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Ground State of N Coupled Nonlinear Schrödinger Equations in R^n, $n \leq 3$

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Abstract: We establish some general theorems for the existence and nonexistence of ground state solutions of steady-state N coupled nonlinear Schrödinger equations. The sign of coupling constants β_{ij}'s is crucial for the existence of ground state solutions. When all β_{ij}'s are positive and the matrix Σ is positively definite, there exists a ground state solution which is radially symmetric. However, if all β_{ij}'s are negative, or one of β_{ij}'s is negative and the matrix Σ is positively definite, there is no ground state solution. Furthermore, we find a bound state solution which is non-radially symmetric when $N = 3$.

1. Introduction

In this paper, we study solitary wave solutions of time-dependent N coupled nonlinear Schrödinger equations given by

$$\begin{cases} -i\frac{\partial}{\partial t}\Phi_j = \Delta\Phi_j + \mu_j|\Phi_j|^2\Phi_j + \sum_{i\neq j}\beta_{ij}|\Phi_i|^2\Phi_j \text{ for } y \in R^n, t > 0, \\ \Phi_j = \Phi_j(y,t) \in \mathbb{C}, \quad j = 1, \dots, N, \\ \Phi_j(y,t) \to 0 \text{ as } |y| \to +\infty, t > 0, \end{cases} \tag{1.1}$$

where $\mu_j > 0$'s are positive constants, $n \leq 3$, and β_{ij}'s are coupling constants. The system (1.1) has applications in many physical problems, especially in nonlinear optics. Physically, the solution Φ_j denotes the j^{th} component of the beam in Kerr-like photorefractive media(cf. [1]). The positive constant μ_j is for self-focusing in the j^{th} component of the beam. The coupling constant β_{ij} is the interaction between the i^{th} and the j^{th} component of the beam. As $\beta_{ij} > 0$, the interaction is attractive, but the interaction is repulsive if $\beta_{ij} < 0$. When the spatial dimension is one, i.e. $n = 1$, the system (1.1) is integrable, and there are many analytical and numerical results on solitary wave solutions of the general N coupled nonlinear Schrödinger equations(cf. [8, 17–19]).

From physical experiment(cf. [23]), two dimensional photorefractive screening solitons and a two dimensional self-trapped beam were observed. It is natural to believe that there are two dimensional N-component($N \geq 2$) solitons and self-trapped beams. However, until now, there is no general theorem for the existence of high dimensional N-component solitons. Moreover, some general principles like the interaction and the configuration of two and three dimensional N-component solitons are unknown either. This may lead us to study solitary wave solutions of the system (1.1) for $n = 2, 3$. Here we develop some general theorems for N-component solitary wave solutions of the system (1.1) in two and three spatial dimensions.

To obtain solitary wave solutions of the system (1.1), we set $\Phi_j(y, t) = e^{i\lambda_j t} u_j(y)$ and we may transform the system (1.1) to steady-state N coupled nonlinear Schrödinger equations given by

$$\begin{cases} \Delta u_j - \lambda_j u_j + \mu_j u_j^3 + \sum\limits_{i \neq j} \beta_{ij} u_i^2 u_j = 0 \text{ in } R^n, \\ u_j > 0 \text{ in } R^n, j = 1, \ldots, N, \\ u_j(y) \to 0 \text{ as } |y| \to +\infty, \end{cases} \tag{1.2}$$

where $\lambda_j, \mu_j > 0$ are positive constants, $n \leq 3$, and β_{ij}'s are coupling constants. Here we want to study the existence and the configuration of ground state solutions of the system (1.2). The existence of ground state solutions may depend on coupling constants β_{ij}'s. When all β_{ij}'s are positive and the matrix Σ(defined in (1.9)) is positively definite, there exists a ground state solution which is radially symmetric, i.e. $u_j(y) = u_j(|y|)$, $j = 1, \cdots, N$. Such a radially symmetric solution may support the existence of N circular self-trapped beams. However, if all β_{ij}'s are negative, or one of β_{ij}'s is negative and the matrix Σ is positively definite, there is no ground state solution. Furthermore, we find a bound state solution which is non-radially symmetric when $N = 3$. We will prove these results in the rest of this paper.

Now we give the definition of ground state solutions as follows:

In the one component case ($N = 1$), we may obtain a solution to (1.2) through the following minimization:

$$\inf_{\substack{u \geq 0, \\ u \in H^1(R^n)}} \frac{\int_{R^n} |\nabla u|^2 + \lambda_1 \int_{R^n} u^2}{(\int_{R^n} u^4)^{\frac{1}{2}}}. \tag{1.3}$$

An equivalent formulation, called Nehari's manifold approach (see [6] and [7]), is to consider the following minimization problem:

$$\inf_{u_1 \in N_1} E[u_1],$$

where

$$N_1 = \left\{ u \in H^1(R^n) : u \not\equiv 0 \,, \int_{R^n} |\nabla u|^2 + \lambda_1 \int_{R^n} u^2 = \mu_1 \int_{R^n} u^4 \right\}. \tag{1.4}$$

It is easy to see that (1.3) and (1.4) are equivalent. A solution obtained through (1.4) is called a ground state solution in the following sense: (1) $u > 0$ and satisfies (1.2), (2)

$E[u] \le E[v]$ for any other solution v of (1.2). Hereafter, we extend the definition of ground state solutions to N-component case. To this end, we define first

$$\mathbf{N} = \Big\{ \mathbf{u} = (u_1, \dots, u_N) \in \left(H^1(R^n)\right)^N : u_j \ge 0, u_j \not\equiv 0, \tag{1.5}$$
$$\int_{R^n} |\nabla u_j|^2 + \lambda_j \int_{R^n} u_j^2 = \mu_j \int_{R^n} u_j^4 + \sum_{i \ne j} \beta_{ij} \int_{R^n} u_i^2 u_j^2, \quad j = 1, \dots, N \Big\}$$

and consider the following minimization problem:

$$c = \inf_{u \in \mathbf{N}} E[\mathbf{u}], \tag{1.6}$$

where the associated energy functional is given by

$$E[\mathbf{u}] = \sum_{j=1}^{N} \left(\frac{1}{2} \int_{R^n} |\nabla u_j|^2 + \frac{\lambda_j}{2} \int_{R^n} u_j^2 - \frac{\mu_j}{4} \int_{R^n} u_j^4 \right) \tag{1.7}$$
$$- \frac{1}{4} \sum_{\substack{i,j=1,\\ i \ne j}}^{N} \beta_{ij} \int_{R^n} u_i^2 u_j^2$$

for

$$\mathbf{u} = (u_1, \dots, u_N) \in (H^1(R^n))^N. \tag{1.8}$$

Since $n \le 3$, by Sobolev embedding, $E[\mathbf{u}]$ is well-defined. A minimizer $\mathbf{u}^0 = (u_1^0, \dots, u_N^0)$ of (1.6), if it exists, is called a ground state solution of (1.2), and it may have the following properties:

1. $u_j^0 > 0, \forall j$, and $\mathbf{u}^0$ satisfies (1.2);
2. $E[u_1^0, \dots, u_N^0] \le E[v_1, \dots, v_N]$ for any other solution $(v_1, \dots, v_N)$ of (1.2).

It is natural to ask when the ground state solution exists. As $N = 1$, the existence of the ground state solution is trivial (see [6]). However, the existence of the ground state solution with multi-components is quite complicated.

For general $N \ge 2$, we introduce the following auxiliary matrix:

$$\sum = (|\beta_{ij}|), \quad \text{where we set } \beta_{ii} = \mu_i. \tag{1.9}$$

Our first theorem concerns the all repulsive case:

Theorem 1. *If $\beta_{ij} < 0, \forall i \ne j$, then the ground state solution doesn't exist, i.e. c defined at (1.6) can not be attained.*

Our second theorem concerns the all attractive case.

Theorem 2. *If $\beta_{ij} > 0, \forall i \ne j$, and the matrix $\sum$ (defined at (1.9)) is positively definite, then there exists a ground state solution $(u_1^0, \dots, u_N^0)$. All u_j^0 must be positive, radially symmetric and strictly decreasing.*

When attraction and repulsion coexist, i.e. some of β_{ij}'s are positive but some of them are negative, things become very complicated. Our third theorem shows that if one state is repulsive to all the other states, then the ground state solution doesn't exist.

Theorem 3. *If there exists an i_0 such that*

$$\beta_{i_0 j} < 0, \forall j \neq i_0, \text{ and } \beta_{ij} > 0, \forall i \neq i_0, j \notin \{i, i_0\} \tag{1.10}$$

and assume that the matrix Σ is positively definite, then the ground state solution to (1.2) doesn't exist.

