CLASSIFICATION OF BLOWUP LIMITS FOR $SU(3)$ SINGULAR TODA SYSTEMS

CHANG-SHOU LIN, JUNCHENG WEI, AND LEI ZHANG

ABSTRACT. We prove that for singular $SU(3)$ Toda systems, the weak limits of the energy belong to a finite set. For more general systems we prove a uniform estimate for fully blown-up solutions. Our method uses a selection process and a careful study of the interaction of bubbling solutions.

1. INTRODUCTION

Systems of elliptic equations in two dimensional spaces with exponential nonlinearity are very commonly observed in Physics, Geometry, Chemistry and Biology. In this article we consider the following general system of equations defined in $\mathbb{R}^2$:

$$\Delta u_i + \sum_{j\in I} a_{ij} h_j e^{u_j} = 4\pi\gamma_i\delta_0, \quad \text{in } B_1 \subset \mathbb{R}^2 \quad \text{for } i \in I, \tag{1.1}$$

where $I = \{1,...,n\}$, B_1 is the unit ball in $\mathbb{R}^2$, $h_1,...,h_n$ are smooth functions, $A = (a_{ij})_{n\times n}$ is a constant matrix, $\gamma_i > -1$, δ_0 is the Dirac mass at 0. If $n = 1$ and $a_{11} = 1$, the system (1.1) is reduced to a single Liouville equation, which has vast background in conformal geometry and Physics. The general system (1.1) is used for many models in different disciplines of science. If the coefficient matrix A is non-negative, symmetric and irreducible, (1.1) is called a Liouville system and is related to models in the theory of chemotaxis [12, 22], in the Physics of charged particle beams [2, 14, 23], and in the theory of semi-conductors [37]. See [6, 13, 30] and the reference therein for more applications of Liouville systems. If A is the Cartan matrix A_n:

$$A_n = \begin{pmatrix} 2 & -1 & 0 & ... & 0 \\ -1 & 2 & -1 & ... & 0 \\ 0 & -1 & 2 & & 0 \\ \vdots & \vdots & & \vdots & \\ 0 & \dots & -1 & 2 & -1 \\ 0 & \dots & & -1 & 2 \end{pmatrix},$$

the system (1.1) is called $SU(n+1)$ Toda system and is related to the non-abelian gauge in Chern-Simons theory, see [16, 17, 18, 24, 25, 34, 35, 36, 38, 39, 44, 45]

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and the reference therein. There are also many works on the relationship between $SU(n+1)$ Toda systems and holomorphic curves in $\mathbb{CP}^n$, flat $SU(n+1)$ connection, complete integrability and harmonic sequences, see [3, 4, 5, 11, 15, 19, 25, 28] for references.

After decades of extensive study, the Liouville equation, especially the blowup phemomenon of it, is well understood. However, the understanding of the more general systems (1.1) is not complete. In recent years, much progress has been obtained for two major extensions of the Liouville equation to systems. For Liouville systems, under some strong interaction assumption on A, Lin and Zhang [30, 31] completed a degree counting project for the Liouville system defined on Riemann surfaces. For Toda systems, Jost-Lin-Wang [20] proved some uniform estimates for $SU(3)$ Toda systems using holonomy theory and Lin-Wei-Zhao [29] improved the estimate of Jost-Lin-Wang using the non-degeneracy of the linearized Toda system, which is established among other things in Lin-Wei-Ye [28].

In this article we mainly focus on the the asymptotic behavior of blowup solutions of (1.1) and the weak limit of the energy sequence for $SU(n+1)$ Toda system. More specially let $u^k=(u_1^k,...,u_n^k)$ be a sequence of blowup solutions

$$\Delta u_i^k+\sum_{j=1}^n a_{ij}h_j^k e^{u_j^k}=4\pi\gamma_i^k\delta_0, \quad \text{in} \quad B_1, \quad i=1,...,n \tag{1.2}$$

with 0 being its only blowup point in B_1:

$$\max_{K\subset\subset B_1\setminus\{0\}} u_i^k\le C(K). \quad \max_{x\in B_1, i\in I}\{u_i^k(x)-2\gamma_i^k\log|x|\}\to\infty. \tag{1.3}$$

We assume that $\gamma_i^k\to\gamma_i>-1$, $h_1^k,...,h_n^k$ are positive smooth functions with a uniform bound on their C^3 norms:

$$\frac{1}{C}\le h_i^k\le C, \quad \|h_i^k\|_{C^3(B_1)}\le C, \quad \text{in } B_1, \quad \gamma_i^k\to\gamma_i>-1, \ \forall i\in I; \tag{1.4}$$

and we suppose that u^k has bounded oscillation on ∂B_1 and a uniform bound on its energy (we call $\int_{B_1}h_i^k e^{u_i^k}$ the energy of u_i^k):

$$|u_i^k(x)-u_i^k(y)|\le C, \quad \forall x,y\in\partial B_1, \qquad \int_{B_1}h_i^k e^{u_i^k}\le C, \quad i\in I, \tag{1.5}$$

where C is independent of k.

If $A=A_n$, system (1.2) describes $SU(n+1)$ with sources. Our first main theorem is concerned with the limits of the energy of u^k for $SU(3)$ with sources. Let

$$\sigma_i=\lim_{\delta\to 0}\lim_{k\to\infty}\frac{1}{2\pi}\int_{B_\delta}h_i^k e^{u_i^k}, \quad i=1,2. \tag{1.6}$$

Let

$$\Gamma=\{(\sigma_1,\sigma_2):\ \sigma_1,\sigma_2\ge 0,\ \sigma_1^2-\sigma_1\sigma_2+\sigma_2^2=2(1+\gamma_1)\sigma_1+2(1+\gamma_2)\sigma_2\}$$

then we define Σ as the following finite set of points on Γ:

Definition 1.1. *First we let the following points belong to* Σ*:*

$$(2(1+\gamma_1),0),\ (0,2(1+\gamma_2)),\ (2(1+\gamma_1),2(2+\gamma_1+\gamma_2)),$$
$$(2(2+\gamma_1+\gamma_2),2(1+\gamma_2)),\quad (2(2+\gamma_1+\gamma_2),2(2+\gamma_1+\gamma_2)).$$

Then we determine other points in Σ *as follows: For* $(a,b)\in\Sigma$ *or* $(a,b)=(0,0)$*, intersect* Γ *with* $\sigma_1=a+2N$ *and* $\sigma_2=b+2N$ *(*$N=0,1,2,...$*) and add the point(s) of intersection to E that belong to the upper right part of* (a,b)*. For each newly added member of* Σ*, such a procedure is applied to obtain possible new members.*

Remark 1.1. *We say* (c,d) *is in the upper right part of* (a,b) *if* $c\geq a$ *and* $d\geq b$*.*

Theorem 1.1. *Suppose (1.2), (1.3) and (1.5) hold for* u^k*,* h_i^k *and* γ_i^k *satisfy (1.4),* $A=A_2$*,* σ_i *is defined by (1.6), then* $(\sigma_1,\sigma_2)\in\Sigma$*, which is defined by Definition (1.1).*

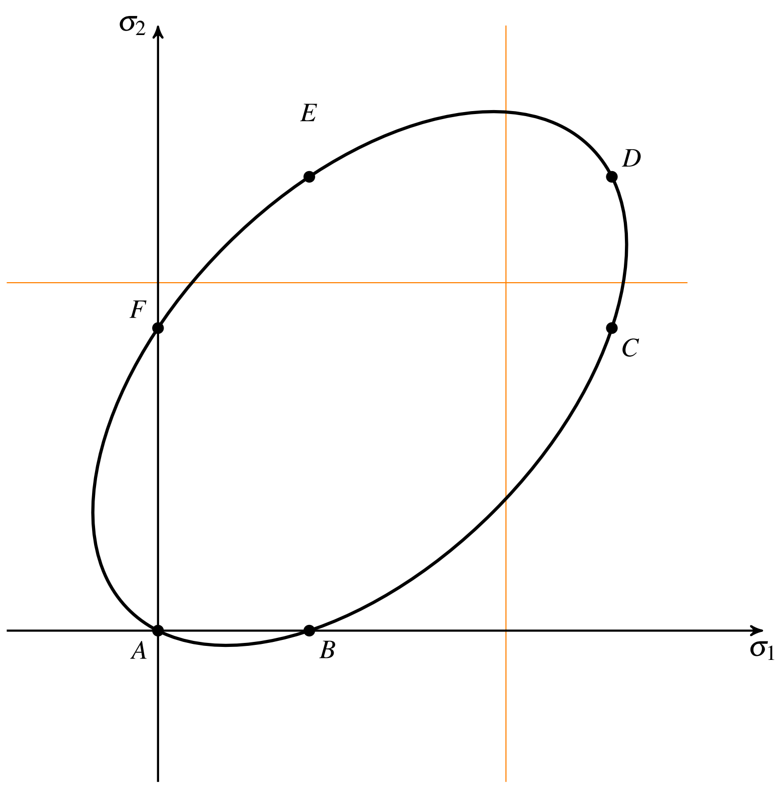
Remark 1.2. *As a special case when* $\gamma_1=\gamma_2=0$*,it is easy to see that*

$$\Sigma=\{(2,0),(0,2),(2,4),(4,2),(4,4)\}.$$

i.e. there is no extra points of intersection. These five types for $SU(3)$ *with no singularity were first discovered by Jost-Lin-Wang* [20]*. A recent work of Pistoia-Musso-Wei* [40] *proved that all cases are possible.*

The following picture indicates how the members of Σ are selected.

Γ



From now on we use $\mu_i = 1+\gamma_i$. In the picture above $A=(0,0)$, $B=(2\mu_1,0)$, $C=(2(\mu_1+\mu_2),2\mu_2)$, $D=(2(\mu_1+\mu_2),2(\mu_1+\mu_2))$, $E=(2\mu_1,2(\mu_1+\mu_2))$, $F=(0,2\mu_2)$. The two lines parallel to σ_1-axis and σ_2-axis, respectively represent $\sigma_1=2$, and $\sigma_2=2$, which are obtained by the increment of $(0,0)$ by 2 in both directions. The intersection of these two lines with Γ add three new members to Σ.

Remark 1.3. *It is easy to observe that the maximum value of σ_1 on Γ is*

$$\frac{4}{3}\mu_1+\frac{2}{3}\mu_2+\frac{4}{3}\sqrt{\mu_1^2+\mu_1\mu_2+\mu_2^2}.$$

The maximum value of σ_2 is

$$\frac{2}{3}\mu_1+\frac{4}{3}\mu_2+\frac{4}{3}\sqrt{\mu_1^2+\mu_1\mu_2+\mu_2^2}.$$

Thus Σ is a finite set. As two special cases, we see that

(1) *If*

$$\frac{4}{3}\mu_1+\frac{2}{3}\mu_2+\frac{4}{3}\sqrt{\mu_1^2+\mu_1\mu_2+\mu_2^2}<2, \quad \textit{and}$$
$$\frac{2}{3}\mu_1/3+\frac{4}{3}\mu_2+\frac{4}{3}\sqrt{\mu_1^2+\mu_1\mu_2+\mu_2^2}<2.$$

then there are only five points in Σ:

$$\Sigma = \Big\{(2\mu_1,0),(0,2\mu_2),(2(\mu_1+\mu_2),2\mu_2), (2\mu_1,2(\mu_1+\mu_2)),(2(\mu_1+\mu_2),2(\mu_1+\mu_2))\Big\}.$$

(2) *For $\gamma_1=\gamma_2=1$, in addition to $(4,0),(0,4),(4,8),(8,4),(8,8)$, Σ has other 14 points.*

Theorem 1.1 manifests drastic differences between $SU(n+1)$ Toda systems and Liouville systems. While the weak energy limits are discrete for $SU(3)$ Toda systems, it is a very different story for Liouville systems. One can see from the work of Chipot-Shafrir-Wolansky [13] and Lin-Zhang [31] that the energy limits for blowup solutions form a hyper-surface. This phenomenon of continuum of energy brings essential difficulties in blowup analysis for Liouville systems [32, 33]. On the other hand, since the energy limits for $SU(3)$ Toda systems are discrete, one can obtain very precise energy estimates for fully blown-up sequences and the asymptotic behavior of blowup solutions.

