_2\cup I_3)}. \tag{3.72}$$

Finally, noting that for $s_2 < s < s_\delta$ we have $(2+s)^\beta \leq C\big(\log\frac{1}{\delta^2}\big)^\beta$ we obtain the following estimate:

$$\|h_m\|_{\mathscr{C}^{2,\mu}_\beta(I_3)} \leq C\big(\log\frac{1}{\delta^2}\big)^{4+\beta}\|g\|_{\mathscr{C}^{0,\mu}_{\beta+1}(I_1\cup I_2\cup I_3)}. \tag{3.73}$$

The outer problem for the operator $\mathfrak{L}_\delta$

Now we will find a solution $\hat{h}_o$ of (3.45) such that

$$\begin{aligned} &\hat{h}_o'' + \hat{p}_\delta \hat{h}_o = \hat{g}, \quad s > s_\delta, \\ &\hat{h}_o(s_\delta) = \hat{h}_m(s_\delta), \quad \hat{h}_o'(s_\delta) = \hat{h}_m'(s_\delta). \end{aligned} \tag{3.74}$$

It is convenient to change variables $s = s_\delta + t$ and regard at first this problem for $t \in \mathbb{R}_+$. We will use the same symbols for the functions involved. Again we will use the variation of parameters formula. To this end, we need to chose two linearly independent solutions of the homogeneous problem such that

$$\hat{h}_{o,1}(t) \to \infty, \quad \text{and } \hat{h}_{o,2}(t) \to 0, \quad t \to \infty.$$

A fundamental set with these properties can be found (for instance see [22]) given that $\hat{p}_\delta(s_\delta + t) = -\frac{1}{4} + o(1)$ as $t \to \infty$. Moreover we can chose $\hat{h}_{o,j}$ in such a way that

$$\begin{aligned} &\hat{h}_{o,1}(0) = 0, \quad \hat{h}_{o,2}(0) = 1, \\ &\hat{h}_{o,1}'(0) = 1, \quad \hat{h}_{o,2}'(0) = -\eta, \end{aligned} \tag{3.75}$$

where $\eta > 0$ is bounded independently on δ. Observe that the Wronskian of these functions is $W(\hat{h}_{o,1}, \hat{h}_{o,2})(t) = -1$. Then we get:

(3.76)
$$\begin{aligned} \hat{h}_o(s_\delta + t) &= [\eta \hat{h}_o(s_\delta) + \hat{h}_o'(s_\delta)]\hat{h}_{o,1}(t) + h_o(s_\delta)\hat{h}_{o,2}(t) \\ &\quad + \hat{h}_{o,1}(t)\int_0^t h_{o,2}(\tau)\hat{g}(s_\delta + \tau)\,d\tau - \hat{h}_{o,2}(t)\int_0^t h_{o,1}(\tau)\hat{g}(s_\delta + \tau)\,d\tau. \end{aligned}$$

Since $\hat{p}_\delta'(s_\delta + t) < 0$ and $\hat{p}_\delta''(s_\delta + t) > 0$ for $t > 0$, therefore by the general theory for second order linear ODEs (see for instance [22], chpt. 9.2) we get that for some $c_j, C_j > 0$, $j = 1, 2$:

(3.77)
$$\begin{aligned} &C_1 \exp\left\{\int_0^t [-\hat{p}_\delta(s_\delta + \tau)]^{1/2}\,d\tau\right\} \le \hat{h}_{o,1}(t) \le C_2 \exp\left\{\int_0^t [-\hat{p}_\delta(s_\delta + \tau)]^{1/2}\,d\tau\right\}, \\ &c_1 \exp\left\{-\int_0^t [-\hat{p}_\delta(s_\delta + \tau)]^{1/2}\,d\tau\right\} \le \hat{h}_{o,2}(t) \le c_2 \exp\left\{-\int_0^t [-\hat{p}_\delta(s_\delta + \tau)]^{1/2}\,d\tau\right\}. \end{aligned}$$

We note that for any $\alpha > 0$, $\nu > 0$ and δ sufficiently small, the functions:

(3.78)
$$\begin{aligned} &(s_\delta + t)^\alpha \big(\log \frac{s_\delta + t}{\delta^2}\big)^{\nu+1} \exp\left\{\int_0^t \big([-\hat{p}_\delta(s_\delta + \tau)]^{1/2} - \frac{1}{2}a(s_\delta + \tau)\big)\,d\tau\right\} \\ &(s_\delta + t)^\alpha \big(\log \frac{s_\delta + t}{\delta^2}\big)^{\nu+1} \exp\left\{\int_0^t \big(-[-\hat{p}_\delta(s_\delta + \tau)]^{1/2} - \frac{1}{2}a(s_\delta + \tau)\big)\,d\tau\right\} \end{aligned}$$

are monotone decreasing for $t>0$, hence using that $s_\delta=\mathscr{O}(\log\frac{1}{\delta})$ and denoting

$$\omega_{\beta,\nu+1}(s_\delta+t)=(s_\delta+t)^\beta\big(\log\frac{s_\delta+t}{\delta^2}\big)^{\nu+1}\exp\Big\{-\int_1^{s_\delta+t}a(\tau)\,d\tau\Big\},$$

we get by (3.73):

$$\begin{aligned}&\omega_{\beta,\nu+1}(s_\delta+t)|[\eta\hat h_o(s_\delta)+\hat h_o'(s_\delta)]\hat h_{o,1}(t)+\hat h_o(s_\delta)\hat h_{o,2}(t)|\\ &\qquad\le C\big(\log\frac{1}{\delta}\big)^{5+2\beta+\nu}\|g\|_{\mathscr{C}^{0,\mu}_{\beta+1,0}(0,s_\delta)}\\ &\qquad\le C\big(\log\frac{1}{\delta}\big)^{4+2\beta}\|g\|_{\mathscr{C}^{0,\mu}_{\beta+1,\nu+1}(0,s_\delta)}.\end{aligned}\tag{3.79}$$

On the other hand, for any $\beta>0,\nu>0$ and δ sufficiently small, the functions:

$$\begin{aligned}&(s_\delta+t)^{-\beta-1}\big(\log\frac{s_\delta+t}{\delta^2}\big)^{-\nu-1}\exp\Big\{\int_0^t\big(-[-\hat p_\delta(s_\delta+\tau)]^{1/2}+\frac12 a(s_\delta+\tau)\big)\,d\tau\Big\},\\ &(s_\delta+t)^{-\beta-1}\big(\log\frac{s_\delta+t}{\delta^2}\big)^{-\nu-1}\exp\Big\{\int_0^t\big([-\hat p_\delta(s_\delta+\tau)]^{1/2}+\frac12 a(s_\delta+\tau)\big)\,d\tau\Big\},\end{aligned}\tag{3.80}$$

are monotone increasing for $t>0$. Then, assuming $\|g\|_{\mathscr{C}^{0,\mu}_{\beta+1,\nu+1}(\mathbb{R}_+)}<\infty$, we get that the functions

$$\begin{aligned}y_1(t)&=\hat h_{o,1}(t)\int_0^t h_{o,2}(\tau)\hat g(s_\delta+\tau)\,d\tau\\ y_2(t)&=\hat h_{o,2}(t)\int_0^t h_{o,1}(\tau)\hat g(s_\delta+\tau)\,d\tau,\end{aligned}$$

satisfy:

$$\omega_{\beta,\nu+1}(s_\delta+t)(|y_1(t)|+|y_2(t)|)\le C\|g\|_{\mathscr{C}^{0,\mu}_{\beta+1,\nu+1}(\mathbb{R}_+)}\tag{3.81}$$

We recall that

$$h_o(s_\delta+t)=\hat h_o(s_\delta+t)\exp\Big\{-\frac12\int_1^{s_\delta+t}a(\tau)\,d\tau\Big\}.$$

Thus, by the variation of parameters formula (3.76) and (3.79)–(3.81) it follows that:

$$\|h_o\|_{\mathscr{C}^{0,\mu}_{\beta,\nu+1}(s_\delta,\infty)}\le C\big(\log\frac{1}{\delta}\big)^{4+2\beta}\|g\|_{\mathscr{C}^{0}_{\beta+1,\nu}(\mathbb{R}_+)}.\tag{3.82}$$

To estimate the Hölder norms of the derivatives we write the equation for h_o in the form:

$$\Big(h_o'\exp\Big\{\int_{s^*}^s a(\tau)\,d\tau\Big\}\Big)'=\exp\Big\{\int_{s^*}^s a(\tau)\,d\tau\Big\}(g-p_\delta h),$$

where $s^* < s_\delta$ is large and fixed independently on δ. Integrating this equation from s^* to $s > s_\delta$ we get

(3.83)

$$\Big|h_o'(s)\exp\Big\{\int_{s^*}^s a(\tau)\,d\tau\Big\}\Big| \le |h_o'(s^*)| + \Big|\int_{s^*}^s \exp\Big\{\int_{s^*}^\sigma a(\tau)\,d\tau\Big\}(g-p_\delta h)\,d\sigma\Big|$$
$$\le |h_o'(s^*)| + C(\|g\|_{\mathscr{C}^0_{\beta+1,\nu}(\mathbb{R}_+)} + \|h_o\|_{\mathscr{C}^{0,\mu}_{\beta,\nu}(\mathbb{R}_+)})\int_{s^*}^s \tilde\omega_{\beta,\nu}(\sigma)\,d\sigma,$$

where

$$\tilde\omega_{\beta,\nu}(\sigma) = (2+\sigma)^{-\beta-1}\big(\log\frac{2+\sigma}{\delta^2}\big)^{-\nu}\exp\Big\{\int_{s^*}^\sigma a(\tau)\,d\tau\Big\}.$$

When s^* is taken sufficiently large we have for $\sigma > s^*$

$$\tilde\omega_{\beta,\nu}(\sigma) \le C(2+\sigma)^{-\beta-1}\big(\log\frac{2+\sigma}{\delta^2}\big)^{-\nu}\exp\Big\{\int_{s^*}^s a(\tau)\,d\tau\Big\}.$$

Using this for $s\in(s^*,s^*+1)$ we find by (3.83):

$$(2+s)^{\beta+1}\big(\log\frac{2+s}{\delta^2}\big)^{\nu}|h_o'(s)| \le C\big(\log\frac{1}{\delta}\big)^{4+2\beta}\|g\|_{\mathscr{C}^0_{\beta+1,\nu}(\mathbb{R}_+)},$$

by the previous argument. Then we argue inductively considering intervals of the form (s^*+k,s^*+k+1) to show, that for $s\in(s^*+k,s^*+k+1)$ we have an analogous estimate. This gives at the end:

$$\|h_o'\|_{\mathscr{C}^{0,\mu}_{\beta+1,\nu}((s_\delta,\infty))} \le C(\log\frac{1}{\delta})^{4+2\beta}\|g\|_{\mathscr{C}^{0,\mu}_{\beta+1,\nu+1}(\mathbb{R}_+)}. \tag{3.84}$$

Then we estimate h_o'' using the equation directly.

Now the solution of (3.41) can be written in the form

$$h = h_i\chi_{I_1} + h_n\chi_{I_2} + h_m\chi_{I_3} + h_o\chi_{(s_\delta,\infty)},$$

where χ_I is the characteristic function of the interval I. We conclude the proof of the Lemma 3.6 by combining estimates (3.54), (3.67), (3.73) and (3.84). For future purposes we will denote the right inverse of $\mathfrak{L}_\delta$ by $\mathfrak{L}_\delta^{-1}$. According to the statement of the Lemma 3.6 we have in particular:

(3.85)

$$\|\mathfrak{L}_\delta^{-1}(g)\|_{\mathscr{C}^{0,\mu}_{\beta,\nu}(\mathbb{R}_+)} + \|(\mathfrak{L}_\delta^{-1}(g))'\|_{\mathscr{C}^{1,\mu}_{\beta+1,\nu}(\mathbb{R}_+)} \le C(\log\frac{1}{\delta^2})^{4+2\beta}\|g\|_{\mathscr{C}^{0,\mu}_{\beta+1,\nu+1}(\mathbb{R}_+)}.$$

Conclusion of the proof of Proposition 3.2

We will now use the theory of the previous two sections to solve (3.12)–(3.13) and thereby complete the proof of the Proposition 3.2.

Notice that the existence of the function v_ε solving (3.13) has been established already. Thus we only need to consider (3.12). We will use a fixed point argument for the nonlinear operator $\mathscr{S}_\delta$. Let $k>0$ be fixed and take the approximate solution

v_k, see Lemma 3.5. We define $\mathrm{u}_k(s) = \mathrm{v}_k(r(s))$. Then, the result of Lemma 3.5 reads:

$$|\mathscr{S}_\delta[\mathrm{u}_k](s)| \leq \frac{C}{\left(\log \frac{2+s}{\delta^2}\right)^k} \frac{1}{(1+s)^{(k+1)/2}}.$$

We will look for a solution in the form $u = \mathrm{u}_k + \phi$. We will write:

$$\mathscr{S}_\delta[\mathrm{u}_k + \phi] = \mathfrak{L}_\delta[\phi] + \mathscr{S}_\delta[\mathrm{u}_k] + \mathfrak{N}_\delta(\phi),$$

where

$$\begin{aligned}\mathfrak{N}_\delta(\phi) &= -\frac{1}{\delta^2} e^{-\mathrm{u}_k}(e^{-\phi} - 1 + \phi) = -b(s)u_0\big[1 + \mathscr{O}\big(\frac{1}{\log \frac{2+s}{\delta^2}}\big)\big](e^{-\phi} - 1 + \phi) \\ &\sim \frac{1}{2+s} \log\big(\frac{2+s}{\delta^2}\big)(e^{-\phi} - 1 + \phi),\end{aligned}$$

is a nonlinear function with quadratic growth in its argument. Thus, we need to solve:

$$\mathfrak{L}_\delta[\phi] + \mathscr{S}_\delta[\mathrm{u}_k] + \mathfrak{N}_\delta(\phi) = h_\delta,$$

Now given the right inverse of $\mathfrak{L}_\delta$ we can put the above equation in the form of a fixed point problem for:

$$\mathscr{T}_\delta[\phi] := -\mathfrak{L}_\delta^{-1}[\mathscr{S}_\delta[\mathrm{u}_k] + \mathfrak{N}_\delta(\phi) - h_\delta].$$

Given the result of Lemma 3.6 and (3.85) the existence of ϕ can be established. To see this let us fix real numbers β, ν, $\gamma > 0$ and a positive integer k, which satisfy in addition:

$$\frac{1}{2} > \beta, \quad \nu > 6 + 2\beta + \gamma, \quad k > 4 + 2\beta + \nu + \gamma.$$

With this choice one can verify that

$$\begin{aligned}\big(\log \frac{1}{\delta^2}\big)^{4+2\beta} \|\mathscr{S}_\delta[\mathrm{u}_k]\|_{\mathscr{C}^{0,\mu}_{\beta+1,\nu+1}(\mathbb{R}_+)} &\leq C\big(\log \frac{1}{\delta^2}\big)^{-\gamma}, \\ \big(\log \frac{1}{\delta^2}\big)^{4+2\beta} \|\mathfrak{N}_\delta(\phi)\|_{\mathscr{C}^{0,\mu}_{\beta+1,\nu+1}(\mathbb{R}_+)} &\leq C\big(\log \frac{1}{\delta^2}\big)^{-\gamma} \|\phi\|^2_{\mathscr{C}^{2,\mu}_{\beta,\nu}(\mathbb{R}_+)} \\ \big(\log \frac{1}{\delta^2}\big)^{4+2\beta} \|h_\delta\|_{\mathscr{C}^{0,\mu}_{\beta+1,\nu+1}(\mathbb{R}_+)} &\leq C\big(\log \frac{1}{\delta^2}\big)^{-\gamma}.\end{aligned} \tag{3.86}$$

Then we see that, for each sufficiently small δ, the map $\mathscr{T}_\delta$ takes the set

$$\{\phi \mid \|\phi\|_{\mathscr{C}^{0,\mu}_{\beta,\nu}(\mathbb{R}_+)} + \|\phi'\|_{\mathscr{C}^{1,\mu}_{\beta+1,\nu}(\mathbb{R}_+)} < \big(\log \frac{1}{\delta^2}\big)^{-\frac{1}{2}\gamma}\}$$

into itself. Also, one can verify in a similar manner that this map is a Lipschitz contraction on this set and thus the proof of the Proposition follows.