Finally, we discuss the existence of bound states, that is, solutions of (1.2) with finite energy. We show that if repulsion is stronger than attraction, there may be non-radial bound states. To simplify our computations, we choose

$$N = 3, \quad \lambda_1 = \lambda_2 = \lambda_3 = \mu_1 = \mu_2 = \mu_3 = 1. \tag{1.11}$$

Theorem 4. *Assume that $N = 3$ and*

$$\beta_{12} = \delta\hat{\beta}_{12} = \beta_{13} = \delta\hat{\beta}_{13} > 0, \quad \beta_{23} = \sqrt{\delta}\hat{\beta}_{23} < 0. \tag{1.12}$$

Then for δ sufficiently small, problem (1.2) admits a non-radial solution $\mathbf{u}^\delta = \left(u_1^\delta, u_2^\delta, u_3^\delta\right)$ with the following properties:

$$u_1^\delta(y) \sim w(y), \quad u_2^\delta(y) \sim w(y - R^\delta e_1), \quad u_3^\delta(y) \sim w(y + R^\delta e_1),$$

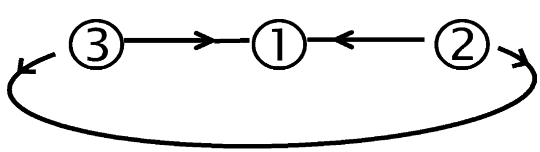
where

$$R^\delta \sim \log \frac{1}{\delta}, \quad e_1 = (1, 0, \dots, 0)^T,$$

and w is the unique solution of the following problem:

$$\begin{cases} \Delta w - w + w^3 = 0 \text{ in } R^n \\ w > 0 \text{ in } R^n, \, w(0) = \max\limits_{y \in R^n} w(y) \\ w(y) \to 0 \text{ as } |y| \to +\infty. \end{cases} \tag{1.13}$$

Graphically, we have



Note that under condition (1.12), there is also a radially symmetric solution $\mathbf{u}^r$ of the following form:

$$\mathbf{u}^r = (u_1^r, u_2^r, u_3^r),\ u_j^r = \sqrt{\xi_j} w(y),\ j = 1, 2, 3,$$

where ξ_j satisfies

$$\xi_j + \sum_{i \neq j} \beta_{ij} \xi_i = 1, \quad j = 1, 2, 3. \tag{1.14}$$

Then we have

Corollary 1. *Assume that $N = 3$ and (1.12) holds. Then for δ sufficiently small, we have*

$$E[\mathbf{u}^\delta] < E[\mathbf{u}^r], \tag{1.15}$$

where $\mathbf{u}^\delta$ is constructed in Theorem 3. As a consequence, if the ground state solution exists, it must be non-radially symmetric.

It is known that (1.2) admits many radially symmetric bound states (see [17] and [18]). Theorem 4 suggests that there are many non-radially symmetric bound states which have lower energy than radially symmetric bound states. We consider the applications of Theorems 1–3 to simple cases $N = 2$ and $N = 3$.

For the case $N = 2$, we have

Corollary 2. *If $N = 2$, then*

1. *for $\beta_{12} < 0$, the ground state solution doesn't exist,*
2. *for $0 < \beta_{12} < \sqrt{\mu_1 \mu_2}$, the ground state solution exists.*

For the case $N = 3$, the matrix Σ becomes

$$\Sigma = \begin{pmatrix} \mu_1 & |\beta_{12}| & |\beta_{13}| \\ |\beta_{12}| & \mu_2 & |\beta_{23}| \\ |\beta_{13}| & |\beta_{23}| & \mu_3 \end{pmatrix}.$$

Assume that $\beta_{ij} \neq 0$. Then we may divide into four cases given by

Case I: all repulsive: $\beta_{12} < 0$, $\beta_{13} < 0$, $\beta_{23} < 0$,
Case II: all attractive: $\beta_{12} > 0$, $\beta_{13} > 0$, $\beta_{23} > 0$,
Case III: two repulsive and one attractive: $\beta_{12} < 0$, $\beta_{13} < 0$, $\beta_{23} > 0$,
Case IV: one repulsive and two attractive: $\beta_{12} > 0$, $\beta_{13} > 0$, $\beta_{23} < 0$.

For Case I–III, we have a complete picture

Corollary 3. *If $N = 3$, then*

1. *for Case I, the ground state solution doesn't exist,*
2. *for Case II and assume Σ is positively definite, the ground state solution exists,*
3. *for Case III and assume Σ is positively definite, the ground state solution doesn't exist.*

It then remains only to consider Case IV. Due to the existence of non-radial bound states in Theorem 4 and non-radial property of ground states in Corollary 1, Case IV becomes very complicated. Our results here will be very useful in the study of (1.2) for bounded domains which relates to multispecies Bose-Einstein condensates, and in the study of solitary wave solutions of N coupled nonlinear Schrödinger equations with trap potentials:

$$\begin{cases} \Delta u_j - V_j(x)u_j + \mu_j u_j^3 + \sum_{i\neq j} \beta_{ij} u_i^2 u_j = 0, \quad x \in R^n, \\ u_j > 0 \ \text{in} \ R^n, j = 1, \ldots, N, \\ u_j(x) \to 0 \ \text{as} \ |x| \to +\infty. \end{cases} \tag{1.16}$$

The main idea in proving Theorem 1–3 is by Nehari's manifold approach and Schwartz symmetrization technique. Theorem 4 is proved by the Liapunov-Schmidt reduction method combined with the variational method. The organization of the paper is as follows: In Sect. 2, we collect some properties of the function w-solution of (1.13) and Schwartz symmetrization. In Sect. 3, we state another equivalent approach of Nehari's method which is more useful in our proofs. It is here that we need that the matrix Σ is positively definite. The proofs of Theorems 1, 2, 3, 4 are given in Sects. 4, 5, 6, 7, respectively. Section 8 contains the proof of Corollary 1.

2. Some Preliminaries

In this section, we analyze some problems in R^n. Recall that w is the unique solution of (1.13). By Gidas-Ni-Nirenberg's Theorem, [14], w is radially symmetric. By a theorem of Kwong [20], w is unique. Moreover, we have

$$w'(|y|) < 0 \ \text{for} \ |y| > 0$$

and

$$w(|y|) = A_n r^{-\frac{n-1}{2}} e^{-r} \left(1 + O\left(\frac{1}{r}\right)\right), \ \text{as} \ r = |y| \to +\infty, \tag{2.1}$$

$$w'(|y|) = -A_n r^{-\frac{n-1}{2}} e^{-r} \left(1 + O\left(\frac{1}{r}\right)\right), \ \text{as} \ r = |y| \to +\infty. \tag{2.2}$$

We denote the energy of w as

$$I[w] = \frac{1}{2}\int_{R^n} |\nabla w|^2 + \frac{1}{2}\int_{R^n} w^2 - \frac{1}{4}\int_{R^n} w^4. \tag{2.3}$$

Let $w_{\lambda,\mu}$ be the unique solution to the following problem:

$$\begin{cases} \Delta w_{\lambda,\mu} - \lambda w_{\lambda,\mu} + \mu w_{\lambda,\mu}^3 = 0 \ \text{in} \ R^n, \\ w_{\lambda,\mu} > 0, w_{\lambda,\mu}(0) = \max_{y\in R^n} w_{\lambda,\mu}(y), \\ w_{\lambda,\mu}(y) \to 0 \ \text{as} \ |y| \to +\infty. \end{cases}$$

It is easy to see that

$$w_{\lambda,\mu}(y) = \sqrt{\frac{\lambda}{\mu}} w\left(\sqrt{\lambda}|y|\right), \tag{2.4}$$

and

$$\frac{1}{2}\int_{R^n} |\nabla w_{\lambda,\mu}|^2 + \frac{\lambda}{2}\int_{R^n} w_{\lambda,\mu}^2 - \frac{\mu}{4}\int_{R^n} w_{\lambda,\mu}^4 = \lambda^{\frac{4-n}{2}}\mu^{-1} I[w]. \tag{2.5}$$

We now collect some of the properties of $w_{\lambda,\mu}$.

Lemma 1. *(1) $w(|y|)$ is the unique solution to the following minimization problem:*

$$\inf_{\substack{u\in H^1(R^n),\\ u\geq 0}} \frac{\int_{R^n} |\nabla u|^2 + \int_{R^n} u^2}{(\int_{R^n} u^4)^{\frac{1}{2}}}. \tag{2.6}$$

(2) The following eigenvalue problem:

$$\begin{cases} \Delta\phi - \lambda\phi + 3\mu w_{\lambda,\mu}^2\phi = \beta\phi \\ \phi \in H^2(R^n) \end{cases} \tag{2.7}$$

admits the following set of eigenvalues:

$$\beta_1 > 0 = \beta_2 = \ldots = \beta_{n+1} > \beta_{n+2} \geq \ldots,$$

where the eigenfunctions corresponding to the zero eigenvalue are spanned by

$$K_0 := span\left\{\frac{\partial w_{\lambda,\mu}}{\partial y_j}, j = 1, \ldots, n\right\} = C_0. \tag{2.8}$$

As a result, the following map:

$$L_{\lambda,\mu}\phi := \Delta\phi - \lambda\phi + 3\mu w_{\lambda,\mu}^2\phi$$

is invertible from $K_0^\perp \to C_0^\perp$ where

$$K_0^\perp = \left\{u \in H^2(R^n)\middle|\int_{R^n} u\frac{\partial w_{\lambda,\mu}}{\partial y_j} = 0, j = 1, \cdots, n\right\}, \tag{2.9}$$

$$C_0^\perp = \left\{u \in L^2(R^n)\middle|\int_{R^n} u\frac{\partial w_{\lambda,\mu}}{\partial y_j} = 0, j = 1, \cdots, n\right\}. \tag{2.10}$$

Proof. (1) follows from the uniqueness of w(cf. [20]). (2) follows from Theorem 2.12 of [22] and Lemma 4.2 of [24].