In the proof Theorem 1.1 we shall introduce a selection process suitable for describing bubbling solutions of general systems. The selection process has been widely used for prescribing curvature type equations (see ([42], [27], [8], etc) and we modify it for general systems of n equations to locate the bubbling area which consists of finite disks. In each of the disks, the blowup solutions have roughly the energy of a globally defined system. If the system has n equations, we say the disk has a fully blown-up picture. Otherwise we call it a partially blown-up picture. We shall study the interaction of these bubbling disks. After locating the bubbling area, we shall prove a Harnack type result (Lemma 2.1) to show that away from the bubbling disks the oscillation of blowup solutions is very small. Next in each bubbling disk we prove that the energy of at least one component can be precisely determined. Once we know the energy limits around each blowup area, we combine areas closest to each other. In this article we use the term "group" to describe bubbling disks closest to each other and relatively far away from other disks. The key idea in this part is to consider a group as one blowup point with singular sources. Since we have only finite bubbling disks, this procedure of putting bubbles in groups only needs to be done finite times.

The organization of the article is as follows. In section two we introduce the selection process mentioned before and in section three we prove a Pohozaev identity crucial for the proof of Theorem 1.1. Note that the selection process requires little on the coefficient matrix A for the system of blowup solutions. The Pohozaev identity only requires A to be symmetric and invertible. Then in Section four we prove a uniform estimate for fully blowup solutions (Theorem 4.2 and Theorem 4.1). The main results in this section are of independent interest and extend some previous works of Jost-Lin-Wang [20] and Lin-Zhang [30]. Then in section five and section six we finish the proof of Theorem 1.1 according to the idea mentioned before.

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2. A SELECTION PROCESS FOR SYSTEMS OF n EQUATIONS

In this section we describe a selection process for (1.2) that plays a crucial role in the proof of Theorem 1.1. The selection process does not require the coefficient matrix A to be invertible or symmetric. Even though the selection process for the system with or without singular sources is almost the same. We present it for the case without singular sources ($\gamma_i^k = 0, \forall i$) first for simplicity.

2.1. Case one: $\gamma_1^k = ... = \gamma_n^k = 0$.

Let $A = (a_{ij})_{n\times n}$ be a real valued matrix (note that we do not require A to be invertible or symmetric). Let $u^k = (u_1^k, .., u_n^k)$ be a sequence of solutions to (1.2) with $\gamma_1^k = .. = \gamma_n^k = 0$ such that (1.5) and (1.3) hold. Also we assume that (1.4) holds for h_i^k.

Proposition 2.1. *Let A be $n\times n$ real valued matrix, $u^k = (u_1^k, .., u_n^k)$ be a sequence of blowup solutions to (1.2) (without sources) that satisfy (1.5) and (1.3). Suppose h^k satisfies (1.4), then there exist finite sequences of points $\Sigma_k := \{x_1^k, ..., x_m^k\}$ (all $x_j^k \to 0, j = 1, ..., m$) and positive numbers $l_1^k, ..., l_m^k \to 0$ such that*

(1) $\max_{i\in I}\{u_i^k(x_j^k)\} = \max_{B(x_j^k, l_j^k), i\in I}\{u_i^k\}$ *for all* $j = 1, .., m$.
(2) $exp(\frac{1}{2}\max_{i\in I}\{u_i^k(x_j^k)\}) l_j^k \to \infty, \quad j = 1, ..., m$.
(3) *In each* $B(x_j^k, l_j^k)$ *one of the following two alternatives holds*
(a): There exists $J \subsetneq I$ *such that for all* $j \in J$, v_j^k *converge to* v_j *in* $C^2_{loc}(\mathbb{R}^n)$,

and for $j\in I\setminus J$, v_j^k tends to minus infinity over all compact subsets of $\mathbb{R}^2$. Thus the limit is a system of $|J|$ equations. For all $i\in J$,

$$\lim_{k\to\infty}\int_{B(x_j^k,l_j^k)}\sum_{t\in J}a_{it}h_t^ke^{v_t^k}>4\pi.$$

(b): $(v_1^k,...,v_n^k)$ *converge in* $C^2_{loc}(\mathbb{R}^2)$ *to* $(v_1,...,v_n)$ *which satisfies*

$$\Delta v_i+\sum_{j\in I}a_{ij}h_je^{v_j}=0,\quad \mathbb{R}^2,\quad i\in I.$$

$$\lim_{k\to\infty}\int_{B(x_j^k,l_j^k)}\sum_{t\in I}a_{it}h_t^ke^{u_t^k}>4\pi,\quad i\in I.$$

(4) *There exists $C_1>0$ independent of k such that*

$$u_i^k(x)+2\log\ dist(x,\Sigma_k)\le C_1,\quad \forall x\in B_1,\quad i\in I$$

where dist stands for distance.

Remark 2.1. *In this article we don't distinguish sequence and subsequence.*

Remark 2.2. *For each $x_j^k\in\Sigma_k$ suppose $2t_j^k$ is the distance from x_j^k to $\Sigma_k\setminus\{x_j^k\}$. Then $t_j^k/l_j^k\to\infty$ as $k\to\infty$ if l_j^k is suitably chosen.*

Proof of Proposition 2.1:

Without loss of generality we assume

$$u_1^k(x_1^k)=\max_{i\in I,x\in B_1}u_i^k(x).$$

Clearly $x_1^k\to 0$. Let $(\tilde v_1^k,..,\tilde v_n^k)$ be the re-scaled functions of u^k:

$$\tilde v_i^k(y)=u_i^k(e^{-\frac12 u_1^k(x_1^k)}y+x_1^k)-u_1^k(x_1^k),\quad i\in I,\quad e^{-\frac12 u_1^k(x_1^k)}y+x_1^k\in B_1.$$

Since $\tilde v_i^k\le 0$, we have $|\Delta\tilde v_i^k|$ is bounded. By standard estimates in elliptic PDEs, $|\tilde v_i^k(z)-\tilde v_i^k(0)|$ is uniformly bounded in any compact subset of $\mathbb{R}^2$. Thus, by $\tilde v_1^k(0)=0$, at least $\tilde v_1^k$ converges in $C^2_{loc}(\mathbb{R}^2)$ to a function v_1. For other components of $\tilde v^k=(\tilde v_1^k,..,\tilde v_n^k)$, either some of them tend to minus infinity over all compact subsets of $\mathbb{R}^2$, or all of them converge to a system of n equations. Let $J\subset I$ be the set of indices corresponding to those convergent components. That is, for all $i\in J$, $\tilde v_i^k$ converges to v_i in $C^2_{loc}(\mathbb{R}^2)$ and for all $j\in I\setminus J$, $\tilde v_i^k$ tends to minus infinity over all compact subsets of $\mathbb{R}^2$. The limit of $\tilde v^k$ is the following system:

$$\Delta v_i+\sum_{j\in J}a_{ij}h_je^{v_j}=0,\quad \text{in }\mathbb{R}^2,\quad \forall i\in J \tag{2.1}$$

where $h_j=\lim_{k\to\infty}h_j^k(x_j^k)$. From (1.5) we have

$$\int_{\mathbb{R}^2}h_je^{v_j}<C,\quad j\in J.$$

Therefore by standard elliptic estimate (see [30])

$$\sum_{j\in J}\int_{\mathbb{R}^2}a_{ij}h_je^{v_j}>4\pi,\quad i\in J \tag{2.2}$$

and

$$v_j(x) = -\frac{1}{2\pi}(\int_{\mathbb{R}^2}\sum_{j\in J} a_{ij}h_j e^{v_j})\log|x| + O(1), \quad |x|>2, \quad i\in J.$$

Thus for any index $i\in I$ we can find $R_k\to\infty$ such that

$$\tilde{v}_i^k(y) + 2\log|y| \le C, \quad |y|\le R_k, \quad \text{for } i\in I. \tag{2.3}$$

Equivalently for u^k there exist $l_1^k\to 0$ such that

$$u_i^k(x) + 2\log|x-x_1^k| \le C, \quad |x|\le l_1^k, \quad \text{for } \ i\in I$$

and

$$e^{\frac{1}{2}u_1^k(x_i^k)}l_1^k \to\infty, \quad i\in J, \quad \text{as } \ k\to\infty.$$

Next we let q_k be the maximum point of $\max_{|x|<1, i\in I} u_i^k(x) + 2\log|x-x_1^k|$. If

$$\max_{|x|\le 1, i\in I} u_i^k(x) + 2\log|x-x_1^k| \to\infty,$$

we let j be the index such that

$$u_j^k(q_k) + 2\log|q_k - x_1^k| = \max_{i\in I} u_i^k(x) + 2\log|x-x_1^k| \to\infty.$$

Set

$$d_k = \frac{1}{2}|q_k - x_1^k|$$

and

$$S_i^k(x) = u_i^k(x) + 2\log\left(d_k - |x-q_k|\right) \text{in} \quad B(q_k,d_k).$$

Then clearly $S_i^k\to -\infty$ as x tends to $\partial B(q_k,d_k)$. On the other hand, at least for the index j we have

$$S_j^k(q_k) = u_j^k(q_k) + 2\log d_k \to\infty.$$

Let p_k be where

$$\max_i \max_{x\in\bar{B}(q_k,d_k)} S_i^k$$

is attained. Let i_0 be the index corresponding to where the the maximum is taken:

$$u_{i_0}^k(p_k) + 2\log\left(d_k - |p_k - q_k|\right) \ge S_j^k(q_k)\to\infty. \tag{2.4}$$

Let

$$l_k = \frac{1}{2}(d_k - |p_k - q_k|).$$

Then for $y\in B(p_k,l_k)$, by the choice of p_k and l_k,

$$u_i^k(y) + 2\log(d_k - |y-q_k|) \le u_{i_0}^k(p_k) + 2\log(2l_k).$$

On the other hand, by the definition of l_k we have

$$d_k - |y-q_k| \ge d_k - |p_k - q_k| - |y-p_k| \ge l_k.$$

Thus

$$u_i^k(y) \le u_{i_0}^k(p_k) + 2\log 2, \quad \forall y\in B(p_k,l_k).$$

Let

$$R_k = e^{\frac{1}{2}u_{i_0}^k(p_k)}l_k$$

and

$$\tilde{v}_i^k(y) = u_i^k(p_k + e^{-\frac{1}{2}u_{i_0}^k(p_k)}y) - u_{i_0}^k(p_k), \quad \text{for } i \in I.$$

By (2.4), $R_k \to \infty$. Obviously $|\Delta \tilde{v}_i^k|$ is bounded in $B(0,R_k)$. By the similar method of the above, there exists $\emptyset \neq J \subset I$ such that for all $i \in J$, $\tilde{v}_i^k$ converges to the limit system (2.1). On the other hand $\tilde{v}_i^k$ converges uniformly to $-\infty$ over all compact subsets of $\mathbb{R}^2$ for all $i \in I \setminus J$. Clearly (2.3) holds. Going back to u^k we have

$$u_i^k(x) + 2\log|x - x_2^k| \le C, \text{ for } |x - x_2^k| \le l_2^k$$

where $x_2^k = p_k$, $l_2^k = l_k$. Clearly $B(x_1^k, l_1^k) \cap B(x_2^k, l_2^k) = \emptyset$.