4 Setting up the infinite dimensional reduction

4.1 Construction of the approximation

Let Γ be the eternal solution of the mean curvature flow with $c = 1$ and let Γ_ε be the corresponding surface translating with speed $c = \varepsilon \ll 1$. We will use the natural representation of Γ as a graph of the radial function $x_{N+1} = F(r)$. The scaled surface is given by $\Gamma_\varepsilon = \{x_{N+1} = F_\varepsilon(r) \mid F_\varepsilon(r) = \varepsilon^{-1}F(\varepsilon r)\}$. In general we will take advantage of the radial symmetry of the eternal solution and employ the infinite dimensional Lyapunov-Schmidt reduction method to reduce the original PDE:

$$\Delta u + \varepsilon \partial_{x_{N+1}} u + u - u^3 = 0, \quad \text{in } \mathbb{R}^{N+1}, \tag{4.1}$$

to a one dimensional system of two equations whose independent variable is the radial variable r. This will be in fact the Jacobi-Toda system treated above.

We will now proceed to define an approximation of a solution of (4.1) which depends on the radial variable r and the signed distance z to Γ_ε. We will use the notation introduced in Sections 2.1–2.2, with obvious modifications taking into account the fact that Γ_ε is radially symmetric and thus has a globally defined parametrization.

A model for the multicomponent traveling wave near Γ_ε

In the sequel it will be useful to keep in mind that a global system of coordinates on Γ and Γ_ε can be defined by:

$$\Gamma = \{(r\Theta, F(r)) \mid r > 0, \Theta \in S^{N-1}\}, \quad \Gamma_\varepsilon = \{(r\Theta, \frac{1}{\varepsilon}F(\varepsilon r)) \mid r > 0, \Theta \in S^{N-1}\}.$$

There are other ways to introduce local coordinates on Γ. For instance around each point $y \in \Gamma$ we have the normal geodesic coordinates. It is not hard to show that there exists $\delta_0 > 0$ such that these coordinates are well defined for each $y \in \Gamma$ at least in a neighborhood of y of the form $U_{y,\delta_0} = B(y, \delta_0) \cap \Gamma$. A similar statement can be made when $y \in \Gamma_\varepsilon$ are considered, now with $U_{y,\delta_0/\varepsilon} = B(y, \delta_0/\varepsilon) \cap \Gamma$.

We chose an orientation $\nu(y) = \frac{(-\nabla F(r(y)),1)}{\sqrt{1+|\nabla F(r(y))|^2}}$ on Γ and take $z = z(x) = \operatorname{dist}(x, \Gamma)$ compatible with this orientation. Let us introduce the following weight functions:

$$\omega(x) = 2 + |F_r(r)|^2, \quad \omega_\varepsilon(x) = 2 + |F_r(\varepsilon r)|^2, \quad x = (x', x_{N+1}), r = |x'|.$$

We recall here that $F_r(r) \sim r$, $r \gg 1$. Also in what follows we will write $\omega(r)$, $\omega_\varepsilon(r)$, understanding that $r = r(x) = |x'|$.

It is not hard to show that there exists an $\eta_0 > 0$ such that for all points x such that $|z(x)| \le \eta_0 \log \omega(r)$ the map

$$x \mapsto y + z\nu(y), \quad y \in \Gamma,$$

is a diffeomorphism. We denote this diffeomorphism by $X(x) = (y, z)$ and for a function u given in a neighborhood of Γ we set $(X^*u)(y, z) = (u \circ X^{-1})(y, z)$. The coordinates (y,z) above are called Fermi coordinates of Γ. Similar claims are true when we consider Γ_ε and points x such that $|z(x)| \leq \frac{\eta_0}{\varepsilon}\log(\omega_\varepsilon(r))$. Taking this into account we introduce the following neighborhood of Γ_ε:

$$U_{\Gamma_\varepsilon}(M) = \big\{x \in \mathbb{R}^{N+1} \mid |z(x)| = |\mathrm{dist}\,(x,\Gamma_\varepsilon)| \leq M\log\big(\frac{\omega_\varepsilon(r)}{\varepsilon^2}\big)\big\}.$$

Clearly Fermi coordinates are well defined in $U_{\Gamma_\varepsilon}(M)$ for all $M>0$ large and $\varepsilon>0$ small. If by X_ε we denote the diffeomorphism in $U_{\Gamma_\varepsilon}(M)$ defined by $X_\varepsilon(x) = (y,z)$ then for a function u defined in this neighborhood we set:

$$(X_\varepsilon^* u)(y,z) = (u\circ X_\varepsilon^{-1})(y,z).$$

We will describe functions f_j representing the leading order for the location of the nodal set of our traveling wave. To this end we appeal to the results of Proposition 3.2 and let the functions f_j, $j=1,2$ to be solutions of the Jacobi-Toda system (3.9) with $h_j\equiv 0$. We get that functions f_j satisfy:

$$f_j(r) = \frac{(-1)^j}{2\sqrt{2}}\log\frac{2\sqrt{2}}{\varepsilon^2\alpha_0|A_\Gamma(r)|^2} + \mathscr{O}(\log\log\frac{1}{\varepsilon^2|A_\Gamma(r)|^2}). \tag{4.2}$$

In addition we have $f_1 = -f_2$.

In the sequel we will use scaled versions of these functions, namely $f_{\varepsilon,j}\colon \Gamma_\varepsilon\to\mathbb{R}$, defined by:

$$f_{\varepsilon,j}(r) = f_j(\varepsilon r), \quad r = r(y) = |y'|, \quad y = (y', y_{N+1})\in\Gamma_\varepsilon.$$

We recall here that $\varepsilon^2|A_\Gamma(\varepsilon r)|^2 = |A_{\Gamma_\varepsilon}(r)|^2$.

In reality functions $f_{\varepsilon,j}$ give only the leading order behavior of the traveling fronts and thus we further need two functions, which will be for a moment unknown parameters to be determined in the course of the Lyapunov-Schmidt scheme we use.

Thus we let h_j, $j=1,2$ be functions of the radial variable r on Γ such that for some $\beta,\tau\in(0,1)$ we have:

$$\|h_j\|_{\mathscr{C}^{2,\mu}_\beta(\Gamma)} \leq \varepsilon^\tau. \tag{4.3}$$

As before we introduce scaled versions of these functions $h_{\varepsilon,j}\colon \Gamma_\varepsilon\to\mathbb{R}$ defined by $h_{\varepsilon,j}(r) = h_j(\varepsilon r)$. Let us make an elementary observation about the relation between the weighted norms on Γ and Γ_ε. Defining the $\mathscr{C}^{2,\mu}_\beta(\Gamma_\varepsilon)$ norm in a natural way, namely using the weight function $\omega^\beta_\varepsilon(r) = \omega^\beta(\varepsilon r)$ and letting $h_\varepsilon(y) = h(\varepsilon y)$, for $y\in\Gamma_\varepsilon$ we get:

$$\|h_\varepsilon\|_{\mathscr{C}^{2,\mu}_\beta(\Gamma_\varepsilon)} \leq \|h\|_{\mathscr{C}^{2,\mu}_\beta(\Gamma)} \leq \varepsilon^{-2-\mu}\|h_\varepsilon\|_{\mathscr{C}^{2,\mu}_\beta(\Gamma_\varepsilon)}.$$

In particular we get from this and (4.3):

$$\|h_{\varepsilon,j}\|_{\mathscr{C}^{2,\mu}_\beta(\Gamma_\varepsilon)} \leq \varepsilon^\tau, \quad j=1,2. \tag{4.4}$$

Given the functions $f_{\varepsilon,j}$ and $h_{\varepsilon,j}$ as described above we will denote:

$$\mathbf{f}_\varepsilon = (f_{\varepsilon,1}, f_{\varepsilon,2}), \quad \mathbf{h}_\varepsilon = (h_{\varepsilon,1}, h_{\varepsilon,2}),$$

etc.

To define a model for the traveling profile we first recall that by H we have denoted the unique, odd, and monotonically increasing solution of $H'' + H(1 - H^2) = 0$. Next we consider a cut off function:

$$\chi(t) = \begin{cases} 0, & |t| < 1, \\ 1, & |t| > 2. \end{cases}$$

Now, let $M > 0$ be a fixed large number and let

$$\chi_\varepsilon(x) = \chi\Big(\frac{z(x)}{M\log(\frac{\omega_\varepsilon(r)}{\varepsilon^2})}\Big), \quad z(x) = \operatorname{dist}(x, \Gamma_\varepsilon). \tag{4.5}$$

Taking M large and ε small we define the initial approximation of the solution in the support of χ_ε by

$$(X_\varepsilon^* u_\varepsilon)(r,z) = H\big(z - f_{\varepsilon,1}(r) - h_{\varepsilon,1}(r)\big) - H\big(z - f_{\varepsilon,2}(r) - h_{\varepsilon,2}(r)\big) - 1. \tag{4.6}$$

Next we define the initial approximation globally in $\mathbb{R}^{N+1}$ by:

$$w_\varepsilon(x) = \big(1 - \chi_\varepsilon(x)\big)u_\varepsilon(x) - \chi_\varepsilon(x). \tag{4.7}$$

4.2 Reduction to the nonlinear projected problem

We look for a solution of

$$S(u) = \Delta u + \varepsilon \partial_{x_{N+1}} u + u(1 - u^2) = 0,$$

in the form $u = w_\varepsilon + \varphi_\varepsilon$, where φ_ε is a small function. We write:

$$S(w_\varepsilon + \varphi_\varepsilon) = S(w_\varepsilon) + L\varphi_\varepsilon + N(\varphi_\varepsilon),$$

where

$$L\varphi_\varepsilon = \Delta\varphi_\varepsilon + \varepsilon\partial_{x_{N+1}}\varphi_\varepsilon + (1 - 3w_\varepsilon^2)\varphi_\varepsilon,$$
$$N(\varphi_\varepsilon) = -3w_\varepsilon\varphi_\varepsilon^2 - \varphi_\varepsilon^3.$$

We will decompose our nonlinear problem into a system suitable to apply an infinite dimensional Lyapunov-Schmidt reduction scheme. To this end we recall that we have: given functions $\mathbf{f}_\varepsilon$, and also unknown functions $\mathbf{h}_\varepsilon$.

Given a large number M as in the definition of w_ε above we consider smooth cutoff functions $\zeta_j \geq 0$, $j = 1, 2$ which satisfy the following conditions

$$\zeta_1(t) + \zeta_2(t) = \begin{cases} 1, & |t| \leq M, \\ 0, & |t| \geq 2M, \end{cases} \qquad \zeta_1(t) = \begin{cases} 1, & -M < t < -1, \\ 0, & t > 1. \end{cases} \tag{4.8}$$

We define cutoff functions $\zeta_{\varepsilon,j}$ by:

$$(X_\varepsilon^* \zeta_{\varepsilon,j})(r,z) = \zeta_j\big(z - (\tfrac{1}{2} + \delta)|f_{\varepsilon,1}(r) - f_{\varepsilon,2}(r)|\big), \tag{4.9}$$

where δ is a small constant. Note that with this definition we have

$$\zeta_{\varepsilon,1}+\zeta_{\varepsilon,2}=1,\quad |z|<M+(\frac{1}{2}+\delta)|f_{\varepsilon,1}(r)-f_{\varepsilon,2}(r)|,$$
$$\zeta_{\varepsilon,1}+\zeta_{\varepsilon,2}=0,\quad |z|>2M+(\frac{1}{2}+\delta)|f_{\varepsilon,1}(r)-f_{\varepsilon,2}(r)|.$$

Also we have

$$\zeta_{\varepsilon,j}(r,(f_{\varepsilon,j}+h_{\varepsilon,j})(\varepsilon r))=1.$$

Furthermore we chose cut off functions $\tilde{\zeta}_{\varepsilon,j}$ such that

$$\operatorname{supp}\zeta_{\varepsilon,1}=\{-3M-\frac{1}{2}|f_{\varepsilon,1}(r)-f_{\varepsilon,2}(r)|<z<(\frac{1}{2}+2\delta)|f_{\varepsilon,1}(r)-f_{\varepsilon,2}(r)|\},$$
$$\operatorname{supp}\zeta_{\varepsilon,2}=\{3M+\frac{1}{2}|f_{\varepsilon,1}(r)-f_{\varepsilon,2}(r)|>z>-(\frac{1}{2}+2\delta)|f_{\varepsilon,1}(r)-f_{\varepsilon,2}(r)|\},$$

and additionally

$$\tilde{\zeta}_{\varepsilon,j}\zeta_{\varepsilon,j}=\zeta_{\varepsilon,j}.$$

Now we look for a solution of our problem φ_ε in the form:

$$\varphi_\varepsilon=\sum_{j=1,2}\zeta_{\varepsilon,j}\phi_{\varepsilon,j}+\psi_\varepsilon.$$

The functions $\phi_{\varepsilon,j}$, ψ_ε must still be determined from a system of equations that we will now describe. First we introduce functions $H'_{\varepsilon,j}$ defined by:

$$(X_\varepsilon^* H'_{\varepsilon,j})(y,z)=H'(z-f_{\varepsilon,j}(\varepsilon r)),\quad r=|y'|.$$

We also introduce new unknowns $\mathtt{c}_{\varepsilon,j}$, $j=1,2$, which are functions on Γ_ε. Next, we ask that the functions $\phi_{\varepsilon,j}$, ψ_ε, $\mathtt{c}_{\varepsilon,j}$ be solutions of the following coupled system of equations:

(4.10)

$$\tilde{\zeta}_{\varepsilon,j}L\phi_{\varepsilon,j}=\tilde{\zeta}_{\varepsilon,j}\{-(S(w_\varepsilon)+N)-(L-\Delta-\varepsilon\partial_{x_{N+1}}+2)\psi_\varepsilon-[L,\zeta_{\varepsilon,j}]\phi_{\varepsilon,j}+\mathtt{c}_{\varepsilon,j}H'_{\varepsilon,j}\},$$

(4.11)

$$\begin{aligned}(\Delta+\varepsilon\partial_{x_{N+1}}-2)\psi_\varepsilon=&-\big(1-\sum_{i=1,2}\zeta_{i,\varepsilon}\big)\big\{S(w_\varepsilon)+N+[L,\zeta_{\varepsilon,i}]\phi_{\varepsilon,i}\big\}\\&-\big(1-\sum_{i=1,2}\zeta_{\varepsilon,i}\big)(L-\Delta-\varepsilon\partial_{X_{N+1}}+2)\psi_\varepsilon,\end{aligned}$$

where $N=N(\sum_{j=1,2}\phi_{\varepsilon,j}\zeta_{\varepsilon j}+\psi_\varepsilon)$. Note that after multiplying (4.10) by $\zeta_{\varepsilon,j}$, $j=1,2$, using the fact that $\zeta_{\varepsilon,j}\tilde{\zeta}_{\varepsilon,j}\equiv 1$, and adding the resulting expression and (4.11) we obtain:

$$L\varphi_\varepsilon+S(w_\varepsilon)+N(\varphi_\varepsilon)=\sum_{j=1,2}\mathtt{c}_{\varepsilon,j}H'_{\varepsilon,j}\zeta_{\varepsilon,j}. \tag{4.12}$$

As is usual in a Lyapunov-Schmidt reduction approach, the functions $\mathtt{c}_{\varepsilon,j}$ will be initially determined in such a way that (4.10) has a solution for any given parameter function $\mathbf{h}_\varepsilon$. Later we will adjust the traveling front, whose location is represented by $\mathbf{f}_\varepsilon + \mathbf{h}_\varepsilon$, so that $\mathtt{c}_{\varepsilon,j} \equiv 0$. After this is done we will get the solution of our original problem.