Set also

$$I_{\lambda,\mu}[u] = \frac{1}{2}\int_{R^n} |\nabla u|^2 + \frac{\lambda}{2}\int_{R^n} u^2 - \frac{\mu}{4}\int_{R^n} u^4. \tag{2.11}$$

We then have

Lemma 2. $\inf_{u\in N_{\lambda,\mu}} I_{\lambda,\mu}[u]$ *is attained only by* $w_{\lambda,\mu}$,
where

$$N_{\lambda,\mu} = \left\{u \in H^1(R^n)\middle| \int_{R^n} |\nabla u|^2 + \lambda \int_{R^n} u^2 = \mu \int_{R^n} u^4\right\}. \tag{2.12}$$

Proof. It is easy to see that $\inf_{u\in N_{\lambda,\mu}} I_{\lambda,\mu}[u]$ is equivalent to

$$\inf_{\substack{u\geq 0,\\ u\in H^1(R^n)}} \frac{\int_{R^n} |\nabla u|^2 + \lambda \int_{R^n} u^2}{(\int_{R^n} u^4)^{\frac{1}{2}}}.$$

The rest follows from (1) of Lemma 1.

The next lemma is not so trivial.

Lemma 3. $\inf_{u\in N'_{\lambda,\mu}} I_{\lambda,\mu}[u]$ *is also attained only by* $w_{\lambda,\mu}$,
where

$$N'_{\lambda,\mu} = \left\{u \in H^1(R^n)\middle| \int_{R^n} |\nabla u|^2 + \lambda \int_{R^n} u^2 \leq \mu \int_{R^n} u^4\right\}. \tag{2.13}$$

Proof. Let u_k be a minimizing sequence and u_k^* be its Schwartz symmetrization. Then by the property of symmetrization

$$\int_{R^n} |\nabla u_k^*|^2 + \lambda \int_{R^n} (u_k^*)^2 \leq \int_{R^n} |\nabla u_k|^2 + \lambda \int_{R^n} u_k^2 \leq \mu \int_{R^n} u_k^4 = \mu \int_{R^n} (u_k^*)^4, \tag{2.14}$$

and

$$I_{\lambda,\mu}[u_k^*] \leq I_{\lambda,\mu}[u_k]. \tag{2.15}$$

Hence, we may assume that u_k is radially symmetric and decreasing. Since $u_k \in H^1(R^n)$, and u_k is strictly decreasing, it is well-known that

$$u_k(r) \leq Cr^{-\frac{N-1}{2}} \|u_k\|_{H^1}. \tag{2.16}$$

So $u_k \to u_0$(up to a subsequence) in $L^4(R^n)$, where u_0 is also radially symmetric and decreasing. Moreover, by Fatou's Lemma, $u_0 \in N'_{\lambda,\mu}$. Hence $\inf_{u\in N'_{\lambda,\mu}} I_{\lambda,\mu}[u]$ can be attained by u_0.

We then claim that

$$\int_{R^n} |\nabla u_0|^2 + \lambda \int_{R^n} u_0^2 = \mu \int_{R^n} u_0^4. \tag{2.17}$$

Suppose not. That is

$$\int_{R^n} |\nabla u_0|^2 + \lambda \int_{R^n} u_0^2 < \mu \int_{R^n} u_0^4.$$

Then $u_0 \in (N'_{\lambda,\mu})^0$ - the interior of $N'_{\lambda,\mu}$. By standard elliptic theory, u_0 is a critical point of $I_{\lambda,\mu}[u]$, i.e.

$$\nabla I_{\lambda,\mu}[u_0] = 0, \tag{2.18}$$

where "∇" means the derivative.

Multiplying (2.18) by u_0, we have $\int_{R^n} |\nabla u_0|^2 + \lambda \int_{R^n} u_0^2 = \mu \int_{R^n} u_0^4$ a contradiction. Hence $u_0 \in N_{\lambda,\mu}$. By Lemma 2, $u_0 = \sqrt{\frac{\lambda}{\mu}} w(\sqrt{\lambda}|y|) = w_{\lambda,\mu}(y)$.

We present another characterization of $w_{\lambda,\mu}$:

Lemma 4.

$$\inf_{u \in N_{\lambda,\mu}} I_{\lambda,\mu}[u] = \inf_{\substack{u \geq 0, \\ u \in H^1(R^n)}} \sup_{t>0} I_{\lambda,\mu}[tu].$$

Proof. This follows from a simple scaling.

Finally, we recall the following well-known result, whose proof can be found in Theorem 3.4 of [21].

Lemma 5. *Let $u \geq 0$, $v \geq 0$, $u, v \in H^1(R^n)$ and u^*, v^* be their Schwartz Symmetrization. Then*

$$\int_{R^n} uv \leq \int_{R^n} u^* v^*.$$

Our last lemma concerns some integrals.

Lemma 6. *Let $y_1 \neq y_2 \in R^n$. Then as $|y_1 - y_2| \to +\infty$, we have for $\lambda_1 < \lambda_2$,*

$$\int_{R^n} w^2_{\lambda_1,\mu_1}(y - y_1) w^2_{\lambda_2,\mu_2}(y - y_2) \sim w^2_{\lambda_1,\mu_1}(y_1 - y_2) \int_{R^n} w^2_{\lambda_2,\mu_2}(z) e^{2\sqrt{\lambda_1}\left\langle z, \frac{y_1 - y_2}{|y_1 - y_2|}\right\rangle} dz. \tag{2.19}$$

If $\lambda_1 = \lambda_2$, then

$$w^{2+\sigma}_{\lambda_1,\mu_1}(y_1 - y_2) \leq \int_{R^n} w^2_{\lambda_1,\mu_1}(y - y_1) w^2_{\lambda_2,\mu_2}(y - y_2) \leq w^{2-\sigma}_{\lambda_1,\mu_1}(y_1 - y_2) \tag{2.20}$$

for any $0 < \sigma < 1$.

Proof. Let $y = y_2 + z$. Then from (2.1), we have

$$\begin{aligned}
&w^2_{\lambda_1,\mu_1}(y - y_1) w^2_{\lambda_2,\mu_2}(y - y_2) \\
&= w^2_{\lambda_1,\mu_1}(y_2 - y_1 + z) w^2_{\lambda_2,\mu_2}(y - y_2) \\
&= w^2_{\lambda_1,\mu_1}(y_2 - y_1) e^{2\sqrt{\lambda_1}(|y_2 - y_1| - |y_2 - y_1 + z|)}(1 + o(1)) w^2_{\lambda_2,\mu_2}(y - y_2) \\
&= w^2_{\lambda_1,\mu_1}(y_1 - y_2)(1 + o(1)) e^{2\sqrt{\lambda_1}\left\langle z, \frac{y_1 - y_2}{|y_1 - y_2|}\right\rangle} w^2_{\lambda_2,\mu_2}(z)\,.
\end{aligned}$$

Hence by Lebesgue Dominated Convergence Theorem gives (2.19). The proof of (2.20) is similar.

3. Nehari's Manifold Approach

In this section, we consider the relation between two minimization problems

Problem 1.

$$c = \inf_{\mathbf{u}\in\mathbf{N}} E[\mathbf{u}], \tag{3.1}$$

where

$$\mathbf{N} = \left\{ u \in (H^1(R^n))^N \middle| \int_{R^n} |\nabla u_j|^2 + \lambda_j \int_{R^n} u_j^2 = \mu_j \int_{R^n} u_j^4 + \sum_{i\neq j} \beta_{ij} \int_{R^n} u_i^2 u_j^2,\ j = 1, \ldots, N \right\}.$$

Problem 2.

$$m = \inf_{\mathbf{u}\geq 0} \sup_{t_1,\ldots,t_N>0} E[\sqrt{t_1}u_1, \ldots, \sqrt{t_N}u_N]. \tag{3.2}$$

We have

Theorem 5. *Suppose either $\beta_{ij} < 0, \forall i \neq j$, or the matrix Σ defined by*

$$\Sigma = (|\beta_{ij}|) \ \ with \ \ \beta_{ii} = \mu_i$$

is positively definite. Then

$$c = \inf_{\mathbf{u}\in\mathbf{N}} E[\mathbf{u}] = m = \inf_{\mathbf{u}\geq 0} \sup_{t_1,\ldots,t_N>0} E[\sqrt{t_1}u_1, \ldots, \sqrt{t_N}u_N].$$

Proof. We consider the following function

$$\beta(t_1, \ldots, t_N) = E[\sqrt{t_1}u_1, \ldots, \sqrt{t_N}u_N].$$

First we assume $\mathbf{u} \in \mathbf{N}$.