To continue the selection process, we let $\Sigma_{k,2} := \{x_1^k, x_2^k\}$ and consider

$$\max_{i\in I, x\in B_1} u_i^k(x) + 2\log dist(x, \Sigma_{k,2}).$$

If along a subsequence, the quantity above tends to infinity we apply the same procedure to get x_3^k and l_3^k. Since after each selection of bubbles we contribute a positive energy and we have a uniform bound on the energy (1.5), the process stops after finite steps. Eventually we let

$$\Sigma_k = \{x_1^k, ..., x_m^k\}$$

and it holds

$$u_i^k(x) + 2\log d(x, \Sigma_k) \le C, \quad i \in I. \tag{2.5}$$

Proposition 2.1 is established. □

2.2. **Case two: Singular case $\exists \gamma_i \neq 0$.** First the selection process is almost the same. The difference is instead of taking the maximum of u_i^k over B_1 we let $0 \in \Sigma_k$. Then we consider the maximum of $u_i^k(x) + 2\log dist(x, \Sigma_k) = u_i^k(x) + 2\log|x|$ and the selection goes the same as before. Therefore in the singular case $\Sigma_k = \{0, x_1^k, ..., x_m^k\}$.

Lemma 2.1. *Let Σ_k be the blowup set (Thus if all $\gamma_i^k = 0$, $\Sigma_k = \{x_1^k, ..., x_m^k\}$, if the system is singular, $\Sigma_k = \{0, x_1^k, ..., x_m^k\}$). In either case for all $x_0 \in B_1 \setminus \Sigma_k$, there exists C_0 independent of x_0 and k such that*

$$|u_i^k(x_1) - u_i^k(x_2)| \le C_0, \quad \forall x_1, x_2 \in B(x_0, d(x_0, \Sigma_k)/2), \textit{for all } i \in I.$$

Proof of Lemma 2.1: We can assume $|x| < \frac{1}{10}$ because it is easy to see from the Green's representation formula that the oscillation of u_i^k on $B_1 \setminus B_{1/10}$ is finite. Let

$$\tilde{u}_i^k(x) = u_i^k(x) - 2\gamma_i^k \log|x|$$

Then in B_1 we have

$$\Delta \tilde{u}_i^k(x) + \sum_{j\in I} a_{ij} h_j^k(x) |x|^{2\gamma_j^k} e^{\tilde{u}_j^k(x)} = 0, \quad B_1, \quad i \in I.$$

Let σ_k be the distance between x_0 and Σ_k. Clearly, for $x_0 \in B_1 \setminus \Sigma_k$ and $x_1, x_2 \in B(x_0, d(x_0, \Sigma_k)/2)$,

$$\begin{aligned} & u_i^k(x_1) - u_i^k(x_2) \\ = \; & \tilde{u}_i^k(x_1) - \tilde{u}_i^k(x_2) + O(1) \\ = \; & \int_{B_1} (G(x_1, \eta) - G(x_2, \eta)) \sum_{j \in I} a_{ij} h_j^k(\eta) |\eta|^{2\gamma_j^k} e^{\tilde{u}_j^k(\eta)} d\eta + O(1). \end{aligned}$$

Here G is the Green's function on B_1. The last term on the above is $O(1)$ because it is the difference of two points of a harmonic function that has bounded oscillation on ∂B_1. Since both $x_1, x_2 \in B_{1/10}$, it is easy to use the uniform bound on the energy (1.5) to obtain

$$\int_{B_1} (\gamma(x_1, \eta) - \gamma(x_2, \eta)) \sum_{j \in I} a_{ij} h_j^k(\eta) |\eta|^{2\gamma_j^k} e^{\tilde{u}_j^k(\eta)} d\eta = O(1)$$

where $\gamma(\cdot, \cdot)$ the regular part of G. Therefore we only need to show

$$\int_{B_1} \log \frac{|x_1 - \eta|}{|x_2 - \eta|} \sum_j a_{ij} h_j^k |\eta|^{2\gamma_j} e^{\tilde{u}_j} d\eta = O(1).$$

If $\eta \in B_1 \setminus B(x_0, \frac{3}{4}\sigma_k)$, we have $\log(|x_1 - \eta|/|x_2 - \eta|) = O(1)$, then the integration over $B_1 \setminus B(x_0, \frac{3}{4}\sigma_k)$ is uniformly bounded. Therefore we only need to show

$$\begin{aligned} & \int_{B(x_0, \frac{3}{4}\sigma_k)} \log \frac{|x_1 - \eta|}{|x_2 - \eta|} \sum_j a_{ij} h_j^k |\eta|^{2\gamma_j} e^{\tilde{u}_j^k} d\eta \\ = \; & \int_{B(x_0, \frac{3}{4}\sigma_k)} \log \frac{|x_1 - \eta|}{|x_2 - \eta|} \sum_j a_{ij} h_j^k e^{u_j^k} d\eta = O(1). \end{aligned}$$

To this end, let

$$v_i^k(y) = u_i^k(x_0 + \sigma_k y) + 2\log \sigma_k, \quad i \in I, \quad y \in B_{3/4}. \tag{2.6}$$

Then we just need to show

$$\int_{B_{3/4}} \log \frac{|y_1 - \eta|}{|y_2 - \eta|} \sum_j a_{ij} h_j^k(x_0 + \sigma_k \eta) e^{v_j^k(\eta)} d\eta = O(1). \tag{2.7}$$

We assume, without loss of generality that e_1 is the image of the closest blowup point in Σ_k. Thus by the selection process

$$\tilde{v}_i^k(\eta) \le -2\log|\eta - e_1| + C.$$

Therefore

$$e^{\tilde{v}_i^k(\eta)} \le C|\eta - e_1|^{-2}.$$

With this estimate we observe that $|\eta - e_1| \ge C > 0$ for $\eta \in B_{3/4}$. Thus for $j = 1, 2$ and any fixed $i \in I$,

$$\int_{B_{3/4}} \left| \log|y_j - \eta| \right| e^{v_i^k(\eta)} d\eta \le C \int_{B_{3/4}} \frac{\left| \log|y_j - \eta| \right|}{|\eta - e_1|^2} d\eta \le C.$$

Lemma 2.1 is established. □

3. POHOZAEV IDENTITY AND RELATED ESTIMATES ON THE ENERGY

In this section we derive a Pohozaev identity for u^k satisfying (1.2),(1.5), (1.3), h_i^k and γ_i^k satisfying (1.4). For the matrix A we only require it to be symmetric and invertible.

Proposition 3.1. *Let A by a symmetric and invertible matrix, σ_i be defined by (1.6). Suppose $u^k=(u_1^k,...,u_n^k)$ satisfy (1.2), (1.5) and (1.3), h^k and γ_i^k satisfy (1.4). Then we have*

$$\sum_{i,j\in I} a_{ij}\sigma_i\sigma_j = 4\sum_{i=1}^n (1+\gamma_i)\sigma_i.$$

Proof of Proposition 3.1:

The following lemma is important for evaluating different terms in the Pohozaev identity later.

Lemma 3.1. *Given any $\varepsilon_k\to 0$ such that $\Sigma_k\subset B(0,\varepsilon_k/2)$, there exist $l_k\to 0$ satisfying $l_k\ge 2\varepsilon_k$ and*

$$\bar u_i^k(l_k)+2\log l_k\to -\infty, \text{ for all } i\in I, \text{ where } \bar u_i^k(r):=\frac{1}{2\pi r}\int_{\partial B_r} u_i^k. \tag{3.1}$$

Proof of Lemma 3.1: Since $\Sigma_k\subset B(0,\varepsilon_k/2)$, we have, by the fourth statement of Proposition 2.1,

$$u_i^k(x)+2\log|x|\le C,\quad |x|\ge\varepsilon_k. \tag{3.2}$$

Given $\varepsilon_{k,1}\ge\varepsilon_k$ that tends to 0 as well, then for each fixed i there exists $r_{k,i}\ge\varepsilon_{k,1}$ such that

$$u_i^k(x)+2\log r_{k,i}\to-\infty,\quad \forall x\in\partial B_{r_{k,i}},\quad i=1,...,n. \tag{3.3}$$

In fact, otherwise we have some $\varepsilon_{k,1}\to 0$ with $\varepsilon_{k,1}\ge\varepsilon_k$ such that for all $r\ge\varepsilon_{k,1}$,

$$\sup_{x\in\partial B_r}(u_i^k(x)+2\log|x|)\ge -C$$

for some $C>0$. By Lemma 2.1 u_i^k has bounded oscillation on each ∂B_r. Thus

$$u_i^k(x)+2\log|x|\ge -C$$

for some C and all $x\in\partial B_r$, $r\ge\varepsilon_{k,1}$, and then

$$e^{u_i^k(x)}\ge C|x|^{-2},\quad \varepsilon_{k,1}\le|x|\le 1.$$

Integrating $e^{u_i^k}$ on $B_1\setminus B_{\varepsilon_{k,1}}$ we get a contradiction on the uniform energy bound of $\int_{B_1} h_i^k e^{u_i^k}$. (3.3) is established.

First for u_1^k, we find $\tilde r_k\ge\varepsilon_k$ so that

$$\bar u_1^k(\tilde r_k)+2\log\tilde r_k\to-\infty.$$

Then by Lemma 2.1, if $\tilde r_k$ does not approach 0, we can find $\hat r_k\to 0$ such that

$$\bar u_1^k(r)+2\log r\to-\infty\quad \hat r_k\le r\le\tilde r_k. \tag{3.4}$$

Indeed, suppose
$$u_1^k(x)+2\log|x|\le -N_k,\quad |x|=\tilde r_k$$
for some $N_k\to\infty$. Then by Lemma 2.1,
$$u_1^k(x)+2\log|x|\le -N_k+C,\quad \tilde r_k/2<|x|<\tilde r_k,$$
$$u_1^k(x)+2\log|x|\le -N_k+2C,\quad \tilde r_k/4<|x|<\tilde r_k/2.$$
Then it is easy to see that $\hat r_k$ can be found so that $\hat r_k/\tilde r_k\to 0$ and (3.4) holds. If $\tilde r_k\to 0$, we can use Lemma 2.1 to find $s_k>\tilde r_k$ and $s_k\to 0$ and $s_k/\tilde r_k\to\infty$ such that
$$\bar u_1^k(r)+2\log r\to -\infty,\quad \tilde r_k\le r\le s_k.$$
Then between $\hat r_k$ and $\tilde r_k$ in the first case, or $\tilde r_k$ and s_k in the second case, there is a r_k such that
$$\bar u_2^k(r_k)+2\log r_k\to -\infty,$$
because otherwise we would have
$$\bar u_2^k(r)+2\log r\ge -C,\ \text{for } \hat r_k\le r\le \tilde r_k \text{ or } \tilde r_k\le r\le s_k.$$
In either case by $\tilde r_k/\hat r_k\to\infty$ or $s_k/\tilde r_k\to\infty$ we get a contradiction to the uniform bound on the energy. By the same procedure for other components of u^k we see that $l_k\to 0$ can be found to satisfy (3.1). Lemma 3.1 is established. □

Now we continue with the proof of Proposition 3.1.