In fact a slight modification of (4.10), which we will describe now, is needed. We introduce the following functions:

$$(X_\varepsilon^* w_{\varepsilon,j})(y,z) = H(z - f_{\varepsilon,j}(\varepsilon r)), \quad j = 1,2, \ r = |y'|,$$

and check that we have, say in the set $\tilde{\zeta}_{\varepsilon,j} \equiv 1$,

$$\begin{aligned} L\phi_{\varepsilon,j} = \Delta_{\Gamma_\varepsilon} &+ \partial_z^2 \phi_{\varepsilon,j} + f'(w_{\varepsilon,j})\phi_{\varepsilon,j} \\ &+ [f'(w_\varepsilon) - f'(w_{\varepsilon,j})]\phi_{\varepsilon,j} + [\Delta_{\Gamma_{\varepsilon,z}} - \Delta_{\Gamma_\varepsilon}]\phi_{\varepsilon,j} \\ &- (H_{\Gamma_{\varepsilon,z}} - \varepsilon \nu_{\Gamma_\varepsilon,N+1})\partial_z \phi_{\varepsilon,j} + \varepsilon \nabla_{\Gamma_{\varepsilon,z}}(\pi_{\varepsilon,N+1}) \cdot \nabla_{\Gamma_{\varepsilon,z}} \phi_{\varepsilon,j}. \end{aligned}$$

Then, we can write (4.10) in the form:

$$\Delta_{\Gamma_\varepsilon} \phi_{\varepsilon,j} + \partial_z^2 \phi_{\varepsilon,j} + f'(w_{\varepsilon,j})\phi_{\varepsilon,j} = \mathfrak{g}_{\varepsilon,j} + \mathtt{c}_{\varepsilon,j} H'_{\varepsilon,j}, \tag{4.13}$$

at least when $\tilde{\zeta}_{\varepsilon,j} \equiv 1$. However, it is convenient to view this problem in the set $\Gamma_\varepsilon \times \mathbb{R}$. Indeed the operator $L_{\varepsilon,j} = \Delta_{\Gamma_\varepsilon} + \partial_z^2 + f'(w_{\varepsilon,j})$ is defined on functions whose domain is $\Gamma_\varepsilon \times \mathbb{R}$, while the right hand side is a function supported on a set $\operatorname{supp} \tilde{\zeta}_{\varepsilon,j}$. More precisely we have:

$$\begin{aligned} \mathfrak{g}_{\varepsilon,j} = \tilde{\zeta}_{\varepsilon,j}(S(w_\varepsilon) + N) &- \tilde{\zeta}_{\varepsilon,j}(L - \Delta - \varepsilon \partial_{x_{N+1}} + 2)\psi_\varepsilon - \tilde{\zeta}_{\varepsilon,j}[L, \zeta_{\varepsilon,j}]\phi_{\varepsilon,j} \\ &+ \tilde{\zeta}_{\varepsilon,j}[f'(w_\varepsilon) - f'(w_{\varepsilon,j})]\phi_{\varepsilon,j} + \tilde{\zeta}_{j,\varepsilon}[\Delta_{\Gamma_{\varepsilon,z}} - \Delta_{\Gamma_\varepsilon}]\phi_{\varepsilon,j} \\ &+ \tilde{\zeta}_{\varepsilon,j}[(H_{\Gamma_{\varepsilon,z}} - \varepsilon \nu_{\Gamma_\varepsilon,N+1})\partial_z \phi_{\varepsilon,j} - \varepsilon \nabla_{\Gamma_{\varepsilon,z}}(\pi_{\varepsilon,N+1}) \cdot \nabla_{\Gamma_{\varepsilon,z}} \phi_{\varepsilon,j}]. \end{aligned} \tag{4.14}$$

Again, multiplying (4.13) by $\zeta_{\varepsilon,j}$ and adding the resulting equations and (4.11) we get (4.12).

For future purposes we write (4.11) in the form

$$(\Delta + \varepsilon \partial_{x_{N+1}} - 2)\psi_\varepsilon = \mathfrak{h}_\varepsilon, \tag{4.15}$$

where by $\mathfrak{h}_\varepsilon$ we have denoted the right hand side of (4.11). Note that if we assume that $\phi_{\varepsilon,j}$ and ψ_ε are functions of (r, x_{N+1}) only with $r = |x'|$, then so are the functions $\mathfrak{g}_{\varepsilon,j}$ and $\mathfrak{h}_\varepsilon$. Conversely, if we consider more generally problems of the form (4.13) and (4.15) with $\mathfrak{g}_{\varepsilon,j}$ and $\mathfrak{h}_\varepsilon$ depending on (r, x_{N+1}) only, then the solutions of these problems $\phi_{\varepsilon,j}$ and ψ_ε will also depend on (r, x_{N+1}) only.

4.3 Further modification of (4.13)

Let us look now at the equation (4.13) more closely. We have in general the following system to solve:

$$[\Delta_{\Gamma_\varepsilon} + \partial_z^2 + f'(w_{\varepsilon,j})]\phi_{\varepsilon,j} = \mathfrak{g}_{\varepsilon,j}, \quad \text{in } \Gamma_\varepsilon \times \mathbb{R}, \quad j = 1,2.$$

It is convenient to rewrite this system in the following way: first, we introduce shifted Fermi coordinates

$$\mathtt{t}_j = z - f_{\varepsilon,j}(r), \quad j = 1,2.$$

Second, we write each of the operators above in terms of these new coordinates:

$$\begin{aligned}\Delta_{\Gamma_\varepsilon} + \partial_z^2 + f'(w_{\varepsilon,j}) = {} & \Delta_{\Gamma_\varepsilon} + \partial_{\mathtt{t}_j}^2 + f'(H(\mathtt{t}_j)) \\ & - \Delta_{\Gamma_\varepsilon} f_{\varepsilon,j}\partial_{\mathtt{t}_j} - \nabla_{\Gamma_\varepsilon} f_{\varepsilon,j}\cdot\nabla_{\Gamma_\varepsilon}\partial_{\mathtt{t}_j} + |\nabla_{\Gamma_\varepsilon} f_{\varepsilon,j}|^2\partial_{\mathtt{t}_j}^2.\end{aligned}$$

Usually the second line above is relatively small in the sense that its norm can be controlled by the norm of the solution times a small factor and thus we can absorb it on the right hand side of the corresponding equation. Note also that variables $\mathtt{t}_j$ are related through the formula:

$$\mathtt{t}_1 - \mathtt{t}_2 = f_{\varepsilon,2} - f_{\varepsilon,1}. \tag{4.16}$$

Then letting

$$\tilde{\mathfrak{g}}_{\varepsilon,j}(y,\mathtt{t}_j) = \mathfrak{g}_{\varepsilon,j} + \tilde{\zeta}_{\varepsilon,j}\left[\Delta_{\Gamma_\varepsilon} f_{\varepsilon,j}\partial_{\mathtt{t}_j} + \nabla_{\Gamma_\varepsilon} f_{\varepsilon,j}\cdot\nabla_{\Gamma_\varepsilon}\partial_{\mathtt{t}_j} - |\nabla_{\Gamma_\varepsilon} f_{\varepsilon,j}|^2\partial_{\mathtt{t}_j}^2\right]\phi_{\varepsilon,j},$$

we obtain the following system:

$$[\Delta_{\Gamma_\varepsilon} + \partial_{\mathtt{t}_j}^2 + f'(H(\mathtt{t}_j))]\phi_{\varepsilon,j} = \tilde{\mathfrak{g}}_{\varepsilon,j}(y,\mathtt{t}_j) + \mathtt{c}_{\varepsilon,j}H'(\mathtt{t}_j), \quad j = 1,2, \tag{4.17}$$

where now, with some abuse of notation, $\phi_{\varepsilon,j} = \phi_{\varepsilon,j}(y,\mathtt{t}_j)$. This system can be considered as a system for functions defined on two copies $\Gamma_\varepsilon \times \mathbb{R}$, and it looks at first sight as being decoupled. However in reality we have, in the original setting:

$$\tilde{\mathfrak{g}}_{\varepsilon,j} = \tilde{\mathfrak{g}}_{\varepsilon,j}(y,z;\phi_{\varepsilon,1},\phi_{\varepsilon,2},\psi_\varepsilon).$$

Therefore when considering for instance the equation for $\phi_{\varepsilon,1}$ in the shifted variable $\mathtt{t}_1$ we need to use the above relation between $\mathtt{t}_1$ and $\mathtt{t}_2$ to express all functions involved in terms of $y \in \Gamma_\varepsilon$ and $\mathtt{t}_1 \in \mathbb{R}$. Of course the same must be done with the second equation. As a result we will obtain a nonlinear and nonlocal system for $\phi_{\varepsilon,j}$, $j = 1,2$. The advantage of making this transformation is that we always work with the same, basic linearized operator on the left hand side. Again we point out that all the functions involved depend on y through the radial variable $r = |y'|$.

5 Linear theory

We recall that we have denoted $\omega(r(y)) = 1 + |\nabla F(r(y))|^2$, $\omega_\varepsilon(r) = \omega(\varepsilon r)$. Given a $\mathscr{C}^{2,\mu}(\Gamma_\varepsilon\times\mathbb{R})$ function u we define its weighted norms by:
(5.1)

$$\begin{aligned}\|u\|_{\mathscr{C}^{0,\mu}_{\beta,\eta}(\Gamma_\varepsilon\times\mathbb{R})} &= \sup_{(y,z)\in\Gamma_\varepsilon\times\mathbb{R}} (\cosh z)^\eta\,\omega_\varepsilon^\beta(r(y))\|u\|_{\mathscr{C}^{0,\mu}(B(y,1)\cap\Gamma_\varepsilon\times(z-1,z+1))} \\ \|u\|_{\mathscr{C}^{2,\mu}_{\beta,\eta}(\Gamma_\varepsilon\times\mathbb{R})} &= \|u\|_{\mathscr{C}^{0,\mu}_{\beta,\eta}(\Gamma_\varepsilon\times\mathbb{R})} + \|\nabla_{\Gamma_\varepsilon\times\mathbb{R}}u\|_{\mathscr{C}^{0,\mu}_{\beta,\eta}(\Gamma_\varepsilon\times\mathbb{R})} + \|D^2_{\Gamma_\varepsilon\times\mathbb{R}}u\|_{\mathscr{C}^{0,\mu}_{\beta,\eta}(\Gamma_\varepsilon\times\mathbb{R})}.\end{aligned}$$

Above $\nabla_{\Gamma_\varepsilon\times\mathbb{R}}$ and $D^2_{\Gamma_\varepsilon\times\mathbb{R}}$ denote the gradient and second derivative on the manifold $\Gamma_\varepsilon\times\mathbb{R}$ equipped with a natural product metric and the associated Levi-Civita connection.

In this section we will consider the following basic linearized operator:

$$\Delta_{\Gamma_\varepsilon}\phi+\partial_z^2\phi+f'(H(z))\phi\equiv \mathrm{L}_\varepsilon\phi.$$

We note that

$$\partial_z^2 H'+f'(H)H'=0.$$

In fact H' is the unique bounded element in the kernel of $\partial_z^2+f'(H)$. In particular we have, with some $\nu_0>0$:

$$\int_{\mathbb{R}}|\phi'(z)|^2-f'(H(z))|\phi(z)|^2\geq \nu_0\int_{\mathbb{R}}|\phi(z)|^2,$$

whenever ϕ satisfies:

$$\int_{\mathbb{R}}\phi(z)H'(z)\,dz=0.$$

In general we will consider the following problem:

$$\begin{aligned}&\Delta_{\Gamma_\varepsilon}\phi+\partial_z^2\phi+f'(H)\phi=\mathfrak{g},\quad \text{in } \Gamma_\varepsilon\times\mathbb{R},\\ &\int_{-\infty}^{\infty}\phi(y,z)H'(z)\,dz=0,\quad y\in\Gamma_\varepsilon.\end{aligned}\tag{5.2}$$

We will assume that

$$\|\mathfrak{g}\|_{\mathscr{C}^{0,\mu}_{\beta,\eta}(\Gamma_\varepsilon\times\mathbb{R})}\leq\infty,$$

with some $\beta,\eta>0$. In the case at hand we have $\beta\in(0,1)$ and $\eta\in(0,\sqrt{2})$.

5.1 A priori estimates

Most of what will be said in this section follows the argument of [8] and so we will only outline the main points.

First we need the following:

Lemma 5.1. *The only bounded solutions of*

$$\Delta\phi+\partial_z^2\phi+f'(H(z))\phi=0,\quad \textit{in } \mathbb{R}^{N+1},\quad N\geq 0,$$

are of the form $\phi=cH'(z)$, with some constant c.

This lemma is proven in [11] (see also [9]) .

Next, we show the following *a priori* estimate:

Lemma 5.2. *Let ϕ be a solution of the problem (5.2). There holds:*

$$\|\phi\|_{\mathscr{C}^{2,\mu}_{\beta,\eta}(\Gamma_\varepsilon\times\mathbb{R})}\leq C\|\mathfrak{g}\|_{\mathscr{C}^{0,\mu}_{\beta,\eta}(\Gamma_\varepsilon\times\mathbb{R})}.\tag{5.3}$$

Proof. The proof of this lemma follows arguments in [11] and [9], with only small changes due to the fact that here we use slightly different norms.

We argue by contradiction. Thus we assume that there exists sequences $\{\varepsilon_n\}$, $\{\phi_{\varepsilon_n}\}$, $\{\mathfrak{g}_{\varepsilon_n}\}$ such that

$$\Delta_{\Gamma_{\varepsilon_n}}\phi_{\varepsilon_n} + \partial_z^2\phi_{\varepsilon_n} + f'(H)\phi_{\varepsilon_n} = \mathfrak{g}_{\varepsilon_n}, \quad \text{in } \Gamma_{\varepsilon_n}\times\mathbb{R},$$
$$\int_{-\infty}^{\infty}\phi_{\varepsilon_n}(y,z)H'(z)\,dz = 0, \quad y\in\Gamma_{\varepsilon_n},$$

and such that as $\varepsilon_n \to 0$:

$$\|\phi_{\varepsilon_n}\|_{\mathscr{C}^{2,\mu}_{\beta,\eta}(\Gamma_{\varepsilon_n}\times\mathbb{R})} = 1, \quad \|\mathfrak{g}_{\varepsilon_n}\|_{\mathscr{C}^{0,\mu}_{\beta,\eta}(\Gamma_{\varepsilon_n}\times\mathbb{R})} \to 0.$$

In particular from the definition of the norm there exists $y_n \in \Gamma_{\varepsilon_n}$ and $z_n \in \mathbb{R}$ such that:

$$(5.4) \qquad (\cosh z_n)^{\eta}\,\omega^{\beta}_{\varepsilon_n}(r(y_n))\|\phi_{\varepsilon_n}\|_{\mathscr{C}^{0,\mu}(B(y_n,1)\cap\Gamma_{\varepsilon_n}\times(z_n-1,z_n+1))} > \frac{1}{2}.$$

We consider 4 cases depending on the behavior of the sequences $\{\varepsilon_n r(y_n)\}$, $\{z_n\}$. The various possibilities are for example: (1) $\varepsilon_n r(y_n)$ and z_n bounded, (2) $\varepsilon_n r(y_n) \to \infty$ while z_n bounded etc. In each of these cases we use essentially the same argument with just slight modifications. This has been done in detail in [11] and [9].

To get the idea of the general scheme we assume for instance that $\{\varepsilon_n r(y_n)\}$ and $\{z_n\}$ are bounded. We take the normal geodesic coordinates on Γ_{ε_n}, which are defined around each y_n at least in the set $U_n = B(y_n, \delta_0/\varepsilon_n)\cap\Gamma_{\varepsilon_n}$, where $\delta_0 > 0$ is a small number independent on y_n. We denote the coordinates of an $y \in U_n$ by $\xi = (\xi_1,\dots,\xi_N)$ and set:

$$\tilde\phi_n(\xi,z) = \phi_{\varepsilon_n}(y,z), \quad (y,z)\in U_n\times\mathbb{R}.$$

In the local coordinates we have

$$\begin{aligned}\Delta_{\Gamma_{\varepsilon_n}}\phi_{\varepsilon_n} + \partial_z^2\phi_{\varepsilon_n} + f'(H)\phi_{\varepsilon_n} &= \Delta\tilde\phi_n + \partial_z^2\tilde\phi_n + f'(H)\tilde\phi_n \\ &\quad + a_{\varepsilon_n,ij}\partial_{ij}\tilde\phi_n + b_{\varepsilon_n,j}\partial_j\tilde\phi_n.\end{aligned}$$

Passing to the limit over compacts we obtain that $\tilde\phi_n \to \tilde\phi$ in $\mathscr{C}^{2,\mu'}_{loc}(\mathbb{R}^N\times\mathbb{R})$, $\mu' < \mu$, where $\tilde\phi(0) > 0$ and $\tilde\phi$ is bounded, and at the same time

$$\Delta\tilde\phi + \partial_z^2\tilde\phi + f'(H)\tilde\phi = 0.$$

Lemma 5.1 implies that $\tilde\phi = cH'$ but this contradicts the fact that we also have

$$\int_R \tilde\phi(\cdot,z)H'(z)\,dz = 0,$$

passing to the limit in the orthogonality condition.