Claim 1. $\beta(t_1, \ldots, t_N)$ attains its global maximum at $t_1 = \ldots = t_N = 1$. In fact,

$$\beta(t_1, \ldots, t_N) = \sum_{j=1}^{N} t_j \left[\int_{R^n} |\nabla u_j|^2 + \lambda_j u_j^2 \right] - Q[t_1, \ldots, t_N],$$

where

$$Q[t_1, \ldots, t_N] = \frac{1}{4} \sum_{j=1}^{N} \mu_j t_j^2 \int_{R^n} u_j^4 + \frac{1}{4} \sum_{\substack{i,j=1,\\ i\neq j}}^{N} \beta_{ij} t_i t_j \int_{R^n} u_i^2 u_j^2$$
$$= \frac{1}{4} \mathbf{t}^T \Sigma' \mathbf{t},$$

where $\mathbf{t} = (t_1, ..., t_N)^T$ and

$$\Sigma' = \left(\beta_{ij} \int_{R^n} u_i^2 u_j^2 \right). \tag{3.3}$$

If $\beta_{ij} < 0, \forall i \neq j$, then since $\mathbf{u} \in \mathbf{N}$, we have

$$\mu_j \int_{R^n} u_j^4 + \sum_{i \neq j} \beta_{ij} \int_{R^n} u_i^2 u_j^2 = \int_{R^n} |\nabla u_j|^2 + \lambda_j \int_{R^n} u_j^2 > 0\,.$$

Moreover, we see that the matrix Σ' is diagonally dominant and hence Σ' is positively definite.

If $\beta_{ij} > 0$ for all $i \neq j$, then for $t_j > 0$, $j = 1, \ldots, N$,

$$\begin{aligned} Q[t_1, \ldots, t_N] = & \frac{1}{4}\left[\sum_{i,j} \beta_{ij} t_i t_j \int_{R^n} u_i^2 u_j^2\right] \\ \ge & \frac{1}{4}\left[\sum_{j=1}^{N} \mu_j t_j^2 \int_{R^n} u_j^4\right] - \frac{1}{4} \sum_{\substack{i,j=1,\\ i\neq j}}^{N} |\beta_{ij}| \left(\int_{R^n} u_i^4\right)^{\frac{1}{2}} \left(\int_{R^n} u_j^4\right)^{\frac{1}{2}} t_i t_j \\ > & \ 0. \end{aligned}$$

Again, $Q[t_1, \ldots, t_N]$ is positively-definite. Thus $\beta(t_1, \ldots, t_N)$ is concave and hence there exists a unique critical point. Since $\mathbf{u} \in \mathbf{N}$, $(1 \ldots, 1)$ is a critical point. So we complete the proof of Claim 1.

From Claim 1, we deduce that

$$\inf_{\mathbf{u} \in \mathbf{N}} E[\mathbf{u}] \ge \inf_{\mathbf{u} \ge 0} \sup_{t_1, \ldots, t_N} E[\sqrt{t_1} u_1, \ldots, \sqrt{t_N} u_N]. \tag{3.4}$$

On the other hand, suppose that

$$\sup_{t_1, \ldots, t_N} E[\sqrt{t_1} u_1, \ldots, \sqrt{t_N} u_N] = \beta(t_1^0, \ldots, t_N^0) < +\infty\,,$$

where $\mathbf{u} = (u_1, \cdots, u_N) \ge 0$. Certainly, $(t_1^0, \ldots, t_N^0)$ is a critical point of $\beta(t_1, \ldots, t_N)$ and hence $(u_1^0, \ldots, u_N^0) \equiv \left(\sqrt{t_1^0} u_1, \ldots, \sqrt{t_N^0} u_N\right) \in \mathbf{N}$. So

$$E[u_1^0, \ldots, u_N^0] = \beta(t_1^0, \ldots, t_N^0) \ge \inf_{\mathbf{u} \in \mathbf{N}} E[\mathbf{u}]$$

which proves

$$c = \inf_{\mathbf{u} \in \mathbf{N}} E[\mathbf{u}] \le m = \inf_{\mathbf{u} \ge 0} \sup_{t_1, \ldots, t_N} E[\sqrt{t_1} u_1, \ldots, \sqrt{t_N} u_N]. \tag{3.5}$$

Combining (3.4) and (3.5), we obtain Theorem 5.

4. Proof of Theorem 1

In this section, we prove Theorem 1.

First by Theorem 5,

$$c = \inf_{\mathbf{u}\geq 0} \sup_{t_1,\dots,t_N} E[\sqrt{t_1}u_1, \dots, \sqrt{t_N}u_N].$$

Now we choose

$$u_j(y) := w_{\lambda_j,\mu_j}(y - jRe_1), \quad j = 1, \dots, N, \tag{4.1}$$

where $R >> 1$ is a large number and $e_1 = (1, 0, \dots, 0)^T$.

By choosing R large enough and applying Lemma 5, we obtain that

$$\begin{aligned}\int_{R^n} u_i^2 u_j^2 &= \int_{R^n} w^2_{\lambda_i,\mu_i}(y - iRe_1) w^2_{\lambda_j,\mu_j}(y - jRe_1)\\ &= \int_{R^n} w^2_{\lambda_i,\mu_i}(y) w^2_{\lambda_j,\mu_j}(y + (i - j)Re_1)dy \to 0\end{aligned}$$

as $R \to +\infty$.

Let $(t_1^R, \dots, t_N^R)$ be the critical point of $\beta(t_1, \dots, t_N)$. Then we have

$$\int_{R^n} |\bigtriangledown u_j|^2 + \lambda_j \int_{R^n} u_j^2 = \mu_j t_j^R \int_{R^n} u_j^4 + \sum_{i\neq j} \beta_{ij} t_i^R \int_{R^n} u_i^2 u_j^2$$

since the matrix $\left(\beta_{ij} \int_{R^n} u_i^2 u_j^2\right)$ is positively definite (similar to arguments in Sect. 3), by implicit function theorem

$$t_j^R = 1 + o(1).$$

Thus

$$c \leq \lim_{R\to+\infty} \beta(t_1^R, \dots, t_N^R) = \sum_{j=1}^{N} \left[\frac{1}{2}\left(\int_{R^n} |\bigtriangledown w_j|^2 + \lambda_j w_j^2\right) - \frac{\mu_j}{4} \int_{R^n} w_j^4\right]. \tag{4.2}$$

Next we claim that

$$c \geq \sum_{j=1}^{N} \left[\frac{1}{2}\left(\int_{R^n} |\bigtriangledown w_j|^2 + \lambda_j w_j^2\right) - \frac{\mu_j}{4} \int_{R^n} w_j^4\right]. \tag{4.3}$$

In fact, let $(u_1, \dots, u_N) \in \mathbf{N}$, then since $\beta_{ij} < 0, \forall i \neq j$,

$$\begin{aligned}E[u_1, \dots, u_N] &\geq \sum_{j=1}^{n} \left[\frac{1}{2}\left(\int_{R^n} |\bigtriangledown u_j|^2 + \lambda_j u_j^2\right) - \frac{\mu_j}{4} \int_{R^n} u_j^4\right] \qquad (4.4)\\ &= \sum_{j=1}^{n} I_{\lambda_j,\mu_j}[u_j],\end{aligned}$$

and

$$\int_{R^n} |\nabla u_j|^2 + \lambda_j u_j^2 \le \mu_j \int_{R^n} u_j^4 . \tag{4.5}$$

By Lemma 2,

$$\begin{aligned} E[u_1, \dots, u_N] &\ge \sum_{j=1}^{n} I_{\lambda_j,\mu_j}[u_j] \\ &\ge \sum_{j=1}^{n} \inf_{w \in \mathbf{N}'_{\lambda_j,\mu_j}} I_{\lambda_j,\mu_j}[w] \\ &= \sum_{j=1}^{N} I_{\lambda_j,\mu_j}[w_{\lambda_j,\mu_j}] \end{aligned} \tag{4.6}$$

which proves (4.3). Hence

$$c = \sum_{j=1}^{N} I_{\lambda_j,\mu_j}[w_{\lambda_j,\mu_j}]. \tag{4.7}$$

If c is attained by some $(u_1^0, \dots, u_N^0)$, then= $(u_1^0, \dots, u_N^0) \in \mathbf{N}$ and u_j^0 is a solution of (1.2). By the Maximum Principle, $u_j^0 > 0$, $j = 1, \dots, N$. Then we have

$$c = E[u_1^0, \dots, u_N^0] > \sum_{j=1}^{N} I_{\lambda_j,\mu_j}[u_j^0] \ge \sum_{j=1}^{N} I_{\lambda_j,\mu_j}[w_{\lambda_j,\mu_j}] \tag{4.8}$$

which contradicts (4.7), and we may complete the proof of Theorem 1.

5. Proof of Theorem 2

Now we prove Theorem 2 in this section. Our main idea is by Schwartz symmetrization.

For $u_j \ge 0, u_j \in H^1(R^n)$, we denote u_j^* as its Schwartz symmetrization. By Lemma 6, for $i \ne j$

$$\int_{R^n} u_i^2 u_j^2 \le \int_{R^n} (u_i^*)^2 (u_j^*)^2. \tag{5.1}$$

Hence

$$E[u_1^*, \dots, u_N^*] \le E[u_1, \dots, u_N]. \tag{5.2}$$

The new function $\mathbf{u}^* = (u_1^*, \dots, u_N^*)$ will satisfy the following inequalities:

$$\int_{R^n} |\nabla u_j^*|^2 + \lambda_1 \int_{R^n} (u_j^*)^2 - \sum_{i \ne j} \beta_{ij} \int_{R^n} (u_i^*)^2 (u_j^*)^2 \tag{5.3}$$

$$\le \mu_j \int_{R^n} (u_j^*)^4$$

(by (5.1) and the fact that $\beta_{ij} > 0$).