Case one: $\gamma_i^k\equiv 0$

Choose $l_k\to 0$ such that $\Sigma_k\subset B(0,l_k/2)$, (3.1) holds and
$$\frac{1}{2\pi}\int_{B_{l_k}} h_i^k e^{u_i^k}=\sigma_i+o(1),\quad \text{for } i\in I. \tag{3.5}$$
Indeed, such l_k exists, we can first choose l_k to make (3.5) hold. Then apply Lemma 3.1 (taking l_k as ε_k) we can further assume that l_k satisfies (3.1). Let
$$v_i^k(y)=u_i^k(l_k y)+2\log l_k,\quad i\in I.$$
Then clearly we have
$$\begin{cases} \Delta v_i^k(y)+\sum_{j=1}^n a_{ij}H_j^k(y)e^{v_j^k(y)}=0, & |y|\le 1/l_k,\quad i\in I\\ \bar v_i^k(1)\to -\infty, \end{cases} \tag{3.6}$$
where
$$H_i^k(y)=h_i^k(l_k y),\quad i\in I,\quad |y|\le 1/l_k.$$
The Pohozaev identity we use is
$$\begin{aligned} &\sum_i\int_{B_{\sqrt{R_k}}}(x\cdot\nabla H_i^k)e^{v_i^k}+2\sum_i\int_{B_{\sqrt{R_k}}}H_i^k e^{v_i^k}\\ =\ &\sqrt{R_k}\int_{\partial B_{\sqrt{R_k}}}\sum_i H_i^k e^{v_i^k}+\sqrt{R_k}\int_{\partial B_{\sqrt{R_k}}}\sum_{i,j}\left(a^{ij}\partial_\nu v_i^k\partial_\nu v_j^k-\frac{1}{2}a^{ij}\nabla v_i^k\nabla v_j^k\right) \end{aligned} \tag{3.7}$$
where $R_k\to\infty$ will be chosen later, (a^{ij}) is the inverse matrix of (a_{ij}). The key point of the following proof is to choose R_k properly in order to estimate ∇v_i^k on

$\partial B_{\sqrt{R_k}}$. In the estimate of $\partial B_{\sqrt{R_k}}$, the procedure is to get rid of not important parts and prove that the radial part of ∇v_i^k is the leading term. To estimate all the terms of the Pohozaev identity we first write (3.7) as

$$\mathscr{L}_1+\mathscr{L}_2=\mathscr{R}_1+\mathscr{R}_2+\mathscr{R}_3$$

where $\mathscr{L}_1$ stands for "the first term on the left". Other terms are understood similarly. First we choose $R_k\to\infty$ such that $R_k^{3/2}=o(l_k^{-1})$, then by using $l_k\to 0$ to show that $\mathscr{L}_1=o(1)$. To evaluate $\mathscr{L}_2$, we observe that by Lemma 2.1, $v_i^k(y)\to-\infty$ over all compact subsets of $\mathbb{R}^2\setminus B_{1/2}$. Thus we further require R_k to satisfy

$$\int_{B_{R_k}\setminus B_{3/4}} H_i^k e^{v_i^k}=o(1) \tag{3.8}$$

and for $i\in I$, by (3.6) and Lemma 2.1

$$v_i^k(y)+2\log|y|\to-\infty,\ \text{uniformly in}\quad 1<|y|\le R_k. \tag{3.9}$$

By the choice of l_k we clearly have

$$\frac{1}{2\pi}\int_{B_1} H_i^k e^{v_i^k}=\frac{1}{2\pi}\int_{B_{l_k}} h_i^k e^{u_i^k}=\sigma_i+o(1),\quad i\in I.$$

By (3.8) we have

$$\mathscr{L}_2=4\pi\sum_{i=1}^n\sigma_i+o(1).$$

For $\mathscr{R}_1$ we use (3.9) to have $\mathscr{R}_1=o(1)$.

Therefore we are left with the estimates of $\mathscr{R}_2$ and $\mathscr{R}_3$, for which we shall estimate ∇v_i^k on ∂B_{R_k}. Let

$$G_k(y,\eta)=-\frac{1}{2\pi}\log|y-\eta|+\gamma_k(y,\eta)$$

be the Green's function on $B_{l_k^{-1}}$ with respect to Dirichlet boundary condition. Clearly

$$\gamma_k(y,\eta)=\frac{1}{2\pi}\log\frac{|y|}{l_k^{-1}}\Big|\frac{l_k^{-2}y}{|y|^2}-\eta\Big|$$

and we have

$$\nabla_y\gamma_k(y,\eta)=O(l_k),\quad y\in\partial B_{\sqrt{R_k}},\quad \eta\in B_{l_k^{-1}}. \tag{3.10}$$

We first estimate ∇v_i^k on $\partial B_{R_k^{1/2}}$. By Green's representation formula

$$v_i^k(y)=\int_{B_{l_k^{-1}}} G(y,\eta)\sum_{j=1}^n a_{ij}H_i^k e^{v_j^k}d\eta+H_{ik},$$

where H_{ik} is the harmonic function satisfying $H_{ik} = v_i^k$ on $\partial B_{l_k^{-1}}$. Since $H_{ik} - c_k = O(1)$ for some c_k, $|\nabla H_{ik}(y)| = O(l_k)$.

$$\begin{aligned} \nabla v_i^k(y) &= \int_{B_{l_k^{-1}}} \nabla_y G_k(y,\eta) \sum_{j=1}^n a_{ij} H_j^k e^{v_j^k} d\eta + \nabla H_{ik}(y) \\ &= -\frac{1}{2\pi} \int_{B_{l_k^{-1}}} \frac{y-\eta}{|y-\eta|^2} \sum_{j=1}^n a_{ij} H_j^k e^{v_j^k} d\eta + O(l_k). \end{aligned} \tag{3.11}$$

We estimate the integral in (3.11) over a few subregions. First the integral over $B_{l_k^{-1}} \setminus B_{R_k^{2/3}}$ is $o(1) R_k^{-\frac{1}{2}}$ because over this region $1/|y-\eta| \sim 1/|\eta| \le o(R_k^{-1/2})$. For the integral over B_1, we use

$$\frac{y-\eta}{|y-\eta|^2} = \frac{y}{|y|^2} + O(1/|y|^2)$$

to obtain

$$-\frac{1}{2\pi} \int_{B_1} \frac{y-\eta}{|y-\eta|^2} \sum_{j=1}^n a_{ij} H_j^k e^{v_j^k} = \left(-\frac{y}{|y|^2} + O(1/|y|^2)\right)\left(\sum_{j=1}^n a_{ij}\sigma_j + o(1)\right).$$

This is the leading term. For the integral over region $B(0, \sqrt{R_k}/2) \setminus B_1$, we use $1/|y-\eta| \sim 1/|y|$ and (3.8) to get

$$\int_{B_{R_k^{1/2}/2} \setminus B_1} \frac{y-\eta}{|y-\eta|^2} \sum_{j=1}^n a_{ij} H_j^k e^{v_j^k} = o(1)|y|^{-1}.$$

By similar argument we also have

$$\int_{B_{R_k^{2/3}} \setminus (B_{R_k^{1/2}/2} \cup B(y, \frac{|y|}{2}))} \frac{y-\eta}{|y-\eta|^2} \sum_{j=1}^n a_{ij} H_j^k e^{v_j^k} = o(1)|y|^{-1}.$$

Finally over the region $B(y, \frac{|y|}{2})$ we use $e^{v_i^k(\eta)} = o(1)|\eta|^{-2}$ to get

$$\int_{B(y, \frac{|y|}{2})} \frac{y-\eta}{|y-\eta|^2} \sum_{j=1}^n a_{ij} H_j^k e^{v_j^k} = o(1)|y|^{-1}.$$

Combining the estimates on all the subregions mentioned above we have

$$\nabla v_i^k(y) = \left(-\frac{y}{|y|^2}\right)\left(\sum_{j=1}^n a_{ij}\sigma_j + o(1)\right) + o(|y|^{-1}), \quad |y| = R_k^{\frac{1}{2}}.$$

Using the above in $\mathscr{R}_2$ and $\mathscr{R}_3$ we have

$$\sum_{i,j=1}^n a_{ij}\sigma_i\sigma_j = 4\sum_{i=1}^n \sigma_i + \circ(1).$$

Proposition 3.1 is established for the non-singular case.

Case two: Singular case: $\exists \gamma_i \neq 0$.

Lemma 3.2. *For $\sigma \in (0,1)$, the following Pohozaev identity holds:*

$$\begin{aligned}&\sigma \int_{\partial B_\sigma} \sum_{i,j\in I} a^{ij}\big(\partial_\nu u_i^k \partial_\nu u_j^k - \frac{1}{2}\nabla u_i^k\cdot \nabla u_j^k\big) + \sum_{i\in I}\sigma \int_{\partial B_\sigma} h_i^k e^{u_i^k} \\ = \quad & 2\sum_{i\in I}\int_{B_\sigma} h_i^k e^{u_i^k} + \sum_{i\in I}\int_{B_\sigma}(x\cdot\nabla h_i^k)e^{u_i^k} + 4\pi \sum_{i,j\in I} a^{ij}\gamma_i^k\gamma_j^k.\end{aligned}$$

Proof of Lemma 3.2:
First, we claim that for each fixed k,

$$\nabla u_i^k(x) = 2\gamma_i^k x/|x|^2 + O(1) \quad \text{near the origin.} \tag{3.12}$$

Indeed, let

$$\tilde{u}_i^k(x) = u_i^k(x) - 2\gamma_i^k \log|x|, \quad i\in I. \tag{3.13}$$

We have

$$\Delta \tilde{u}_i^k(x) + \sum_j |x|^{2\gamma_j^k} h_j^k(x) e^{\tilde{u}_j^k(x)} = 0 \quad B_1.$$

By the argument of Lemma 4.1 in [30], for fixed k, $\tilde{u}_i^k$ is bounded above near 0, then elliptic estimate leads to (3.12).

Let $\Omega = B_\sigma \setminus B_\varepsilon$. Then standard Pohozaev identity on Ω is

$$\begin{aligned}&\sum_{i\in I}\left(\int_\Omega (x\cdot\nabla h_i^k)e^{u_i^k} + 2h_i^k e^{u_i^k}\right) \\ = \quad & \int_{\partial\Omega}\left(\sum_i (x\cdot\nu)h_i^k e^{u_i^k} + \sum_{i,j} a^{ij}(\partial_\nu u_j^k(x\cdot\nabla u_i^k) - \frac{1}{2}(x\cdot\nu)(\nabla u_i^k\cdot\nabla u_j^k))\right).\end{aligned}$$

Let $\varepsilon \to 0$, then the integration over Ω extends to B_σ by the integrability of $h_i^k e^{u_i^k}$ and (1.4). For the terms on the right hand side, clearly $\partial\Omega = \partial B_\sigma \cup \partial B_\varepsilon$. Thanks to (3.12), the integral on ∂B_ε is $-4\pi \sum_{i,j} a^{ij}\gamma_i^k\gamma_j^k$. Lemma 3.2 is established. □

Let

$$\sigma_i^k(r) = \frac{1}{2\pi}\int_{B_r} h_i^k e^{u_i^k}, \quad i\in I,$$

then we have

Lemma 3.3. *Let $\varepsilon_k \to 0$ such that $\Sigma_k \subset B(0,\varepsilon_k/2)$ and*

$$u_i^k(x) + 2\log|x| \to -\infty, \quad |x| = \varepsilon_k, \quad i\in I. \tag{3.14}$$

Then we have

$$\sum_{i,j\in I} a_{ij}\sigma_i^k(\varepsilon_k)\sigma_j^k(\varepsilon_k) = 4\sum_{i\in I}(1+\gamma_i^k)\sigma_i^k(\varepsilon_k) + o(1). \tag{3.15}$$

Proof of Lemma 3.3: First the existence of ε_k that satisfies (3.14) is guaranteed by Lemma 2.1. In B_{ε_k}, we let $\tilde{u}_i^k(x)$ be defined as in (3.13). Then

$$v_i^k(y) = \tilde{u}_i^k(\varepsilon_k y) + 2(1+\gamma_i^k)\log\varepsilon_k.$$