To get an idea of how the other cases are handled let us consider the case $\varepsilon_n r(y_n) \to \infty$ while $\{z_n\}$ remains bounded. Then we proceed in a similar manner as above letting

$$\tilde{\phi}_n(\xi,z) = \omega_\varepsilon^\beta(r(y))\phi_{\varepsilon_n}(y,z).$$

For the remaining cases we refer the reader to [11] (see also [9]). □

5.2 An existence result for the model linear problem

Proposition 5.3. *Given* $\mathfrak{g} \in \mathscr{C}^{0,\mu}_{\beta,\eta}(\Gamma_\varepsilon \times \mathbb{R})$ *such that* $\int_{\mathbb{R}} \mathfrak{g}(\cdot,z)H'(z)\,dz = 0$*, there exists a unique solution of (5.2).*

Proof. We will argue by approximations. Let us replace $\mathfrak{g}$ in (5.2) by a function $\mathfrak{g}_R(y,z) = \mathfrak{g}(y,z)\chi_{(0,R)}(y)$ where we will take $R \to \infty$ later on. With this right hand side we can give to the problem (5.2) a weak formulation in the closed subspace the Sobolev space $H^1(\Gamma_\varepsilon \times \mathbb{R})$ of functions which satisfy the orthogonality conditions in (5.2). Thus we have

$$\begin{gathered} \Delta_{\Gamma_\varepsilon}\phi_R + \partial_z^2\phi_R + f'(H(z))\phi_R = \mathfrak{g}_R, \\ \int_{\mathbb{R}} \phi_R(y,z)H'(z)\,dz = 0, \end{gathered} \tag{5.5}$$

With this operator we associate the bilinear form

$$\mathfrak{a}_R(\phi,\psi) = \int_{\Gamma_\varepsilon\times\mathbb{R}} [\nabla_{\Gamma_\varepsilon}\phi\cdot\nabla_{\Gamma_\varepsilon}\psi + \partial_z\phi\partial_z\psi - f''(H(z))\phi\psi]\,dV(\Gamma_\varepsilon)dz.$$

Then we say that that ϕ_R is a weak solution of this problem if for all tests functions ψ we have

$$\mathfrak{a}_R(\phi_R,\psi) = \int_{\Gamma_\varepsilon\cap B(0,R)\times\mathbb{R}} \mathfrak{g}_R\psi\,dV(\Gamma_\varepsilon)dz.$$

Since we have as well, by our assumption,

$$\int_{\mathbb{R}} \mathfrak{g}_R(y,z)H'(z)\,dz = 0, \quad \forall y \in \Gamma_\varepsilon,$$

and, under the orthogonality conditions, the bilinear form $\mathfrak{a}_R(\psi,\eta)$ is actually positive definite, it follows that there exists a unique $\phi_R \in H^1(\Gamma_\varepsilon \times \mathbb{R})$ which satisfies weakly the equation and the orthogonality condition.

Letting $R \to +\infty$ and using the uniform a priori estimates valid for the approximations completes the proof of the Proposition. □

5.3 *A priori* estimates and existence for (4.11)

In this section we will consider the following problem:

$$\left(\Delta + \partial^2_{x_{N+1}} + \varepsilon\partial_{x_{N+1}} - 2\right)\psi = \mathfrak{h}, \tag{5.6}$$

We observe that if $\mathfrak{h}$ depends on $r = |x'|$, $x' \in \mathbb{R}^{N-1}$ and x_{N+1} only, so does ψ.

We will use the following weighted norms:

$$\|\mathfrak{h}\|_{\mathscr{C}^{0,\mu}_\beta(\mathbb{R}^N\times\mathbb{R})} = \sup_{x'\in\mathbb{R}^N} (1+\varepsilon^2|x'|^2)^\beta \|\mathfrak{h}\|_{\mathscr{C}^{0,\mu}(B(x',1)\times\mathbb{R})}, \quad \beta > 0.$$

The weighted Hölder norms $\mathscr{C}^{2,\mu}_{\beta}(\mathbb{R}^N\times\mathbb{R})$ are defined similarly. Note that the definition of the norm implies in particular that

$$\|\mathfrak{h}\|_{\mathscr{C}^{0,\mu}_{\beta}(\mathbb{R}^N\times\mathbb{R})}<\infty \Longrightarrow \|\mathfrak{h}\|_{\mathscr{C}^{0,\mu}(\mathbb{R}^N\times\mathbb{R})}<\infty,$$

and thus, by a standard argument, we obtain the existence of a solution to (5.6), $\psi\in\mathscr{C}^{2,\mu}(\mathbb{R}^N\times\mathbb{R})$. Now, to show that in fact

$$\|\psi\|_{\mathscr{C}^{2,\mu}_{\beta}(\mathbb{R}^N\times\mathbb{R})}\le C\|\mathfrak{h}\|_{\mathscr{C}^{0,\mu}_{\beta}(\mathbb{R}^N\times\mathbb{R})}.$$

one can use a comparison argument based on the fact that the reciprocal of the weight function $(1+\varepsilon^2|x'|^2)^{\beta}$ is a positive supersolution of (5.6). Details are left to the reader.

6 Infinite dimensional reduction

6.1 Estimates for the error of the initial approximation

Our first goal is to estimate the functions $\tilde{\mathfrak{g}}_{\varepsilon,j}$, defined in (4.14) and (4.16). Whenever convenient we will indicate the fact that these functions depend on their functional arguments by writing $\tilde{\mathfrak{g}}_{\varepsilon,j}=\tilde{\mathfrak{g}}_{\varepsilon,j}(\phi_{\varepsilon,1},\phi_{\varepsilon,2},\psi_\varepsilon,\mathbf{h}_\varepsilon)$. In general, besides the assumptions on $\mathbf{h}_\varepsilon$, which we have made in (4.3)–(4.4) we will also assume that for some $\sigma\in(0,\sqrt{2})$ and $K>0$ we have, with $\beta_\sigma=1-\frac{\sigma}{\sqrt{2}}$,

$$\|\phi_{\varepsilon,j}\|_{\mathscr{C}^{2,\mu}_{\beta_\sigma,\sigma}(\Gamma_\varepsilon\times\mathbb{R})}\le K\varepsilon^{2-\sigma\sqrt{2}} \tag{6.1}$$

About the function ψ_ε we assume that, with some $\kappa>3$, we have

$$\|\psi\|_{\mathscr{C}^{2,\mu}_{\kappa}(\mathbb{R}^N\times\mathbb{R})}\le K\varepsilon^3. \tag{6.2}$$

Lemma 6.1. *Under the preceding hypothesis there exists a $\sigma\in(0,\sqrt{2})$ such that the following estimate holds:*

(6.3)
$$\|\tilde{\mathfrak{g}}_{\varepsilon,j}\|_{\mathscr{C}^{0,\mu}_{\beta_\sigma,\sigma}(\Gamma_\varepsilon\times\mathbb{R})}\le C\{\varepsilon^{2-\sigma\sqrt{2}}+o(1)\sum_{j=1,2}\|\phi_{\varepsilon,j}\|_{\mathscr{C}^{2,\mu}_{\beta_\sigma,\sigma}(\Gamma_\varepsilon\times\mathbb{R})}+\|\psi_\varepsilon\|_{\mathscr{C}^{2,\mu}_{\kappa}(\mathbb{R}^N\times\mathbb{R})}\}.$$

The function $\mathfrak{g}_{\varepsilon,j}$ is a Lipschitz function of its arguments and we have:

$$\begin{aligned}&\|\tilde{\mathfrak{g}}_{\varepsilon,j}(\phi^{(1)}_{\varepsilon,1},\phi^{(1)}_{\varepsilon,2},\psi^{(1)}_{\varepsilon},\mathbf{h}^{(1)}_{\varepsilon})-\tilde{\mathfrak{g}}_{\varepsilon,j}(\phi^{(2)}_{\varepsilon,1},\phi^{(2)}_{\varepsilon,2},\psi^{(2)}_{\varepsilon},\mathbf{h}^{(2)}_{\varepsilon})\|_{\mathscr{C}^{0,\mu}_{\beta_\sigma,\sigma}(\Gamma_\varepsilon\times\mathbb{R})}\\ &\le C\{\varepsilon^{2-\sigma\sqrt{2}}\|\mathbf{h}^{(1)}_{\varepsilon}-\mathbf{h}^{(2)}_{\varepsilon}\|_{\mathscr{C}^{2,\mu}_{\beta_\sigma}(\Gamma_\varepsilon)}+o(1)\sum_{j=1,2}\|\phi^{(1)}_{\varepsilon,j}-\phi^{(2)}_{\varepsilon,j}\|_{\mathscr{C}^{2,\mu}_{\beta_\sigma,\sigma}(\Gamma_\varepsilon\times\mathbb{R})}\\ &\quad+\|\psi^{(1)}_{\varepsilon}-\psi^{(2)}_{\varepsilon}\|_{\mathscr{C}^{2,\mu}_{\kappa}(\mathbb{R}^N\times\mathbb{R})}\}.\end{aligned} \tag{6.4}$$

Proof. The proof of this lemma follows by a somewhat tedious but rather straightforward calculation. Similar calculations can be also found in [7] and [8]. We will outline the main part which is the estimate of the term involving $S(w_\varepsilon)$. Note that

$\tilde{\zeta}_{\varepsilon,j}S(w_\varepsilon) = \tilde{\zeta}_{\varepsilon,j}S(u_\varepsilon)$, (see (4.6)–(4.7)). Let us denote $\tilde{u}_\varepsilon(y,z) = (X_\varepsilon^* u_\varepsilon)(y,z)$. We expand Δ near Γ_ε in terms of the Fermi coordinates to get:

(6.5)
$$(X_\varepsilon^* S(u_\varepsilon)) = \Delta_{\Gamma_\varepsilon}\tilde{u}_\varepsilon + [\partial_z^2 \tilde{u}_\varepsilon + f(\tilde{u}_\varepsilon)] + [\varepsilon\partial_z(\pi_{\varepsilon,N+1}) - H_{\Gamma_\varepsilon}]\partial_z\tilde{u}_\varepsilon - z|A_{\Gamma_\varepsilon}|^2\partial_z\tilde{u}_\varepsilon$$
$$+\varepsilon\nabla_{\Gamma_\varepsilon}(\pi_{\varepsilon,N+1})\cdot\nabla_{\Gamma_\varepsilon}\tilde{u}_\varepsilon + \mathbb{A}_\varepsilon[\tilde{u}_\varepsilon] + \mathbb{B}_\varepsilon[\tilde{u}_\varepsilon] - z^2 R_{\Gamma_\varepsilon}\partial_z\tilde{u}_\varepsilon.$$

Above, $\mathbb{A}_\varepsilon$ and $\mathbb{B}_\varepsilon$ are linear differential operators of second and first order, respectively, whose expressions in terms of local coordinates on Γ_ε are given in section 2.1. Most of the terms in (6.5) are estimated directly. The leading order term is in fact given by:

$$\partial_z^2\tilde{u}_\varepsilon + f(\tilde{u}_\varepsilon) = f(\tilde{u}_\varepsilon) - f(H(z - f_{\varepsilon,1} - h_{\varepsilon,1})) - f(-H(z - f_{\varepsilon,2} - h_{\varepsilon,2})).$$

Using this, the definition of $\tilde{u}_\varepsilon$ and (2.19), we can estimate, taking $\sigma \in (0,\sqrt{2})$:

$$\begin{aligned}|\partial_z^2\tilde{u}_\varepsilon + f(\tilde{u}_\varepsilon)| &\le C\{H'(z - f_{\varepsilon,1} - h_{\varepsilon,1})[1 + H(z - f_{\varepsilon,2} - h_{\varepsilon,2})] \\ &\quad + H'(z - f_{\varepsilon,2} - h_{\varepsilon,1})[1 - H(z - f_{\varepsilon,1} - h_{\varepsilon,1})]\} \\ &\le C\max_j\{e^{-\sigma|z-f_{\varepsilon,j}|}\}\exp\Big(-\frac{\sqrt{2}-\sigma}{\sqrt{2}}\log\frac{\omega_\varepsilon}{\varepsilon^2}\Big) \\ &\le C\varepsilon^{2-\sigma\sqrt{2}}\max_j\{e^{-\sigma|z-f_{\varepsilon,j}|}\}\omega_\varepsilon^{-\beta_\sigma}.\end{aligned}$$

Since we have $\varepsilon\partial_z(\pi_{\varepsilon,N+1}) - H_{\Gamma_\varepsilon} = 0$ the remaining non-zero term in the first line in (6.5) is

$$\begin{aligned}\Delta_{\Gamma_\varepsilon}\tilde{u}_\varepsilon - z|A_{\Gamma_\varepsilon}|^2\partial_z\tilde{u}_\varepsilon &= \sum_{j=1,2}(-1)^j H'(z - f_{\varepsilon,j} - h_{\varepsilon,j})\Delta_{\Gamma_\varepsilon}(f_{\varepsilon,j} + h_{\varepsilon,j}) \\ &\quad + |A_{\Gamma_\varepsilon}|^2\sum_{j=1,2}(f_{\varepsilon,j} + h_{\varepsilon,j})H'(z - f_{\varepsilon,j} - h_{\varepsilon,j}) \\ &\quad + \sum_{j=1,2}H''(z - f_{\varepsilon,j} - h_{\varepsilon,j})|\nabla_{\Gamma_\varepsilon}(f_{\varepsilon,j} - h_{\varepsilon,j})|^2 \\ &\quad + |A_{\Gamma_\varepsilon}|^2\sum_{j=1,2}(z - f_{\varepsilon,j} - h_{\varepsilon,j})H'(z - f_{\varepsilon,j} - h_{\varepsilon,j})\end{aligned} \tag{6.6}$$

We note that

$$|A_{\Gamma_\varepsilon}(r)|^2 = \varepsilon^2|A_\Gamma(\varepsilon r)|^2 \le C\varepsilon^2\omega_\varepsilon^{-2}(r). \tag{6.7}$$

Each term in (6.6) is then estimated directly. The second line in (6.5) is seen easily to be smaller relative to the terms we have just considered. As for the terms involving functions $\phi_{\varepsilon,j}$ we observe that the largest among them is:

$$[L,\zeta_{\varepsilon,j}]\phi_{\varepsilon,j} = \Delta(\zeta_{\varepsilon,j}\phi_{\varepsilon,j}) - \zeta_{\varepsilon,j}\Delta\phi_{\varepsilon,j}.$$

Using the fact that $\Delta\zeta_{\varepsilon,j} = o(1)$ and $\nabla\zeta_{\varepsilon,j} = o(1)$, which follows from the choice of the cutoff functions $\zeta_{\varepsilon,j}$, we can estimate this term by $o(1)\|\phi_{\varepsilon,j}\|_{\mathscr{C}^{2,\mu}_{\beta_\sigma,\sigma}(\Gamma_\varepsilon\times\mathbb{R})}$.

The rest of the proof is straightforward and we leave the details to the reader. □

Going back to the system (4.17), and taking into account the theory of the preceding section we see that the functions $\mathtt{c}_{\varepsilon,j}$ need to be determined from the formula:

$$\mathtt{c}_{\varepsilon,j} = \frac{\int_{\mathbb{R}} \tilde{\mathfrak{g}}_{\varepsilon,j}(y,\mathtt{t}_j)H'(\mathtt{t}_j)\,d\mathtt{t}_j}{\int_{\mathbb{R}} (H'(\mathtt{t}_j))^2\zeta_{\varepsilon,j}(y,\mathtt{t}_j)\,d\mathtt{t}_j}. \tag{6.8}$$

Using Lemma 6.1 we see that statements analogous to (6.3) and (6.4) hold when we replace $\tilde{\mathfrak{g}}_{\varepsilon,j}$ by $\tilde{\mathfrak{g}}_{\varepsilon,j} + \mathtt{c}_{\varepsilon,j}H'(\mathtt{t}_j)\zeta_{\varepsilon,j}$.