Therefore, we have

$$c = \inf_{\mathbf{u}\in\mathbf{N}} E[\mathbf{u}] \geq \inf_{\mathbf{u}\in\mathbf{N}'} E[\mathbf{u}] := c',$$

where

$$\mathbf{N}' = \left\{\mathbf{u} \in (H^1(R^n))^N \middle| \int_{R^n} |\nabla u_j|^2 + \lambda_j u_j^2 \right. \\ \left. \leq \mu_j \int_{R^n} u_j^4 + \sum_{i\neq j} \beta_{ij} \int_{R^n} u_i^2 u_j^2,\ j = 1, \ldots, N \right\}. \tag{5.4}$$

We first study c' and then we show that $c' = c$.

By the previous argument, we may assume any minimizing sequence $(u_1, \ldots, u_N)$ of c' must be radially symmetric and decreasing. We follow the proof of Lemma 2 to conclude that a minimizer for c' exists and must be radially symmetric and decreasing. Moreover, we have

$$\int_{R^n} |\nabla u_j|^2 + \lambda_j u_j^2 \leq \mu_j \int_{R^n} u_j^4 + \sum_{i\neq j} \beta_{ij} \int_{R^n} u_i^2 u_j^2, \quad j = 1, \ldots, N. \tag{5.5}$$

If all the inequalities of (5.5) are strict, then as for the proof of Lemma 2, we may have a contradiction. So we may assume at least one of (5.5) is an equality. Without loss of generality, we may assume that

$$G_j[u] := \int_{R^n} |\nabla u_j|^2 + \lambda_j \int_{R^n} u_j^2 - \mu_j \int_{R^n} u_j^4 \\ - \sum_{i\neq j} \beta_{ij} \int_{R^n} u_i^2 u_j^2 = 0, \quad j = 1, \ldots, k < N. \tag{5.6}$$

Then we have

$$\nabla E[u_1, \ldots, u_N] + \sum_{j=1}^{k} \Lambda_j \nabla G_j[u_1, \ldots, u_N] = 0, \tag{5.7}$$

where G_j is defined at (5.6). We assume that $\Lambda_{k+1} = \ldots = \Lambda_N = 0$ and we write (5.7) as

$$\nabla E[u_1, \ldots, u_N] + \sum_{j=1}^{N} \Lambda_j \nabla G_j[u_1, \ldots, u_N] = 0. \tag{5.8}$$

From (5.6), we obtain

$$\sum_{j=1}^{N} \Lambda_j \langle \nabla G_j, u_j \rangle = 0$$

which is equivalent to

$$\sum_{j=1}^{N} \left(\beta_{ij} \int_{R^n} u_i^2 u_j^2 \right) \Lambda_j = 0$$

since the matrix Σ' is positively define, the matrix

$$\left(\beta_{ij} \int_{R^n} u_i^2 u_j^2\right) \text{ is non-singular}$$

and hence $\Lambda_j = 0$, $j = 1, \ldots, N$. As for the proof of Lemma 1, $\mathbf{u} \in \mathbf{N}$. Hence $c = c'$ and c can be achieved by radially symmetric pairs $(u_1^0, \ldots, u_N^0)$. Hence $(u_1^0, \ldots, u_N^0)$ must satisfy (1.2).

By the maximum principle, $u_j^0 > 0$, since u_j^0 satisfies

$$\Delta u_j^0 - \lambda_j u_j^0 + \mu_j (u_j^0)^3 + \sum_{i \neq j} \beta_{ij} (u_i^0)^2 u_j^0 = 0, \quad \beta_{ij} > 0$$

by the moving plane method for cooperative systems (cf. [27]), u_j^0 must be radially symmetric and strictly decreasing. Therefore we may complete the proof of Theorem 2.

6. Proof of Theorem 3

In this section, we prove Theorem 3. The proof combines those of Theorem 1 and Theorem 2.

Assume $\mathbf{u} = (u_1, \ldots, u_N) \in \mathbf{N}$. Without loss of generality, we may assume that $i_0 = 1$.
Then

$$\beta_{1j} < 0, \forall j > 0, \quad \text{and} \quad \beta_{ij} > 0, \forall i > 1, j \notin \{1, i\}.$$

We may divide the energy $E[u_1, \ldots, u_N]$ into two parts

$$\begin{aligned} E[u_1, \ldots, u_N] = \frac{1}{2} \int_{R^n} |\nabla u_1|^2 + \frac{\lambda_1}{2} \int_{R^n} u_1^2 - \frac{\mu_1}{4} \int_{R^n} u_1^4 \\ -\frac{1}{2} \sum_{j=2}^{N} \beta_{1j} \int_{R^n} u_1^2 u_j^2 + E'[u_2, \ldots, u_N], \end{aligned} \tag{6.1}$$

where

$$\begin{aligned} E'[u_2, \ldots, u_N] = \sum_{j=2}^{N} \left(\frac{1}{2} \int_{R^n} |\nabla u_j|^2 + \frac{\lambda_j}{2} \int_{R^n} u_j^2 - \frac{\mu_j}{4} \int_{R^n} u_j^4 \right) \\ -\frac{1}{4} \sum_{\substack{i,j=2, \\ i \neq j}}^{N} \beta_{ij} \int_{R^n} u_i^2 u_j^2. \end{aligned} \tag{6.2}$$

Since $\beta_{1j} < 0$, for $j > 1$,

$$E[u_1, \ldots, u_N] \geq I_{\lambda_1, \mu_1}[u_1] + E'[u_2, \ldots, u_N]. \tag{6.3}$$

On the other hand, u_1 satisfies

$$\int_{R^n} |\nabla u_1|^2 + \lambda_1 \int_{R^n} u_1^2 - \mu_1 \int_{R^n} u_1^4 = \sum_{j=2}^{N} \beta_{1j} \int_{R^n} u_1^2 u_j^2 \leq 0 \tag{6.4}$$

and u_j, $j = 2, \ldots, N$ satisfies

$$\int_{R^n} |\nabla u_j|^2 + \lambda_1 \int_{R^n} u_j^2 \le \mu_j \int_{R^n} u_j^4 + \sum_{\substack{i=2,\\ i\neq j}}^{N} \beta_{ij} \int_{R^n} u_i^2 u_j^2. \tag{6.5}$$

Here we have used the system (1.2) and the fact that $\beta_{1j} < 0$, for $j > 1$. By the proof of Theorem 2,

$$E'[u_2, \ldots, u_N] \ge \inf_{(u_2,\ldots,u_N)\in \mathbf{N}_1} E'[u_2, \ldots, u_N] = c_1, \tag{6.6}$$

where

$$\mathbf{N}_1 = \Big\{ \mathbf{u}' = (u_2, \ldots, u_N) \Big| \int_{R^n} |\nabla u_j|^2 + \lambda_j \int_{R^n} u_j^2 = \mu_j \int_{R^n} u_j^4 + \sum_{\substack{i=2,\\ i\neq j}}^{N} \beta_{ij} \int_{R^n} u_i^2 u_j^2 \Big\}.$$

On the other hand, by Lemma 3,

$$I_{\lambda_1,\mu_1}[u_1] \ge I_{\lambda_1,\mu_1}[w_{\lambda_1,\mu_1}]. \tag{6.7}$$

Hence

$$\inf_{\mathbf{u}\in\mathbf{N}} E[\mathbf{u}] \ge I_{\lambda_1,\mu_1}[w_{\lambda_1,\mu_1}] + c_1. \tag{6.8}$$

Now we claim that

$$\inf_{\mathbf{u}\in\mathbf{N}} E[\mathbf{u}] = I_{\lambda_1,\mu_1}[w_{\lambda_1,\mu_1}] + c_1. \tag{6.9}$$

In fact, by Theorem 5,

$$c = \inf_{\mathbf{u}\in\mathbf{N}} E[\mathbf{u}] = \inf_{\mathbf{u}\ge 0} \sup_{t_1,\ldots,t_N\ge 0} E[\sqrt{t_1}u_1, \ldots, \sqrt{t_N}u_N].$$

Now we choose

$$u_1 = w_{\lambda_1,\mu_1}(y - Re_1)$$

and $u_j = u_j^0$ for $j \ge 2$, where $(u_2^0, \ldots, u_N^0)$ is a minimizer of c_1 at (6.6). Then

$$\int_{R^n} u_1^2 (u_j^0)^2 \to 0 \quad \text{as } R \to +\infty, \forall j > 1.$$

Thus if we set

$$\beta(t_1^R, \ldots, t_N^R) = \sup_{t_1,\ldots,t_N\ge 0} E[\sqrt{t_1}u_1, \ldots, \sqrt{t_N}u_N],$$

then $t_j^R = 1 + o(1)$ and

$$c \le \lim_{R\to+\infty} \beta(t_1^R, \ldots, t_N^R) = I_{\lambda_1,\mu_1}[w_{\lambda_1,\mu_1}] + c_1. \tag{6.10}$$

This, combined with (6.8), proves that

$$c = I_{\lambda_1,\mu_1}[w_{\lambda_1,\mu_1}] + c_1.$$

Finally, we show that c is not attained. In fact, if c is attained by some $(u_1^0, \dots, u_N^0)$, $u_j^0 > 0$, then

$$\begin{aligned} c = E[u_1^0, \dots, u_N^0] &> I_{\lambda_1,\mu_1}[u_1^0] + E'[u_2^0, \dots, u_N^0] \\ &\geq I_{\lambda_1,\mu_1}[w_{\lambda_1,\mu_1}] + c_1. \end{aligned}$$

A contradiction!