Using $v_i^k \to -\infty$ on ∂B_1, we obtain, by Green's representation formula and standard estimates,

$$\nabla v_i^k(y) = (\sum_{j\in I} a_{ij}\sigma_j^k(\varepsilon_k) + o(1))y, \quad y \in \partial B_1.$$

After translating the above to estimates of u_i^k, we have

$$\nabla u_i^k(x) = (\sum_{j\in I} a_{ij}\sigma_j^k(\varepsilon_k) - 2\gamma_j^k)x/|x|^2 + o(1)/|x|, \quad |x| = \varepsilon_k. \tag{3.16}$$

As we observe the Pohozaev identity in Lemma 3.2 with $\sigma = \varepsilon_k$, we see easily that the second term on the LHS and the second term on the RHS are both $o(1)$. The first term on the RHS is clearly $4\pi\sum_i \sigma_i^k(\varepsilon_k)$. Therefore we only need to evaluate the first term on the LHS, for which we use (3.16). Lemma 3.3 is established by similar estimates as in the nonsingular case. □

Proposition 3.1 is established for the singular case as well. □

Remark 3.1. *The proof of Proposition 3.1 clearly indicates the following statements. Let $B(p_k, l_k)$ be a circle centered at p_k with radius l_k. Let Σ_k' be a subset of Σ_k. Suppose $dist(\Sigma_k', \partial B(p_k, l_k)) = o(1) dist(\Sigma_k \setminus \Sigma_k', \partial B(p_k, l_k)$. Then if $p_k = 0$, we have*

$$\sigma_1^k(l_k)^2 - \sigma_1^k(l_k)\sigma_2^k(l_k)^2 + \sigma_2^k(l_k) = 2\mu_1\sigma_1^k(l_k) + 2\mu_2\sigma_2^k(l_k) + o(1).$$

Here we recall that $\mu_i = 1 + \gamma_i$ for $i = 1, 2$. If $0 \in \Sigma_k \setminus \Sigma_k'$, then

$$\tilde{\sigma}_1^k(l_k)^2 - \tilde{\sigma}_1^k(l_k)\tilde{\sigma}_2^k(l_k) + \tilde{\sigma}_2^k(l_k)^2 = 2\tilde{\sigma}_1^k(l_k) + 2\tilde{\sigma}_2^k(l_k) + o(1)$$

where $\tilde{\sigma}_i^k(l_k) = \frac{1}{2\pi}\int_{B(p_k,l_k)} h_i^k e^{u_i^k}$.

4. Fully blown-up systems

Next we consider a typical blowup situation for systems: Fully blown-up solutions. First let $\gamma_i^k \equiv 0$ for all $i \in I$. Let

$$\lambda^k = \max\{\max_{B_1} u_1^k, ..., \max_{B_1} u_n^k\} \tag{4.1}$$

and $x^k \to 0$ be where λ^k is attained. Let

$$v_i^k(y) = u_i^k(x_k + e^{-\frac{1}{2}\lambda^k} y) - \lambda^k, \quad i \in I, \quad y \in \Omega_k \tag{4.2}$$

where $\Omega_k = \{y; \quad e^{-\frac{1}{2}\lambda^k} y + x_k \in B_1\}$. The fully blown-up picture means, along a subsequence

$$\{v_1^k, ..., v_n^k\} \text{ converge in } C^2_{loc}(\mathbb{R}^2) \text{ to } (v_1, ..., v_n) \tag{4.3}$$

that satisfies

$$\Delta v_i + \sum_{j\in I} a_{ij} h_j e^{v_j} = 0, \quad \mathbb{R}^2, \quad i \in I \tag{4.4}$$

where $h_i = \lim_{k\to\infty} h_i^k(0)$. Our next theorem is concerned with the closeness between $u^k = (u_1^k, ..., u_n^k)$ and $v = (v_1, .., v_n)$ if A satisfies the following assumption:

$$A \text{ be a } n\times n \text{ symmetric and invertible matrix,} \qquad (4.5)$$
$$\text{either } a_{ij} \geq 0 \text{ for all } i, j \in I \text{ or } A \text{ is positive definite.}$$

Note that if $A = A_n$, then A satisfies (4.5).

Theorem 4.1. *Let A satisfy (4.5), u^k be a sequence of solutions to (1.2) with $\gamma_i^k = 0, \forall i \in I$. Suppose u^k satisfies (1.3) and (1.5), h^k satisfies (1.4), λ^k, x^k, v^k are described by (4.1), (4.2), respectively. Suppose the fully blown-up picture occurs, then there exists $C > 0$ independent of k such that*

$$|u_i^k(e^{-\frac{1}{2}\lambda^k} y + x^k) - \lambda^k - v_i(y)| \leq C + o(1)\log(1+|y|), \quad \textit{for } x \in \Omega_k,\ i \in I. \qquad (4.6)$$

We note that if A is the Cartan matrix then A satisfies (4.5). If A is nonnegative, i.e. the system is Liouville system, Theorem 4.2 is established in [30].

If $\exists \gamma_i \neq 0$, we let

$$\tilde{u}_i^k(x) = u_i^k(x) - 2\gamma_i^k \log|x|,$$

and

$$\tilde{\lambda}_k = \max\{\frac{\max_{B_1} \tilde{u}_1^k}{(1+\gamma_1^k)}, .., \frac{\max_{B_1} \tilde{u}_n^k}{(1+\gamma_n^k)}\}.$$

Set

$$\tilde{v}_i^k(y) = \tilde{u}_i^k(e^{-\frac{1}{2}\tilde{\lambda}^k} y) - \tilde{\lambda}^k, \quad \text{for} \quad i \in I, \text{ and } y \in \Omega_k := \{y;\ e^{-\frac{1}{2}\tilde{\lambda}^k} y \in B_1\}.$$

We assume

$$\{\tilde{v}_1^k, ..., \tilde{v}_n^k\} \text{ converge in } C^2_{loc}(\mathbb{R}^2) \text{ to } (\tilde{v}_1, ..., \tilde{v}_n) \qquad (4.7)$$

that satisfies

$$\Delta \tilde{v}_i + \sum_{j=1}^n a_{ij}|x|^{2\gamma_j} h_j e^{\tilde{v}_j} = 0 \quad \mathbb{R}^2, \quad i \in I \qquad (4.8)$$

where h_i are limits of $h_i^k(0)$.

Theorem 4.2. *Let A satisfy (4.5), $\tilde{u}^k$, $\tilde{v}^k$, $(\tilde{v}_1, .., \tilde{v}_n)$, $\tilde{\lambda}^k$, ε_k and Ω_k be described as above, h_i^k and γ_i^k satisfy (1.4), then under assumption (4.7) there exists $C > 0$ independent of k such that*

$$|\tilde{u}_i^k(e^{-\frac{1}{2}\tilde{\lambda}^k} y) - \tilde{\lambda}^k - \tilde{v}_i(y)| \leq C + o(1)\log(1+|y|), \quad \textit{for } x \in \Omega_k. \qquad (4.9)$$

Proof of Theorem 4.1:

Recall that σ_i is defined in (1.6). By Proposition 3.1 we have

$$\sum_{i,j\in I} a_{ij}\sigma_i\sigma_j = 4\sum_{i\in I}\sigma_i. \qquad (4.10)$$

On the other hand, let

$$\sigma_{iv} := \frac{1}{2\pi}\int_{\mathbb{R}^2} h_i e^{v_i}, \quad \text{for } i = 1, .., n$$

where $v=(v_1,..,v_n)$ is the limit of the fully blown up sequence after scaling. Clearly $\sigma_v=(\sigma_{1v},...,\sigma_{nv})$ also satisfies (4.10). We claim that

$$\sigma_i=\sigma_{iv},\quad \text{for } i=1,..,n. \tag{4.11}$$

To prove (4.11) let $s_i=\sigma_i-\sigma_{vi}$. Then we have $s_i\ge 0$. The difference between σ and σ_v on (4.10) gives

$$\sum_{i,j\in I}a_{ij}s_is_j+2\sum_{i\in I}(\sum_{j\in I}a_{ij}\sigma_{vj}-2)s_i=0. \tag{4.12}$$

First by Proposition 2.1 we have $\sum_{j\in I}a_{ij}\sigma_{vj}-2>0$. Next if either A is nonnegative ($a_{ij}\ge 0$ for all $i,j=1,..,n$) or A is positive definite, we have $\sum_{i,j\in I}a_{ij}s_is_j\ge 0$. Then (4.12) and $s_i\ge 0$ imply (4.11).

From the convergence from v_i^k to v_i in $C^2_{loc}(\mathbb{R}^2)$ we can find $R_k\to\infty$ such that

$$|v_i^k(y)-v_i(y)|=o(1),\quad |y|\le R_k.$$

For $|y|>R_k$, let

$$\bar{v}_i^k(r)=\frac{1}{2\pi r}\int_{\partial B_r}v_i^k(y)dS_y.$$

Then

$$\frac{d}{dr}\bar{v}_i^k(r)=\frac{1}{2\pi r}\int_{B_r}\Delta v_i^k=-\frac{1}{2\pi r}\int_{B_r}\sum_{j\in I}a_{ij}h_j^ke^{v_j^k}=-\frac{\sum_j a_{ij}\sigma_j+o(1)}{r}.$$

Hence

$$\bar{v}_i^k(r)=-(\sum_{j\in I}a_{ij}\sigma_j+o(1))\log r+O(1),\quad \text{for all } r>2.$$

Since $v_i^k(y)=\bar{v}_i^k(|y|)+O(1)$, we see that Theorem 4.2 for the case $\gamma_i\equiv 0$ ($i\in I$) is established. □

Proof of Theorem 4.2: By (3.15) we have

$$\sum_{i,j\in I}a_{ij}\sigma_i\sigma_j=4\sum_{i\in I}(1+\gamma_i)\sigma_i. \tag{4.13}$$

Recall that $v=(v_1,...,v_n)$ satisfies (4.8). Let

$$\sigma_{iv}=\frac{1}{2\pi}\int_{\mathbb{R}^2}h_i|x|^{2\gamma_i}e^{v_i}.$$

Then one hand, $(\sigma_{1v},...,\sigma_{iv})$ also satisfies (4.13), on the other hand, the integrability of $|x|^{2\gamma_i}e^{v_i}$ implies, by standard estimates (see [10] for example),

$$\sum_{j\in I}a_{ij}\sigma_{jv}>2+2\gamma_i,\quad i\in I. \tag{4.14}$$

Let $s_i=\sigma_i-\sigma_{iv}$ ($i\in I$), then (4.13), which is satisfied by both $(\sigma_1,..,\sigma_n)$ and $(\sigma_{1v},...,\sigma_{nv})$, gives

$$\sum_{i,j\in I}a_{ij}s_is_j+2\sum_{i\in I}(\sum_{j\in J}a_{ij}\sigma_{jv}-2-2\gamma_i)s_i=0.$$

By (4.14) and the assumption on A, we have $s_i=0$ for all $i\in I$. The remaining part of the proof is exactly like the last part of the proof of Theorem 4.1. Theorem 4.2 is established. □

5. Asymptotic behavior of solutions in each simple blowup area

In this section we derive some results on the energy classification around each blowup point. First we let $A = A_n$ (the Cartan Matrix) and consider

The neighborhood around 0.

Since 0 is postulated to belong to Σ_k first, it means there may not be a bubbling picture in a neighborhood of 0.