Next we will consider the right hand side of the equation (4.15). We have:

Lemma 6.2. *Under the same hypothesis as in Lemma 6.1, and assuming that the constant $M > 0$ in (4.5) and (4.8) is large enough, there exist $\kappa > 3$ and $\gamma > 1$ such that we have*

$$\begin{aligned} \|\mathfrak{h}_\varepsilon\|_{\mathscr{C}^{0,\mu}_\kappa(\mathbb{R}^N\times\mathbb{R})} \le C\{\varepsilon^3 + \varepsilon^\gamma \sum_{j=1,2} \|\phi_{\varepsilon,j}\|_{\mathscr{C}^{2,\mu}_{\beta\sigma,\sigma}(\Gamma_\varepsilon\times\mathbb{R})} \\ + o(1)\|\psi_\varepsilon\|_{\mathscr{C}^{2,\mu}_\kappa(\mathbb{R}^N\times\mathbb{R})}\}. \end{aligned} \tag{6.9}$$

Considering $\mathfrak{h}_\varepsilon$ as a function of $(\phi_{\varepsilon,1},\phi_{\varepsilon,2},\psi_\varepsilon,\mathbf{h}_\varepsilon)$

$$\begin{aligned} &\|\mathfrak{h}_\varepsilon(\phi^{(1)}_{\varepsilon,1},\phi^{(1)}_{\varepsilon,2},\psi^{(1)}_\varepsilon,\mathbf{h}^{(1)}_\varepsilon) - \mathfrak{h}_\varepsilon(\phi^{(2)}_{\varepsilon,1},\phi^{(2)}_{\varepsilon,2},\psi^{(2)}_\varepsilon,\mathbf{h}^{(2)}_\varepsilon)\|_{\mathscr{C}^{0,\mu}_{\beta\sigma+\sigma\delta}(\mathbb{R}^N\times\mathbb{R})} \\ &\le C\{\varepsilon^3\|\mathbf{h}^{(1)}_\varepsilon - \mathbf{h}^{(2)}_\varepsilon\|_{\mathscr{C}^{2,\mu}_{\beta\sigma}(\Gamma_\varepsilon)} + \varepsilon^\gamma \sum_{j=1,2}\|\phi^{(1)}_{\varepsilon,j} - \phi^{(2)}_{\varepsilon,j}\|_{\mathscr{C}^{2,\mu}_{\beta\sigma,\sigma}(\Gamma_\varepsilon\times\mathbb{R})} \\ &\quad + o(1)\|\psi^{(1)}_\varepsilon - \psi^{(2)}_\varepsilon\|_{\mathscr{C}^{2,\mu}_\kappa(\mathbb{R}^N\times\mathbb{R})}\}. \end{aligned} \tag{6.10}$$

A proof of this estimates is omitted, since similar results are proven in [7] or [8] and no essentially new elements are needed to carry out the argument in the present case. We only point out that the support of the function $\mathfrak{h}_\varepsilon$ is in the set where $|z - f_{\varepsilon,j}| > M\log\frac{\omega_\varepsilon}{\varepsilon^2}$, from which it follows that all exponentially decaying terms are very small, like $\mathscr{O}(\varepsilon^3)$ at least.

6.2 Projected nonlinear problem

Our objective in this section is to solve (4.13)–(4.15). Given the linear theory available and the results of the preceding section, we will achieve this by a simple fixed point argument.

Let functions $\tilde{\phi}_{\varepsilon,j}$, $j = 1,2$ and $\tilde{\psi}_\varepsilon$ satisfying assumptions (6.1)–(6.2) be fixed. We will also chose $\mathbf{h}_\varepsilon$ satisfying (4.4). We first use the linear theory of Section 5 to solve the following system:

(6.11)

$$\begin{aligned} &\big(\Delta_{\Gamma_\varepsilon} + \partial^2_{\mathtt{t}_j} + f'(H(\mathtt{t}_j))\big)\phi_{\varepsilon,j} = \tilde{\mathfrak{g}}_{\varepsilon,j}(y,\mathtt{t}_j;\tilde{\phi}_{\varepsilon,1},\tilde{\phi}_{\varepsilon,2},\tilde{\psi}_\varepsilon,\mathbf{h}_\varepsilon) + \mathtt{c}_{\varepsilon,j}H'(\mathtt{t}_j), \quad j = 1,2, \\ &\int_{\mathbb{R}} \phi_{\varepsilon,j}(y,\mathtt{t}_j)H'(\mathtt{t}_j)\,d\mathtt{t}_j = 0, \quad j = 1,2, \end{aligned}$$

(6.12)

$$(\Delta + \varepsilon\partial_{x_{N+1}} - 2)\psi_\varepsilon = \mathfrak{h}_\varepsilon(x;\tilde{\phi}_{\varepsilon,1},\tilde{\phi}_{\varepsilon,2},\tilde{\psi}_\varepsilon,\mathbf{h}_\varepsilon),$$

This is equivalent to (4.13)–(4.15) when $\tilde{\phi}_{\varepsilon,j} = \phi_{\varepsilon,j}$ and $\tilde{\psi}_\varepsilon = \psi_\varepsilon$. In fact, using Lemma 6.1 and Lemma 6.2, we obtain existence of such a fixed point satisfying (6.1)–(6.2) by the Banach fixed point theorem. To do this we first solve (6.12) for ψ_ε as a function of $(\tilde{\phi}_{\varepsilon,1}, \tilde{\phi}_{\varepsilon,2}, \mathbf{h}_\varepsilon)$. Existence of ψ_ε follows follows by a fixed point argument using Lemma 6.2 and the results in section 5.3. We have in fact:

$$\|\psi_\varepsilon\|_{\mathscr{C}^{2,\mu}_\kappa(\mathbb{R}^N\times\mathbb{R})} \leq C\{\varepsilon^3 + \varepsilon^\gamma \sum_{j=1,2} \|\phi_{\varepsilon,j}\|_{\mathscr{C}^{2,\mu}_{\beta\sigma,\sigma}(\Gamma_\varepsilon\times\mathbb{R})}\},$$

with a similar estimate showing Lipschitz character of ψ_ε. Given this we solve (6.11) using again the Banach fixed point theorem. Let us summarize this:

Lemma 6.3. *Under the above hypothesis there exists a unique solution* $(\phi_{\varepsilon,1}, \phi_{\varepsilon,2}, \psi_\varepsilon)$ *of (6.11) and (6.12) satisfying (6.1) and (6.2).*

6.3 Solution of the reduced problem

At this point we are left with the task of adjusting $\mathbf{h}_\varepsilon$ in such a way that $\mathtt{c}_{\varepsilon,j} \equiv 0$. For this let us observe that the map

$$(\tilde{\phi}_{\varepsilon,1}, \tilde{\phi}_{\varepsilon,2}, \tilde{\psi}_\varepsilon; \mathbf{h}_\varepsilon) \longmapsto (\phi_{\varepsilon,1}, \phi_{\varepsilon,2}, \psi_\varepsilon).$$

is a uniform contraction (with a small Lipschitz constant) with respect to $\mathbf{h}_\varepsilon$. It follows that $(\phi_{\varepsilon,1}, \phi_{\varepsilon,2}, \psi_\varepsilon)$ are Lipschitz functions of $\mathbf{h}_\varepsilon$ with small Lipschitz constants. This last fact can be seen easily from Lemma 6.1 and Lemma 6.2. Another important fact is that since we have assumed initially that $f_{\varepsilon,j}$ and $h_{\varepsilon,j}$ are functions of r, where $r = |x'|$, $(x', x_{N+1}) \in \mathbb{R}^{N+1}$ therefore $(\phi_{\varepsilon,1}, \phi_{\varepsilon,2}, \psi_\varepsilon)$ are functions of (r, z) only, at least near Γ_ε, i.e. where the Fermi coordinates are defined. In fact, instead of working in an abstract setting, which does not refer to the rotational symmetry of Γ_ε, we could have reduced the whole problem to the one in the half plane $\mathbb{R}^2_+ = (r, x_{N+1})$, and think of Γ_ε as a curve, with (r, z) as its Fermi coordinates. Then the end result, from the point of view of existence of $(\phi_{\varepsilon,1}, \phi_{\varepsilon,2}, \psi_\varepsilon)$, would be of course the same. Summarizing, all functions involved depend on $x = (x', x_{N+1})$, through $r(x) = |x'|$ and x_{N+1}, and when expressed in Fermi coordinates (y, z) they depend on $r(y) = |y'|$ and z only.

Now, we will find the exact conditions for $\mathbf{h}_\varepsilon$ which guarantee that $\mathtt{c}_{\varepsilon,j} \equiv 0$. We will show that they result in a non-homogeneous and nonlocal Jacobi-Toda system, quite similar to the one already studied in Section 3. From the theory developed in this section the existence of $\mathbf{h}_\varepsilon$ will follow immediately, thus completing the proof of Theorem 1.1. Our first task is then to justify rigorously the formal calculations in section (2.2). In fact, with the notations as in the previous sections we need to adjust $\mathbf{h}_\varepsilon$ so that

$$\int_{\mathbb{R}} \tilde{\mathfrak{g}}_{\varepsilon,j}(r, \mathtt{t}_j) H'(\mathtt{t}_j)\, d\mathtt{t}_j = 0, \quad j = 1, 2,$$

Let us recall that $\tilde{\mathfrak{g}}_{\varepsilon,j}$ depends on $(\phi_{\varepsilon,1}, \phi_{\varepsilon,2}, \psi_\varepsilon, \mathbf{h}_\varepsilon)$, that $(\phi_{\varepsilon,1}, \phi_{\varepsilon,2}, \psi_\varepsilon)$ depend non-locally on $\mathbf{h}_\varepsilon$, and that this dependence involves second derivatives of $\mathbf{h}_\varepsilon$. Thus its projection onto $H'(\mathtt{t}_j)$ will be a non-local, second order ODE in terms of the radial variable r.

Let us write

$$\tilde{\mathfrak{g}}_{\varepsilon,j} = \tilde{\zeta}_{\varepsilon,j}S(w_\varepsilon) + \hat{\mathfrak{g}}_{\varepsilon,j}, \quad \hat{\mathfrak{g}}_{\varepsilon,j} = \hat{\mathfrak{g}}_{\varepsilon,j}(\phi_{\varepsilon,1}, \phi_{\varepsilon,2}, \psi_\varepsilon, \mathbf{h}_\varepsilon).$$

Examining the expression for $S(u_\varepsilon)$ in (6.5) we see that as a function of $(r, \mathtt{t}_j)$ it has general form (say, where $\tilde{\zeta}_{\varepsilon,j} \equiv 1$), $S(u_\varepsilon)(r, \mathtt{t}_j) = S(u_\varepsilon)(r, \mathtt{t}_j - h_{\varepsilon,j})$. It is therefore more convenient to integrate $\tilde{\mathfrak{g}}_{\varepsilon,j}$ against $H'(\mathtt{t}_j - h_{\varepsilon,j})$ rather $H'(\mathtt{t}_j)$. It is easily seen that $\mathtt{c}_{j,\varepsilon} = 0$ when:

$$\begin{aligned}
\int_{\mathbb{R}} \tilde{\mathfrak{g}}_{\varepsilon,j}(r(y), \mathtt{t}_j) H'(\mathtt{t}_j - h_{\varepsilon,j})\, d\mathtt{t}_j &= \int_{\mathbb{R}} \tilde{\zeta}_{\varepsilon,j} S(w_\varepsilon)(r(y), \mathtt{t}_j) H'(\mathtt{t}_j - h_{\varepsilon,j})\, \mathtt{t}_j \\
&\quad + \int_{\mathbb{R}} \hat{\mathfrak{g}}_{\varepsilon,j} H'(\mathtt{t}_j - h_{\varepsilon,j})\, d\mathtt{t}_j \\
&= \mathit{\Pi}_{\varepsilon,j} + \hat{\mathit{\Pi}}_{\varepsilon,j} = 0.
\end{aligned} \tag{6.13}$$

As we have argued in section (2.2) the main term in the above integral (remembering that by definition $w_\varepsilon = u_\varepsilon$ in $\operatorname{supp} \zeta_{\varepsilon,j}$) comes from:

$$\mathit{\Pi}_{\varepsilon,j} = \int_{\mathbb{R}} \zeta_{\varepsilon,j} S(u_\varepsilon)(r(y), \mathtt{t}_j) H'(\mathtt{t}_j - h_{\varepsilon,j})\, \mathtt{t}_j,$$

while the remaining part of the projection, denoted by $\hat{\mathit{\Pi}}_{\varepsilon,j}$ is a lower order term.

Repeating calculations in section 2.2 and taking into account formula (6.5) one can derive the following expression:

$$\mathit{\Pi}_{\varepsilon,j} = \alpha_0 J_{\Gamma_\varepsilon}(f_{\varepsilon,j} + h_{\varepsilon,j}) + \mathscr{T}_j(\mathbf{f}_\varepsilon + \mathbf{h}_\varepsilon) + q_{\varepsilon,j}(\mathbf{f}_\varepsilon + \mathbf{h}_\varepsilon), \tag{6.14}$$

where, for a vector function $\mathbf{v} = (v_1, v_2)$, on Γ_ε we have denoted:

$$\begin{aligned}
J_{\Gamma_\varepsilon}(v_j) &= \Delta_{\Gamma_\varepsilon} v_j + |A_{\Gamma_\varepsilon}|^2 v_j + \varepsilon \nabla_{\Gamma_\varepsilon}(\pi_{\varepsilon,N+1}) \cdot \nabla_{\Gamma_\varepsilon} v_j, \\
\mathscr{T}_j(\mathbf{v}) &= -e^{\sqrt{2}(v_{j-1} - v_j)} + e^{\sqrt{2}(v_j - v_{j+1})}.
\end{aligned} \tag{6.15}$$

We observe that the main order term in $q_{\varepsilon,j}$ (see (6.5)) comes from

$$z^2 \tilde{\zeta}_{\varepsilon,j} R_{\Gamma_\varepsilon} \partial_z \tilde{u}_\varepsilon \approx (\mathtt{t}_j - f_{\varepsilon,j})^2 \sum_{\ell=1}^{N} \mathbb{k}^3_{\Gamma_\varepsilon,\ell} H'(\mathtt{t}_j - h_{\varepsilon,j}),$$

where $\mathbb{k}_{\Gamma_\varepsilon,\ell}$ are the principal curvatures of Γ_ε. Direct calculations show that

$$|\mathbb{k}^3_{\Gamma_\varepsilon,\ell}| \approx \varepsilon^3 \omega_\varepsilon^{-3/2}.$$

Taking into account the assumptions we have made at the beginning on $\mathbf{f}_\varepsilon$, and $\mathbf{h}_\varepsilon$ in (4.2)–(4.3), we see that there exist $\beta > 0$ and $\rho > 0$ such that

$$\|q_{\varepsilon,j}\|_{\mathscr{C}^{0,\mu}_{1+\beta}(\Gamma_\varepsilon)} \le C\varepsilon^{2+\rho}.$$

Identifying functions on Γ_ε and Γ by $v_\varepsilon(r) = v(\varepsilon r)$, so that $q_{\varepsilon,j}(r) = q_j(\varepsilon r)$ we get from the above:

$$\|q_j\|_{\mathscr{C}^{0,\mu}_{1+\beta}(\Gamma)} \le C\varepsilon^{2+\rho-\mu}.$$

Function q_j now depends on the functions $\mathbf{f}$ and $\mathbf{h}$ defined on Γ. Similar statements hold for the remaining term in (6.13), namely we have:

$$\|\hat{\Pi}_{\varepsilon,j}\|_{\mathscr{C}^{0,\mu}_{1+\beta}(\Gamma_\varepsilon)} \le C\varepsilon^{2+\rho},$$

and, scaling back to Γ, we can write:

$$\|\hat{\Pi}_j\|_{\mathscr{C}^{0,\mu}_{1+\beta}(\Gamma)} \le C\varepsilon^{2+\rho-\mu}.$$

We let $\mu > 0$ be a small number and set $\kappa = \rho - \mu > 0$, also choosing it in such a way that $\tau < \kappa$ (see (4.3)). Denoting by J_Γ the scaled operator in (6.15), and setting $\hat{q}_j = q_j + \hat{\Pi}_j$ we get then:

$$\alpha_0 \varepsilon^2 J_\Gamma(f_j + h_j) + \mathscr{T}_j(\mathbf{f} + \mathbf{h}) = \hat{q}_j. \tag{6.16}$$

This is a Jacobi-Toda system, which can be solved using the theory we developed in the proof of Proposition 3.2 and in particular the result of Lemma 3.6. In fact $\hat{q}_j$ is a Lipschitz function of $\mathbf{h}$ since it follows from the Lipschitz character of $S(w_\varepsilon), \phi_{\varepsilon,j}, \psi_\varepsilon$ as functions of $\mathbf{h}$ that:

$$\|\hat{q}_j(\mathbf{h}^{(1)}) - \hat{q}_j(\mathbf{h}^{(2)})\|_{\mathscr{C}^{0,\mu}_{1+\beta}(\Gamma)} \le C\varepsilon^{2+\kappa}\|\mathbf{h}^{(1)} - \mathbf{h}^{(2)}\|_{\mathscr{C}^{2,\mu}_{\beta}(\Gamma)}.$$

Defining

$$\mathscr{T}_j(\mathbf{f} + \mathbf{h}) - \mathscr{T}_j(\mathbf{f}) - \mathscr{T}_j'(\mathbf{f})\mathbf{h} = \mathscr{N}_j(\mathbf{h}),$$

we also have:

$$\|\mathscr{N}_j(\mathbf{h})\|_{\mathscr{C}^{0,\mu}_{1+\beta}(\Gamma)} \le C\varepsilon^{2+\tau}\|\mathbf{h}\|_{\mathscr{C}^{2,\mu}_{\beta}(\Gamma)}.$$

Similarly, $\mathscr{N}_j(\mathbf{h})$ is a Lipschitz function of $\mathbf{h}$. Since we have chosen $\mathbf{f}$ to be a solution of the homogeneous version of (6.16) we are left with:

$$\alpha_0 \varepsilon^2 J_\Gamma(h_j) + \mathscr{T}_j'(\mathbf{f})(\mathbf{h}) = \tilde{q}_j, \quad \tilde{q}_j = \hat{q}_j - \mathscr{N}_j. \tag{6.17}$$

The left hand side of this equation is the linearized Jacobi-Toda system, and now Lemma 3.6 can be employed directly to solve (6.16) using Banach fixed point theorem. As similar arguments can be found for instance in [7] and [8] we omit the details here. With this last step we complete our proof.