Remark 1. Theorem 3 also holds if β_{ij} satisfies

$$\beta_{ij} < 0, \quad \text{for } i = i_1, \dots, i_k, \, j \neq i_1, \dots, i_k$$

and

$$\beta_{ij} > 0, \quad \text{for } i \notin \{i_1, \dots, i_k\}, \, j \neq i.$$

7. Proof of Theorem 4

In this section, we construct non-radial bound state of (1.2) in the following case:

$$N = 3, \quad \lambda_1 = \lambda_2 = \lambda_3 = \mu_1 = \mu_2 = \mu_3 = 1, \tag{7.1}$$

$$\beta_{23} = \sqrt{\delta}\hat{\beta}_{23} < 0\,, \tag{7.2}$$

$$\beta_{12} = \delta\hat{\beta_{12}} = \beta_{13} = \delta\hat{\beta_{13}} > 0. \tag{7.3}$$

As we shall see, assumption (7.1) is not essential and it is just for simplification of our computations. The assumption (7.3) imposes some sort of symmetry which means that the role of u_2 and u_3 can be exchanged.

We shall make use of the so-called Liapunov-Schmidt reduction process and variational approach. The Liapunov-Schmidt reduction method was first used in nonlinear Schrödinger equations by Floer and Weinstein [13] in one-dimension, later was extended to higher dimension by Oh [25, 26]. Later it was refined and used in a lot of papers. See [2–5, 15, 16, 25, 26, 28, 29] and the references therein. A combination of the Liapunov-Schmidt reduction method and the variational principle was used in [3, 10, 11, 15] and [16]. Here we follow the approach used in [15].

Let us first introduce some notations: let

$$S_j[u] = \Delta u_j - u_j + u_j^3 + \sum_{i \neq j} \beta_{ij} u_i^2 u_j, \tag{7.4}$$

$$\mathbf{S[u]} = \begin{pmatrix} S_1[u] \\ \vdots \\ S_N[u] \end{pmatrix},$$

$$\begin{aligned}
\mathbb{X} =\ & (H^2(R^n) \cap \{u \mid u(x_1, x') = u(x_1, |x'|)\})^3 \\
& \cap \{(u_1, u_2, u_2) \mid u_2(x_1, x') = u_3(-x_1, x')\}, \\
\mathbb{Y} =\ & (L^2(R^n) \cap \{u \mid u(x_1, x') = u(x_1, |x'|)\})^3 \\
& \cap \{(u_1, u_2, u_2) \mid u_2(x_1, x') = u_3(-x_1, x')\}, \\
w^{R_j}(y) =\ & w\left(y - R_j e_1\right), \\
\mathbf{X} =\ & (H^2(R^n) \cap \{u \mid u(x_1, x') = u(x_1, |x'|)\})^3, \\
\mathbf{X}_0 =\ & \mathbf{X} \cap \{(u_1, u_2, u_3) \mid u_2(x_1, x') = u_3(-x_1, x'), u_1(x_1, x') = u_1(-x_1, x')\}, \\
\mathbf{Y} =\ & (L^2(R^n) \cap \{u \mid u(x_1, x') = u(x_1, |x'|)\})^3, \\
\mathbf{Y}_0 =\ & \mathbf{Y} \cap \{(u_1, u_2, u_3) \mid u_2(x_1, x') = u_3(-x_1, x'), u_1(x_1, x') = u_1(-x_1, x')\}.
\end{aligned} \tag{7.5}$$

Note that $\mathbf{S}[\mathbf{u}]$ is invariant under the map

$$\begin{aligned}
& \mathbf{T} : (u_1(x_1, x'), u_2(x_1, x'), u_3(x_1, x')) \\
& \to (u_1(-x_1, x'), u_3(-x_1, x'), u_2(-x_1, x')).
\end{aligned} \tag{7.6}$$

Thus $\mathbf{S}$ is map from $\mathbf{X}_0$ to $\mathbf{Y}_0$.

Fix $R \in \Lambda_\delta$, where

$$\Lambda_\delta = \{R \mid w(R) < \delta^{\frac{1}{4}-\sigma}\}. \tag{7.7}$$

Here we may choose

$$\sigma = \frac{1}{1000}.$$

We define

$$\begin{aligned}
\mathbf{u}^R :=\ & (w(y), w(y - Re_1), w(y + Re_1))^T \\
=\ & (w, w^R, w^{-R})^T.
\end{aligned} \tag{7.8}$$

We begin with

Lemma 7. *The map*

$$\mathbf{L}_0 \Phi = \begin{pmatrix} \Delta\phi_1 - \phi_1 + 3w^2\phi_1 \\ \Delta\phi_2 - \phi_2 + 3(w^R)^2\phi_2 \\ \Delta\phi_3 - \phi_3 + 3(w^{-R})^2\phi_3 \end{pmatrix} : \mathbf{X}_0 \to \mathbf{Y}_0 \tag{7.9}$$

has its kernel

$$\mathbf{K}_0 = span\left\{\left(0, \frac{\partial w^R}{\partial y_1}, -\frac{\partial w^{-R}}{\partial y_1}\right)^T\right\} \tag{7.10}$$

and cokernel

$$\mathbf{C}_0 = span\left\{\left(0, \frac{\partial w^R}{\partial y_1}, -\frac{\partial w^{-R}}{\partial y_1}\right)^T\right\}. \tag{7.11}$$

Proof. In fact, $\mathbf{L}_0\Phi = 0$. Then we have by Lemma 1 (2),

$$\phi_1 = \sum_{j=1}^{n} c_{1,j}\frac{\partial w}{\partial y_j}, \quad \phi_2 = \sum_{j=1}^{n} c_{2,j}\frac{\partial w^R}{\partial y_j}, \quad \phi_3 = \sum_{j=1}^{n} c_{3,j}\frac{\partial w^{-R}}{\partial y_j}. \tag{7.12}$$

Since $(\phi_1, \phi_2, \phi_3)^T \in \mathbf{X}_0$, we have $\phi_1(x_1, |x'|) = \phi_1(-x_1, |x'|) = \phi_1(x_1, x')$. This forces $\phi_1 = 0$. Similarly, we have $c_{2,2} = \cdots = c_{2,n} = 0, c_{3,2} = \cdots = c_{3,n} = 0$. On the other hand, $\phi_2(x_1, x^{'}) = \phi_3(-x_1, x^{'})$. So we have $c_{2,1} = -c_{3,1}$. This proves (7.10). Since $\mathbf{L}_0$ is a self-adjoint operator, (7.11) follows from (7.10).

From Lemma 7, we deduce that

Lemma 8. *The map*

$$\mathbf{L} := \mathbf{S}'[\mathbf{u}^R] \tag{7.13}$$

is uniformly invertible from

$$\mathbf{L} := \mathbf{K}_0^{\perp} \to \mathbf{C}_0^{\perp}. \tag{7.14}$$

Proof. We may write

$$\mathbf{L} = \mathbf{L}_0 + \sqrt{\delta}\mathbf{B}, \tag{7.15}$$

where $\mathbf{B}$ is a bounded and compact operator. Since $\mathbf{L}_0^{-1}$ exists, by standard perturbation theory, $\mathbf{L}$ is also invertible for δ sufficiently small.