Let $\tau_k = \frac{1}{2} dist(0, \Sigma_k \setminus \{0\})$ we consider the energy limits of $h_i^k e^{u_i^k}$ in B_{τ_k}. By the selection process and Lemma 2.1,

$$u_i^k(x) + 2\log|x| \le C, \quad u_i^k(x) = \bar{u}_i^k(|x|) + O(1) \quad i \in I, \quad |x| \le \tau_k \tag{5.1}$$

where $\bar{u}_i^k(|x|)$ is the average of u_i^k on $\partial B_{|x|}$. Let $\tilde{u}_i^k$ be defined by (3.13). Then we have

$$\Delta \tilde{u}_i^k(x) + \sum_{j\in I} a_{ij}|x|^{2\gamma_j} h_j^k(x) e^{\tilde{u}_j^k(x)} = 0, \quad |x| \le \tau_k.$$

It is easy to see from (5.1) and the super-harmonicity of the equation above that

$$\max_{B(0,\tau_k)} \tilde{u}_i^k \le \tilde{u}_i^k(0) + C.$$

Indeed, suppose $\max_{\bar{B}(0,\tau_k)} \tilde{u}_i^k$ is attained at $x_k \not\emptyset$. Then the selection process implies

$$\min_{\partial B(0,|x_k|)} \tilde{u}_i^k > \max_{\partial B(0,|x_k|)} \tilde{u}_i^k - C = \tilde{u}_i^k(x_k) - C.$$

On the other hand, the super-harmonicity of $\tilde{u}_i^k$ implies $\tilde{u}_i^k(0) \ge \min_{\partial B(0,|x_k|)} \tilde{u}_i^k$.

Let

$$v_i^k(y) = \tilde{u}_i^k(\delta_k y) + 2(1+\gamma_i^k)\log\delta_k, \quad |y| \le \tau_k/\delta_k \tag{5.2}$$

where

$$-2\log\delta_k = \max_{i\in I} \frac{\tilde{u}_i^k(0)}{1+\gamma_i^k}.$$

Thus the equation for v_i^k is

$$\Delta v_i^k(y) + \sum_{j\in I} a_{ij}|y|^{2\gamma_j^k} h_j^k(\delta_k y) e^{v_j^k(y)} = 0, \quad |y| \le \tau_k/\delta_k.$$

Then we consider two trivial cases. First: $\tau_k/\delta_k \le C$. This is a case that there is no entire bubble after scaling.

Let f_i^k solve

$$\begin{cases} \Delta f_i^k + \sum_{j\in I} a_{ij}|y|^{2\gamma_j^k} h_j^k(\delta_k y) e^{v_j^k} = 0, & |y| \le \tau_k/\delta_k, \\ f_i^k = 0, \quad \text{on} \quad |y| = \tau_k/\delta_k. \end{cases}$$

Clearly $|f_i^k| \le C$ on $B(0,\tau_k/\delta_k)$. Since $v_i^k - f_i^k$ is harmonic and v_i^k has bounded oscillation on $\partial B(0,\tau_k/\delta_k)$, we have

$$v_i^k(x) = \bar{v}_i^k(\partial B(0,\tau_k/\delta_k)) + O(1), \quad \forall x \in B(0,\tau_k/\delta_k) \tag{5.3}$$

where $\bar{v}_i^k(\partial B(0,\tau_k/\delta_k))$ stands for the average of v_i^k on $\partial B(0,\tau_k/\delta_k)$. Direct computation shows that

$$\int_{B(0,\tau_k)} e^{u_i^k(x)}dx = \int_{B(0,\tau_k/\delta_k)} e^{v_i^k(y)}|y|^{2\gamma_i^k}dy.$$

Therefore

$$\int_{B_{\tau_k}} h_i^k e^{u_i^k}dx = O(1)e^{\bar{v}_i^k(\partial B(0,\tau_k/\delta_k))}. \tag{5.4}$$

So if $\bar{v}_i^k(\partial B(0,\tau_k/\delta_k)) \to -\infty$, $\int_{B_{\tau_k}} h_i^k e^{u_i^k}dx = o(1)$.

The second trivial case is the fully blown-up picture. Clearly we now have

$$\tau_k/\delta_k \to \infty \tag{5.5}$$

and we assume that $(v_1^k,...,v_n^k) \to (v_1,...,v_n)$ uniformly over all compact subsets of $\mathbb{R}^2$. Clearly

$$\Delta v_i + \sum_{j=1}^n a_{ij}|x|^{2\gamma_j}h_j e^{v_j} = 0 \quad \mathbb{R}^2, \quad i \in I$$

where $h_i = \lim_{k\to\infty} h_i^k(0)$. By the classification theorem of Lin-Wei-Ye [28], we have

$$\frac{1}{2\pi}\sum_{j\in I} a_{ij}\int_{\mathbb{R}^2}|y|^{2\gamma_j}e^{v_j}h_j dy = 2(2+\gamma_i+\gamma_{n+1-i})$$

and

$$v_i(y) = -(4+2\gamma_{n+1-i})\log|y| + O(1), \quad |y|>1, \quad i\in I.$$

By the proof of Theorem 4.2 that there is only one bubble.

The final case we consider is a partially blown-up picture. Note that (5.5) is assumed. For the following two propositions we assume $n=2$. i.e. we consider $SU(3)$ Toda systems.

Proposition 5.1. *Suppose (1.2), (1.3), (1.4) and (1.5) hold for u^k, h_i^k and γ_i etc. The matrix $A = A_2$. (5.5) also holds. Suppose $s_k \in (0,\tau_k)$ satisfies*

$$u_i^k(x) \le -2\log|x| - N_k, \quad i=1,2$$

for all $|x| = s_k$ and some $N_k \to \infty$. Then $(\sigma_1^k(s_k), \sigma_2^k(s_k))$ is a $o(1)$ perturbation of one of the following five types:

$$(2\mu_1,0), \quad (0,2\mu_2), \quad (2(\mu_1+\mu_2),2\mu_2),$$
$$(2\mu_1, 2\mu_1+\mu_2), \quad (2\mu_1+2\mu_2, 2\mu_1+2\mu_2).$$

On $\partial B(0,\tau_k)$, for each i either

$$u_i^k(x) + 2\log|x| \ge -C, \quad |x| = \tau_k$$

for some $C>0$ or

$$u_i^k(x) + 2\log|x| < -(2+\delta)\log|x| + \delta\log\delta_k, \quad |x| = \tau_k \tag{5.6}$$

for some $\delta > 0$. If (5.6) holds for some i, then

$$\sigma_i^k(\tau_k) = o(1), 2\mu_i + o(1), \text{ or } 2\mu_1 + 2\mu_2 + o(1).$$

Moreover, there exists at least one i_0 such that (5.6) holds for i_0.

Similarly for bubbles away from the origin we have

Proposition 5.2. *Suppose (1.2), (1.3), (1.4) and (1.5) hold for u^k, h_i^k and γ_i etc. The matrix $A = A_2$. Let $x_k \in \Sigma_k \setminus \{0\}, \bar{\tau}_k = \frac{1}{2}dist(x_k, \Sigma_k \setminus \{0, x_k\})$ and*

$$\bar{\delta}_k = exp\left(-\frac{1}{2}\max_{i=1,2; x \in B(x_k, \bar{\tau}_k)} u_i^k(x)\right).$$

Then for all $s_k \in (0, \bar{\tau}_k)$, if

$$u_i^k(x) + 2\log|x - x_k| \le -N_k, \quad \forall i \in I, \quad |x - x_k| = s_k, \quad i = 1,2$$

for some $N_k \to \infty$, then $(\frac{1}{2\pi}\int_{B(x_k,s_k)} h_1^k e^{u_1^k}, \frac{1}{2\pi}\int_{B(x_k,s_k)} h_2^k e^{u_2^k})$ is a $o(1)$ perturbation of one of the following five types:

$$(2,0), (0,2), (2,4), (4,2), (4,4).$$

On $\partial B(x_k, \bar{\tau}_k)$, for each i either

$$u_i^k(x) + 2\log\bar{\tau}_k \ge -C, \quad \forall x \in \partial B(x_k, \bar{\tau}_k)$$

or

$$u_i^k(x) \le -(2+\delta)\log\bar{\tau}_k + \delta\log\bar{\delta}_k, \quad \forall x \in \partial B(x_k, \bar{\tau}_k). \tag{5.7}$$

If (5.7) holds for some i, then $\frac{1}{2\pi}\int_{B(x_k,\bar{\tau}_k)} h_i^k e^{u_i^k}$ is $o(1)$, $2 + o(1)$ or $4 + o(1)$. Moreover, there exists at least one i_0 such that (5.7) holds for i_0.

We shall only prove Proposition 5.1 as the proof for Proposition 5.2 is similar.

Proof of Proposition 5.1:

Let v_i^k be defined by (5.2). Since we only need to consider a partially blown-up situation, without loss of generality we assume v_1^k converges to v_1 in $C^2_{loc}(\mathbb{R}^2)$ and v_2^k tends to minus infinity over any compact subset of $\mathbb{R}^2$. The equation for v_1 is

$$\Delta v_1 + 2h_1|y|^{2\gamma_1}e^{v_1} = 0, \quad \mathbb{R}^2, \quad \int_{\mathbb{R}^2} h_1|y|^{2\gamma_1}e^{v_1} < \infty.$$

where $h_1 = \lim_{k\to\infty} h_1^k(0)$. By the classification result of Prajapat-Tarantello [41] we have

$$2\int_{\mathbb{R}^2} h_1|y|^{2\gamma_1}e^{v_1} = 8\pi\mu_1$$

and

$$v_1(y) = -4\mu_1\log|y| + O(1), \quad |y| > 1.$$

Thus we can find $R_k \to \infty$ (without loss of generality $R_k = o(1)\tau_k/\delta_k$) such that

$$\frac{1}{2\pi}\int_{B_{R_k}} h_1^k(\delta_k y)|y|^{2\gamma_1^k}e^{v_1^k} = 2\mu_1 + o(1),$$

and

$$\int_{B_{R_k}} h_2^k(\delta_k y)|y|^{2\gamma_2^k}e^{v_2^k} = o(1).$$

For $r \geq R_k$, we observe that

$$\sigma_i^k(\delta_k r) = \frac{1}{2\pi}\int_{B_r} h_i^k(\delta_k y)|y|^{2\gamma_i^k} e^{v_i^k} dy$$

and

$$\begin{aligned}
\frac{d}{dr}\bar{v}_1^k(r) &= \frac{-2\sigma_1^k(\delta_k r) + \sigma_2^k(\delta_k r)}{r}, \\
\frac{d}{dr}\bar{v}_2^k(r) &= \frac{\sigma_1^k(\delta_k r) - 2\sigma_2^k(\delta_k r)}{r} \quad R_k \leq r \leq \tau_k/\delta_k.
\end{aligned}$$

Clearly we have

$$R_k \frac{d}{dr}\bar{v}_1^k(R_k) = -4\mu_1 + o(1), \quad R_k \frac{d}{dr}\bar{v}_2^k(R_k) = 2\mu_1 + o(1). \tag{5.8}$$

Lemma 5.1. *Suppose $L_k \in (R_k, \tau_k/\delta_k)$ satisfies*

$$v_i^k(y) + 2\gamma_i^k \log|y| \leq -2\log|y| - N_k, \quad R_k \leq |y| \leq L_k, \quad i = 1,2 \tag{5.9}$$

for some $N_k \to \infty$, then

$$\sigma_i^k(\delta_k R_k) = \sigma_i^k(\delta_k L_k) + o(1), \quad i = 1,2.$$

Proof of Lemma 5.1: We aim to prove that σ_i^k does not change much from $\delta_k R_k$ to $\delta_k L_k$. Suppose this is not the case, then there exists i such that $\sigma_i^k(\delta_k L_k) > \sigma_i^k(\delta_k R_k) + \delta$ for some $\delta > 0$. Let $\tilde{L}_k \in (R_k, L_k)$ such that

$$\max_{i=1,2}(\sigma_i^k(\delta_k \tilde{L}_k) - \sigma_i^k(\delta_k R_k)) = \varepsilon, \quad \forall i = 1,2 \tag{5.10}$$

where $\varepsilon > 0$ is sufficiently small. Then for v_1^k,

$$\frac{d}{dr}\bar{v}_1^k(r) \leq \frac{-4(1+\gamma_1)+\varepsilon}{r} \leq -\frac{2(1+\gamma_1)+\varepsilon}{r}. \tag{5.11}$$