7 An example of a traveling wave with a non-convex front

In this section we will prove Theorem 1.2. We will begin with some preliminary facts about the asymptotic form of the non-convex traveling front.

7.1 Traveling, catenoid-like surface

We will summarize here an existence result proven in [5].

Proposition 7.1. *For each $R > 0$ there exist a rotationally symmetric, graphical solutions to the mean curvature flow, given by $F_R^{\pm}\colon \mathbb{R}^N \setminus B_R(0) \times \mathbb{R} \to \mathbb{R}$, and translating with speed $c = 1$, where:*

$$F_R^{\pm}(r,t) = t + W_R^{\pm}(r).$$

The functions $W_R^{\pm}$ satisfy

$$W_R^{\pm}(r) = \frac{r^2}{2(N-1)} - \log r + C^{\pm} + O(r^{-1}), \quad r \to \infty. \tag{7.1}$$

Moreover, the union of these graphs forms a complete non-convex translating solution to the mean curvature flow.

In what follows by Σ we denote the surface obtained by taking the union of the graphs of $W_R^{\pm}$, and by Σ_ε we denote its scaled version. The individual graphs of each function $W_{\mathbb{R}}^{\pm}$ will be referred to as the ends of Σ and will be denoted by $\Sigma^{\pm}$, respectively, with a similar notation for the scaled versions. We assume that the constants $C^{\pm}$ appearing in (7.1) are such that $C^- < C^+$ and we will call Σ^- (Σ^+) the lower (the upper) end of Σ. Also, in order to not to complicate notations we will not indicate explicitly the dependence of the surface Σ on R. Nevertheless the reader should keep in mind that our results are valid for the whole family of traveling catenoids parametrized by R.

The surface Σ is an embedded, rotationally symmetric and genus 0 surface in $\mathbb{R}^N$, and in some sense it is a counterpart of the usual catenoid, now in the context of the eternal solutions of the mean curvature flow. Another important, obvious property is its non convexity.

Comparing the asymptotic formula (7.1) with the asymptotic formula for F we notice that as $r \to \infty$ the ends of Σ remain at a constant distance from Γ. Indeed, we have:

$$|F(r) - 1 - W_R^{\pm}(r) + C^{\pm}| = O(r^{-1}), \quad r \to \infty. \tag{7.2}$$

This is important in calculation of various geometric characteristics of Σ. In fact formula (7.2) says that the mean curvature H_Σ, the second fundamental form A_Σ, ∇_Σ, and Δ_Σ are, for r sufficiently large, very close to their counterparts on Γ. Thus in the sequel we may omit many of the explicit calculations and appeal to the calculation we have already done for Γ.

7.2 An improvement of the initial profile

The fact that the ends of Σ are asymptotically parallel means that if we want to use its scaled version Σ_ε as a model for a traveling wave with the speed $c = \varepsilon$ we must perturb the ends of the surface. To see this let us denote the signed distance to Σ_ε by $z = z(x)$, for $x \in \mathbb{R}^{N+1}$ close to Σ_ε. Then it is natural to take $u_\varepsilon = H(z)$

as the first approximation to the solution. A short calculation will convince the reader that, since the ends of Σ_ε are parallel, the error $S(u_\varepsilon)$ of this approximation contains a term of order $\mathscr{O}(e^{-\frac{1}{\varepsilon}})$. This means that $S(u_\varepsilon)$ is globally a very small function of ε but in is *not* a decaying function of $r = |x'|$ along Σ_ε.

To remedy this situation we will consider an improvement of the initial profile Σ_ε. In general we want a new surface $\tilde{\Sigma}_\varepsilon$ to be a normal graph over Σ_ε, to be identical with Σ_ε on a compact set and to have ends that are diverging from one another as $r \to \infty$.

To give a formal definition we need to introduce some notations. We let χ be a smooth cutoff function such that $\chi(t) = 0$, $t \le 1$, and $\chi(t) = 1$, $t \ge 2$. By r_ε we denote a number to be determined later on and about which we assume initially that, with some $c < C$,

$$r_\varepsilon \gg e^{\frac{c}{\varepsilon}}, \quad \text{and } r_\varepsilon \ll e^{\frac{C}{\varepsilon}}. \tag{7.3}$$

Next, we will fix an orientation on Σ in such a way that a unit normal n is interior to this component of $\mathbb{R}^{N+1} \setminus \Sigma$ which contains the origin. By $n_\varepsilon(y) = n(\varepsilon y)$ we denote the corresponding normal on Σ_ε and by $n^\pm$ and $n^\pm_\varepsilon$ we denote the restrictions of n and n_ε to the ends of Σ. Finally by $\Theta \in S^{N-1}$ we denote both points on S^{N-1}.

The new surface $\tilde{\Sigma}_\varepsilon$ will be a union of its lower and upper ends $\tilde{\Sigma}^\pm_\varepsilon$ given by:

$$\tilde{\Sigma}^\pm_\varepsilon = \Big\{ \big(r\Theta, \frac{1}{\varepsilon} W^\pm_R(\varepsilon r)\big) + \chi\big(\frac{r}{r_\varepsilon}\big) n^\pm(\varepsilon r, \Theta) f^\pm(\varepsilon r) \mid r \ge R, \Theta \in S^{N-1} \Big\}, \tag{7.4}$$

where the radial functions $f^\pm \colon \Sigma \to \mathbb{R}$ are still to be determined.

Construction of $f^\pm$

Choosing the functions $f^\pm$ is a subtle point of our problem. To give some motivation let us recall how in the preceding considerations we have determined the functions $f_1, f_2 \colon \Gamma \to \mathbb{R}$, which model the traveling fronts near Γ_ε. Restricting our attention to $r \gg 1$ we observe that, to main order we needed to solve an algebraic equation:

$$|A_\Gamma|^2 u = \frac{e^{-u}}{\delta^2}, \quad \delta = \frac{\varepsilon\sqrt{\alpha_0}}{2^{3/4}}, \tag{7.5}$$

and then we obtained, to main order,

$$f_1 \approx -\frac{1}{2\sqrt{2}} u, \quad f_2 \approx \frac{1}{2\sqrt{2}} u.$$

Equation (7.5) describes a balance between the interactions of the ends due to the exponential decay of the heteroclinic to the stable phases ± 1 and the geometry of the moving front Γ. Now we need to discover the analog of (7.5) with Γ replaced by Σ. The natural guess would be to take $|A_\Sigma|^2$ on the right hand side leaving the exponential function on the left. However the story is not so simple because, altering the ends of Σ by adding normal perturbations as described above, we have changed the character of the surface—the new surface is not a translating solution of the mean curvature flow anymore. To take this into account we solve (instead of (7.5)) the following problem:

$$\frac{F_{rr}}{1+|F_r|^2}u = \frac{e^{-u}}{\tilde{\delta}^2}, \quad \tilde{\delta} = \tilde{\alpha}\varepsilon, \tag{7.6}$$

where $\tilde{\alpha} > 0$ is a constant to be specified later. In the sequel we will show that we can chose $\tilde{\alpha}$ in such a way that defining

$$f^{\pm} = \frac{1}{2\sqrt{2}}u, \tag{7.7}$$

and using the modified surface as a model for the traveling wave we can achieve the following:

(1) If the approximate solution is defined by $u_\varepsilon = H(z)$, where z is the signed distance from $\tilde{\Sigma}_\varepsilon$ then, at least near this surface, the error of the approximation $S(u_\varepsilon)$ is a small function of ε, and also it decays as $r \to \infty$ at an algebraic rate in r.
(2) The projection of the error onto $H'(z)$, namely $\int_{\mathbb{R}} S(u_\varepsilon)H'(z)$ is a function that behaves like $\frac{\varepsilon^{2+\kappa}}{(1+\varepsilon^2 r^2)^{1+\beta}}$ as $r \to \infty$.

These two claims, which we will make more precise later, are sufficient to implement a Lyapunov-Schmidt construction quite similar to the one presented in the previous sections and, as a result, prove Theorem 1.2.

Let us go back to equation (7.6). Based on the known asymptotic behavior of the function $F(r)$ and its derivatives one can prove the following:

Lemma 7.2. *Let $u = u(r)$ be the solution of (7.5) and let $f^{\pm} = f^{\pm}(r)$ be the functions defined in (7.7). There exist $r_0 > R$ and $C > 0$ such that for all $r > r_0$ it holds:*

$$(f^{+}(r) + f^{-}(r)) \geq \frac{2}{\sqrt{2}}\log\Big(\frac{1+r^2}{\varepsilon^2}\Big) - C\log\log\Big(\frac{1+r^2}{\varepsilon^2}\Big). \tag{7.8}$$

From now on $\tilde{\Sigma}_\varepsilon$ will be the surface we defined in (7.4) with $f^{\pm}$ as in the Lemma 7.2. By $\tilde{n}_\varepsilon$ we will denote its unit normal, and by $\tilde{n}$ the unit normal of its scaled version $\tilde{\Sigma}$. These vectors are chosen in such a way that $\tilde{n}_\varepsilon$ is interior with respect to the connected component of $\mathbb{R}^{N+1} \setminus \tilde{\Sigma}_\varepsilon$ which contains the origin.

7.3 Construction of the initial approximation

We will consider the Fermi coordinates associated with the surface $\tilde{\Sigma}_\varepsilon$:

$$x \longmapsto (y,z), \quad y \in \tilde{\Sigma}_\varepsilon, \quad z = \mathrm{dist}\,(x, \tilde{\Sigma}_\varepsilon),$$

in a neighborhood of $\mathscr{U}_\varepsilon$ of this surface. We let $\mathscr{U}_\varepsilon$ to be such that this map is a diffeomorphism, namely we define:

(7.9)
$$\mathscr{U}_\varepsilon := \left\{ x\in\mathbb{R}^{N+1} \;\middle|\; \begin{array}{c} |z|\leq \frac{C(\Sigma)}{\varepsilon}\left[1-\chi\left(\frac{r}{r_\varepsilon}\right)\right]+\frac{1}{2}\chi\left(\frac{r}{r_\varepsilon}\right)(f^+(\varepsilon r)+f^-(\varepsilon r)), \\ x=y+z\tilde n_\varepsilon(y), r=r(y) \end{array} \right\}.$$

The constant $C(\Sigma)>0$ depends on Σ only. As before, for $u\colon \mathscr{U}_\varepsilon\to\mathbb{R}^k$, by $(X_\varepsilon^* u)(y,z)$ we denote the pullback of f by this diffeomorphism. At this point we will chose conveniently the value of r_ε by letting it be a solution of the following equation:

$$\frac{C(\Sigma)}{\varepsilon}=\frac{1}{2}(f^+(2\varepsilon r_\varepsilon)+f^-(2\varepsilon r_\varepsilon)) \Longrightarrow r_\varepsilon\sim e^{\frac{\tilde c}{\varepsilon}}. \tag{7.10}$$

As a next step we define a smooth cutoff function ρ_ε which is supported in $\mathscr{U}_\varepsilon$ and such that

$$(X_\varepsilon^*\rho_\varepsilon)(y,z)=1, \quad \mathrm{dist}\,(x,\partial\mathscr{U}_\varepsilon)\leq 1, \quad x=y+z\tilde n_\varepsilon(y).$$

To be more precise we take for instance a smooth cutoff function $\rho(t)$ such that $\rho(t)=1, t\leq -1$ and $\rho(t)=0, t\geq 0$ and set:

$$(X_\varepsilon^*\rho_\varepsilon)(y,z)=\rho\left(|z|-\frac{C(\Sigma)}{\varepsilon}[1-\chi(\frac{r}{r_\varepsilon})]-\frac{1}{2}\chi(\frac{r}{r_\varepsilon})|f^+(\varepsilon r)+f^-(\varepsilon r)|\right). \tag{7.11}$$

In order to use a Lyapunov-Schmidt reduction procedure we have to allow possible further perturbations of the surface $\tilde\Sigma_\varepsilon$. They will be given as normal graphs over $\tilde\Sigma_\varepsilon$ of $\mathscr{C}^{2,\mu}_\beta(\tilde\Sigma_\varepsilon)$ functions. More precisely we start with radial functions $h\colon \tilde\Sigma\to\mathbb{R}$ such that

$$\|h\|_{\mathscr{C}^{2,\mu}_\beta(\tilde\Sigma)}\leq\varepsilon^\tau, \quad \text{some } \tau>0,\beta>0. \tag{7.12}$$

We will also make the usual identification $h_\varepsilon(r)=h(\varepsilon r)$ and consider normal graphs of these functions over $\tilde\Sigma_\varepsilon$ as admissible perturbations. Numbers $\tau,\beta>0$ will be specified later on.

We denote the two components of $\mathbb{R}^{N+1}\setminus\Sigma_\varepsilon$ by $D_\varepsilon^\pm$ respectively. We agree that D_ε^+ is the component containing the set $\mathscr{U}_\varepsilon\cap\{z>0\}$, and D_ε^- is "interior" to $\tilde\Sigma_\varepsilon$. Finally by $\chi_{D_\varepsilon^\pm}$ we denote the characteristic functions of these sets.

With these notations we set:

$$(X_\varepsilon^* u_\varepsilon)(y,z)=H(z-h_\varepsilon(r)), \quad r=|y'|,$$

and define the approximate solution

$$w_\varepsilon(x)=\rho_\varepsilon(x)u_\varepsilon(x)+\left(1-\rho_\varepsilon(x)\right)\left(\chi_{D_\varepsilon^+}(x)-\chi_{D_\varepsilon^-}(x)\right). \tag{7.13}$$

7.4 The error of the approximation

In this section we will compute the error of the approximation, namely:

$$S(w_\varepsilon) = \Delta w_\varepsilon + \varepsilon \partial_{x_{N+1}} w_\varepsilon + w_\varepsilon(1 - w_\varepsilon^2).$$

Using (7.13) we can write:

$$\begin{aligned} S(w_\varepsilon) = \rho_\varepsilon S(u_\varepsilon) + \underbrace{w_\varepsilon(1-w_\varepsilon^2) - \rho_\varepsilon u_\varepsilon(1-u_\varepsilon^2)}_{\mathscr{I}} \\ + \underbrace{[\Delta, \rho_\varepsilon]u_\varepsilon - (\Delta\rho_\varepsilon)(\chi_{D_\varepsilon^+} - \chi_{D_\varepsilon^-}) + \varepsilon \partial_{x_{N+1}} \rho_\varepsilon (u_\varepsilon - \chi_{D_\varepsilon^+} + \chi_{D_\varepsilon^-})}_{\mathscr{J}}. \end{aligned} \tag{7.14}$$

As in (6.5) we write $S(u_\varepsilon)$ in Fermi coordinates and denote for brevity $(X_\varepsilon^* u)(y,z) = \tilde{u}_\varepsilon(y,z)$. Thus we get:

(7.15)

$$\begin{aligned} (X_\varepsilon^* S(u_\varepsilon)) = \Delta_{\tilde{\Sigma}_\varepsilon} \tilde{u}_\varepsilon + [\partial_z^2 \tilde{u}_\varepsilon + f(\tilde{u}_\varepsilon)] + [\varepsilon \partial_z(\pi_{\varepsilon,N+1}) - H_{\tilde{\Sigma}_\varepsilon}]\partial_z \tilde{u}_\varepsilon - z|A_{\tilde{\Sigma}_\varepsilon}|^2 \partial_z \tilde{u}_\varepsilon \\ + \varepsilon \nabla_{\tilde{\Sigma}_\varepsilon}(\pi_{\varepsilon,N+1}) \cdot \nabla_{\tilde{\Sigma}_\varepsilon} \tilde{u}_\varepsilon + \mathbb{A}_\varepsilon[\tilde{u}_\varepsilon] + \mathbb{B}_\varepsilon[\tilde{u}_\varepsilon] - z^2 R_{\tilde{\Sigma}_\varepsilon} \partial_z \tilde{u}_\varepsilon. \end{aligned}$$

To proceed we need to calculate various geometric quantities appearing in (7.15) in terms of the parametrization of $\tilde{\Sigma}_\varepsilon$ given in (7.4). These are standard computations and we will only summarize the most important points in the form of a lemma.