Using Lemma 8, we derive the following proposition:

Proposition 1. *For δ sufficiently small, and $R \in \Lambda_\delta$, there exists a unique solution $\mathbf{v}^R = (v_1^R, v_2^R, v_3^R)$ such that*

$$S_1[\mathbf{u}^R + \mathbf{v}^R] = 0, \tag{7.16}$$

$$S_2[\mathbf{u}^R + \mathbf{v}^R] = c_R\frac{\partial w^R}{\partial y_1}, \tag{7.17}$$

$$S_3[\mathbf{u}^R + \mathbf{v}^R] = -c_R\frac{\partial w^{-R}}{\partial y_1}, \tag{7.18}$$

for some constant c_R. Moreover, $\mathbf{v}^R$ is of C^1 in R and we have

$$\|\mathbf{v}^R\|_{H^2(R^n)} \leq c\delta^{1-2\sigma}. \tag{7.19}$$

Proof. Let $R_1 = 0$, $R_2 = R$, $R_2 = -R$ and

$$w^{R_j} = w(y - R_j e_1).$$

We choose $v \in B$, where

$$B = \{\mathbf{v} \in \mathbf{X} \mid \|\mathbf{v}\|_{H^2} < \delta^{1-2\sigma}\} \tag{7.20}$$

and then expand

$$\begin{aligned} S_1[\mathbf{u}^R + \mathbf{v}] = &\ \Delta v_1 - v_1 + 3(w^{R_1})^2 v_1 + [(w^{R_1} + v_1)^3 - (w^{R_1})^3 - 3(w^{R_1})^2 v_1] \\ &+\delta[\hat{\beta}_{12}(w^{R_2} + v_2)^2 + \hat{\beta}_{13}(w^{R_3} + v_3)^2](w^{R_1} + v_1) \\ = &\ L_1 v_1 + H_1[v_1] + E_1, \end{aligned}$$

where

$$L_1 v_1 = \Delta v_1 - v_1 + 3(w^{R_1})^2 v_1,$$

$$E_1 = \delta[\hat{\beta}_{12}(w^{R_2} + v_2)^2 + \hat{\beta}_{13}(w^{R_3} + v_3)^2](w^{R_1} + v_1),$$

and

$$H_1[v_1] = [(w^{R_1} + v_1)^3 - (w^{R_1})^3 - 3(w^{R_1})^2 v_1] = O(|v_1|^2).$$

Here we have

$$\begin{aligned} E_1 &= O(\delta)(w^{R_2} w^{R_1} + w^{R_3} w^{R_1}) \\ &= O(\delta)(w(|R_1 - R_2|) + w(|R_1 - R_3|) \\ &= O(\delta^{\frac{5}{4} - \sigma}). \end{aligned} \tag{7.21}$$

Similarly,

$$S_2[\mathbf{u}^R + \mathbf{v}] = L_2 v_2 + H_2[v_2] + E_2,$$

$$S_3[\mathbf{u}^R + \mathbf{v}] = L_3 v_3 + H_3[v_3] + E_3,$$

where

$$\begin{aligned} L_2 v_2 &= \Delta v_2 - v_2 + 3(w^{R_2})^2 v_2, \\ L_3 v_3 &= \Delta v_3 - v_3 + 3(w^{R_3})^2 v_3, \\ E_2 &= O(1)[\delta\hat{\beta}_{12}(w^{R_1})^2 + \sqrt{\delta}\hat{\beta}_{23}(w^{R_3})^2]w^{R_2}, \\ &= O(\delta^{\frac{5}{4} - \sigma} + \delta^{1-\sigma}) = O(\delta^{1-\sigma}), \\ E_3 &= O(1)[\delta\hat{\beta}_{13}(w^{R_1})^2 + \sqrt{\delta}\hat{\beta}_{23}(w^{R_2})^2]w^{R_3} = O(\delta^{1-\sigma}), \\ H_2[v_2] &= [(w^{R_2} + v_2)^3 - (w^{R_2})^3 - 3(w^{R_2})^2 v_2] = O(|v_2|^2), \\ H_3[v_3] &= [(w^{R_3} + v_3)^3 - (w^{R_3})^3 - 3(w^{R_3})^2 v_3] = O(|v_3|^2). \end{aligned} \tag{7.22}$$

Since $\mathbf{L} : \mathbf{K}_0^{\perp} \to \mathbf{C}_0^{\perp}$ is invertible, solving (7.16)–(7.18) is equivalent to solving

$$\Pi \circ [\mathbf{L}\mathbf{v} + \mathbf{H}[\mathbf{v}] + \mathbf{E}] = 0, \quad \mathbf{v} \in \mathbf{K}_0^{\perp}, \tag{7.23}$$

where Π is the orthogonal projection on $\mathbf{C}_0^{\perp}$ and $\mathbf{v}=(v_1, v_2, v_3)^T$, $\mathbf{H}[\mathbf{v}]=(H_1, H_2, H_3)^T$, $\mathbf{E} = (E_1, E_2, E_3)^T$. Equation (7.23) can be written in the following form:

$$\mathbf{v} = G[\mathbf{v}] := (\Pi \circ \mathbf{L} \circ \Pi^{'})^{-1}[-\mathbf{H}[\mathbf{v}] - \mathbf{E}], \tag{7.24}$$

where $\Pi^{'}$ is the orthogonal projection on $\mathbf{K}_0^{\perp}$.

Since $\mathbf{H}[\mathbf{v}] = O(|\mathbf{v}|^2)$ and $\mathbf{E} = O(\delta^{1-\sigma})$, it is easy to see that the map G defined at (7.24) is a contraction map from B to B. By the contraction mapping theorem, (7.23) has a unique solution $\mathbf{v}^R = (v_1^R, v_2^R, v_3^R) \in \mathbf{K}_0^{\perp}$ with the property that

$$\begin{aligned} \|\mathbf{v}^R\|_{H^2(R^n)} &\leq C\|\mathbf{E}\|_{L^2(R^n)}^{1-\sigma} \\ &\leq C(\delta^{(1-\sigma)(1-\sigma)}) \\ &\leq C\delta^{1-2\sigma}. \end{aligned} \tag{7.25}$$

The C^1 property of $\mathbf{v}^R$ follows from the uniqueness of $\mathbf{v}^R$. See a similar proof in Lemma 3.5 of [15].

Now we let

$$M[R] = E[\mathbf{u}^R + \mathbf{v}^R] : \Lambda_\delta \to R^1,$$

where $\mathbf{v}^R$ is given by Proposition 1. We have

Lemma 9. *For $R \in \Lambda_\delta$ and δ sufficiently small, we have*

$$\begin{aligned} M[R] = &\; 3I[w] \\ &-\frac{1}{2}\left[\sqrt{\delta}\hat{\beta}_{23}\int_{R^n}(w^R)^2(w^{-R})^2 + 2\delta\hat{\beta}_{12}\int_{R^n} w^2(w^R)^2\right] \\ &+O\left(\delta^{\frac{3}{2}+\frac{\sigma}{2}}\right). \end{aligned} \tag{7.26}$$

Proof. We may calculate that

$$\begin{aligned} M[\mathbf{R}] = &\; E[\mathbf{u}^R + \mathbf{v}^R] \\ = &\; \sum_{j=1}^{3}\left\{\frac{1}{2}\left[\int_{R^n} |\nabla (w^{R_j} + v_j^R)|^2 + \int_{R^n}(w^{R_j} + v_j^R)^2\right] - \frac{1}{4}\int_{R^n}(w^{R_j} + v_j^R)^4\right\} \\ &-\frac{1}{4}\sum_{\substack{i,j \\ i\neq j}}\beta_{ij}\int_{R^n}(w^{R_i} + v_i^R)^2(w^{R_j} + v_j^R)^2 \\ = &\; E[\mathbf{u}^R] + \sum_{j=1}^{3}\frac{1}{2}\left[\int_{R^n}(|\nabla v_j^R|^2 + (v_j^R)^2) - 3(w^{R_j})^2(v_j^R)^2\right] \\ &-\frac{1}{2}\sum_{\substack{i,j \\ i\neq j}}\beta_{ij}\int_{R^n}\left[w^{R_i}v_i^R(w^{R_j})^2 + w^{R_j}v_j^R(w^{R_i})^2\right] + O(\delta^{2-4\sigma}) \\ = &\; E[\mathbf{u}^R] + O(\delta^{2-4\sigma}) - \left[\beta_{23}\left(\int_{R^n} v_2^R w^{R_2}(w^{R_3})^2 + \int_{R^n} v_3^R(w^{R_3})(w^{R_2})^2\right)\right] \\ = &\; E[\mathbf{u}^R] + O(\delta^{\frac{3}{2}+\frac{\sigma}{2}}). \end{aligned}$$

Here we have used the assumption (1.12), Eq. (1.13), and Proposition 1.

Since

$$E[\mathbf{u}^R] = 3I[w] - \frac{1}{2}\left[\sqrt{\delta}\hat{\beta}_{23}\int_{R^n}(w^{R_2})^2(w^{R_3})^2 \right.$$
$$\left. +\delta\hat{\beta}_{12}\int_{R^n}(w^{R_1})^2(w^{R_2})^2 + \delta\hat{\beta}_{13}\int_{R^n}(w^{R_1})^2(w^{R_3})^2\right],$$

and $\beta_{12} = \beta_{13}$, $R_2 = -R_3$, we obtain (7.26).