Then it is easy to see from Lemma 2.1 that

$$\int_{B_{\tilde{L}_k}\setminus B_{R_k}} |y|^{2\gamma_1^k} e^{v_1^k} = o(1),$$

which is $\sigma_1^k(\delta_k \tilde{L}_k) = \sigma_1^k(\delta_k R_k) + o(1)$. Indeed, by Lemma 2.1

$$\int_{B_{L_k}\setminus B_{R_k}} |y|^{2\gamma_1^k} e^{v_1^k} = O(1)\int_{B_{L_k}\setminus B_{R_k}} |y|^{2\gamma_1^k} e^{\bar{v}_1^k} = o(1).$$

The second equality above is because by (5.11)

$$\bar{v}_1^k(r) + 2\gamma_1^k \log r \leq -N_k - 2\log R_k + (-2-\varepsilon/2)\log r, \quad R_k \leq r \leq L_k.$$

Thus $\sigma_2^k(\delta_k \tilde{L}_k) = \sigma_2^k(\delta_k R_k) + \varepsilon$. However, since (5.9) holds, by Remark 3.1 we have

$$\lim_{k\to\infty}(\sigma_1^k(\delta_k \tilde{L}_k), \sigma_2^k(\delta_k \tilde{L}_k) \in \Gamma.$$

The two points on Γ that have the first component equal to $2\mu_1$ are $(2\mu_1, 0)$ and $(2\mu_1, 2(\mu_1+\mu_2))$. Thus (5.10) is impossible. Lemma 5.1 is established. $\square$

From Lemma 5.1 and (5.8) we see that for $r \geq R_k$, either

$$v_i^k(y) + 2\gamma_i^k \log|y| \leq -2\log|y| - N_k, \quad R_k \leq |y| \leq \tau_k/\delta_k, \quad i = 1,2 \tag{5.12}$$

or there exists $L_k \in (R_k, \tau_k/\delta_k)$ such that

$$v_2^k(y) + 2\gamma_2^k \log L_k \geq -2\log L_k - C \quad |y| = L_k \tag{5.13}$$

for some $C > 0$, while for $R_k \leq |y| \leq L_k$,

$$v_1^k(y) + 2\gamma_1^k \log|y| \leq -(2+\delta)\log|y|, \quad R_k \leq |y| \leq L_k. \tag{5.14}$$

Indeed, from (5.8) we see that if the energy has to change, σ_2^k has to change first. L_k can be chosen so that $\sigma_2^k(\delta_k L_k) - \sigma_2^k(\delta_k R_k) = \varepsilon$ for some $\varepsilon > 0$ small.

Lemma 5.2. *Suppose there exist $L_k \geq R_k$ such that (5.13) and (5.14) hold. For L_k we assume $L_k = o(1)\tau_k/\delta_k$. Then there exists $\tilde{L}_k = o(1)\tau_k/\delta_k$ and $N_k \to \infty$ such that*

$$v_i^k(y) + 2(1+\gamma_i^k)\log|y| \leq -N_k, \quad |y| = \tilde{L}_k, \quad i = 1,2. \tag{5.15}$$

In particular

$$v_1^k(y) + 2(1+\gamma_1^k + \frac{\delta}{4})\log|y| \leq 0, \quad |y| = \tilde{L}_k. \tag{5.16}$$

$$\sigma_1^k(\delta_k \tilde{L}_k) = 2\mu_1 + o(1), \quad \sigma_2^k(\delta_k \tilde{L}_k) = 2\mu_1 + 2\mu_2 + o(1). \tag{5.17}$$

Proof of Lemma 5.2: First we observe that by Lemma 5.1 the energy does not change if both components satisfy (5.12). Thus we can assume that $\sigma_2^k(\delta_k L_k) \leq \varepsilon$ for some $\varepsilon > 0$ small. Consequently

$$\frac{d}{dr}\bar{v}_1^k(r) \leq \frac{-4(1+\gamma_1)+2\varepsilon}{r}, \quad r \geq R_k.$$

Now we claim that there exists $N > 1$ such that

$$\sigma_2^k(\delta_k(L_k N)) \geq 2 + \gamma_1 + \gamma_2 + o(1). \tag{5.18}$$

If this is not true, we would have $\varepsilon_0 > 0$ and $\tilde{R}_k \to \infty$ such that

$$\sigma_2^k(\delta_k \tilde{R}_k L_k) \leq 2 + \gamma_1 + \gamma_2 - \varepsilon_0. \tag{5.19}$$

On the other hand $\tilde{R}_k$ can be chosen to tend to infinity slowly so that, by Lemma 2.1

$$v_1^k(y) + 2(1+\gamma_1^k)\log|y| \leq -\delta\log|y|, \quad L_k \leq |y| \leq \tilde{R}_k L_k. \tag{5.20}$$

Clearly (5.20) implies $\sigma_1^k(\delta_k L_k) = \sigma_1^k(\delta_k \tilde{R}_k L_k) + o(1)$. Thus by (5.19)

$$\frac{d}{dr}\bar{v}_2^k(r) \geq \frac{-2-2\gamma_2+\varepsilon_0/2}{r}. \tag{5.21}$$

Using (5.21) and

$$v_2^k(y) = (-2-2\gamma_2^k)\log|y| + O(1), \quad |y| = L_k$$

we see easily that

$$\int_{B(0,\tilde{R}_k L_k)\setminus B(0,L_k)} |y|^{2\gamma_2^k} e^{v_2^k} \to \infty,$$

a contradiction to (1.5).Therefore (5.18) holds.

By Lemma 2.1

$$v_i^k(y)+2\log L_k=\bar v_i^k(NL_k)+2\log(NL_k)+O(1),\quad i=1,2,\quad |y|=NL_k.$$

Thus we have

$$\bar v_1^k(NL_k)\le(-2-2\gamma_1^k-\delta/2)\log(NL_k),$$
$$\bar v_2^k(NL_k)\ge(-2-2\gamma_2^k)\log(NL_k)-C.$$

Consequently

$$\bar v_2^k((N+1)L_k)\ge(-2-2\gamma_2^k)\log L_k-C,$$

leads to

$$\int_{B(0,(N+1)L_k)}h_2^k(\delta_k y)|y|^{2\gamma_2^k}e^{v_2^k(y)}dy\ge 2+\gamma_1+\gamma_2+\varepsilon_0$$

for some $\varepsilon_0>0$. Going back to the equation for $\bar v_2^k$ we have

$$\frac{d}{dr}\bar v_2^k(r)\le-\frac{2+2\gamma_2+\varepsilon_0}{r},\quad r=(N+1)L_k.$$

Therefore we can find $\tilde R_k\to\infty$ such that $\tilde R_k L_k=o(1)\tau_k/\delta_k$ and

$$v_2^k(y)\le(-2-2\gamma_2^k-\varepsilon_0)\log|y|-N_k,\quad |y|=\tilde R_k L_k,$$
$$v_1^k(y)\le(-2-2\gamma_1^k-\delta/4)\log|y|,\quad L_k\le|y|\le\tilde R_k L_k.$$

Obviously

$$\sigma_1^k(\delta_k\tilde R_k L_k)=\sigma_1^k(\delta_k L_k)+o(1)=\sigma_1^k(\delta_k R_k)+o(1)=2(1+\gamma_1)+o(1).$$

By computing the Pohozaev identity on $\tilde R_k L_k$ we have

$$\sigma_2^k(\delta_k\tilde R_k L_k)=2\mu_1+2\mu_2+o(1).$$

Letting $\tilde L_k=\tilde R_k L_k$ we have proved Lemma 5.2. □

To finish the proof of Proposition 5.1 we need to consider the region $\tilde L_k\le|y|\le\tau_k/\delta_k$ if $L_k=o(1)\tau_k/\delta_k$ (in which case $\tilde L_k$ can be made as $o(1)\tau_k/\delta_k$), or $L_k=O(1)\tau_k/\delta_k$. First we consider the region $\tilde L_k\le|y|\le\tau_k/\delta_k$ when $\tilde L_k=o(1)\tau_k/\delta_k$. It is easy to verify that

$$\frac{d}{dr}\bar v_1^k(r)=-\frac{2\gamma_1-2\gamma_2}{r}+o(1)/r,\quad r=\tilde L_k,$$
$$\frac{d}{dr}\bar v_2^k(r)=-\frac{6+2\gamma_1+4\gamma_2+o(1)}{r},\quad r=\tilde L_k.$$

The second equation above implies

$$\frac{d}{dr}\bar v_2^k(r)\le-\frac{2\mu_2+\delta}{r},\quad r=\tilde L_k$$

for some $\delta>0$. So $\sigma_2^k(r)$ does not change for $r\ge\tilde L_k$ unless σ_1^k changes. By the same argument as before, either v_1^k rises to $-2\log|y|+O(1)$ on $|y|=\tau_k/\delta_k$ or there is $\hat L_k=o(1)\tau_k/\delta_k$ such that

$$\sigma_i^k(\delta_k\hat L_k)=2\mu_1+2\mu_2+o(1),\quad i=1,2.$$

Since this is the energy of a fully blowup system, we have in this case both

$$v_i^k(y) \le -(2\mu_i+\delta)\log|y|, \quad |y|=\tau_k\delta_k, \quad i=1,2$$

for some $\delta>0$.

If $L_k=O(1)\tau_k/\delta_k$. In this case it is easy to use Lemma 2.1 to see that one component is $-2(1+\gamma_i^k)\log|y|+O(1)$ and the other component has the fast decay. Proposition 5.1 is established. □

6. Combination of bubbling areas

The following definition plays an important role:

Definition 6.1. *Let $Q_k=\{p_1^k,..,p_q^k\}$ be a subset of Σ_k with more than one point in it. Q_k is called a group if*

(1)
$$dist(p_i^k,p_j^k)\sim dist(p_s^k,p_t^k),$$
where p_i^k,p_j^k,p_s^k,p_t^k are any points in Q_k such that $p_i^k\ne p_j^k$ and $p_t^k\ne p_s^k$.