Lemma 7.3. *Let $n_\varepsilon^\pm$ be the unit normal, $g_{\varepsilon,ij}^\pm$ be the coefficients of the metric and $\mathbb{k}_{\varepsilon,j}^\pm$ be the principal curvatures of the ends $\Sigma_\varepsilon^\pm$ of the surface Σ_ε and let $\tilde{n}_\varepsilon^\pm$, $\tilde{g}_{\varepsilon,ij}^\pm$ and $\tilde{\mathbb{k}}_{\varepsilon,j}^\pm$ be the corresponding quantities on $\tilde{\Sigma}_\varepsilon^\pm$, expressed in terms of the local coordinates $(r,\Theta) \in \mathbb{R}_+ \times S^{N-1}$.*

Then, it holds:

(7.16)

$$\tilde{n}_\varepsilon(r,\Theta) = n_\varepsilon(r,\Theta) \mp \big(0, \varepsilon\chi\big(\frac{r}{r_\varepsilon}\big)\frac{\partial_r^2 W_R^\pm(\varepsilon r) f^\pm(\varepsilon r)}{1 + |\partial_r W_R^\pm(\varepsilon r)|^2}\big) + \varepsilon\chi\big(\frac{r}{r_\varepsilon}\big) O\big(\frac{|f^\pm(\varepsilon r)|}{(1+\varepsilon^2 r^2)^{3/2}}\big).$$

Furthermore, the matrices $g_{\varepsilon,ij}^\pm$ and $\tilde{g}_{\varepsilon,ij}^\pm$ are diagonal and we have the following formulas:

$$\tilde{g}_{\varepsilon,ij} = g_{\varepsilon,ij}\Big(1 + \varepsilon\chi\big(\frac{r}{r_\varepsilon}\big) O\big(\frac{|f^\pm(\varepsilon r)|}{(1+\varepsilon^2 r^2)^{1/2}}\big)\Big).$$

The principal curvatures satisfy:

$$\begin{aligned} \tilde{\mathbb{k}}_{\varepsilon,1}^\pm &= \mathbb{k}_{\varepsilon,1}^\pm \Big(1 + \varepsilon\chi\big(\frac{r}{r_\varepsilon}\big) O\big(\frac{|f^\pm(\varepsilon r)|}{(1+\varepsilon^2 r^2)^{1/2}}\big)\Big), \\ \tilde{\mathbb{k}}_{\varepsilon,j}^\pm &= \mathbb{k}_{\varepsilon,j}^\pm \Big(1 + \varepsilon\chi\big(\frac{r}{r_\varepsilon}\big) O\big(\frac{|f^\pm(\varepsilon r)|}{(1+\varepsilon^2 r^2)}\big)\Big), \quad j = 2, \dots, N. \end{aligned}$$

Let us recall that asymptotically, as $r \to \infty$, the ends of Σ_ε are parallel to Γ_ε. As a result in the above formulas we can replace $g_{\varepsilon,ij}^\pm$ and $\mathbb{k}_{\varepsilon,j}^\pm$ in the right hand side by the coefficients of the metric and principal curvatures computed on Γ_ε. The error created this way will be very small. Another observation we make is that if we take r_ε as in (7.10), then we have

$$\chi\big(\frac{r}{r_\varepsilon}\big)\frac{|f^\pm(\varepsilon r)|}{(1+\varepsilon^2 r^2)^\beta} \le \frac{C\varepsilon^2}{(1+\varepsilon^2 r^2)^{\beta'}}$$

for all $\beta' < \beta$ provided that ε is taken sufficiently small.

Then, straightforward calculations show that the error of the initial approximation is essentially of the same size in $\mathscr{C}^{0,\mu}_{\beta,\sigma}(\tilde{\Sigma}_\varepsilon)$ sense. Namely, we can show the exact analog of the Lemma 6.1 for this part of the error:

$$\|\rho_\varepsilon S(u_\varepsilon)\|_{\mathscr{C}^{0,\mu}_{\beta\sigma,\sigma}(\tilde{\Sigma}_\varepsilon)} \le C\varepsilon^{2-\sigma\sqrt{2}}, \quad \beta_\sigma = 1-\sigma\sqrt{2}. \tag{7.17}$$

Now we will estimate the second term in (7.14) denoted by $\mathscr{I}$. For future purposes it is convenient to have an explicit formula:

$$\mathscr{I} = \begin{cases} 3(u_\varepsilon+1)^2\rho_\varepsilon(\rho_\varepsilon-1)+(u_\varepsilon+1)^3\rho_\varepsilon(1-\rho_\varepsilon^2), & \text{in } D_\varepsilon^-, \\ 3(u_\varepsilon-1)^2\rho_\varepsilon(1-\rho_\varepsilon)+(u_\varepsilon-1)^3\rho_\varepsilon(1-\rho_\varepsilon^2), & \text{in } D_\varepsilon^-. \end{cases} \tag{7.18}$$

From this, using $H(t) = \pm 1 + O(e^{-\sqrt{2}|t|})$, and also the asymptotic formula (7.8), we find, with some $\sigma > 0$:

$$\|\mathscr{I}\|_{\mathscr{C}^{0,\mu}_{\beta\sigma,\sigma}(\tilde{\Sigma}_\varepsilon)} \le C\varepsilon^{2-\sigma\sqrt{2}}. \tag{7.19}$$

Our final calculation involves the third term in (7.14) denoted by $\mathscr{J}$. This term is quite important since it represents the interactions between the ends of $\tilde{\Sigma}_\varepsilon$. We write:

$$\mathscr{J} = (\Delta\rho_\varepsilon + \varepsilon\partial_{x_{N+1}}\rho_\varepsilon)(u_\varepsilon - \chi_{D_\varepsilon^+} + \chi_{D_\varepsilon^-}) + 2\nabla\rho_\varepsilon\cdot\nabla u_\varepsilon.$$

Since $H'(t) = O(e^{-\sqrt{2}|t|})$ we can estimate:

$$\begin{aligned} |\mathscr{J}| &\le Ce^{-\sqrt{2}|t|}\chi_{\{0<\rho_\varepsilon<1\}} \\ &\le Ce^{-\sigma|z|}\exp\Big(-(\sqrt{2}-\sigma)\big\{\frac{C(\Sigma)}{\varepsilon}[1-\chi\big(\frac{r}{r_\varepsilon}\big)] + \frac{1}{2}\chi\big(\frac{r}{r_\varepsilon}\big)|f^+(\varepsilon r)-f^-(\varepsilon r)|\big\}\Big). \end{aligned}$$

By (7.8) we have:

$$\|\mathscr{J}\|_{\mathscr{C}^{0,\mu}_{\beta\sigma,\sigma}(\tilde{\Sigma}_\varepsilon)} \le C\varepsilon^{2-\sigma\sqrt{2}}. \tag{7.20}$$

We will summarize (7.17)–(7.20).

Lemma 7.4. *Let w_ε be the approximate solution defined in (7.13). For any $\sigma \in (0,1)$ the error of this approximation $S(w_\varepsilon)$ satisfies the following estimate:*

$$\|S(w_\varepsilon)\|_{\mathscr{C}^{0,\mu}_{\beta\sigma,\sigma}(\tilde{\Sigma}_\varepsilon)} \le C\varepsilon^{2-\sigma\sqrt{2}}, \quad \beta_\sigma = 1-\sigma\sqrt{2}. \tag{7.21}$$

Assuming that the admissible perturbation of $\tilde{\Sigma}_\varepsilon$ satisfies (7.12), the constant C appearing above depends on σ but not on this perturbation.

In addition, as a function of the admissible perturbations, $S(w_\varepsilon)$ is a Lispchitz function from $\mathscr{C}^{2,\mu}_{\beta}(\tilde\Sigma_\varepsilon)$ into $\mathscr{C}^{0,\mu}_{\beta\sigma,\sigma}(\tilde\Sigma_\varepsilon)$ with a Lispchitz constant proportional to $\varepsilon^{2-\sigma\sqrt{2}}$.

7.5 An outline of the Lyapunov-Schmidt reduction

Given the results of Lemma 7.4 it is rather straightforward to implement a Lyapunov-Schmidt reduction procedure similar to the one used in the proof of Theorem 1.1. In fact large parts are simply repetitions with some natural changes. Thus we will only give a brief outline of the general scheme. As before we look for a solution of the problem

$$S(u) = \Delta u + \varepsilon\partial_{x_{N+1}} u + u(1-u^2) = 0, \quad \text{in } \mathbb{R}^{N+1},$$

in the form $u = w_\varepsilon + \varphi_\varepsilon$. Now we write

$$\varphi_\varepsilon = \rho_\varepsilon \phi_\varepsilon + \psi_\varepsilon,$$

and decompose the original problem into a system as described in section 4.2. As a result we get the following analog of (4.13)–(4.15):

$$\Delta_{\tilde\Sigma_\varepsilon}\phi_\varepsilon + \partial_z^2\phi_\varepsilon + f'(u_\varepsilon)\phi_\varepsilon = \mathfrak{g}_\varepsilon + \mathsf{c}_\varepsilon H'(z-h_\varepsilon), \quad \text{in } \tilde\Sigma_\varepsilon \times \mathbb{R}, \tag{7.22}$$

$$(\Delta + \varepsilon\partial_{x_{N+1}} - 2)\psi_\varepsilon = \mathfrak{h}_\varepsilon, \quad \text{in } \mathbb{R}^{N+1}. \tag{7.23}$$

The functions $\mathfrak{g}_\varepsilon$ and $\mathfrak{h}_\varepsilon$ are similar to their counterparts in 4.2 and it can be proven that they have all the properties described in 6.1. Also all the linear theory needed is a verbatim repetition of the content of section 5. This leads us to the existence result for the nonlinear projected problem as in section 6.2. Namely, we have a solution of the system (7.22)–(7.23), with

$$\mathsf{c}_\varepsilon = \frac{\int_{\mathbb{R}} \mathfrak{g}_\varepsilon H'(z-h_\varepsilon)\,dz}{\int_{\mathbb{R}} [H'(z-h_\varepsilon)]^2\,dz}.$$

At this point all that remains to be done is to find h_ε such that $\mathsf{c}_\varepsilon = 0$. Next we will address this problem.

7.6 Solution of the reduced problem

We note that the leading terms in the projection of $\mathfrak{g}_\varepsilon$ onto $H'(z-h_\varepsilon)$ come from the projection of the error of the approximation $S(w_\varepsilon)$. To prove this requires somewhat tedious calculations that we omit. Thus we concentrate on

$$\int_{\mathbb{R}} S(w_\varepsilon)H'(z-h_\varepsilon)\,dz = \int_{\mathbb{R}} \rho_\varepsilon S(u_\varepsilon)H'(z-h_\varepsilon)\,dz + \int_{\mathbb{R}} (\mathscr{I} + \mathscr{J})H'(z-h_\varepsilon)\,dz. \tag{7.24}$$

Using (7.14) and analyzing the terms involved we observe that

$$\begin{aligned}\int_{\mathbb{R}} \rho_\varepsilon S(u_\varepsilon)H'(z-h_\varepsilon)\,dz = -c_0\mathscr{J}_{\tilde\Sigma_\varepsilon}(h_\varepsilon) + \int_{\mathbb{R}} \rho_\varepsilon[\varepsilon\partial_z(\pi_{\varepsilon,N+1}) - H_{\tilde\Sigma_\varepsilon}](H'(z-h_\varepsilon))^2\,dz \\ + \Xi_\varepsilon(h_\varepsilon),\end{aligned} \tag{7.25}$$

where $\mathscr{J}_{\tilde{\Sigma}_\varepsilon}$ is essentially the Jacobi operator on $\tilde{\Sigma}_\varepsilon$:

$$\mathscr{J}_{\tilde{\Sigma}_\varepsilon}(h_\varepsilon) = \Delta_{\tilde{\Sigma}_\varepsilon} h_\varepsilon + \varepsilon \nabla_{\tilde{\Sigma}_\varepsilon}(\pi_{\varepsilon,N+1}) \cdot \nabla_{\tilde{\Sigma}_\varepsilon} h_\varepsilon + |A_{\tilde{\Sigma}_\varepsilon}|^2 h_\varepsilon,$$

and $\Xi_\varepsilon(h_\varepsilon)$ is a small term for all admissible functions h_ε, in the sense that we have:

$$\|\Xi_\varepsilon\|_{\mathscr{C}^{0,\mu}_\beta(\tilde{\Sigma}_\varepsilon)} \leq C\varepsilon^{2+\tau}, \quad \text{some } \beta >, \tau > 0.$$

It remains to calculate the second term on the right hand side of (7.25). We observe that since Σ_ε is a translating solution to the mean curvature flow this term would have been zero if we had not modified Σ_ε to $\tilde{\Sigma}_\varepsilon$. Using the fact that $\partial_z(\pi_{\varepsilon,N+1}) = \tilde{n}_{\varepsilon,N+1}$, i.e. it is simply the $(N+1)$th component of the normal on $\tilde{\Sigma}_\varepsilon$ we get, by (7.16) in Lemma 7.3:

(7.26)
$$\begin{aligned}
&\int_{\mathbb{R}} \rho_\varepsilon[\varepsilon\partial_z(\pi_{\varepsilon,N+1}) - H_{\tilde{\Sigma}_\varepsilon}][H'(z-h_\varepsilon)]^2 \\
&= -\varepsilon^2 \int_{\mathbb{R}} \rho_\varepsilon^{\pm} \chi\Big(\frac{r}{r_\varepsilon}\Big) \frac{\partial_r^2 W_R^{\pm}(\varepsilon r) f^{\pm}(\varepsilon r)}{1+|\partial_r W_R^{\pm}(\varepsilon r)|^2} [H'(z-h_\varepsilon)]^2\, dz + \mathscr{O}_{\mathscr{C}^{0,\mu}_\beta(\tilde{\Sigma}_\varepsilon)}(\varepsilon^{2+\tau}) \\
&= -\varepsilon^2 \int_{\mathbb{R}} \rho_\varepsilon^{\pm} \chi\Big(\frac{r}{r_\varepsilon}\Big) \frac{\partial_r^2 F(\varepsilon r) f^{\pm}(\varepsilon r)}{1+|\partial_r F(\varepsilon r)|^2} [H'(z-h_\varepsilon)]^2\, dz + \mathscr{O}_{\mathscr{C}^{0,\mu}_\beta(\tilde{\Sigma}_\varepsilon)}(\varepsilon^{2+\tau}) \\
&= -a_0 \varepsilon^2 \chi\Big(\frac{r}{r_\varepsilon}\Big) \frac{\partial_r^2 F(\varepsilon r)[f^+(\varepsilon r) + f^-(\varepsilon r)]}{1+|\partial_r F(\varepsilon r)|^2} + \mathscr{O}_{\mathscr{C}^{0,\mu}_\beta(\tilde{\Sigma}_\varepsilon)}(\varepsilon^{2+\tau}).
\end{aligned}$$

where $a_0 > 0$ is a constant and

$$\rho_\varepsilon^{\pm}(y,z) = \begin{cases} \rho_\varepsilon(y,z), & y \in \tilde{\Sigma}_\varepsilon^{\pm}, \\ 0, & \text{otherwise.} \end{cases}$$

In (7.26) we have omitted terms that are at most of a size comparable with $\varepsilon^{2+\tau}$ in the sense of $\mathscr{C}^{0,\mu}_\beta(\tilde{\Sigma}_\varepsilon)$, as indicated by the notation. We observe as well that the error in replacing $\partial_r^2 W_R^{\pm}(\varepsilon r)$ and $\partial_r W_R^{\pm}(\varepsilon r)$ by $\partial_r^2 F(\varepsilon r)$ and $\partial_r F(\varepsilon r)$, respectively, again results in a lower order term. This justifies the third line in (7.26).