Next we have

Lemma 10. *If $R^\delta \in (\Lambda_\delta)^0$ – the interior of Λ_δ is a critical point of $M[R]$, then the corresponding solution $\mathbf{u}^\delta = \mathbf{u}^{R^\delta} + \mathbf{v}^{R^\delta}$ is a critical point of $E[\mathbf{u}]$.*

Proof. Since $R^\delta \in (\Lambda_\delta)^0$ – the interior of Λ_δ is a critical point of $M[R]$, we then have

$$\frac{d}{dR}M[R]\bigg|_{R=R^\delta} = 0$$

which is equivalent to

$$< \nabla E[\mathbf{u}^R + \mathbf{v}^R], \frac{d}{dR}(\mathbf{u}^R + \mathbf{v}^R) > \bigg|_{R=R^\delta} = 0.$$

Using Proposition 1, we obtain

$$c_R\int_{R^n}\frac{\partial w^R}{\partial y_1}\frac{d}{dR}(w^R + v_2^R) - c_R\int_{R^n}\frac{\partial w^{-R}}{\partial y_1}\frac{d}{dR}(w^{-R} + v_3^R) = 0 \tag{7.27}$$

for $R = R^\delta$. Note that since $\mathbf{v} \in \mathbf{K}_0^\perp$, we have

$$\int_{R^n}\left[\frac{\partial w^R}{\partial y_1}v_2^R - \frac{\partial w^{-R}}{\partial y_1}v_3^R\right] = 0. \tag{7.28}$$

Differentiating (7.28) with respect to R, we obtain that

$$\int_{R^n}\left[\frac{\partial w^R}{\partial y_1}\frac{d}{dR}v_2^R - \frac{\partial w^{-R}}{\partial y_1}\frac{d}{dR}v_3^R\right]$$

$$= -\int_{R^n}\left[\frac{\partial^2 w^R}{\partial R\partial y_1}v_2^R - \frac{\partial^2 w^{-R}}{\partial R\partial y_1}v_3^R\right] = O(\delta^{1-2\sigma}). \tag{7.29}$$

On the other hand, we see that

$$\int_{R^n}\left[\frac{\partial w^R}{\partial y_1}\frac{d}{dR}(w^R) - \frac{\partial w^{-R}}{\partial y_1}\frac{d}{dR}(w^{-R})\right] = -2\int_{R^n}\left(\frac{\partial w}{\partial y_1}\right)^2. \tag{7.30}$$

From (7.27), (7.29) and (7.30), we deduce that

$$c_R = 0, \text{ for } R = R^\delta, \tag{7.31}$$

which then implies that the corresponding solution $\mathbf{u}^\delta = \mathbf{u}^{R^\delta} + \mathbf{v}^{R^\delta}$ is a critical point of $E[\mathbf{u}]$.

Finally, we prove Theorem 4.

Proof of Theorem 4. We consider the following minimization problem:

$$M_0 = \min_{R \in \bar{\Lambda}_\delta} M[R], \tag{7.32}$$

since $M[R]$ is continuous and $\bar{\Lambda}_\delta$ is closed, $M[R]$ attains its minimum at a $R^\delta \in \bar{\Lambda}_\delta$.

We claim that $R^\delta \notin \partial\bar{\Lambda}_\delta$. Suppose not. That is $R^\delta \in \partial\bar{\Lambda}_\delta$. Then we have $w(R^\delta) = \delta^{\frac{1}{4}-\sigma}$.

Let

$$\rho(R) = \int_{R^n} w^2(y) w^2(y - Re_1). \tag{7.33}$$

Then from Lemma 9, we have

$$M[R] = 3I[w] - \frac{1}{2}\sqrt{\delta}\hat{\beta}_{23}\rho(2R) - \delta\hat{\beta}_{12}\rho(R) + O(\delta^{\frac{3}{2}+\frac{\sigma}{2}}). \tag{7.34}$$

By Lemma 6, $\rho(2R) \geq (\rho(R))^{2+\frac{\sigma}{4}}$ and $\rho(R) \geq w^2(R)$, we have for $R = R^\delta$,

$$\begin{aligned} -\sqrt{\delta}\hat{\beta}_{23}\rho(2R^\delta) - 2\delta\hat{\beta}_{12}\rho(R^\delta) &\geq -\sqrt{\delta}\hat{\beta}_{23}(\rho(R^\delta))^{2+\frac{\sigma}{4}} - 2\delta\hat{\beta}_{12}\rho(R^\delta) \\ &\geq \rho(R^\delta)\left[\sqrt{\delta}(-\hat{\beta}_{23})\rho^{1+\frac{\sigma}{4}}(R^\delta) - 2\delta\hat{\beta}_{12}\right] \\ &\geq \rho(R^\delta)\left[\sqrt{\delta}\delta^{(\frac{1}{2}-2\sigma)(1+\frac{\sigma}{4})}(-\hat{\beta}_{23}) - 2\delta\hat{\beta}_{12}\right] \\ &> 2\rho(R^\delta)\delta^{1-\sigma}, \end{aligned} \tag{7.35}$$

and thus by (7.34)

$$M[R^\delta] > 3I[w] + \rho(R^\delta)\delta^{1-\sigma}. \tag{7.36}$$

On the other hand, by choosing $\bar{R} \in \Lambda_\delta$ such that

$$\sqrt{\delta}\hat{\beta}_{23}\rho(2\bar{R}) + \delta\hat{\beta}_{13}\rho(\bar{R}) = 0, \tag{7.37}$$

then we have

$$M[R^\delta] \leq M[\bar{R}] \leq 3I[w] - \delta\hat{\beta}_{12}\rho(\bar{R}) + O(\delta^{\frac{3}{2}+\frac{\sigma}{2}}) \leq 3I[w], \tag{7.38}$$

a contradiction to (7.36).

It remains to show that (7.37) is possible since from (7.37) we have

$$w^2(\bar{R}) \leq \rho(\bar{R}) \leq C\delta^{\frac{1}{2}(1+\frac{\sigma}{4})^{-1}} < \delta^{\frac{1}{2}-2\sigma}$$

and hence it is possible to have $\bar{R} \in \Lambda_\delta$, where C is a positive constant depending only on $\hat{\beta}_{13}$ and $\hat{\beta}_{23}$. Here we have used the fact that

$$\rho(2R) \geq (\rho(R))^{2+\frac{\sigma}{4}} \quad \text{and} \quad \rho(R) \geq w^2(R).$$

This proves that $R^\delta \in (\bar{\Lambda}_\delta)^0$. So R^δ is a critical point of $M[R]$. By Lemma 10, $\mathbf{u}^\delta = \mathbf{u}^{R^\delta} + \mathbf{v}^{R^\delta}$ is a critical point of $E[\mathbf{u}]$ and hence a bound state of (1.2).

8. Proof of Corollary 1

In this section, we prove Corollary 1.

First, substituting $u_j = \sqrt{\xi_j} w$ into Eq. (1.2), we obtain the following algebraic equation

$$\xi_j + \sum_{i \neq j} \beta_{ij} \xi_i = 1, \quad j = 1, 2, 3. \tag{8.1}$$

Since by our assumption $|\beta_{ij}| << 1$, we see that solution to (8.1) exists and moreover, we have

$$\xi_j = 1 - \sum_{i \neq j} \beta_{ij} + O(\delta), \quad j = 1, 2, 3. \tag{8.2}$$

Now we compute

$$\begin{aligned} E[\mathbf{u}^r] &= \left[\sum_{j=1}^{3} \left(\frac{\xi_j}{2} - \frac{\xi_j^2}{4}\right) - \frac{1}{4} \sum_{\substack{i,j \\ i \neq j}} \beta_{ij} \xi_i \xi_j \right] \int_{R^n} w^4 \\ &\geq \frac{3}{4} \int_{R^n} w^4 - \frac{1}{4} \sum_{\substack{i,j \\ i \neq j}} \beta_{ij} \int_{R^n} w^4 + O(\delta) \\ &= \frac{3}{4} \int_{R^n} w^4 + \frac{1}{2} \sqrt{\delta} |\hat{\beta}_{23}| \int_{R^n} w^4 + O(\delta)\,. \end{aligned} \tag{8.3}$$

On the other hand, by (7.38), we have

$$E[\mathbf{u}^\delta] < \frac{3}{4} \int_{R^n} w^4. \tag{8.4}$$

From (8.3) and (8.4), we arrive at the following:

$$E[\mathbf{u}^\delta] < E[\mathbf{u}^r]. \tag{8.5}$$

Now if we have a ground state solution $\mathbf{u}_\delta$ which is radially symmetric, we have to show that for δ small, $\mathbf{u}_\delta = \mathbf{u}^r$. In fact, since $\mathbf{u}_\delta$ is a ground state solution, we have that $\mathbf{u}_\delta$ is uniformly bounded. Letting $\delta \to 0$, we see that $\mathbf{u}_\delta \to \mathbf{u}_0 = (w, w, w)^T$. Thus $\mathbf{u}_\delta - \mathbf{u}^r = o(1)$ as $\delta \to 0$.

To show that $\mathbf{u}_\delta = \mathbf{u}^r$, we let $\mathbf{u}_\delta = \mathbf{u}^r + \mathbf{v}_\delta$. Then it is easy to see that $\mathbf{v}_\delta$ satisfies

$$\Delta v_{\delta,j} - v_{\delta,j} + 3w^2 v_{\delta,j} + O(\sqrt{\delta} w |\mathbf{v}_\delta| + |\mathbf{v}_\delta|^2) = 0, \quad j = 1, 2, 3. \tag{8.6}$$

Since the operator $L_{1,1}$ is uniformly invertible in radially symmetric function class (by Lemma 1) and $\mathbf{v}_\delta = o(1)$, we see that

$$v_{\delta,j} = L_{1,1}^{-1} \circ O(\sqrt{\delta} w |\mathbf{v}_\delta| + |\mathbf{v}_\delta|^2) = O(\sqrt{\delta} w |\mathbf{v}_\delta| + |\mathbf{v}_\delta|^2), \quad j = 1, 2, 3,$$

and hence $\mathbf{v}_\delta = 0$ for δ small.

This proves Corollary 1. □

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