(2) *For any $p_k\in\Sigma_k\setminus Q_k$, $\frac{dist(p_i^k,p_j^k)}{dist(p_i^k,p_k)}\to 0$ for all $p_i^k,p_j^k\in Q_k$ with $p_i^k\ne p_j^k$.*

Proof of Theorem 1.1: Let $2\tau_k$ be the distance between 0 and $\Sigma_k\setminus\{0\}$. For each $z_k\in\Sigma_k\cap\partial B(0,2\tau_k)$, if $dist(z_k,\Sigma_k\setminus\{z_k\})\sim\tau_k$, let G_0 be the group that contains the origin. On the other hand, if there exists $z_k'\in\partial B(0,2\tau_k)$ such that $\tau_k/dist(z_k',\Sigma_k\setminus z_k')\to\infty$ we let G_0 be 0 itself. By the definition of group, all members of G_0 are in $B(0,N\tau_k)$ for some N independent of k. Let

$$\tilde v_i^k(y)=u_i^k(\tau_k y)+2\log\tau_k, \quad |y|\le\tau_k^{-1}.$$

Then we have

$$\Delta\tilde v_i^k(y)+\sum_{j=1}^2 a_{ij}h_j^k(\tau_k y)e^{\tilde v_j^k(y)}=4\pi\gamma_i^k\delta_0, \quad |y|\le\tau_k^{-1}. \tag{6.1}$$

Let 0, Q_1,...,Q_m be the images of members of G_0 after the scaling from y to $\tau_k y$. Then all $Q_i\in B_N$. By Proposition 5.1 and Proposition 5.2 at least one component decays fast on ∂B_1. Without loss of generality we assume

$$\tilde v_1^k\le -N_k \quad \text{on} \quad \partial B_1$$

for some $N_k\to\infty$ and

$$\sigma_1^k(\tau_k)=o(1), 2\mu_1+o(1) \text{ or } 2\mu_1+2\mu_2+o(1).$$

Specifically, if $\tau_k/\delta_k\le C$, $\sigma_1^k(\tau_k)=o(1)$. Otherwise, $\sigma_1^k(\tau_k)$ is equal to the two other cases mentioned above. By Lemma 2.1 $v_1^k\le -N_k+C$ on all $\partial B(Q_t,1)$ ($t=1,...,m$), therefore by Proposition 5.2,

$$\frac{1}{2\pi}\int_{B(Q_t,1)}h_1^k(\tau_k\cdot)e^{\tilde v_1^k}=2m_t+o(1), \quad t=1,...,m$$

where for each t, $m_t=0,1$ or 2. Let $2\tau_k L_k$ be the distance from 0 to the nearest group other than G_0. Then $L_k\to\infty$. By Lemma 2.1 and the proof of Lemma 3.1

we can find $\bar{L}_k \leq L_k$, $\bar{L}_k \to \infty$ such that most of the energy of $\tilde{v}_1^k$ in $B(0,\tilde{L}_k)$ is contributed by bubbles and $\tilde{v}_2^k$ decays faster than $-2\log \tilde{L}_k$ on $\partial B(0,\tilde{L}_k)$:

$$\begin{aligned}
&\frac{1}{2\pi}\int_{B(0,\bar{L}_k)} h_1^k(0)e^{\tilde{v}_1^k} \\
&= \quad 2m+o(1), \quad 2\mu_1+2m+o(1) \quad \text{ or } \quad 2(\mu_1+\mu_2)+2m+o(1)
\end{aligned} \tag{6.2}$$

for some nonnegative integer m, and

$$\tilde{v}_2^k(y)+2\log \tilde{L}_k \to -\infty \quad |y|=\tilde{L}_k. \tag{6.3}$$

Then we evaluate the Pohozaev identity on $B(0,\tilde{L}_k)$. Since (6.3) holds, by Remark 3.1 we have

$$\lim_{k\to\infty}(\sigma_1^k(\tau_k\tilde{L}_k),\sigma_2^k(\tau_k\tilde{L}_k)) \in \Gamma.$$

Moreover, by (6.2) we see that $\lim_{k\to\infty}(\sigma_1^k(\tau_k\tilde{L}_k),\sigma_2^k(\tau_k\tilde{L}_k)) \in \Sigma$ because the limit point is the intersection between the line $\sigma_1 = \lim_{k\to\infty}\sigma_1^k(\tau_k\tilde{L}_k)$ with Γ.

The Pohozaev identity for $(\sigma_1^k(\tau_k\tilde{L}_k),\sigma_2^k(\tau_k\tilde{L}_k))$ can be written as

$$\begin{aligned}
&\sigma_1^k(\tau_k\tilde{L}_k)(2\sigma_1^k(\tau_k\tilde{L}_k)-\sigma_2^k(\tau_k\tilde{L}_k)-4\mu_1) \\
&+\sigma_2^k(\tau_k\tilde{L}_k)(2\sigma_2^k(\tau_k\tilde{L}_k)-\sigma_1^k(\tau_k\tilde{L}_k)-4\mu_2)=o(1).
\end{aligned}$$

Thus either

$$2\sigma_1^k(\tau_k\tilde{L}_k)-\sigma_2^k(\tau_k\tilde{L}_k) \geq 4\mu_1+o(1) \tag{6.4}$$

or

$$2\sigma_2^k(\tau_k\tilde{L}_k)-\sigma_1^k(\tau_k\tilde{L}_k) \geq 4\mu_2+o(1).$$

Moreover, if

$$2\sigma_1^k(\tau_k\tilde{L}_k)-\sigma_2^k(\tau_k\tilde{L}_k) \geq 2\mu_1+o(1) \text{ and } 2\sigma_2^k(\tau_k\tilde{L}_k)-\sigma_1^k(\tau_k\tilde{L}_k) \geq 2\mu_2+o(1),$$

by the proof of Theorem 4.2

$$\int_{B_{l_k}\setminus \tau_k\tilde{l}_k} h_i^k e^{u_i^k} = o(1), \quad i=1,2$$

for any $l_k \to 0$. In this case we have

$$\sigma_i = \lim_{k\to\infty}\sigma_i^k(\tau_k\tilde{L}_k), \quad i=1,2$$

and Theorem 1.1 is proved in this case.

Thus without loss of generality we assume that (6.4) holds. From the equation for u_1^k, this means

$$\bar{u}_1^k(\tau_k\tilde{L}_k) \leq -2\log(\tau_k\tilde{L}_k)-N_k, \quad \frac{d}{dr}\bar{u}_1^k(r) < (-2-\delta)/r, \quad r=\tau_k\tilde{L}_k. \tag{6.5}$$

The property above implies, by the proof of Proposition 5.1, that as r grows from $\tau_k\tilde{L}_k$ to $\tau_k L_k$, the following three situations may occur:

Case one: Both u_i^k satisfy, for some $N_k \to \infty$, that

$$u_i^k(x)+2\log|x| \leq -N_k, \quad \tau_k\tilde{L}_k \leq |x| \leq \tau_k L_k, \quad i=1,2.$$

In this case

$$\sigma_i^k(\tau_k\tilde{L}_k) = \sigma_i^k(\tau_k L_k)+o(1), \quad i=1.2$$

So on $\partial B(0,\tau_k L_k)$, u_1^k is still a fast decaying component.

Case two: There exist $L_{1,k}, L_{2,k} \in (\tilde{L}_k, L_k)$ such that

$$u_2^k(x) \geq -2\log L_{1,k} - C \quad |x| = \tau_k L_{1,k},$$

$$u_i^k(x) \leq -2\log L_{2,k} - N_k \quad |x| = \tau_K L_{2,K}, \quad i=1,2 \tag{6.6}$$

and

$$\sigma_1^k(\tau_k \tilde{L}_k) = \sigma_1^k(\tau_k L_{2,k}) + o(1). \tag{6.7}$$

Since (6.6) holds, by Remark 3.1, $(\lim_{k\to\infty}\sigma_1^k(\tau_k L_{2,k}), \lim_{k\to\infty}\sigma_2^k(\tau_k L_{2,k})) \in \Gamma$. Then we further observe that since (6.7) holds, $\lim_{k\to\infty}(\sigma_1^k(\tau_k L_{2,k}), \sigma_2^k(\tau_k L_{2,k})) \in \Sigma$ because this point is obtained by intersecting Γ with $\sigma_1 = \lim_{k\to\infty}\sigma_1^k(\tau_k \tilde{L}_k)$. In another word, the new point $\lim_{k\to\infty}(\sigma_1^k(\tau_k L_{2,k}), \sigma_2^k(\tau_k L_{2,k}))$ is on the upper right part of the old point $\lim_{k\to\infty}(\sigma_1^k(\tau_k \tilde{L}_k), \sigma_2^k(\tau_k \tilde{L}_k))$.

Case three:

$$u_2^k(x) \geq -2\log \tau_k L_k - C, \quad |x| = \tau_k L_k$$

for some $C>0$ and $\sigma_1^k(\tau_k \tilde{L}_k) = \sigma_1^k(\tau_k L_k) + o(1)$. This means at $\partial B(0,\tau_k L_k)$, u_1^k is still the fast decaying component.

If the second case above happens, the discussion of the relationship between σ_1^k and σ_2^k on $B(0,\tau_k L_k)\setminus B(0,\tau_k L_{2,k})$ is the same as before. In any case on $\partial B(0,\tau_k L_k)$ at least one of the two components has fast decay and has its energy equal to a corresponding component of a point in Σ. For any group not equal to G_0, it is easy to see that the fast decay component has its energy equal to 0, 2 or 4. The combination of bubbles for groups is very similar to the combination of bubbling disks as we have done before. For example, let $G_0, G_1, ..., G_t$ be groups in $B(0,\varepsilon_k)$ for some $\varepsilon_k \to 0$. Suppose the distance between any two of $G_0,..,G_t$ are comparable and

$$dist(G_i,G_j) = o(1)\varepsilon_k, \quad \forall i,j=0,..,t, \quad i\neq j.$$

Also we require $(\Sigma_k \setminus (\cup_{i=0}^t G_i)) \cap B(0,2\varepsilon_k) = \emptyset$. Let $\varepsilon_{1,k} = dist(G_0,G_1)$, then all $G_0,...,G_t$ are in $B(0,N\varepsilon_{1,k})$ for some $N>0$. Without loss of generality let u_1^k be a fast decaying component on $\partial B(0,N\varepsilon_{1,k})$. Then we have

$$\sigma_1^k(N\varepsilon_{1,k}) = \sigma_1^k(\tau_k L_k) + 2m + o(1)$$

where m is a nonnegative integer because by Lemma 2.1, u_1^k is also a fast decaying component for $G_1,...,G_t$. Moreover, by Proposition 5.2, the energy of u_1^k in G_s ($s=1,...,t$) is $o(1), 2+o(1)$ or $4+o(1)$. If u_2^k also has fast decay on $\partial B(0,N\varepsilon_{1,k})$, then $\lim_{k\to\infty}(\sigma_1^k(N\varepsilon_{1,k}), \sigma_1^k(N\varepsilon_{1,k})) \in \Sigma$ because this is a point of intersection between Γ and $\sigma_1 = \lim_{k\to\infty}\sigma_1^k(\tau_k L_k) + 2m$. If

$$u_2^k(x) \geq -2\log N\varepsilon_{1,k} - C, \quad |x| = N\varepsilon_{1,k},$$

then as before we can find $\varepsilon_{3,k}$ in $(N\varepsilon_{1,k}, \varepsilon_k)$ such that, for some $N_k \to \infty$,

$$u_i^k(x) + 2\log \varepsilon_{3,k} \leq -N_k, \quad i=1,2, \quad |x| = \varepsilon_{3,k}$$

and

$$\sigma_1^k(N\varepsilon_{1,k}) = \sigma_1^k(\varepsilon_{3,k}).$$

Thus we have

$$\lim_{k\to\infty}(\sigma_1^k(\varepsilon_{3,k}),\sigma_2^k(\varepsilon_{3,k}))\in\Sigma.$$

because this point is the intersection between Γ and $\sigma_1=\lim_{k\to\infty}\sigma_1^k(N\varepsilon_{1,k})$.

The last possibility on $B(0,\varepsilon_k)\setminus B(0,\varepsilon_{1,k})$ is

$$\sigma_1^k(\varepsilon_k)=\sigma_1^k(N\varepsilon_{1,k})+o(1)$$

and

$$u_2^k(x)+2\log\varepsilon_k\geq -C,\quad |x|=\varepsilon_k.$$

In this case u_1^k is the fast decaying component on $\partial B(0,\varepsilon_k)$.

Such a procedure can be applied to include groups further away from G_0. Since we have only finite blowup disks this procedure only needs to be applied finite times. Finally let $s_k\to 0$ such that

$$\sigma_i=\lim_{k\to\infty}\lim_{s_k\to 0}\sigma_i^k(s_k),\quad i=1,2$$

and, for some $N_k\to\infty$,

$$u_i^k(x)+2\log s_k\leq -N_k,\quad i=1,2,\quad |x|=s_k.$$

Then we see that $(\sigma_1,\sigma_2)\in\Sigma$. Theorem 1.1 is established. □

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