Going back to (7.24) we observe that the projection on $\mathscr{I}$ is again negligible since, by (7.18), we see that $\mathscr{I} \approx e^{-2\sqrt{2}|z|}\chi_{\{0<\rho_\varepsilon<1\}}$. Thus it remains to calculate:

$$\begin{aligned}
&\int_{\mathbb{R}} \mathscr{I} H'(z-h_\varepsilon)\, dz \\
&= \int_{\mathbb{R}} \big[(\Delta\rho_\varepsilon + \varepsilon\partial_{x_{N+1}}\rho_\varepsilon)(u_\varepsilon - \chi_{D_\varepsilon^+} + \chi_{D_\varepsilon^-}) + 2\nabla\rho_\varepsilon \cdot \nabla u_\varepsilon\big] H'(z-h_\varepsilon)\, dz
\end{aligned} \tag{7.27}$$

Using definition of ρ_ε in (7.11), and the identity $1-H^2=\sqrt{2}H'$, after some integrations by parts we get:
(7.28)

$$\begin{aligned}
&\int_{\mathbb{R}} \mathscr{J} H'(z-h_\varepsilon)\,dz\\
&\quad=\int_{\mathbb{R}}[\rho_\varepsilon''(H'(z-h_\varepsilon)-\chi_{D_\varepsilon^+}+\chi_{D_\varepsilon^-})+2\rho_\varepsilon' H'(z-h_\varepsilon)]H'(z-h_\varepsilon)\,dz+\mathscr{O}_{\mathscr{C}_\beta^{0,\mu}(\tilde\Sigma_\varepsilon)}(\varepsilon^{2+\tau})\\
&\quad=2\int_{\mathbb{R}}\rho_\varepsilon'[H'(z-h_\varepsilon)]^2\,dz+\mathscr{O}_{\mathscr{C}_\beta^{0,\mu}(\tilde\Sigma_\varepsilon)}(\varepsilon^{2+\tau})\\
&\quad=a_1\exp\{-2\sqrt{2}\frac{C(\Sigma)\left[1-\chi\left(\frac{r}{r_\varepsilon}\right)\right]}{\varepsilon}\}\exp\{-\sqrt{2}\chi\left(\frac{r}{r_\varepsilon}\right)(f^+(\varepsilon r)+f^-(\varepsilon r))\}e^{-2\sqrt{2}h_\varepsilon}\\
&\qquad+\mathscr{O}_{\mathscr{C}_\beta^{0,\mu}(\tilde\Sigma_\varepsilon)}(\varepsilon^{2+\tau}).
\end{aligned}$$

Summarizing (7.25),(7.26) and (7.28) we get that the reduced problem amounts to solving for h_ε the following equation:
(7.29)

$$\begin{aligned}
&c_0\mathscr{J}_{\tilde\Sigma_\varepsilon}(h_\varepsilon)+\tilde c_1\exp\{-2\sqrt{2}\frac{C(\Sigma)\left[1-\chi\left(\frac{r}{r_\varepsilon}\right)\right]}{\varepsilon}\}\exp\{-\sqrt{2}\chi\left(\frac{r}{r_\varepsilon}\right)(f^+(\varepsilon r)+f^-(\varepsilon r))\}h_\varepsilon\\
&\quad=a_0\varepsilon^2\chi\left(\frac{r}{r_\varepsilon}\right)\frac{\partial_r^2F(\varepsilon r)[f^+(\varepsilon r)+f^-(\varepsilon r)]}{1+|\partial_rF(\varepsilon r)|^2}\\
&\qquad-a_1\exp\{-2\sqrt{2}\frac{C(\Sigma)\left[1-\chi\left(\frac{r}{r_\varepsilon}\right)\right]}{\varepsilon}\}\exp\{-\sqrt{2}\chi\left(\frac{r}{r_\varepsilon}\right)(f^+(\varepsilon r)+f^-(\varepsilon r))\}\\
&\qquad+\mathscr{O}_{\mathscr{C}_\beta^{0,\mu}(\tilde\Sigma_\varepsilon)}(\varepsilon^{2+\tau}).
\end{aligned}$$

This is of course a fixed point problem for h_ε and the term which we have denoted by $\mathscr{O}_{\mathscr{C}_\beta^{0,\mu}(\tilde\Sigma_\varepsilon)}(\varepsilon^{2+\tau})$ depends in a nonlinear and nonlocal way on h_ε. It can be shown that this term in fact is a Lipschitz contraction of h_ε (and consequently of the admissible functions h_ε). This is quite similar as in the previous part. We concentrate on analyzing the invertibility of the linear operator on the right hand side of (7.29).

We make first an observation that by the choice of $f^\pm$ we have that for $r>2r_\varepsilon$:

$$a_0\varepsilon^2\frac{\partial_r^2F(\varepsilon r)[f^+(\varepsilon r)+f^-(\varepsilon r)]}{1+|\partial_rF(\varepsilon r)|^2}-a_1\exp\{-\sqrt{2}(f^+(\varepsilon r)+f^-(\varepsilon r))\}=0. \tag{7.30}$$

Second, when $r\le 2r_\varepsilon$ then by the choice of r_ε we have that the whole right hand side is an $\mathscr{O}_{\mathscr{C}_\beta^{0,\mu}(\tilde\Sigma_\varepsilon)}(\varepsilon^{2+\tau})$ term. As a consequence, arranging some terms suitably, we are left with solving the following problem:

$$-c_0\mathscr{J}_{\tilde\Sigma_\varepsilon}(h_\varepsilon)+\chi\left(\frac{r}{r_\varepsilon}\right)\exp\{-\sqrt{2}(f^+(\varepsilon r)+f^-(\varepsilon r))\}h_\varepsilon=\mathscr{O}_{\mathscr{C}_\beta^{0,\mu}(\tilde\Sigma_\varepsilon)}(\varepsilon^{2+\tau}).$$

Scaling back to the surface $\tilde{\Sigma}$ we are left with the problem of the form:

(7.31)

$$\Delta_{\tilde{\Sigma}} h + \nabla_{\tilde{\Sigma}}(\pi_{N+1}) \cdot \nabla_{\tilde{\Sigma}} h + |A_{\tilde{\Sigma}}|^2 h + \frac{1}{\varepsilon^2}\chi\big(\frac{r}{\varepsilon r_\varepsilon}\big)\exp\{-\sqrt{2}(f^+(r)+f^-(r))\}h$$
$$= \mathscr{O}_{\mathscr{C}^{0,\mu}_{\beta}(\tilde{\Sigma})}(\varepsilon^\tau).$$

Since we consider only the radial perturbations of the original surface Σ as admissible, then $\tilde{\Sigma}$ is also rotationally symmetric, and the above problem reduces to an ODE. Thus we may use a similar technique as in the previous part, namely solve it by variation of parameters formula, gluing various parts. When $r < \varepsilon r_\varepsilon$ our operator is essentially identical with the linearization of the translating graph solution to the mean curvature flow (c.f Lemma 7.3). Inverting this operator is the only more significantly different part of the theory and thus we will present it in some details. Note that when $r > \varepsilon r_\varepsilon$ the operator above resembles the linearized operator $\mathfrak{L}_\delta$, treated extensively in section 3.6. An argument similar to the one in section 3.6 can be used to to control a fundamental set and to write the variation of parameters formula.

7.7 The Jacobi operator of the traveling catenoid Σ

Our goal is to prove the following:

Lemma 7.5. *Let $g \in \mathscr{C}^{0,\mu}_{\beta}(\tilde{\Sigma})$, $\beta > 1$, be a function depending on the radial variable only. There exists a solution $v = v(r)$ of the problem:*

$$[\Delta_{\tilde{\Sigma}} + \nabla_{\tilde{\Sigma}}(\pi_{N+1}) \cdot \nabla_{\tilde{\Sigma}} + |A_{\tilde{\Sigma}}|^2]v + \frac{1}{\varepsilon^2}\chi\big(\frac{r}{\varepsilon r_\varepsilon}\big)\exp\{-\sqrt{2}(f^+(r)+f^-(r))\}v = g,$$

with

$$\|v\|_{\mathscr{C}^{2,\mu}_{\beta-1}(\tilde{\Sigma})} \leq C\|g\|_{\mathscr{C}^{0,\mu}_{\beta}(\tilde{\Sigma})}.$$

In the region where $r < \varepsilon r_\varepsilon$ the surface $\tilde{\Sigma}$ coincides with the original traveling catenoid Σ. This is where our problem is different and we need the following result:

Lemma 7.6. *Let us consider the following problem:*

$$\mathscr{J}_\Sigma(v) = [\Delta_\Sigma + \nabla_\Sigma(\pi_{N+1}) \cdot \nabla_\Sigma + |A_\Sigma|^2]v = g, \tag{7.32}$$

where $g \in \mathscr{C}^{0,\mu}_{1+\beta}(\Sigma)$, with $\beta > 0$, is a function that depends on the radial variable r only. There exist a solution $v = v(r)$ of this problem such that

$$\|v\|_{\mathscr{C}^{0,\mu}_{\beta}(\Sigma)} \leq C\|g\|_{\mathscr{C}^{0,\mu}_{1+\beta}(\Sigma)}. \tag{7.33}$$

Proof. We observe that the Jacobi operator for the surface Σ can not be anymore expressed in terms of the radial variable globally. In fact we need to use three charts on Σ to write conveniently equation (7.31) in local variables. This in fact is the only new element.

Near the point of the traveling catenoid where $r=R$ we will express the surface as a graph over the x_{N+1}-axis. Thus we have, following the results in [5]:

$$\Sigma \cap B_{R_1} = \{(q(z)\Theta, z) \mid \quad z \in (-z_0, z_0)\},$$

where $R_1 > R$ and q satisfies:

$$\big(\frac{N-1}{q} - q'\big)\big(1+(q')^2\big) = q''.$$

With this in mind we express the radial function g on the right hand side of (7.32) in terms of $z = q^{-1}(r)$. We will abuse notation and denote this, and other functions involved, by the same symbols g, v etc.

We write the Jacobi operator $\mathscr{J}_\Sigma$ restricted to functions of $v = v(z)$ in this chart and get the following ODE:

$$\frac{v''}{1+|q'|^2} + \big(\frac{N-1}{h} + \frac{1}{1+|q'|^2}\big)v' + \big[\frac{|q'|^2}{(1+|q'|^2)^3} + \frac{N-1}{q^2(1+|q'|^2)}\big]v = g. \tag{7.34}$$

We multiply this equation through by $1+|q'|^2$ and arrive at the equation in the following form:

$$v'' + p_1(z)v' + p_2(z)v = (1+|q'|^2)g = \tilde{g}.$$

Let ϕ_0 and ϕ_1 be two linearly independent elements of a fundamental set of the operator chosen so that

$$\phi_0(0) = 0, \quad \phi_1(0) = 1$$
$$\phi_0'(0) = 1, \quad \phi_1'(0) = 0.$$

Finally let $P_1(z)$ be a primitive of p_1. Then we can write explicitly:

$$v(z) = \int_{-z_0}^{z} \frac{e^{-P_1(\zeta)}}{\phi_0^2(\zeta)} \int_{-z_0}^{\zeta} \tilde{g}(\zeta')\phi_0(\zeta')e^{P_1(\zeta')}\,d\zeta' + a_0\phi_0(z) + a_1\phi(z). \tag{7.35}$$

Next we write $\mathscr{J}_\Sigma$ on the ends $\Sigma^\pm \setminus B_{r_0}$, where r_0 is chosen so that $R < r_0 < R_1$ and the various local charts overlap. The natural parametrization is of course:

$$\Sigma^\pm \setminus B_{r_0} = \{(r\Theta, W_R^\pm(r)) \mid (\Theta, r) \in S^{N-1} \times \mathbb{R}_+\}.$$

In this chart $\mathscr{J}_\Sigma$ can be written as an ODE in r for each of the two ends. This is very similar to what we did in Lemma 3.4. Denoting by $\phi_0^\pm$, $\phi_1^\pm$ the elements of a fundamental set corresponding to ϕ_0, ϕ_1 in Lemma 3.4, and letting $\tilde{g}^\pm = (1+|\partial_r W_R^\pm|^2)$ we get the following formula:

(7.36)

$$v^\pm(r) = -\phi_0^\pm \int_{r_0}^{r} \frac{\phi_1^\pm(\rho))\tilde{g}^\pm(\rho)}{W^\pm(\rho)}\,d\rho + \phi_1^\pm \int_{r_0}^{r} \frac{\phi_0^\pm(\rho))\tilde{g}^\pm(\rho)}{W^\pm(\rho)}\,d\rho + a_1^\pm\phi_1^\pm(r),$$

for a general solution v in $\mathscr{C}_\beta^{2,\mu}(\Sigma)$. Note that we have

$$\phi_0^\pm(r) \sim \frac{1}{1+r}, \quad \phi_1^\pm(r) \sim re^{-r^2}, \quad r \gg 1,$$

which is the reason why in (7.36) we have included only constant multiplicities of $\phi_1^\pm$.

Next we need to choose the four constants a_0, a_1 and $a_1^\pm$ in such a way that

$$v^\pm(r_0) = v \circ (W_R^\pm)^{-1}(r_0),$$
$$\partial_r v^\pm(r_0) = \partial_r v \circ (W_R^\pm)^{-1}(r_0).$$

This is a matter of solving a simple system of 4 linear equations.

After this is done we have a solution defined now on the whole surface Σ. Estimate (7.33) follows directly from the explicit formulas we have derived. This ends the proof. □

Next, we describe how to solve the linearized problem (7.31). Note that as long as $r < \varepsilon r_\varepsilon$ we are dealing with the Jacobi operator discussed in the Lemma above. Thus, at least up to $r = \varepsilon r_\varepsilon$, we will have no problem in defining a solution v in $\mathscr{C}^{2,\mu}_{\beta-1}(\Sigma \cap \{r < r_\varepsilon\})$ (here we take $\beta > 1$). What is left is to solve a problem of the form:

$$\mathscr{J}_{\tilde\Sigma^\pm}(v^\pm) + \frac{1}{\varepsilon^2}\chi\Big(\frac{r}{\varepsilon r_\varepsilon}\Big)\exp\{-\sqrt{2}(f^+(r)+f^-(r))\}v = g^\pm, \tag{7.37}$$

on each end $\Sigma^\pm$, with $r > r_\varepsilon$ for radial functions $v^\pm$, with initial data given by the solution v, already found, at $r = \varepsilon r_\varepsilon$.

Now, we notice that because of the definition on $f^\pm$ in (7.6)–(7.7), the operator appearing in (7.37) is very similar to the operator $\mathfrak{L}_\delta$ considered in section 3.5. In fact, we can write (7.37) in the form:

$$\frac{(1+o(1))v^\pm_{rr}}{1+|\partial_r W_R^\pm|^2} + \frac{(N-1)(1+o(1))v^\pm_r}{r} + p^\pm_\varepsilon(r)v^\pm = g^\pm, \quad r > \varepsilon r_\varepsilon,$$

where

$$p_\varepsilon(r) \sim \frac{1}{1+r^2}\log\Big(\frac{1+r^2}{\varepsilon^2}\Big), \quad r > \varepsilon r_\varepsilon$$

which is in agreement with the behavior of the function p_δ in (3.37) and the $o(1)$ term above means terms that are small both in ε and r. Since we are interested in this problem only for large values of $r \geq \varepsilon r_\varepsilon \sim e^{\frac{\tilde c}{\varepsilon}}$ we see that the argument in section 3.6 can be repeated verbatim to solve finally our problem. Having the inverse of the operator in (7.31) at hand we proceed in the same way as in the previous case to solve finally a fixed point problem for h. We omit the details.

Acknowledgment.

The research of Manuel del Pino was supported in part by Fondecyt grant 1110181, Proyecto Anillo ACT 125 CAPDE, and Fondo Basal-CMM, Chile. Michał Kowalczyk was partly supported by Fondecyt grant 1090103, Proyecto Anillo ACT 125 CAPDE, and Fondo Basal-CMM, Chile. Juncheng Wei was partly supported by General Research Fund from RGC of Hong Kong.

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Received Month 200X.